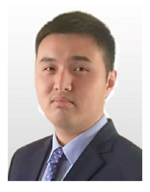
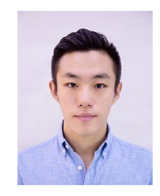
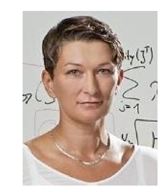
# Can Learning Deteriorate Control? Analyzing Computational Delays in Gaussian Process-Based Event-Triggered Online Learning



Xiaobing Dai*
xiaobing.dai@tum.com


Armin Lederer*
armin.lederer@tum.de


Zewen Yang
zewen.yang@tum.com


Sandra Hirche
hirche@tum.de


## Motivation

Measurement across state space is unsafe without reliable control law

⇕

Closed-loop control performance cannot be evaluated without any data

⇒ **Online learning control with smart data selection**

More accurate model requires more data and computation load [1]
⇒ Consideration of the effects from **computational delays** is necessary!

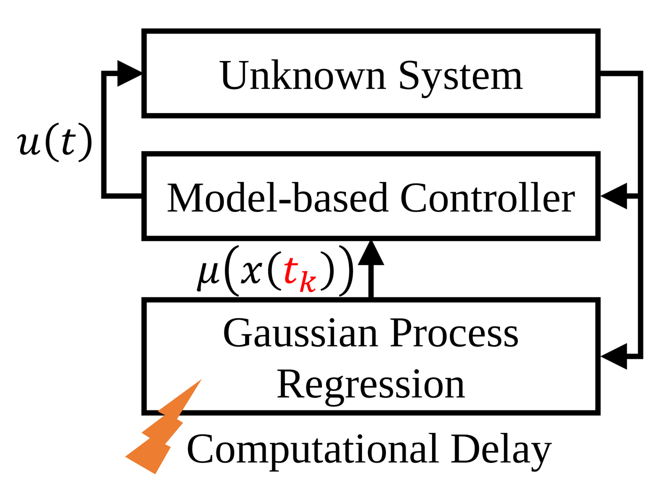
Balance of model accuracy and induced computation time
⇓
Event-triggered online data selection under computational delay

## Problem Setting

Consider a nonlinear system with

$$\begin{aligned}\dot{\boldsymbol{q}}_1(t) &= \boldsymbol{q}_2(t),\\ \dot{\boldsymbol{q}}_2(t) &= \boldsymbol{q}_3(t),\\ &\vdots\\ \dot{\boldsymbol{q}}_m(t) &= \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{u}(t),\end{aligned} \tag{1}$$

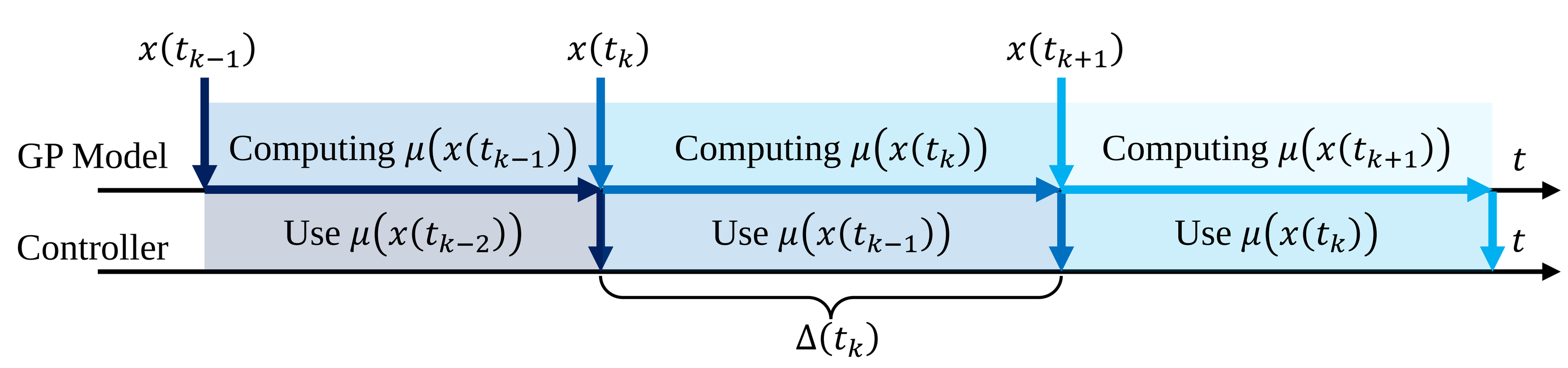
where $\boldsymbol{q}_1,\cdots,\boldsymbol{q}_m \in \mathbb{R}^n$ and $\boldsymbol{x} = [\boldsymbol{q}_1^T,\cdots,\boldsymbol{q}_m^T]^T \in \mathbb{X} \subset \mathbb{R}^{mn}$ with compact $\mathbb{X}$.




