A PROOFS

A.1 Proof of Theorem 4.2

PROOF. Let $x_1, x_2 \in \mathbb{D}_\perp$ be two sensitive values, and $x_1 < x_2$, $x_2 - x_1 = t$. We calculate the probability that the output value o_2 of x_2 is greater than the output value o_1 of x_1. We have

$$\begin{aligned}
Pr[o_2 > o_1] &= \sum_{o_2 \in [L,R]} \sum_{o_1 \in [L,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= \sum_{o_2 \in (L,x_1]} \sum_{o_1 \in [L,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&+ \sum_{o_2 \in (x_1,x_2]} \sum_{o_1 \in [L,x_1)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&+ \sum_{o_2 \in (x_1,x_2]} \sum_{o_1 \in [x_1,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&+ \sum_{o_2 \in (x_2,R]} \sum_{o_1 \in [L,x_1)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&+ \sum_{o_2 \in (x_2,R]} \sum_{o_1 \in [x_1,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2]
\end{aligned}$$

For brevity, we denote $q = e^{-\epsilon/2}$, $D = ((1+q-q^{x_1-L+1}-q^{R-x_1+1}) \cdot (1+q-q^{x_2-L+1}-q^{R-x_2+1}))^{-1}$.

$$\begin{aligned}
&\sum_{o_2 \in (L,x_1]} \sum_{o_1 \in [L,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= \sum_{o_2 \in (L,x_1]} \sum_{o_1 \in [L,o_2)} \frac{q^{|o_1-x_1|}}{\sum_{v_1 \in [L,R]} q^{|v_1-x_1|}} \cdot \frac{q^{|o_2-x_2|}}{\sum_{v_2 \in [L,R]} q^{|v_2-x_2|}} \\
&= D \cdot (1-q)^2 \cdot \sum_{o_2 \in (L,x_1]} \sum_{o_1 \in [L,o_2)} q^{x_2-o_2} \cdot q^{x_1-o_1} \\
&= D \cdot (1-q) \cdot \sum_{o_2 \in (L,x_1]} (q^{x_1+x_2-2o_2+1} - q^{x_1+x_2-L-O_2+1}) \\
&= D \cdot (q^{x_1+x_2-2L+1} - q^{x_2-L+1} + \frac{q^{x_2-x_1+1} - q^{x_1+x_2-2L+1}}{1+q}) \\
&= D \cdot (\frac{q^{x_1+x_2-2L+2} + q^{x_2-x_1+1}}{1+q} - q^{x_2-L+1})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{o_2 \in (x_1,x_2]} \sum_{o_1 \in [L,x_1)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= D \cdot (q - q^{x_1-L+1} - q^{x_2-x_1+1} + q^{x_2-L+1})
\end{aligned}$$

$$\begin{aligned}
&\sum_{o_2 \in (x_1,x_2]} \sum_{o_1 \in [x_1,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= D \cdot (1 - ((1-q)(x_2-x_1)+1) \cdot q^{x_2-x_1})
\end{aligned}$$

$$\begin{aligned}
&\sum_{o_2 \in (x_2,R]} \sum_{o_1 \in [L,x_1)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= D \cdot (q^2 - q^{x_1-L+2} - q^{R-x_2+2} + q^{R-x_2+x_1-L+2})
\end{aligned}$$

$$\begin{aligned}
&\sum_{o_2 \in (x_2,R]} \sum_{o_1 \in [x_1,o_2)} Pr[O = o_1|x_1] \cdot Pr[O = o_2|x_2] \\
&= D \cdot (q - q^{R-x_2+1} - \frac{q^{x_2-x_1+2} + q^{2R-x1-x2+2}}{1+q})
\end{aligned}$$

By summation, we have

$$\begin{aligned}
Pr[o_2 > o_1] = D \cdot ((1+q)^2 + \frac{q^{x_1+x_2-2L+2} + q^{2R-x_1-x_2+2}}{1+q} + \\
q^{R-L-t+2} - q(q+1)(q^{x_1-L} - q^{R-x_2}))
\end{aligned}$$

