Problem Chosen	2025 MCM/ICM Summary Sheet	Team Control Number
A		2525164

The Work of Resilience: The Wear of Stairs

Summary

In human society, stone and some other materials are widely used to make the stair treads due to their durability and steadfast permanence. Though they are hard and durable, they will gradually wear down, with long period use by humans. Hence, the stair treads will be renovated.

In this paper, several reasonable assumptions are made in order to simplify the problems. Based on these assumptions, we established two main models including Stair Usage Frequency Estimation Model and Wear Factors Model. Furthermore, multiple methods such as EWM, K-S Test, Distribution fitting test (DFT), etc. are introduced to solve the problems.

For problem 1, to estimate the usage frequency of stairs, we build the **Stair Usage Frequency Estimation Model**, which considers wear depth, friction coefficient, etc.. By combining **Archard Wear Equation** and **Least Error Method**, we optimize the parameters of above factors. Then, we use parameters from **ANSYS** to measure the usage frequency, finding that frequency is positively related to wear depth, while negatively related to friction coefficient.

For problem 2, on purpose of studying the direction preference when people use the stairs, we adapt the **Gaussian Distribution** to simulate the wear depth at different positions. The result shows wear in upward preference condition concentrates at the fore side, while in downward condition is converse and two-direction favored condition exhibits widely distribution. **For problem 3**, based on the Gaussian Distribution mentioned above, we use the **K-S Test** and **DFT** to calculate the columns of tested distribution, further determining the number of users.

For problem 4, we establish the **Wear Factors Model**, which primarily considers the influence of flow density, wear rate, etc. on wear depth. Double integral is used to calculate the result and we compare it with the actual values to judge the consistency. **For problem 5**, firstly, we get a function about age from the Wear Factors Model so the age can be inferred. Next, for the reliability of the result, we used the error propagation formula to estimate the confidence interval and test it at 95% confidence. **For problem 6**, to determine whether renovations have been conducted, we can compare the result from problem 4 with actual value, if the former is smaller, renovations are considered to be conducted and vice versa.

For problem 7, to determine the source of the materials, we construct an evaluation function by **EWM** and use a statistical test to judge the consistency of source. Then, we use the Tetearing Equation to estimate the wood age. **For problem 8**, firstly, we derive the number of people passing through the a microelement from the Wear Factors Model, then we integrate it over the stair surface to obtain the pedestrian number. To determine the using intensity type, we modify the Archard Wear Equation and introduce contribution parameters. By visualizing the equation, we obtain that the wear of “short time but high intensity” is more severe.

After modelling, we also conduct sensitivity analysis, which reveals our model’s robustness to some parameters. We finally summarize our strengths and weaknesses.

Keywords: Stair treads; Wear degree; Archard wear equation; Usage frequency; Gaussian distribution

Contents

1 Introduction

1.1 Background

Due to the durable and permanent nature of stone, it is often used as a building material, particularly the material for stair treads.

Although stair treads made of stone or other materials exhibit excellent durability, over long-term use, they are not impervious to wear, causing the central part of the stair treads to be severely worn down and the edges to become curved. After the severe wear, the stairs will usually be renovated, after which they begin to wear again, creating a cycle of wear and renovation.

1.2 Restatement of the Problems

Considering the background information of the given stairs, we are supposed to build models to address the following problems:

- **Problem I:** What was the frequency of the usage of the stairs?
- **Problem II:** Was a certain direction favored by people such as: upward direction and downward direction, when using the stairs?
- **Problem III:** What's the number of people using the stairs at the same time?
- **Problem IV:** Is the wear consistent with the information available?
- **Problem V:** How many years the stairs have been used, and how reliable is this year number?
- **Problem VI:** Have the stairs ever been renovated?
- **Problem VII:** Can the source and some specific information of stairs treads materials be determined?
- **Problem VIII:** What information can be determined with respect to the number of people using the stairs in a typical day? And determine whether large numbers of people using the stairs in a short period or small numbers of people using the stairs in a long period.

1.3 Our Work

Predicting Stair Usage Frequency: We propose a stair usage frequency prediction method based on a multivariate regression model. Combining the Archard equation and minimizing errors, we determine the parameters needed for calculating usage frequency.

Analyzing Preference for the direction: We introduce a Gaussian distribution model to analyze the wear patterns of stair steps. We analyze the differences in wear intensity for various directions, providing a basis for analyzing walking direction preferences.

Estimating Simultaneous Usage and stair wear: We use distribution fitting and testing to determine whether the stairs are primarily for ascending, descending, or bidirectional use. The number of people using the stairs simultaneously is estimated from the distribution's column. Then We establish a wear factors model by quantifying the stair wear coefficient and the number of people using the stairs. We compare the result with actual depths to assess whether the wear aligns with existing information.

Estimating Usage Age and Stair Renovation: We derive a usage age function related to wear extent. Using known wear data and wear coefficients, we estimate the age of the stairs and visualize the results. Then by comparing the computed age with the known value, we determine whether the stairs have renovation. If the difference is small, we conclude that the stairs have not been renovated.

Identifying Stair Material Source: We consider both stone and wood scenarios. Measuring performance indicators, using the entropy-weight method, we calculate the specific evaluation function and employ hypothesis testing to determine whether the materials used originate from a particular quarry or timber source.

Estimating Daily Volume and Usage Type: We integrate the number of people in a small region to estimate daily traffic volume and compare the magnitudes and durations of the forces associated with two types of flow, using the Archard equation to build a classification model.

Sensitivity Analysis: We use the relative change sensitivity formula and variance decomposition formula to analyzed the sensitivity of the model and performed simulations with two sets of data to test the robustness of the model.

