Chebyshev Approximation

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This document describes Chebyshev polynomials of the first kind and function approximation using these polynomials.

1 Chebyshev Polynomials

Chebyshev polynomial of the first kind of degree n is defined as

$$T_n(x) = \cos(n \arccos(x)) \tag{1}$$

for $x \in [-1, 1]$. Hence

$$T_0 = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, \ldots$$

The polynomials follows the recurrence relationship

$$T_{n+1} = 2xT_n - T_{n-1} \tag{2}$$

Proof

$$T_{n+1}(x) = \cos((n+1)\arccos(x)) = \cos(n \arccos(x))x - sin(n \arccos(x))sin(\arccos(x))$$

$$T_{n-1}(x) = \cos((n-1)\arccos(x)) = \cos(n \arccos(x))x + sin(n \arccos(x))sin(\arccos(x))$$

Summing up the two equations, we have

$$T_{n+1}(x) + T_{n-1}(x) = 2x\cos(n \arccos(x)) = 2xT_n(x)$$

Hence

$$T_{n+1} = 2xT_n - T_{n-1}$$

Q.E.D.

1.1 Zeros and Extrema

It is easy to see T_n has n zeros at

$$\bar{x}_k = \cos\left(\frac{(k - \frac{1}{2})\pi}{n}\right) \tag{3}$$

for $k = 1, \ldots, n$ and $n + 1$ extrema (maxima and minima) at

$$\hat{x}_k = \cos\left(\frac{k\pi}{n}\right) \tag{4}$$

for $k = 0, \ldots, n$.

1.2 Orthogonality

Chebyshev polynomials are orthogonal with respect to weight function $\sqrt{1-x^2}$ on the interval $[-1,1]$.

Theorem 1

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & i=j=0 \\ \frac{\pi}{2} & i=j>0 \\ 0 & i \neq j \end{cases} \tag{5}$$

Proof For $i \neq j$, we shall show that

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = 0 \tag{6}$$

Let $y = \arccos(x)$, then

$$dx = -\sqrt{1-x^2}dy$$

Hence

$$\begin{aligned} \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx &= \int_0^\pi \cos(iy)\cos(jy)dy \\ &= \frac{1}{2}\int_0^\pi \left[\cos((i+j)y) + \cos((i-j)y)\right] dy \\ &= \frac{1}{2}\left[\frac{sin((i+j)y)}{i+j} + \frac{sin((i-j)y)}{i-j}\right]\Bigg|_{y=0}^{\pi} \\ &= 0 \end{aligned}$$

When $i = j = 0$,

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \int_0^\pi 1dy = \pi$$

When $i = j \neq 0$,

$$\begin{aligned} \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx &= \int_0^\pi (\cos(iy))^2 dy \\ &= \frac{1}{2}\int_0^\pi (\cos(2iy)+1)\, dy \\ &= \frac{\pi}{2} \end{aligned}$$

Q.E.D.

Chebyshev polynomials are also orthogonal at discrete sets of zeros and extrema. The following two lemmas will be useful in the proof of discrete orthogonality.

Lemma 2

$$\sum_{k=1}^{n+1} \cos(k-\frac{1}{2})\theta = \begin{cases} n+1 & \theta = 0 \\ \frac{\sin(n+1)\theta}{2\sin(\frac{\theta}{2})} & 0 < \theta < 2\pi \\ -(n+1) & \theta = 2\pi \end{cases} \tag{7}$$

Proof

It is trivial for $\theta = 0$ or 2π. Now we assume $0 < \theta < 2\pi$, let $z = e^{\theta i}$ where i is the symbol for imaginary number.