$Pr[o_2 > o_1]$ can be regard as a function of x_1, and $Pr[o_2 > o_1]$ minimizes at $x_1 = L$ or $x_1 = R - t$. Without loss of generality, we let $x_1 = L$

$$\begin{aligned}
&Pr[o_2 > o_1] \\
&\geq \frac{(1+q)^2 + \frac{1}{1+q}(q^{t+2} + q^{2R-2L-t+2}) + q^{R-L-t+2}}{(1+q-q-q^{R-L+1}) \cdot (1+q-q^{t+1}-q^{R-L-t+1})} - \\
&\qquad \frac{q(q+1)(1+q^{R-L-t}) + (\frac{2q^2+q+1}{1+q} + (1-q)t) \cdot q^t}{(1+q-q-q^{R-L+1}) \cdot (1+q-q^{t+1}-q^{R-L-t+1})} \\
&= \frac{1+q+\frac{q^{2R-2L-t+2}}{1+q} - q^{R-L-t+1} - (\frac{q^2+q+1}{1+q} + (1-q)t) \cdot q^t}{(1-q^{R-L+1}) \cdot (1+q-q^{t+1}-q^{R-L-t+1})} \\
&\geq \frac{1+q-q^{R-L-t+1} - (\frac{q^2+q+1}{1+q} + (1-q)t) \cdot q^t}{1+q-q^{t+1}-q^{R-L-t+1}} \\
&= 1 - \frac{\frac{1}{1+q} + (1-q) \cdot t}{1+q-q^{t+1}-q^{|D|-t}} \cdot q^t
\end{aligned}$$

Thus,

$$Pr[o_2 > o_1] \geq 1 - \frac{(1-q^2) \cdot t + 1}{(1+q-q^{t+1}-q^{|\mathbb{D}_\perp|-t})(1+q)} \cdot q^t$$

□

A.2 Proof of Theorem 4.3

PROOF. Assume that $x_1, x_2 \in \mathbb{D}_\perp$ are two sensitive values, $o_1, o_2 \in \mathbb{D}_\perp$ are their corresponding output. And w.l.o.g, we let $x_2 - x_1 = t$. The perturbation mechanism of GRR is

$$Pr[GRR(x) = o] = \begin{cases} p_1 = \frac{e^\epsilon}{|\mathbb{D}_\perp|+e^\epsilon-1}, o = x \\ p_2 = \frac{1}{|\mathbb{D}_\perp|+e^\epsilon-1}, o \neq x \end{cases} \quad (2)$$

We have

$$Pr[o_2 > o_1 | x_2 > x_1] = Pr[o_1 = x_1 \wedge o_2 = x_2]$$
$$+ Pr[o_1 = x_1 \wedge o_2 > x_1 \wedge o_2 \neq x_2] + Pr[o_2 = x_2 \wedge o_1 < x_2 \wedge o_1 \neq x_1]$$
$$+ Pr[o_1 < o_2 \wedge o_1 \neq x_1 \wedge o_2 \neq x_2]$$
$$= p_1^2 + p_1 p_2 (R - x_1 - 1) + p_1 p_2 \cdot (x_2 - L - 1)$$
$$+ p_2^2 \cdot (\frac{1}{2} \cdot |\mathbb{D}_\perp|(|\mathbb{D}_\perp| - 1) - (R - x_1 - 1) - (x_2 - L - 1) - 1)$$
$$= p_1^2 + p_1 p_2 \cdot (|\mathbb{D}_\perp| + t - 3) + p_2^2 \cdot (\frac{1}{2}|\mathbb{D}_\perp|(|\mathbb{D}_\perp| - 3) - t + 2)$$