**Task:** Track a differentiable and bounded trajectory $\boldsymbol{x}_d$.

Controller: $\boldsymbol{u}(t) = \dot{\boldsymbol{q}}_{dm}(t) + \sum_{i=1}^{m} \lambda_i(\boldsymbol{q}_i(t) - \boldsymbol{q}_{di}(t)) - \boldsymbol{\mu}(\boldsymbol{x}(t_k))$ with $\lambda_1,\cdots,\lambda_m$ Hurwitz

- Delayed prediction: $\boldsymbol{\mu}(\boldsymbol{x}(t_k))$ with $t_{k+1} = t_k + \Delta(t_k)$ ← computational delay
- Event-triggered data selection: $\mathbb{D} = \{\boldsymbol{x}^{(\iota)}, \boldsymbol{y}^{(\iota)} = \dot{\boldsymbol{q}}_m(t_\iota) + \boldsymbol{w}^{(\iota)}\}_{\iota \in \mathbb{N}}$

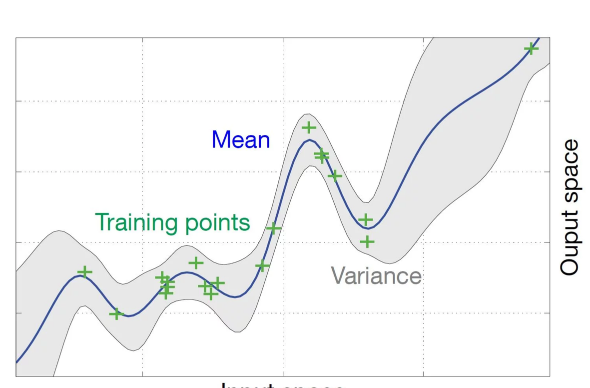



**Assumption 1.** *$\Delta(\cdot): \mathbb{R}_{0,+} \to \mathbb{R}_+$ is bounded, i.e., $\Delta(t) \leq \bar{\Delta}, \forall t \in \mathbb{R}_{0,+}$.*

**Goal:** Design event-triggered online learning such that $\|\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\| \leq \bar{e}$.

## Gaussian Process Regression

Given data set $\mathbb{D} = \{\boldsymbol{x}^{(\iota)}, \boldsymbol{y}^{(\iota)} \in \mathbb{R}^n\}_{\iota=1}^N$, $n$ Gaussian processes are used to provide posterior means $\mu_i(\boldsymbol{x})$ and variances $\sigma_i^2(\boldsymbol{x})$ [2].




**Lemma 1** ([3]). *For every $\delta \in (0,1)$ and $i = 1,\cdots,n$, it holds*

$$\Pr(|f_i(\boldsymbol{x}) - \mu_i(\boldsymbol{x})| \leq \eta_\delta(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{X}) \geq 1 - \delta, \tag{2}$$

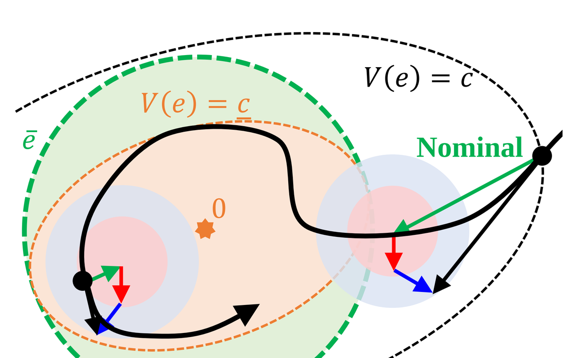
*where $\eta_\delta$ can be explicitly computed.*