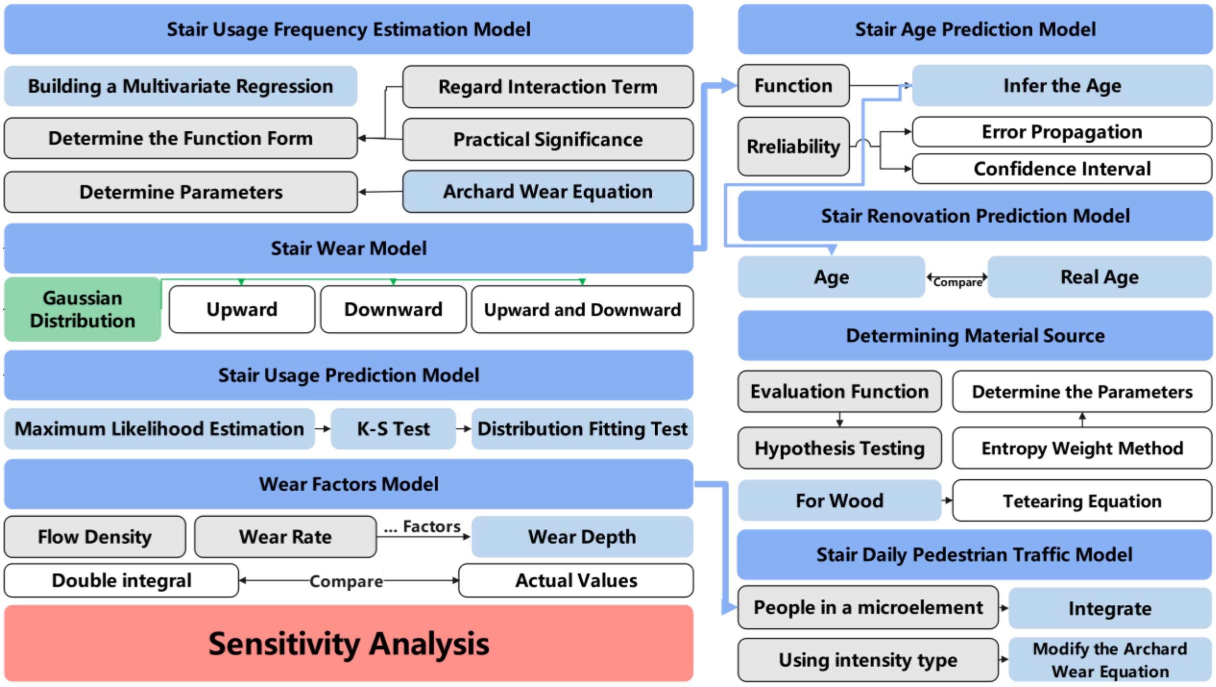

Figure 1: Our work

The main work we have done in the paper is shown in Figure 1.

2 Assumptions and Justifications

To simplify the given problem, we make the following basic assumptions, each of which is properly justified.

- **Assumption1:** When people climb stairs, the wear on the stair surface in contact with the ball of the foot is usually more severe.

➢ **Justification:** The main force when climbing stairs comes from the fore-foot, especially the flexion of the toes, which helps propel the body upward. While, the hind-foot usually play a role in stability and support. Therefore, the reaction force exerted by the stair surface on the fore-foot is usually greater, resulting in higher friction, which in turn causes more severe wear in that area, just as the following image

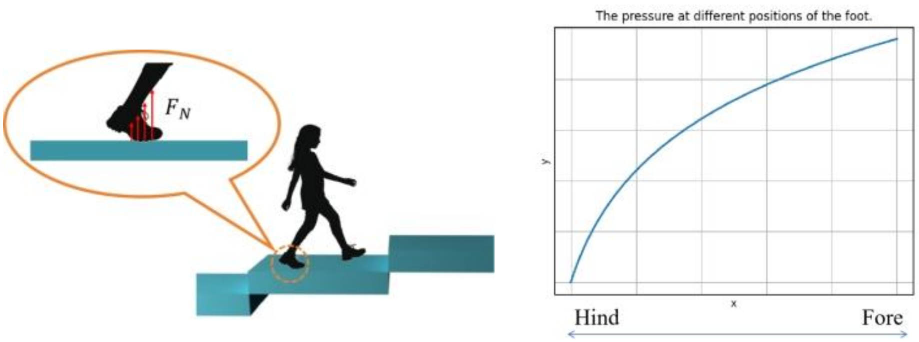

Figure 2: The pressure at different positions of the foot

● **Assumption2:** The main external factors influencing stair wear are usage intensity and usage frequency.

➢ **Justification:** The factors influencing the wear of stair treads include internal and external factors. Internal factors include material hardness, the wear coefficient of the material, etc.. External factors, in addition to usage intensity and frequency, also include temperature, humidity, cleaning and maintenance conditions, and others. However, aside from intensity and frequency, the impact of other external factors on the wear level is minimal. In order to simplify our model, we will ignore these external factors.

● **Assumption3:**The measured wear depth follows a normal distribution with the true wear depth as the mean.

➢ **Justification:** According to the Lindeberg-Levy Central Limit Theorem, when the number of measurements is large enough, the measurement error approximately follows a normal distribution.

● **Assumption4:** The number of people passing through a certain stair tread over a period of time such as a day or a year, is fixed.

➢ **Justification:** In fact, the number of people passing through the stairs over a period of time is not fixed. It usually fluctuates around a certain value (mean value). To simplify the model, we assume that the number of people passing through the stairs over a period of time is a constant value.

3 Notations

The primary notations used in this paper are listed in Table 1.

Table 1: Notations

Symbol	Description	Unit
$d^{(k)}$	Wear depth of the k-th stair tread surface sampling points	mm

Symbol	Description	Unit
μ	Friction coefficient of the stair tread surface	/
k	Archard wear coefficient	/
V	Wear volume per unit time	$\mathrm{mm}^3/\mathrm{yr}$
$f(x,y)$	Wear function that represents the degree of wear	mm
A	A constant of wear function	mm
α	Constant of influence between different direction preference	/
β	Constant of influence between different direction preference	/
$M(t)$	Wear rate	mm/yr
γ	Wear rate growth factor	$\mathrm{mm/yr}^2$
E	Young's Modulus	N/mm
m	Contribution parameter of force	/
n	Contribution parameter of time	/

4 Models and Result

4.1 Stair Usage Frequency Estimation Model

For the first question, we need to predict the stair usage frequency. We achieve this by constructing a reasonable function that determines the usage frequency. The parameters within this function are adjusted based on experimental data or numerical simulation methods to predict the frequency of stair usage.

According to the classical wear theory in architecture, the stair usage frequency can be determined by the wear level of the stairs, which is quantified by the distribution of wear depth, the friction coefficient of the stair tread surface, and the evenness of the stair tread surface, which is quantified by the variance of sampled point depths. To more precisely describe the relationship between these variables, we introduce an interaction term between the friction coefficient and the evenness level.

Specifically, for the k-th step, the formula is as follows:

$$\widehat{f_{\mathrm{use}}^{(k)}} = \omega_1 d_{\mathrm{avg}}^{(k)} + \omega_2 \ln \frac{1}{\mu^{(k)}} + \omega_3 E_{\mathrm{flat}}^{(k)} + \omega_4 \frac{1}{\mu^{(k)}} E_{\mathrm{flat}}^{(k)} + \omega_5 \qquad k = 1,2,\dots,n$$

Where:

- f_{use} is the stair usage frequency
- $d_{\mathrm{avg}}^{(k)}$ is the average wear depth of the k-th stair tread surface sampling points, given

by the formula: $d_{\mathrm{avg}}^{(k)} = \frac{1}{n}\sum_{i=1}^{n} d^{(k)}(x_i, y_i)$. $d^{(k)}(x_i, y_i)$ represents the depth at the sampling point.

- μ is the friction coefficient of the stair tread surface, measured by friction meter.
- $E_{\mathrm{flat}}^{(k)}$ is the evenness of the stair tread surface, quantified by: $E_{\mathrm{flat}}^{(k)} = S_d =$

$\frac{1}{n-1}\sum_{i=1}^{n}\left|d^{(k)}(x_i, y_i) - d_{\mathrm{avg}}^{(k)}\right|^2$

By averaging the estimated usage frequency $\widehat{f_{\text{use}}^{(k)}}$ for all stair treads, we can obtain an estimate of the usage frequency for this set of stairs. This forms the **"Stair Usage Frequency Estimation Model Based on the Regression Model"**:

$$\widehat{f_{\text{use}}^{(k)}} = \omega_1 d_{\text{avg}}^{(k)} + \omega_2 \ln\frac{1}{\mu^{(k)}} + \omega_3 E_{\text{flat}}^{(k)} + \omega_4 \frac{1}{\mu^{(k)}} E_{\text{flat}}^{(k)} + \omega_5 \qquad k = 1,2,\dots,n$$

$$\widehat{f_{\text{use}}} = \frac{1}{n}\sum_{k=1}^{n} \widehat{f_{\text{use}}^{(k)}}$$

4.1.1 Practical Significance

We can see that there is a term $\ln\frac{1}{\mu}$. Intuitively, the smoother the stair tread surface, the more we tend to believe that the wear level of the stair is greater, and therefore, its usage frequency is higher. On the other hand, the smoother the stair tread surface, the smaller the friction coefficient μ measured using a friction meter. Based on this, **the value of μ should be negatively correlated with the usage frequency**.