Note that

$$\textstyle\sum_{k=1}^{n+1} \cos(k-\frac{1}{2})\theta \quad = \quad \Re\left\{z^{\frac{1}{2}} + z^{\frac{3}{2}} + \ldots + z^{(n+\frac{1}{2})}\right\}$$

where $\Re(\cdot)$ indicates the real part of a complex number.

$$\begin{aligned} z^{\frac{1}{2}} + e^{\frac{3}{2}} + \ldots + z^{(n+\frac{1}{2})z} &= z^{\frac{1}{2}}\left(1 + z + \ldots z^n\right) \\ &= \tfrac{z^{n+1}-1}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}} \\ &= \tfrac{(\cos(n+1)\theta-1)+\sin(n+1)\theta i}{2\sin(\frac{\theta}{2})i} \end{aligned}$$

Hence

$$\Re\left\{z^{\frac{1}{2}} + z^{\frac{3}{2}} + \ldots + z^{(n+\frac{1}{2})}\right\} = \frac{\sin(n+1)\theta}{2\sin(\frac{\theta}{2})}$$

and it follows that

$$\sum_{k=1}^{n+1} \cos(k-\frac{1}{2})\theta = \frac{\sin(n+1)\theta}{2\sin(\frac{\theta}{2})}$$

for $0 < \theta < 2\pi$. Q.E.D.

Lemma 3

$$\textstyle\sum_{k=0}^{\circ\, n} \cos k\theta = \begin{cases} n & \theta = 0 \;\; or \;\; 2\pi \\ \frac{\sin(n\theta)\cos(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})} & 0 < \theta < 2\pi \end{cases} \tag{8}$$

Remark 4 *$\overset{\circ}{\sum}$ denotes that both the first and last terms are to be halved. That is*

$$\textstyle\sum_{k=0}^{\circ\, n} \cos k\theta = \frac{1}{2}\cos(0) + \cos(\theta) + \ldots + \cos(n-1)\theta + \frac{1}{2}\cos(n\theta)$$

Proof

$$\textstyle\sum_{k=0}^{\circ\, n} \cos k\theta = \displaystyle\sum_{k=0}^{n} \cos k\theta - \frac{1}{2}(1 + \cos(n\theta))$$

It is easy to see that when $\theta = 0$ or 2π,

$$\textstyle\sum_{k=0}^{\circ\, n} \cos k\theta = n$$

Now we assume $0 < \theta < 2\pi$. Let $z = e^{\theta i}$, then

$$\begin{aligned} \textstyle\sum_{k=0}^{n} \cos k\theta &= \Re\left\{1 + z + \ldots + z^n\right\} \\ &= \Re\left\{\tfrac{z^{n+1}-1}{z-1}\right\} \\ &= \Re\left\{\tfrac{z^{n+\frac{1}{2}}-z^{-\frac{1}{2}}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}\right\} \\ &= \Re\left\{\tfrac{\left(\cos(n+\frac{1}{2})\theta-\cos(\frac{1}{2}\theta)\right)+\left(\sin(n+\frac{1}{2})\theta+\sin(\frac{1}{2}\theta)\right)i}{2\sin(\frac{\theta}{2})i}\right\} \\ &= \tfrac{\sin(n+\frac{1}{2})\theta+\sin(\frac{1}{2}\theta)}{2\sin(\frac{\theta}{2})} \\ &= \tfrac{\sin(n\theta)\cos(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})} + \tfrac{\cos(n\theta)+1}{2} \end{aligned}$$

It then follows that

$$\sum_{k=0}^{\circ\, n} \cos k\theta = \frac{\sin(n\theta)\cos(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})}$$

for $0 < \theta < 2\pi$. Q.E.D.

Now we state the discrete orthogonality of Chebyshev polynomials for zeroes.

Theorem 5

$$\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k) = \begin{cases} n+1 & i=j=0 \\ 0 & i \neq j, i, j \leq n \\ \frac{1}{2}(n+1) & 0 < i = j \leq n \end{cases} \tag{9}$$

where $\bar{x}_k$ are zeros of T_{n+1}.