## Tracking Error Bounds with Delayed Predictions

**Theorem 1.** *Assume $\|\boldsymbol{e}(0)\| = 0$ and $\bar{\Delta} < \bar{\Delta}_{\max} = 1/(2L_f)$, then the tracking error is bounded as*

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\| \leq \bar{e} = \underbrace{\phi_\Delta \bar{\Delta}}_{\text{Computational Delay}} + \underbrace{\phi_\eta \bar{\eta}_\delta}_{\text{Prediction Accuracy}} \tag{3}$$

*for all time $t \in \mathbb{R}_{0,+}$ with parameters $\phi_\eta, \phi_\Delta$ under probability of at least $2(1-\delta)^n - 1$.*

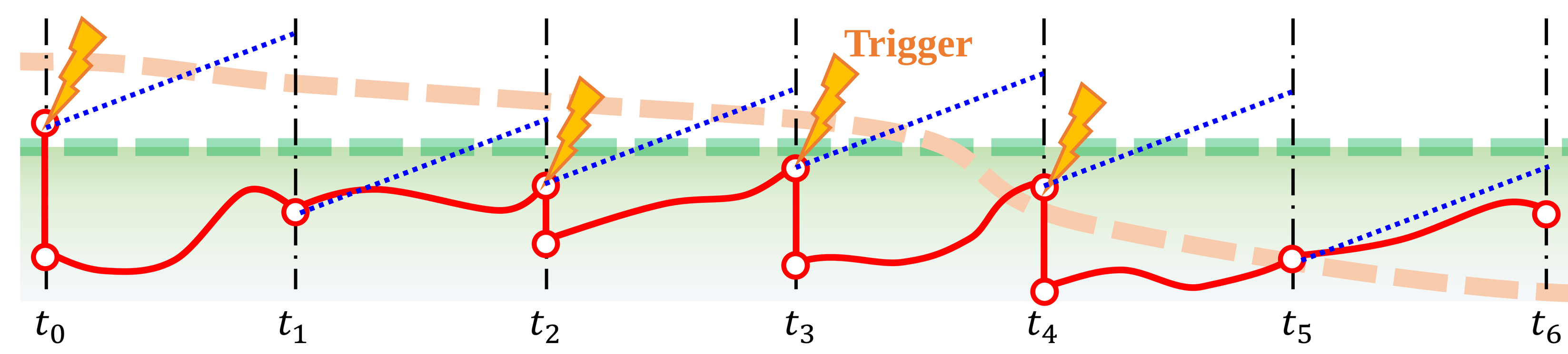



## Event-Triggered Online Learning under Delays

**Theorem 2.** *Assume $\bar{\Delta} < \bar{\Delta}_{\max} = 1/(2L_f)$ and design the event-trigger as*

$$\xi_\eta \|\boldsymbol{\eta}_\delta(\boldsymbol{x}(t_k))\| + \xi_\Delta \bar{\Delta} \geq \max(\xi_e \|\boldsymbol{e}(t_k)\|, \bar{e}) \tag{4}$$

*with positive parameters $\xi_\eta, \xi_\Delta$ and $\xi_e$, then the tracking error $\|\boldsymbol{x} - \boldsymbol{x}_d\|$ is bounded by $\bar{e}$ after some finite time $T \in \mathbb{R}_{0,+}$ with probability of at least $1 - \delta$.*




Online learning ⇒ Smaller prediction error + More computation time

Online learning benefits control performance when additional delay is small.