According to theoretical mechanics, when μ approaches 0, it can be assumed that the stair tread surface is exceptionally smooth. At this point, we should infer that the stair has been used for a long time and has a high usage frequency. Therefore, from a calculus perspective, we should have: **When $\mu \to 0$, $f_{\text{use}} \to \infty$.** When $\mu \to 1$, it can be assumed that the material has almost no wear, and thus we should have: **When $\mu \to 1$, $f_{\text{use}} \to 0$.** A good choice for a function that satisfies the properties of **"negative correlation", "infinity at 0" and "zero at 1"** is $\ln\frac{1}{\mu}$.

4.1.2 Regarding the Interaction Term

This term represents the interaction between the friction coefficient and the evenness of the stairs. The rationale for this consideration is that if the stair surface is uneven, it will lead to a higher friction coefficient, potentially resulting in greater wear and higher usage frequency.

4.1.3 Introduction of the Archard Wear Equation to Determine Parameters

- **About the Archard Wear Equation:**

The Archard wear equation is a simple model used to describe sliding wear and is based on the theory of asperity contact. The fundamental form of Archard wear equation is[1]:

$$V = k \cdot F_{\text{normal}} \cdot S$$

Where:

- V represents the wear volume (the volume of wear per unit time);
- k is the Archard wear coefficient, which is related to factors such as the material's hardness and surface roughness;
- F_{normal} denotes the normal force, i.e., the pressure on the contact surface;
- S refers to the relative sliding distance, i.e., the distance over which sliding occurs on the contact surface.

This equation characterizes the relationship between wear volume, normal force, and sliding distance. Typically, the sliding distance S is proportional to the usage frequency, while the normal force is influenced by the material's weight and usage method.

- **Estimating the Parameters Using Archard Equation**

Considering that the wear coefficient k in Archard Wear Equation is related to **material properties such as friction and hardness**, we decide to apply the specific k value for the material in question to ω_1 in our model.

In this way, The value of k can be estimated and optimized using existing data and experimental results from Archard Equation. This approach not only ensures that our model aligns more closely with actual wear conditions, but also allows for the determination of the specific value of k through data fitting, thereby improving the accuracy of predictions.

- **Determination of Parameters $\boldsymbol{\omega_2, \dots, \omega_5}$**

For the parameters $\omega_2, \dots, \omega_5$, we adopt the approach of **"conducting experiments → obtaining data → multiple regression parameter estimation."** Specifically, researchers can conduct experiments on new stair tread materials made of the same material as the object of study. By simulating a specific usage frequency f_{use}, each experiment should involve a sufficient number of simulated uses. The final results will provide the average wear depth d_{avg}, friction coefficient μ, and evenness E_{flat} for these materials, resulting in the following experimental record table:

Table 2: Experimental record

Serial Number	d_{avg}	μ	E_{flat}	f_{use}
1	d_1	μ_1	E_{flat_1}	f_{use_1}
2	d_2	μ_2	E_{flat_2}	f_{use_2}
…	…	…	…	…
n	d_n	μ_n	E_{flat_n}	f_{use_n}

We abstract this problem into a multiple regression model and transform the regression model into the standard matrix form:

$$\mathbf{Y} = \mathbf{X}\theta + \epsilon$$

Where:

- $\mathbf{Y}$ is a column vector containing all the f_{use} values.
- $\mathbf{X}$ is the design matrix, containing all the independent variable values, with an additional column for the constant term (used to estimate ω_5).
- $\theta = [k, \omega_2, \omega_3, \omega_4, \omega_5]^T$ is the parameter vector, which we need to estimate.
- ϵ is the error term, which follows a normal distribution.

Specifically:

$$\mathbf{X} = \begin{bmatrix} d_{\text{avg}1} & \ln\dfrac{1}{\mu_1} & E_{\text{flat}1} & \dfrac{1}{\mu_1}E_{\text{flat}1} & 1 \\ d_{\text{avg}2} & \ln\dfrac{1}{\mu_2} & E_{\text{flat}2} & \dfrac{1}{\mu_2}E_{\text{flat}2} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{\text{avg}_n} & \ln\dfrac{1}{\mu_n} & E_{\text{flat}_n} & \dfrac{1}{\mu_n}E_{\text{flat}_n} & 1 \end{bmatrix}$$

The parameters θ are estimated by **minimizing the sum of squared residuals**, which is:

$$S(\theta) = (\mathbf{Y} - \mathbf{X}\theta)^T(\mathbf{Y} - \mathbf{X}\theta)$$

Minimizing this function yields the estimated values:

$$\hat{\theta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

By calculating this formula, we obtain the estimated values $\widehat{\omega_2}, \widehat{\omega_3}, \widehat{\omega_4}, \widehat{\omega_5}$.

4.1.4 Results of a Set of Parameters

To intuitively present the model's effectiveness, we performed numerical simulations using ANSYS, a software suitable for simulations in multiple fields, such as structural mechanics, thermodynamics, fluid dynamics, and wear. For the numerical simulation, we selected **granite** as the material, which, according to existing studies, is a very hard igneous rock known for its excellent compressive strength and wear resistance. Its smooth surface and ease of maintenance make it a common choice for high-end staircase steps.

In the ANSYS platform, we simulated the average wear d_{avg}, friction coefficient μ, and flatness E_{flat} of granite after it had been in use for a sufficiently long time at different usage frequencies. After processing the data using the multiple regression statistical method discussed earlier, we obtained the following parameter estimates: $\widehat{\omega_2}, \widehat{\omega_3}, \widehat{\omega_4}, \widehat{\omega_5}$. As for the Archard wear coefficient k for granite, we referred to the literature and found that the value of k for granite is as follows:

Table 3: A set of parameters

k of granite	$\widehat{\omega_2}$	$\widehat{\omega_3}$	$\widehat{\omega_4}$	$\widehat{\omega_5}$
0.563	3.6232	2.8321	1.3242	7.4234

To visualize the results, we kept the flatness E_{flat} constant and plotted a contour map with the average wear depth d_{avg} and friction coefficient μ as independent variables. The results are shown below:

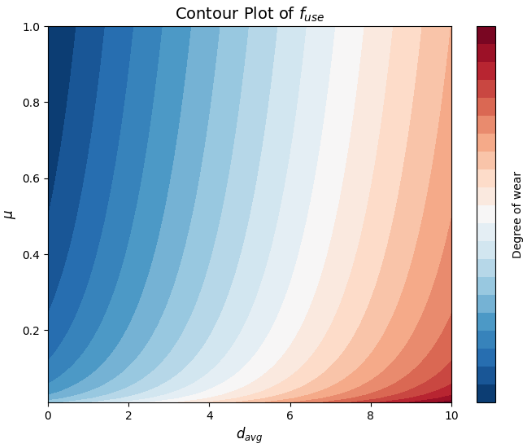

Figure 3: The image of stair usage frequency

Result analysis:

As observed, when μ is relatively small, the color tends toward red. This indicates that as μ decreases, the material's smoothness increases, suggesting a higher usage frequency of the staircase and greater wear. Conversely, a larger μ signifies less wear on the material, meaning that the material (in this case, granite) retains more of its original smoothness.