□

A.3 Proof of Theorem 4.6

Proof. Let $x_1, x_2 \in \mathbb{D}$ be two sensitive data points. Specifically, $x_1 < x_2$, and $t = x_2 - x_1$, $T = \lfloor \frac{t}{\theta} \rfloor$. We calculate the probability that the output value o_2 of x_2 is greater than the output value o_1 of x_1. Denote $\hat{\mathcal{P}}_{m(x_1)}$ as the partition that x_1 is mapped, $\hat{\mathcal{P}}_{m(x_2)}$ as the partition that x_2 is mapped. If $\hat{\mathcal{P}}_{m(x_1)}$ and $\hat{\mathcal{P}}_{m(x_2)}$ are two different partitions and $\hat{\mathcal{P}}_{m(x_1)}$ is on the left of $\hat{\mathcal{P}}_{m(x_2)}$, then it has

$$Pr[o_2 > o_1 | \hat{\mathcal{P}}_{m(x_2)} > \hat{\mathcal{P}}_{m(x_1)}] = 1$$

According to the result of Theorem 4.2, we can directly get the probability that $\hat{\mathcal{P}}_{m(x_1)}$ is on the left of $\hat{\mathcal{P}}_{m(x_2)}$ as

$$Pr[\hat{\mathcal{P}}_{m(x_2)} > \hat{\mathcal{P}}_{m(x_1)}] \geq 1 - \frac{(1-q^2) \cdot T + 1}{(1 + q - q^{T+1} - q^{k-T})(1+q)} \cdot q^T$$

The probability that x_1 and x_2 are mapped to the same partition is

$$Pr[\hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}]$$
$$= \sum_{\mathcal{P}_o \in [\mathcal{P}_1, \mathcal{P}_k]} Pr[RM(x_1) = \mathcal{P}_o] \cdot Pr[RM(x_2) = \mathcal{P}_o]$$
$$= \sum_{\mathcal{P}_o \in [[\mathcal{P}_1, \mathcal{P}_k]]} \frac{q^{|m(x_1) - o|}}{\sum_{\mathcal{P}_c \in [\mathcal{P}_1, \mathcal{P}_k]} q^{|m(x_1) - c|}} \cdot \frac{q^{|m(x_2) - o|}}{\sum_{\mathcal{P}_c \in [\mathcal{P}_1, \mathcal{P}_k]} q^{|m(x_2) - c|}}$$

By calculating the above probability summation formula, we can get

$$Pr[\hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}] \geq \frac{(1-q)^2 (T+1)}{(1+q)^2} \cdot q^T$$

Since $x_2 > x_1$, when o_1 and o_2 are in the same partition, it has

$$Pr[o_2 > o_1 | \hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}] > Pr[o_1 > o_2 | \hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}]$$

So we can approximate the probability $Pr[o_2 > o_1, \hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}]$ as

$$Pr[o_2 > o_1, \hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}] \geq \frac{(1-q)^2 (T+1)}{2(1+q)^2} \cdot q^T$$

Finally, we have

$$Pr[o_2 > o_1]$$
$$= Pr[o_2 > o_1, \hat{\mathcal{P}}_{m(x_2)} = \hat{\mathcal{P}}_{m(x_1)}] + Pr[o_2 > o_1, \hat{\mathcal{P}}_{m(x_2)} > \hat{\mathcal{P}}_{m(x_1)}]$$
$$\geq 1 - \frac{((1-q^2) \cdot T + 1) \cdot q^T}{(1 + q - q^{T+1} - q^{k-T})(1+q)} + \frac{(1-q)^2 (T+1) \cdot q^T}{2(1+q)^2}$$

□

A.4 Proof of Theorem 4.11

Proof. Let v_1 and v_2 are two values with $v_1 - v_2 = t$, $[l, u] \subseteq \mathbb{D}_\perp$ is the range of output value o after adding bounded discrete Laplace noise. Denote N_1 and N_2 as the random noise sampling from bounded discrete Laplace noise $Lap_{\mathbb{Z}}(\frac{1}{\epsilon})$, where $N_1 \in [l - v_1, u - v_1]$ and $N_2 \in [l - v_2, u - v_2]$. We prove that

$$\frac{Pr[v_1 + N_1 = o]}{Pr[v_2 + N_2 = o]} = \frac{Pr[N_1 = o - v_1]}{Pr[N_2 = o - v_2]} \leq e^{2t\epsilon}$$