Proof Let

$$\theta_k = \frac{(k-\frac{1}{2})\pi}{n+1}$$

then

$$\begin{aligned} \textstyle\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k) &= \textstyle\sum_{k=1}^{n+1} \cos(i\theta_k)\cos(j\theta_k) \\ &= \textstyle\frac{1}{2}\sum_{k=1}^{n+1} [\cos((i+j)\theta_k) + \cos((i-j)\theta_k)] \\ &= \textstyle\frac{1}{2}\sum_{k=1}^{n+1} \cos((i+j)\theta_k) + \frac{1}{2}\sum_{k=1}^{n+1} \cos((i-j)\theta_k) \end{aligned}$$

When $i = j = 0$, it is easy to see that

$$\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k) = n+1$$

When $0 < i = j \leq n$, $0 < \frac{(i+j)\pi}{n+1} < 2\pi$. From Lemma (2), we have

$$\sum_{k=1}^{n+1} \cos((i+j)\theta_k) = \sum_{k=1}^{n+1} \cos\left(\frac{(i+j)\pi}{n+1}(k-\frac{1}{2})\right) = \frac{\sin(i+j)\pi}{2\sin\left(\frac{(i+j)\pi}{2(n+1)}\right)} = 0$$

Since

$$\frac{1}{2}\sum_{k=1}^{n+1} \cos((i-j)\theta_k) = \frac{1}{2}(n+1)$$

we have

$$\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k) = \frac{1}{2}(n+1)$$

for the case $0 < i = j \leq n$.

When $i \neq j$ and $i, j \leq n$, from Lemma (2), we have

$$\frac{1}{2}\sum_{k=1}^{n+1} [\cos((i+j)\theta_k) + \cos((i-j)\theta_k)] = \frac{1}{2}\left[\frac{\sin(i+j)\pi}{2\sin\left(\frac{(i+j)\pi}{2(n+1)}\right)} + \frac{\sin(i-j)\pi}{2\sin\left(\frac{(i-j)\pi}{2(n+1)}\right)}\right] = 0$$

Q.E.D.

Theorem 6

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) = \begin{cases} n & i=j=0 \text{ } or \text{ } n \\ 0 & i \neq j, i, j \leq n \\ \frac{n}{2} & 0 < i = j < n \end{cases} \tag{10}$$

where $\hat{x}_k$ are extrema of T_n.

Proof

Note that

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) = \sideset{}{^{\circ}}\sum_{k=0}^{n} \cos(\tfrac{ik\pi}{n})\cos(\tfrac{jk\pi}{n})$$

It is easy to see that when $i = j = 0$ or n,

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) = n$$

To prove the other two cases, we note that

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{ik\pi}{n}\right)\cos\left(\frac{jk\pi}{n}\right) = \frac{1}{2}\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i+j)k\pi}{n}\right) + \frac{1}{2}\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i-j)k\pi}{n}\right)$$

When $0 < i + j < 2n$, from Lemma (3)

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i+j)k\pi}{n}\right) = \frac{\sin((i+j)\pi)\cos(\frac{(i+j)\pi}{2n})}{2\sin(\frac{(i+j)\pi}{2n})} = 0$$

When $i \neq j$

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i-j)k\pi}{n}\right) = \frac{\sin((i-j)\pi)\cos(\frac{(i-j)\pi}{2n})}{2\sin(\frac{(i-j)\pi}{2n})} = 0$$

and when $i = j$

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i-j)k\pi}{n}\right) = n$$

When $i \neq j, i, j \leq n$, we must have $0 < i + j < 2n$ and $i \neq j$, hence

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i+j)k\pi}{n}\right) = 0$$

and

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} \cos\left(\frac{(i-j)k\pi}{n}\right) = 0$$

It then follows

$$\sideset{}{^{\circ}}\sum_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) = 0$$

when $i \neq j, i, j \leq n$.