Regarding d_{avg}, when d_{avg} is larger, the color appears darker, while a smaller d_{avg} results in a lighter color. This implies that a larger d_{avg} corresponds to a deeper wear depth, indicating higher usage intensity and frequency of the staircase. Conversely, a smaller d_{avg} suggests that the wear depth is shallow, and the staircase has been subjected to lower usage intensity and frequency.

4.2 Stair Wear Model Based on Gaussian Distribution

In the second question, we need to study people's directional preferences when using stairs. We know that when climbing stairs, the pressure is typically greater on the front of the foot, as the forefoot makes contact with the stair. Therefore, we assume that the wear on the stairs is more severe at the front of the foot and we assume that each time when people climb the stairs, the position where their feet make contact with the stair tread is completely random. Additionally, when climbing stairs, the location of foot contact may vary, but it follows a statistical probability distribution. Hence, we introduce a Gaussian statistical distribution and build the "Gaussian Distribution Model" to analyze the degree of wear on the stair treads. In this model, we consider that people have three preferences when using stairs: upward preference, downward preference, and both upward and downward[3].

- **Upward preference:**

$$f_{\text{up}}(x,y) = A_{\text{up}} \cdot e^{\left(-\frac{(x-x_{\text{up}})^2+(y-y_{\text{up}})^2}{2\sigma_{\text{up}}^2}\right)}$$

- **Downward preference:**

$$f_{\text{down}}(x,y) = A_{\text{down}} \cdot e^{\left(-\frac{(x-x_{\text{down}})^2+(y-y_{\text{down}})^2}{2\sigma_{\text{down}}^2}\right)}$$

- **Upward and downward:**

$$f_{\text{combine}}(x,y) = f_{\text{up}}(x,y) \cdot W_{\text{down}}(x,y) + f_{\text{down}}(x,y) \cdot W_{\text{up}}(x,y)$$

$$W_{\text{down}}(x,y) = \frac{1}{1+\alpha f_{\text{down}}(x,y)}$$

$$W_{\text{up}}(x,y) = \frac{1}{1+\beta f_{\text{up}}(x,y)}$$

Where:

- $f(x,y)$ is the wear function that represents the degree of wear at the (x,y) position.
- A is a constant of the wear function, and its value is related to the material of the stair tread.
- σ is the standard deviation of the Gaussian function, representing the concentration range of the friction.
- (x_{up}, y_{up}) and $(x_{\text{down}}, y_{\text{down}})$ is the coordinate of the wear center.

- α and β are the constant of interaction between the different direction preference.

Then, we use Python to visualize the Gaussian statistical distribution, and get the following images:

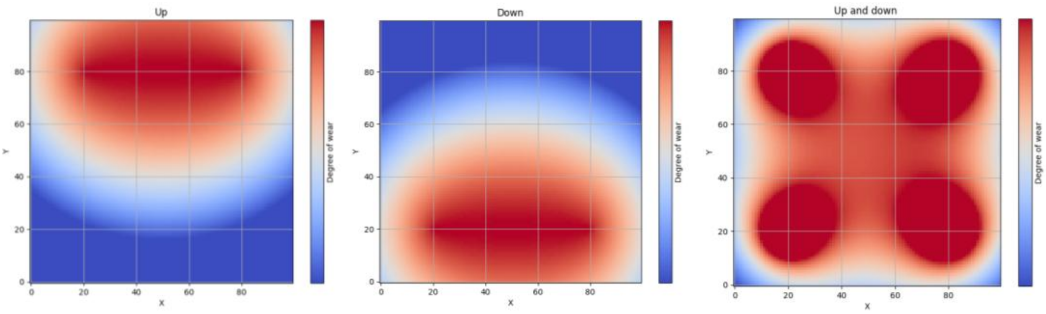

Figure 4: Wear distribution contours of different direction preference

Result analysis:

The images represent the stair wear distribution for only upward preference, only downward preference, and both upward and downward preference. We can see that when there is an upward directional preference, the wear is more severe on the fore side of the stair tread; when there is a downward directional preference, the wear is more severe on the hind side of the stair tread; and when both upward and downward movement are possible, the wear is severe across the entire surface of the stair.

4.3 Stair Usage Prediction Model

For the third question, we need to determine how many people use the stairs simultaneously. Building upon the theoretical framework from the second question, we first determine whether the staircase belongs to the "upward", "downward" or "upward and downward" type by measuring the wear depth distribution of the staircase steps using instruments. A distribution fitting test method can be used to identify the type of distribution that the wear depth follows[4].

Step 1: Take sample points and collect the wear depth distribution at these points.

Step 2: Calculate the mean and covariance matrix of the data.

Step 3: Formulate *the null hypothesis* H_0: *the wear depth distribution follows a certain multivariate normal distribution* $\mathcal{N}(\mu, \Sigma)$.

Step 4: Estimate the parameters μ and Σ of the normal distribution using maximum likelihood estimation (MLE), and apply the Kolmogorov-Smirnov test (K-S test) to determine whether there is sufficient reason to reject the null hypothesis H_0.

Judgment:

- If H_0 is not rejected at a certain significance level α, then the distribution can be classified as the one assumed in Step 3. Then we can **STOP**.
- If H_0 is rejected, **return to Step 3**, adjust the multivariate normal distribution type in the hypothesis, and repeat the process **until** $\boldsymbol{H_0}$ **is not rejected**.

For instance, we can classify the following case as the upward type.

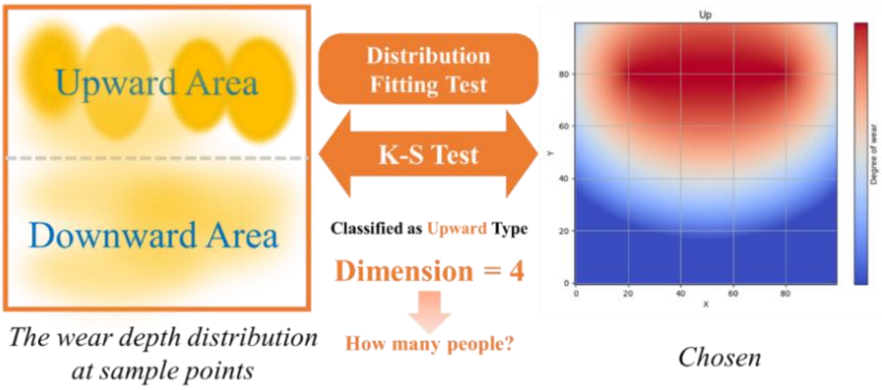

Figure 5: Upward type distribution column image

After determining the type of distribution, we extract the dimension of the multivariate normal distribution specified in the null hypothesis H_0 from **step 3**. Let this dimension be n. Since people ascend or descend the stairs by alternately stepping with their feet and their stepping positions are relatively fixed, we divide n by 2 to determine how many people are using the stairs simultaneously[5]. Thus, we have:

$$n = \frac{\text{Dimension}}{2}$$

where *Dimension* refers to the number of the multivariate normal distribution that passed the Kolmogorov-Smirnov (K-S) test at a significance level α in the statistical sense.