When $v_1 < v_2 < l < u$, it has $l - v_1 > 0$, $u - v_1 > 0$, $l - v_2 > 0$, and $u - v_2 > 0$. Then we have the ratio of $Pr[N_1 = o - v_1]$ and $Pr[N_2 = o - v_2]$ is

$$\frac{Pr[N_2 = o - v_2]}{Pr[N_1 = o - v_1]} = \frac{e^{-(l-v_1)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{e^{-(l-v_2)\epsilon}(1 - e^{-(u-l+1)\epsilon})} \cdot \frac{e^{-|o-v_2|\epsilon}}{e^{-|o-v_1|\epsilon}}$$
$$= e^{((o-v_1)-(o-v_2))\epsilon} \cdot e^{((l-v_2)-(l-v_1))\epsilon}$$
$$= e^{(v_2-v_1)\epsilon} \cdot e^{(v_1-v_2)\epsilon} = 1$$

When $l < u < v_1 < v_2$, it has $l - v_1 < 0$, $u - v_1 < 0$, $l - v_2 < 0$, and $u - v_2 < 0$. Then we have the ratio of $Pr[N_1 = o - v_1]$ and $Pr[N_2 = o - v_2]$ is

$$\frac{Pr[N_1 = o - v_1]}{Pr[N_2 = o - v_2]} = \frac{e^{(u-v_2)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{e^{(u-v_1)\epsilon}(1 - e^{-(u-l+1)\epsilon})} \cdot \frac{e^{-|o-v_1|\epsilon}}{e^{-|o-v_2|\epsilon}}$$
$$= e^{((u-v_2)-(u-v_1))\epsilon} \cdot e^{(-(o-v_2)+(o-v_1))\epsilon}$$
$$= e^{(v_1-v_2)\epsilon} \cdot e^{(v_2-v_1)\epsilon} = 1$$

When $l < v_1 < v_2 < u$, it has $l - v_1 < 0$, $u - l > 0$, $l - v_2 < 0$, and $u - v_2 > 0$. Then we have the ratio of $Pr[N_1 = o - v_1]$ and $Pr[N_2 = o - v_2]$ is

$$\frac{Pr[N_1 = o - v_1]}{Pr[N_2 = o - v_2]}$$
$$= \frac{1 - e^{-(-(l-v_2)+1)\epsilon} - e^{-(u-v_2+1)\epsilon} + e^{-\epsilon}}{1 - e^{-(-(l-v_1)+1)\epsilon} - e^{-(u-v_1+1)\epsilon} + e^{-\epsilon}} \cdot \frac{e^{-|o-v_1|\epsilon}}{e^{-|o-v_2|\epsilon}}$$
$$\leq e^{t\epsilon} \cdot e^{t\epsilon} \cdot \frac{e^{-t\epsilon} - e^{-(-(l-v_2)+1+t)\epsilon} - e^{-(u-v_2+1+t)\epsilon} + e^{-(1+t)\epsilon}}{1 - e^{-(-(l-v_1)+1)\epsilon} - e^{-(u-v_1+1)\epsilon} + e^{-\epsilon}}$$
$$\leq e^{2t\epsilon}$$

When $v_1 < l < v_2 < u$, it has $l - v_1 > 0$, $u - v_1 > 0$, $l - v_2 < 0$, and $u - v_2 > 0$. Then we have the ratio of $Pr[N_1 = o - v_1]$ and