**Problem:** Online learning ⇒ $|\mathbb{D}| \to \infty \Rightarrow \Delta(\cdot) \to \infty$

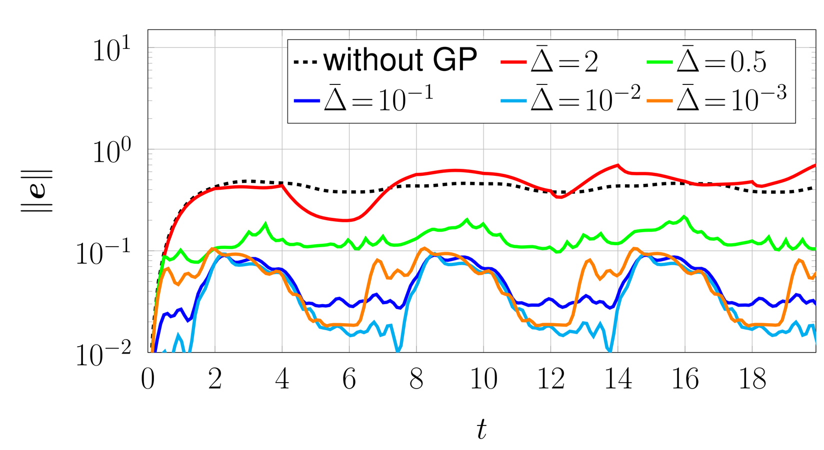
**Corollary 1.** *Suitable data deletion strategy executed before the model update keeps computation time bounded and maintains the tracking performance.*

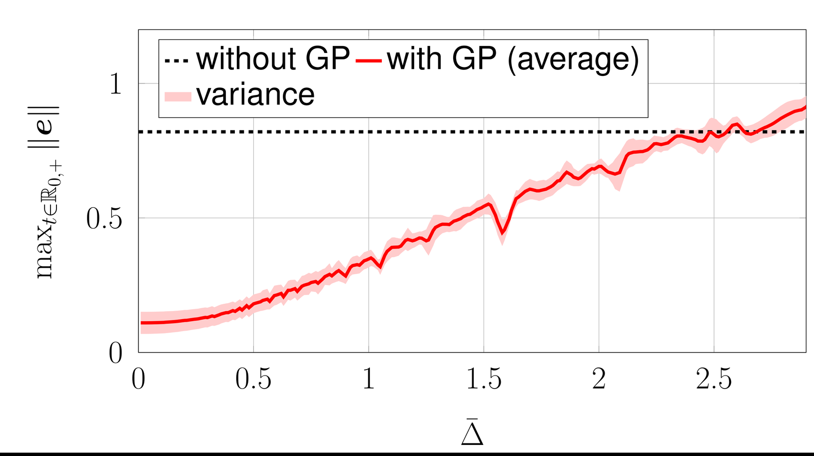
## Simulation: Influence of Delay on Tracking Errors

Dynamical system: $\dot{x}_1 = x_2$, $\dot{x}_2 = f(\boldsymbol{x}) + u$ with $L_f \approx 1$ and

$$f(\boldsymbol{x}) = \sin(x_1) + \frac{1}{2(1 + e^{x_2/10})}. \tag{5}$$

- Offline learning with $|\mathbb{D}| = 100$ uniformly distributed training samples
- Computational delay: $\Delta(t) = \bar{\Delta}, \forall t \in \mathbb{R}_{0,+}$, $\quad \bar{\Delta}_{\max} = 1/(2L_f) \approx 0.5$