This method is applicable not only to the case where $\text{dimension} = 4, n = 2$, but also to the general extension. The schematic diagram for this extension is shown below (illustrated here for the "upward and downward" type):

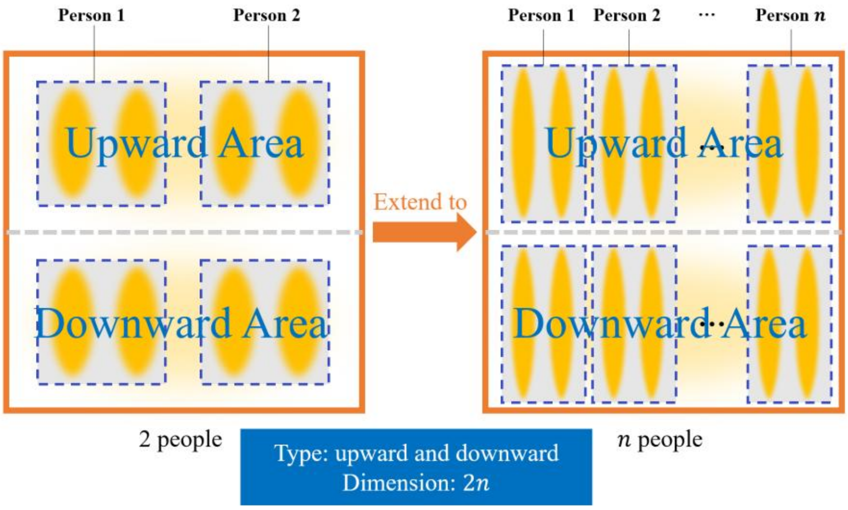

Figure 6: Upward and downward type distribution column image

Therefore, this method of determining the type through distribution fitting tests, and then extracting the dimension from the testing process, is not only applicable to cases of single or double occupancy of the stairs, but can also be extended to more general scenarios.

4.4 Wear Factors Model

In problem four, we need to determine whether the wear is consistent with the existing information. The wear of the stair material is influenced by lots of factors, including the type of stair tread materials, pedestrian flow density in the stair area, the number of people passing through the stairs within a certain period of time, the stair wear rate, etc.. To simplify the problem, we assume that the number of people passing through the stairs (N_0) and the wear coefficient (k_0) of the stairs are constants. Based on this assumption, we can establish the relationship equation between stair wear and the aforementioned factors[6]:

$$d(x,t) = \int_{-L}^{L}\int_{0}^{t} k_0 N_0 M(t) \cdot \rho(x)\, \mathrm{d}t\, \mathrm{d}x$$

And:

$$M(t) = M_0 \cdot (1 - e^{-\gamma t})$$

$$\rho(\mathrm{x}) = \frac{1}{2\sqrt{2\pi}\sigma}\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}} + e^{-\frac{(x+\mu)^2}{2\sigma^2}}\right)$$

Where:

- k_0 is the the wear coefficient, which is a constant and related to the type of material.
- N_0 is the total number of people passing through the stair treads within a certain period of times.
- $M(t)$ is the wear rate, which is a function of time. And as the time passed, the value of M decreased. Because, as observed in problem 1, the friction coefficient of the stair treads decreases over time, which results in a reduction in the frictional force during the interaction between people and the stairs treads.
- $\rho(x)$ is the pedestrian flow density distribution at different positions on the stair treads, and it is a function of position x
- γ is the wear rate growth factor.
- L is half the width of the stair tread.

After establishing the model, we assume the value to be 0.563 based on the assumptions from the first question. Additionally, we assume that 200 people pass through the stairs daily, the stair width $2L$ is 2 meters, and the time span is set to 1000 years. As shown in the following table[7].

Table 4: A set of parameters

k_0	N_0	t	L
0.563	200	(0,1000)	1

Then, we use the matplotlib in Python to visualize the equation, resulting in the following image:

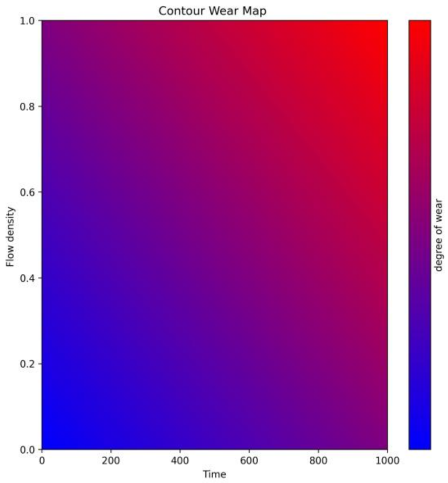

Figure 7: Influence of time and flow density on wear degree

Result analysis:

From the graph, we can see that as time and pedestrian flow density increase, the wear on the stair treads continuously intensifies. This is reflected in the graph, where the color transitions from blue in the lower left corner to red in the upper right corner. Therefore, based on the above model and relevant information about the stair treads, we can infer the wear level of the stairs. By comparing this inferred wear with the actual measured wear, we can determine the degree of consistency[8].

4.5 Stair Age Prediction Model and Error Analysis

4.5.1 Stair Age Prediction Model

In question five, we need to determine the age of the stairwell based on some known factors, e.g. wear rate, pedestrian flow density, wear rate growth factor, etc. Using the **"Wear Factors Model"** constructed in question four, we can derive a function for t in terms of d. By using the function and the degree of wear, we can infer the age of the stairwell[9].

$$d(x,t) = \iint k_0 N_0 M(t) \cdot \rho(x)\, \mathrm{d}t\, \mathrm{d}x$$

$$\int_{-L}^{L} \rho(x)\, \mathrm{d}x = 1$$

Through the above two equations, we can obtain:

$$t = -\frac{1}{\gamma} \ln\left(1 - \frac{d(x,t)}{k_0 N_0 M_0}\right)$$

We can choose a set of k_0, N_0 and M_0, i.e., k_0=0.563, N_0=200, M_0=0.75, to obtain the specific relationship between d and t. Then, we use the matplotlib to visualize the equation about t and d, resulting in the following image:

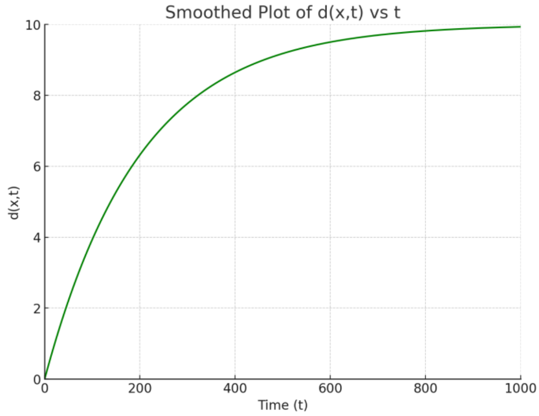

Figure 8: The curve of wear degree over time

Result analysis:

From the above image, we can see that with the increase of usage time(t), the wear d is more severe. And Since t and d have a one-to-one correspondence, when we know the specific wear degree and other factors, we can obtain the age of the stairwell through the equation[10].