$Pr[N_2 = o - v_2]$ is

$$
\begin{aligned}
&\frac{Pr[N_2 = o - v_2]}{Pr[N_1 = o - v_1]} \\
&= \frac{e^{-|o-v_2|\epsilon}}{e^{-|o-v_1|\epsilon}} \cdot \frac{e^{-(l-v_1)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{1 - e^{-(-(l-v_2)+1)\epsilon} - e^{-(u-v_2+1)\epsilon} + e^{-\epsilon}} \\
&= e^{((o-v_1)-|o-v_2|)\epsilon} \cdot \frac{e^{-(l-v_1)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{1 - e^{((l-v_2)-1)\epsilon} - e^{-(u-v_2+1)\epsilon} + e^{-\epsilon}} \\
&\leq e^{t\epsilon} \cdot \frac{e^{-(l-v_1)\epsilon}}{e^{(l-v_2)\epsilon}} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{-(l-v_2)\epsilon} - e^{-\epsilon} - e^{(2v_2-(u+l)-1)\epsilon} + e^{(v_2-l-1)\epsilon}} \\
&= e^{t\epsilon} \cdot e^{(v_1+v_2-2l)\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{-(l-v_2)\epsilon} - e^{-\epsilon} - e^{2v_2-(u+l)-1} + e^{(v_2-l-1)\epsilon}} \\
&\leq e^{2t\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{-(l-v_2)\epsilon} - e^{-\epsilon} - e^{(2v_2-(u+l)-1)\epsilon} + e^{(v_2-l-1)\epsilon}} \\
&\leq e^{2t\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{1 - e^{-(u-l+1)\epsilon}} = e^{2t\epsilon}
\end{aligned}
$$

When $l < v_1 < u < v_2$, it has $l - v_1 < 0$, $u - v_1 > 0$, $l - v_2 < 0$, and $u - v_2 < 0$. Then we have the ratio of $Pr[N_1 = o - v_1]$ and $Pr[N_2 = o - v_2]$ is

$$
\begin{aligned}
&\frac{Pr[N_1 = o - v_1]}{Pr[N_2 = o - v_2]} \\
&= \frac{e^{-|o-v_1|\epsilon}}{e^{-|o-v_2|\epsilon}} \cdot \frac{e^{(u-v_2)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{1 - e^{-(-(l-v_1)+1)\epsilon} - e^{-(u-v_1+1)\epsilon} + e^{-\epsilon}} \\
&= e^{((v_2-o)-|o-v_1|)\epsilon} \cdot \frac{e^{(u-v_2)\epsilon}(1 - e^{-(u-l+1)\epsilon})}{1 - e^{((l-v_1)-1)\epsilon} - e^{-(u-v_1+1)\epsilon} + e^{-\epsilon}} \\
&\leq e^{t\epsilon} \cdot \frac{e^{(u-v_2)\epsilon}}{e^{-(u-v_1)\epsilon}} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{(u-v_1)\epsilon} - e^{(u+l-2v_1-1)\epsilon} - e^{-\epsilon} + e^{(u-v_1-1)\epsilon}} \\
&= e^{t\epsilon} \cdot e^{(2u-v_1-v_2)\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{(u-v_1)\epsilon} - e^{(u+l-2v_1-1)\epsilon} - e^{-\epsilon} + e^{(u-v_1-1)\epsilon}} \\
&\leq e^{2t\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{e^{(u-v_1)\epsilon} - e^{(u+l-2v_1-1)\epsilon} - e^{-\epsilon} + e^{(u-v_1-1)\epsilon}} \\
&\leq e^{2t\epsilon} \cdot \frac{1 - e^{-(u-l+1)\epsilon}}{1 - e^{-(u-l+1)\epsilon}} = e^{2t\epsilon}
\end{aligned}
$$

In summary, for any output range $[l, u] \subseteq \mathbb{D}_{\perp}$, the ratio of $Pr[N_1 = o - v_1]$ and $Pr[N_2 = o - v_2]$ satisfies

$$\frac{Pr[v_1 + N_1 = o]}{Pr[v_2 + N_2 = o]} = \frac{Pr[N_1 = o - v_1]}{Pr[N_2 = o - v_2]} \leq e^{2t\epsilon}$$

$\square$