When $0 < i = j < n$, we must have $0 < i + j < 2n$ and $i = j$, then

$$\sum_{k=0}^{\circ\, n} \cos\left(\frac{(i+j)k\pi}{n}\right) = 0$$

and

$$\sum_{k=0}^{\circ\, n} \cos\left(\frac{(i-j)k\pi}{n}\right) = n$$

It follows that

$$\sum_{k=0}^{\circ\, n} T_i(\hat{x}_k)T_j(\hat{x}_k) = \frac{n}{2}$$

Q.E.D.

2 Chebyshev Approximation

Chebyshev polynomials can be used as basis for function approximation. Given a function f, our goal is to approximate f with a linear combination of Chebyshev polynomials, i.e.,

$$f(x) \approx \sum_{i=0}^{n} c_i T_i(x)$$

The criteria to choose c_i is to match f at a certain set of fixed points. There are two options we have for the choice of points: zeros and extrema. In the following two subsections, we discuss the two approaches.

2.1 Function Approximation with Zeros

In this approach, we match $f(x)$ at the zeros $\bar{x}_k$ of T_{n+1} for $k = 1, \ldots, n+1$. That is

$$f(\bar{x}_k) = \sum_{i=0}^{n} c_i T_i(\bar{x}_k) \tag{11}$$

for $k = 1, \ldots, n+1$.

To solve c_i, we multiply both sides of the above equation by $T_j(\bar{x}_k), 0 \leq j \leq n$ and sum over k

$$\begin{aligned} \textstyle\sum_{k=1}^{n+1} f(\bar{x}_k)T_j(\bar{x}_k) &= \textstyle\sum_{k=1}^{n+1} \left(\sum_{i=0}^{n} c_i T_i(\bar{x}_k)T_j(\bar{x}_k)\right) \\ &= \textstyle\sum_{i=0}^{n} c_i \left(\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k)\right) \end{aligned}$$

From Theorem (5), the RHS is

$$\sum_{i=0}^{n} c_i \left(\sum_{k=1}^{n+1} T_i(\bar{x}_k)T_j(\bar{x}_k)\right) = \begin{cases} (n+1)c_j & j = 0 \\ \frac{n+1}{2} c_j & j > 0 \end{cases}$$

Combining with the fact that $T_0 = 1$, we have

$$c_j = \begin{cases} \frac{1}{n+1}\sum_{k=1}^{n+1} f(\bar{x}_k) & j = 0 \\ \frac{2}{n+1}\sum_{k=1}^{n+1} f(\bar{x}_k)T_j(\bar{x}_k) & j > 0 \end{cases}$$

where

$$T_j(\bar{x}_k) = \cos\left(\frac{j(k-\frac{1}{2})\pi}{n+1}\right)$$

2.2 Function Approximation with Extrema

Similar to approximation with zeros, we can also choose to match f at extrema. Since there are $n+1$ extrema for T_n, we match f at extrema $\hat{x}_k$ of T_n for $k = 0, \ldots, n$. That is, we require

$$f(\hat{x}_k) = \sum_{i=0}^{n} c_i T_i(\hat{x}_k) \tag{12}$$

for $k = 0, \ldots, n$.

For any $0 \le j \le n$, multiply the above equation by $T_j(\hat{x}_k)$ and sum over k, we have

$$\begin{aligned} \mathop{{\sum}^{\circ}}_{k=0}^{n} f(\hat{x}_k)T_j(\hat{x}_k) &= \mathop{{\sum}^{\circ}}_{k=0}^{n}\sum_{i=0}^{n} c_i T_i(\hat{x}_k)T_j(\hat{x}_k) \\ &= \sum_{i=0}^{n} c_i \mathop{{\sum}^{\circ}}_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) \end{aligned}$$

[1] From Theorem (6),

$$\sum_{i=0}^{n} c_i \mathop{{\sum}^{\circ}}_{k=0}^{n} T_i(\hat{x}_k)T_j(\hat{x}_k) = \begin{cases} n a_j & j = 0 \text{ or } n \\ \frac{n}{2} a_j & 0 < j < n \end{cases}$$