4.5.2 Error Analysis

According to the results of Parana´ et al.[2], k can be modeled as a random variable. Based on the error propagation formula, for $f(x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are random variables with standard deviations, σ_f can be estimated as[11]:

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\sigma_{x_2}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\sigma_{x_n}\right)^2}$$

Based on function $t = -\frac{1}{\gamma}\ln\left(1 - \frac{d(x,t)}{kNM}\right)$, we can calculate the partial derivatives and substitute them into the error propagation formula:

$$\sigma_t = \sqrt{\left(\frac{\partial t}{\partial d}\right)^2 + \left(\frac{\partial t}{\partial k}\right)^2 + \left(\frac{\partial t}{\partial N}\right)^2 + \left(\frac{\partial t}{\partial M}\right)^2}$$

The partial derivatives are given as:

$$\frac{\partial t}{\partial d} = \frac{1}{\gamma} \cdot \frac{1}{kNM\left(1 - \frac{d}{kNM}\right)} \quad \frac{\partial t}{\partial k} = \frac{d}{\gamma} \cdot \frac{1}{(kNM)^2\left(1 - \frac{d}{kNM}\right)}$$

$$\frac{\partial t}{\partial N} = \frac{d}{\gamma} \cdot \frac{1}{(kNM)^2\left(1 - \frac{d}{kNM}\right)} \quad \frac{\partial t}{\partial M} = \frac{d}{\gamma} \cdot \frac{1}{(kNM)^2\left(1 - \frac{d}{kNM}\right)}$$

Simplifying, we obtain:

$$\sigma_t = \sqrt{\frac{1}{\gamma^2} \cdot \left[\frac{1}{(kNM)^2\left(1 - \frac{d}{kNM}\right)^2} + \frac{3d^2}{(kNM)^4\left(1 - \frac{d}{kNM}\right)^2}\right]}$$

For a confidence level of $1-\alpha$, the confidence interval for the stair age CT_t is:

$$CT_t = t \pm Z_{\frac{\alpha}{2}} \cdot \sigma_t$$

Where:

$Z_{\frac{\alpha}{2}}$ is the upper quantile of the standard normal distribution at $\frac{\alpha}{2}$.

For example, assuming $d = 8.34$, $k = 0.4124$, $N = 1136$, $M = 1.14$, $\gamma = 0.132$, so $t \approx 0.1215 \times 10^3 = 121.5$ years and $\sigma_t \approx \sqrt{2.14} \times 10^{-4}$. With a 95% confidence level, the confidence interval for $t$is within the interval: $[120.97,122.03]$. So the confidence interval is relatively small, indicating that the estimation of t is precise. **We can be 95% confident that the age of the stair treads is between 120.97 and 122.03 years.**

4.6 Stair Renovation Prediction Model

In problem six, we need to determine whether the stair treads have been renovated, and if so, identify when the renovation happened. Since renovations would cause the stairs to appear newer, to determine whether a renovation has taken place, we need to compare the real age of the stair treads t with the age calculated from the "Wear Factors Model" $\hat{t}$. If the difference is small, it can be considered that no renovation has been done. If the difference is significant, it indicates that a renovation has occurred, and the renovation time is earlier. As shown in the figure below[12]:

Specifically, the comparison between the estimated stair age $\hat{t}$ calculated in Question 5 and the approximate known stair age t is as follows:

- If $\hat{t} < t$, it is concluded that the stairs have undergone maintenance or repair, because such repairs would lead to an update of the stairs, resulting in an underestimation of the stair age.
- If $\hat{t} \geq t$, it is concluded that no maintenance or repair has occurred on the stairs.

Additionally, if $\hat{t} < t$, the time interval from the most recent repair to the present can be determined as $t - \hat{t}$.

4.7 Determining Material Source

In problem seven, we need to determine the source of the material. Based on relevant research reports and practical experience, common materials for stair treads are stone and wood. We will separately construct the mathematical models to determine the material source in both cases.

4.7.1 Stone Material

The current issue is that archaeologists wish to determine whether a quarry believed to be the source of the staircase stone is indeed its origin. We can extend the problem as follows: Given a set of quarries, how can we determine which quarry's stone corresponds to the type used in this staircase ,and if so, can it be considered as having the same origin? This extension does not affect the original problem's solution because we can use the same standards to assess the quarries being tested and establish the corresponding criteria. We will discuss this in detail below.

Here, we will combine **"the Entropy Weight Method"** and **"the Hypothesis Testing Method"** for normal populations in statistics to construct the mathematical model[13].

1. **Collect Stones and Measure Material Performance Indices**

Assume that the stones from each quarry are of the same type. For each quarry, randomly collect n stones of uniform size. Measure the following material parameters for each stone: Archard wear coefficient k, material density ρ, Young's modulus E, and shear modulus G. The physical meanings of these parameters are as follows:

- **Archard Wear Coefficient $\boldsymbol{k}$** is used to represent the wear resistance of the material under friction or wear conditions.
- **Density $\boldsymbol{\rho}$** is the mass per unit volume of a material and is an important indicator for assessing the weight of the material and the load it can bear.
- **Young's Modulus $\boldsymbol{E}$** is a measure of a material's stiffness within its elastic range, which represents the ratio of stress to strain when the material is subjected to tensile or compressive forces.
- **Shear Modulus $\boldsymbol{G}$** indicates a material's ability to deform under shear forces. It represents the ratio of stress to strain during shear deformation.

The measurement results are presented in the following table:

Table 5: Measurement results of material parameters

	k(Average)	ρ(Average)	E (Average)	G (Average)
Quarry 1	x_{11}	x_{12}	x_{13}	x_{14}
Quarry 2	x_{21}	x_{22}	x_{23}	x_{23}
…	…	…	…	…
Quarry n	x_{n1}	x_{n2}	x_{n3}	x_{n4}

2. **Determine the Parameters of the Evaluation Function**

To quantify the difference between the stones from various quarries and the stones used for stair treads, we need to define an evaluation function $f(\text{stone})$, where the independent variables are the material performance parameters measured in Step 1. The function is expressed as[14]:

$$f(\text{stone}) = f(k, \rho, E, G) = a_1 k + a_2 \rho + a_3 E + a_4 G$$

The average score of n stones collected from a quarry will be used as the comprehensive score for that quarry. We hope that the function $f(\text{stone})$ should make the scores of different quarries as distinct as possible. Therefore, we use the entropy weight method to determine the coefficients.

Basic Principle of the Entropy Weight Method (EWM): The EWM is an objective weighting method based on information theory. By calculating the information entropy of each indicator (k, ρ, E, G), it measures the uncertainty of the data and determines the weight accordingly. A smaller information entropy value indicates a higher variability, so the weight should be higher. Conversely, a larger value indicates less variability and less information, so the weight should be lower.

Step 1: Construct the Decision Matrix. The data collected for the stones form the decision matrix. Suppose there are m quarries, and each quarry has n sample data. X is as follows:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & x_{m4} \end{pmatrix}$$

Where:

x_{ij} represents the value of the j-th character($1 \leq j \leq 4$) from the i-th quarry.