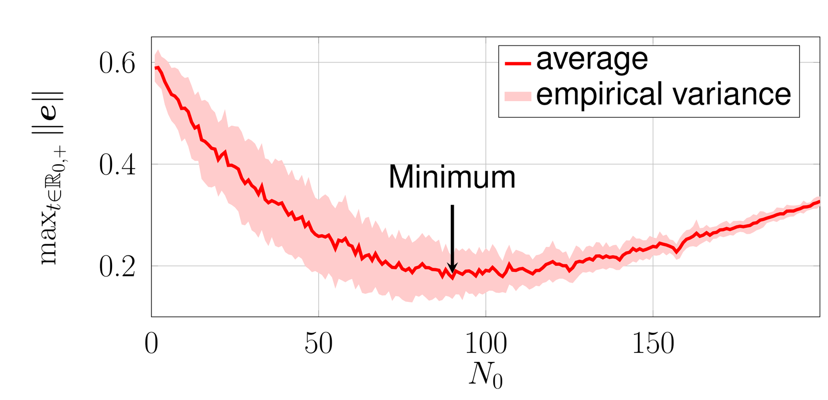



Computational delays ↑ ⇒ Tracking performances ↓

## Simulation: Accuracy-Delay Trade-Off

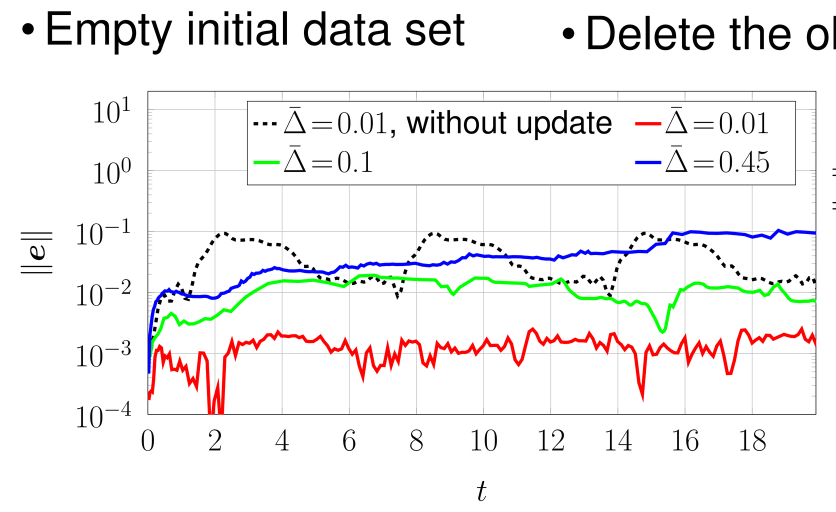
- Offline training: $|\mathbb{D}| = N_0$
- Delay: $\Delta(\cdot) = \bar{\Delta} = cN_0$
- Optimal size of $\mathbb{D}$: $N_0^* \approx 90$

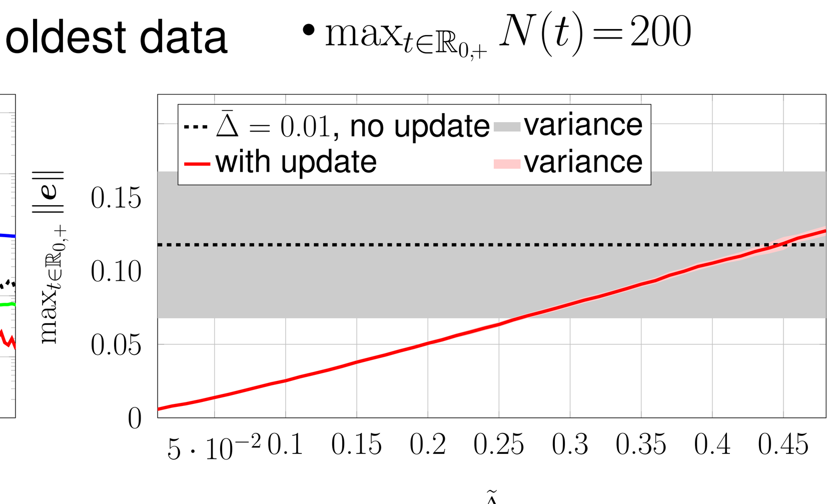



Negative effects from increasing $\bar{\Delta}$ can overcome the benefits from larger $\mathbb{D}$!

## Simulation: Event-triggered Online Learning

- Empty initial data set
- Delete the oldest data
- $\max_{t \in \mathbb{R}_{0,+}} N(t) = 200$




Sufficiently small update times ⇒ Performance increase

### References


[1] J. Umlauft and S. Hirche, “Feedback Linearization based on Gaussian Processes with Event-triggered Online Learning,” *IEEE Transactions on Automatic Control*, vol. 65, no. 10, pp. 4154–4169, 2019.

[2] C. K. Williams and C. E. Rasmussen, *Gaussian Processes for Machine Learning*. MIT press Cambridge, MA, 2006, vol. 2, no. 3.

[3] A. Lederer, J. Umlauft, and S. Hirche, “Uniform Error Bounds for Gaussian Process Regression with Application to Safe Control,” *Advances in Neural Information Processing Systems*, vol. 32, 2019.