Hence

$$a_j = \begin{cases} \frac{1}{n}\mathop{{\sum}^{\circ}}_{k=0}^{n} f(\hat{x}_k)T_j(\hat{x}_k) & j = 0 \text{ or } n \\ \frac{2}{n}\mathop{{\sum}^{\circ}}_{k=0}^{n} f(\hat{x}_k)T_j(\hat{x}_k) & 0 < j < n \end{cases}$$

where

$$T_j(\hat{x}_k) = \cos\left(\frac{jk\pi}{n}\right)$$

To be more explicit

$$a_j = \begin{cases} \frac{1}{n}\mathop{{\sum}^{\circ}}_{k=0}^{n} f(\hat{x}_k) & j = 0 \\ \frac{2}{n}\mathop{{\sum}^{\circ}}_{k=0}^{n} f(\hat{x}_k)\cos\left(\frac{jk\pi}{n}\right) & 0 < j < n \\ \frac{1}{n}\mathop{{\sum}^{\circ}}_{k=0}^{n} (-1)^k f(\hat{x}_k) & j = n \end{cases}$$

[1] Note that in the summation, the first and last term are halved.

3 Clenshaw Algorithm

Once the coefficients c_i are calculated, in principle we can evaluate the approximation at any point x using the formula

$$f(x) \approx \sum_{i=0}^{n} c_i T_i(x)$$

Clenshaw algorithm provides a more efficient and numerically stable way for it. The algorithm is based on the recurrence relation of Chebyshev polynomials

$$T_{n+1} = 2xT_n - T_{n-1}$$

Note that

$$\begin{aligned}
f(x) &\approx c_0T_0 + c_1T_1 \ldots + c_{n-2}T_{n-2} + c_{n-1}T_{n-1} + c_nT_n \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-2} - c_n)T_{n-2} + (c_{n-1} + 2xc_n)T_{n-1} \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-2} - c_n)T_{n-2} + b_{n-1}T_{n-1} \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-3} - b_{n-1})T_{n-3} + (c_{n-2} + 2xb_{n-1} - c_n)T_{n-2} \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-3} - b_{n-1})T_{n-3} + b_{n-2}T_{n-2} \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-4} - b_{n-2})T_{n-4} + (c_{n-3} + 2xb_{n-2} - b_{n-1})T_{n-3} \\
&= c_0T_0 + c_1T_1 \ldots + (c_{n-4} - b_{n-2})T_{n-4} + b_{n-3}T_{n-3}
\end{aligned}$$

where we define

$$b_{n-1} = c_{n-1} + 2xc_n$$

$$b_{n-2} = c_{n-2} + 2xb_{n-1} - c_n$$

and

$$b_{n-3} = c_{n-3} + 2xb_{n-2} - b_{n-1}$$

Note that if we define $b_{n+1} = 0$ and $b_n = a_n$, then

$$b_i = c_i + 2xb_{i+1} - b_{i+2}$$

for $i = n-3, n-2, n-1$.

If we continue this process and have

$$f(x) \approx (c_0 - b_2)T_0 + b_1T_1 = c_0 - b_2 + b_1x$$

where b_i is obtained recursively from

$$b_i = c_i + 2xb_{i+1} - b_{i+2}$$

for $i = n-1, n-2, \ldots, 1$.

We summarize Clenshaw algorithm below

1. Set $b_{n+1} = 0$ and $b_n = c_n$
2. Compute $b_i = c_i + 2xb_{i+1} - b_{i+2}$ for $i = n-1, n-2, \ldots, 1$
3. $f(x) \approx c_0 + b_1x - b_2$

References

[1] J. C. Mason, D. Handscomb: *Chebyshev Polynomials*, CHAPMAN & HALL/CRC