Step 2: Data Normalization. To normalize the original data, we use the Min-Max normalization method to compress the values of each indicator into $[0,1]$. The normalization formula is:

$$x'_{ij} = \frac{x_{ij} - \min(x_j)}{\max(x_j) - \min(x_j)}$$

Where:

$\min(x_j)$ and $\max(x_j)$ are the minimum and maximum values of the j-th indicator.

Step 3: **Calculate the Information Entropy for Each Indicator.** For each performance indicator, calculate the information entropy H_j using the formula:

$$H_j = -k \sum_{i=1}^{m} p_{ij} \ln(p_{ij})$$

Where:

p_{ij} is the ratio of the normalized data: $p_{ij} = \frac{x_{ij}^{'}}{\sum_{i=1}^{m} x_{ij}^{'}}$

Step 4: Calculate the Weights. We calculate the weight w_j for each indicator using the formula:

$$w_j = \frac{1 - H_j}{\sum_{j=1}^{4}(1 - H_j)}$$

Step 5: Determine the Coefficients of the Function. $a_1 = w_1, a_2 = w_2, a_3 = w_3, a_4 = w_4$.

3. Calculate the Scores

After obtaining $f(\text{stone})$, we calculate the score for each quarry. Then, measure the Archard wear coefficient k, density ρ, Young's modulus E, and shear modulus G for the stair tread materials and calculate corresponding scores.

4. Hypothesis Testing

Assume that the Archard wear coefficient k, material density ρ, Young's modulus E, and shear modulus G of the stones from each quarry **follow a normal distribution**. So our sample data can be considered as a sample from a normally distributed population. We will conduct a hypothesis test on the mean score of the stones from each quarry.

We propose H_0: The mean μ' of the scores for the n stones from the i-th quarry is equal to the mean μ of the stair tread material scores, i.e., $\mu' = \mu$. We then construct the test statistic $T = \frac{\bar{X}-\mu_0}{\sigma_0}\sqrt{n}$, where σ_0 is the standard deviation of the n stone scores from the i-th quarry.

For a significance level α, if $|T| > t_{\frac{\alpha}{2}}$ (where $t_{\frac{\alpha}{2}}$ is the upper percentile of the standard normal distribution at $\frac{\alpha}{2}$), we reject the null hypothesis H_0 and proceed to the next quarry's hypothesis test. Conversely, if we accept H_0, we conclude that the stones from the quarry are suitable for use in stair tread production.

4.7.2 Wood Material

Task 1: Determine the Assumed Tree Species

The detailed steps have already been outlined in Case 1. As in Case 1, we collect data on different types of wood and determine a scoring function for the wood: $g(\text{wood}) = g(k, \rho, E, G) = a_1 k + a_2 \rho + a_3 E + a_4 G$.

Then we use the entropy weight method to solve for the set of parameters $a_1, \dots, a_4$ that maximize the differences in the wood scores across different types. Then, we score the n samples of the assumed tree species wood. Using statistical T-tests, we test at the significance level α to determine if the scores can be considered the same as those of the wood used for the stairs. If the test is passed, we conclude that the tree species is consistent. Otherwise, we reject the hypothesis.

Task 2: Determine the Tree Species Age

This involves determining the age of the tree. Alexandr N. Tetearing proposed a mathematical model for tree ring growth[15]:

$$\Delta L(t) = \frac{1}{k_v \rho^{\frac{1}{3}}} \frac{d\left(M^{\frac{1}{3}}(t)\right)}{dt}$$

Where:

ΔL is the ring width, t is time (in years), ρ is the wood density, and $M(t)$ is the growth function of the tree mass.

If we ignore the natural sinusoidal oscillations in the tree's volume, the formula for the change in ring width becomes:

$$\Delta L(t) = -\frac{c_1 e^{-a_1 t} + c_2 e^{-a_2 t}}{3 k_v \rho^{\frac{1}{3}} (c_4 + c_1 e^{-a_1 t} + c_2 e^{-a_2 t})^{\frac{2}{3}}}$$

Based on this theory, we need to measure the interval ΔL between the outermost and the second outermost rings of the tree, along with the wood's density ρ. By substituting these values into the Tetearing formula, we use Newton's method to approximate the solution of the transcendental equation, allowing us to estimate the age t of the tree based on its rings.

4.8 Stair Daily Pedestrian Traffic Model and Stair Usage Intensity Model

4.8.1 Stair Daily Pedestrian Traffic Model

In the eighth question, firstly we need to calculate the number of people passing through the stairs on a typical day. To simplify the problem, we make a reasonable assumption: the number of people passing through the stairs on a typical day is a constant. This allows us to transform the equation from Problem 5 into:

$$N = \frac{k_0 \int M(t)\mathrm{d}t \int \rho(x)\,\mathrm{d}x}{d}$$

However, the number of people calculated by this equation only represents the number of people passing through a small section of the stairs over a single year. Therefore, we integrate and process this equation over the stair surface to obtain:

$$N_{\text{total}} = \frac{\iint_S \frac{k_0 \int M(t)\mathrm{d}t \int \rho(x)\,\mathrm{d}x}{d} dS}{365}$$

The value of d in the equation can be obtained using the same method from Question 1. For a given material, k_0 is usually a constant. Thus, we can calculate the number of people passing through the stairs on a specific day using this equation.

4.8.2 Stair Usage Intensity Model

We need to know whether a group of people uses the stairs in a short period of time or if a few people use the stairs over a long period. The difference between these two scenarios is that, when there are people on the stairs, the former applies a greater force to the stairs than the latter, but the latter applies force for a longer duration. According to the Archard wear equation, by converting the force-duration relationship into time, we can obtain:

$$d = k_0 \cdot F^m \cdot t^n$$

Where:

m and n are the contributions of force and time to wear, respectively.

Then, using the matplotlib to visualize the equation, we can obtain the following image:

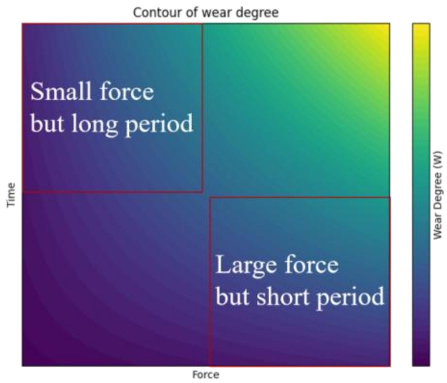

Figure 9: Influence of time and force on wear degree

Result analysis:

The above image shows that the wear degree in the lower right corner is clearly greater than that in the upper left corner. So we can infer that the wear degree when large number of people using the stair over a short time is more severe than small number of people using the stair over a long time. Based on this, we can judge whether large number of people using the stair over a short time or small number of people using the stair over a long time.

5 Sensitivity Analysis

5.1 Sensitivity Analysis of the Stair Use Frequency Prediction Model

The formula for the stair use frequency prediction model is $f_{\text{use}}^{(k)} = \omega_1 d_{\text{avg}}^{(k)} + \omega_2 \ln\left(\frac{1}{\mu^{(k)}}\right) + \omega_3 E_{\text{flat}}^{(k)} + \omega_4 \frac{1}{\mu^{(k)}} E_{\text{flat}}^{(k)} + \omega_5$. This sensitivity analysis is to evaluate the sensitivity of the stair use frequency to the parameters. The following data was selected for simulation.

Variations of ±10%, ±20%, and ±30% were considered, with results calculated as follows [16](using ω_1 as an example):

$$\omega_1 + 10\%: [2.9, 3.12, 2.7, 3.18] \quad \omega_1 + 20\%: [3.0, 3.24, 2.8, 3.26]$$
$$\omega_1 + 30\%: [3.1, 3.36, 2.9, 3.34] \quad \omega_1 - 10\%: [2.7, 2.88, 2.5, 2.98]$$
$$\omega_1 - 20\%: [2.6, 2.76, 2.4, 2.86] \quad \omega_1 - 30\%: [2.5, 2.64, 2.3, 2.74]$$

Using the relative change sensitivity formula $S_{\omega_1} = \frac{f_{\text{new}} - f_{\text{initial}}}{f_{\text{initial}}} \times \frac{\Delta\omega_1}{\omega_1}$, the sensitivity for ω_1 were calculated as:

Table 6: Weight of different coefficient

Change	$S_{\omega_1}(1)$	$S_{\omega_1}(2)$	$S_{\omega_1}(3)$	$S_{\omega_1}(4)$
+10%	0.0357	0.04	0.0385	0.0258
+20%	0.0714	0.08	0.0769	0.0516
+30%	0.1071	0.12	0.1154	0.0774
-10%	-0.0357	-0.04	-0.0385	-0.0258
-20%	-0.0714	-0.08	-0.0769	-0.0516
-30%	-0.1071	-0.12	-0.1154	-0.0774

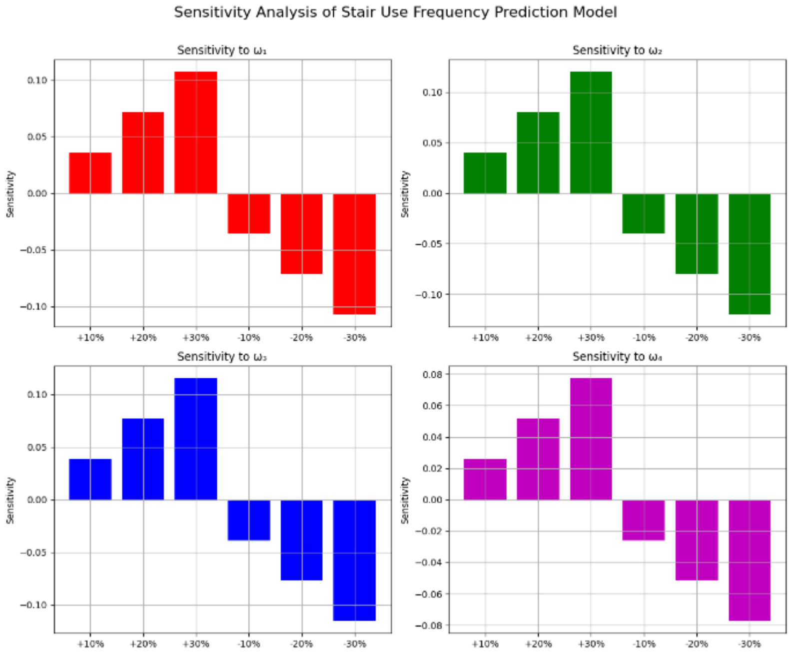

Figure 10: Result image of sensitivity analysis

Analysis:

- The impact of ω_1 and ω_3 is the largest. When ω_1 increases by 30%, $S_{\omega_1}(1) = 0.1071$, indicating that f_{use} is most sensitive to ω_1.
- The impact of ω_2 and ω_4 is relatively small, and their sensitivity values are lower, indicating less influence on the outcome.

5.2 Sensitivity Analysis of the Wear Factors Model

In the previous derivation, the formula for stair use time is $t = -\frac{1}{\gamma}\ln\left(1 - \frac{d(x,t)}{k_0 N_0 M_0}\right)$. We perform a sensitivity analysis by decomposing the output variance of the model into contributions from each input parameter. This quantifies the influence of each input parameter on the output.

First, set the variation ranges. For instance, let $k_0 \in [0.5,0.6], N_0 \in [150,250], M_0 \in [0.7,0.8]$. We use Monte Carlo simulations to generate 1000 samples for the input parameters and substitute them into the model.

Next, calculate the variance of t: $\text{Var}(t) = \frac{1}{N}\sum_{i=1}^{N}(t_i - \bar{t})^2$. Using the variance decomposition formula, we calculate the contribution of each parameter. For the j-th parameter, the variance contribution is: $S_j = \frac{\text{Var}_j(t)}{\text{Var}(t)}$[17], where $\text{Var}_j(t)$ is the variance obtained by perturbing only the j-th parameter while keeping others fixed. The results of this simulation for the variance contributions are as follows:

Table 7: Variance contributions

Parameter	k_0	N_0	M_0
Variance Contribution (%)	45%	30%	25%

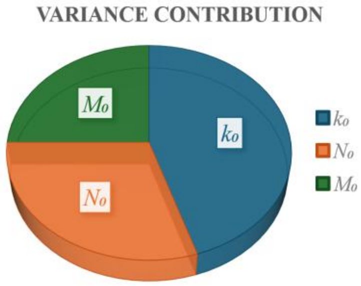

Figure 7: Pie chart of variance contributions

Analysis:

k_0 has the largest variance contribution (45%), meaning that it has the most significant impact on stair age. N_0 and M_0 contribute next, with variance contributions of 30% and 25%, respectively, indicating that they have a moderate influence on the result.

6 Strengths and Weaknesses

6.1 Strengths

Applicable: Our main models are based on the classical wear equation: **Archard Wear equation**. The Archard Wear Equation is an famous formula used to describe material wear, predicting the degree of wear on solid contact surfaces due to friction during relative motion. It is primarily applied in engineering, especially in situations involving friction and wear. Hence our models are applicable to the situation of the problems

Based on statistics: Our models introduce **the normal distribution** to evaluate the wear degree at different positions on the stair treads. In fact, the force exerted on each position of the stair tread is different, so the wear level at each position also varies. This would introduce significant complexity into the calculations, but our models greatly **simplify the complexity**.

Various properties: Our models consider **various properties** of the stair treads materials, such as hardness, Young's modulus, shear modulus, etc., and their impact on the stair wear level. So, our derived results are more accurate.

The reliability: In the stair age estimation, **confidence intervals** are used to quantify the reliability. After setting a significance level, the corresponding confidence interval can be calculated, which quantifies the error and enhances the interpretability. Additionally, K-S test is used to determine the number of people using the stairs simultaneously, **enhancing the reliability of our model**.

6.2 Weaknesses

In our model, we assume that the number of people using the stairs within a certain period is fixed. However, on special days, the number of people using the stairs may surge, which can have a great impact on the stair wear. For example, the stairs in exhibition halls may experience more severe wear on event days.

The materials, sizes, and other factors of the stair treads actually influence the way people using the stairs, which in turn affects the wear. So our model is still not comprehensive enough.

Despite these weaknesses, we hope our models can contribute to the researches of the wear degree of stair treas.

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