P1. a. $\alpha \models \beta$ iff $Mods(\alpha) \subseteq Mods(\beta)$.

$\alpha \wedge \neg\beta$ is inconsistent iff $Mods(\alpha \wedge \neg\beta) = \emptyset$

It should be clear that $Mods(\alpha) \subseteq Mods(\beta)$ iff the set $Mods(\alpha) \cap Mods(\beta) = \emptyset$. By properties of the Mods function, we have $Mods(\alpha) \cap Mods(\beta)$ is equal to $Mods(\alpha \wedge \neg\beta)$. Hence, $Mods(\alpha) \subseteq Mods(\beta)$ iff $Mods(\alpha \wedge \neg\beta) = \emptyset$, which proves the theorem.

b. Assuming we have accepted the Refutation theorem, and by showing that $\alpha \Rightarrow \beta$ is valid iff $\alpha \wedge \neg\beta$ is inconsistant.

Since sentence α is valid iff its negation $\neg\alpha$ is inconsistent.

$Mods(\alpha) = \Omega$ iff $Mods(\neg\alpha) = \emptyset$, then $Mods(\neg\alpha) = \overline{Mods(\alpha)}$

All we have to show now is that $\alpha \Rightarrow \beta$ is equivalent to the negation of $\alpha \wedge \neg\beta$, which proves that $\alpha \Rightarrow \beta$ is valid iff $\alpha \wedge \neg\beta$ is Inconsistent.

$\neg(\alpha \wedge \neg\beta)$ is equivalent to $\neg\alpha \vee \beta$ by law and double negation.

$\neg\alpha \vee \beta$ is equivalent to $\alpha \Rightarrow \beta$ by definition of implication.

This completes the proof.

P1. c. By Deduction theorem, if $\alpha \models \beta$, then $\alpha \Rightarrow \beta$ is valid

so, $\alpha \Rightarrow \alpha \wedge \beta$ is valid. Besides, $\alpha \wedge \beta \Rightarrow \alpha$ is valid

so. $\alpha \wedge \beta \Leftrightarrow \alpha$. $\alpha \wedge \beta$ is equivalent to α.

d. By Deduction theorem, if $\alpha \models \beta$, then $\alpha \Rightarrow \beta$ is valid.

so $\alpha \vee \beta \Rightarrow \beta$ is valid. Besides $\beta \Rightarrow \alpha \vee \beta$ is valid.

So $\alpha \vee \beta \Leftrightarrow \beta$. $\alpha \vee \beta$ is equivalent to β.

P2. if $Pr(\alpha | \beta \wedge \gamma) = Pr(\alpha | \gamma)$

then $$Pr(\alpha | \gamma) Pr(\beta | \gamma) = Pr(\alpha | \beta \wedge \gamma) Pr(\beta | \gamma)$$

$$= \frac{Pr(\alpha \wedge \beta \wedge \gamma)}{Pr(\beta | \gamma) \cdot Pr(\gamma)} \cdot Pr(\beta | \gamma)$$

$$= \frac{Pr(\alpha \wedge \beta \wedge \gamma)}{Pr(\gamma)}$$

$$= Pr(\alpha \wedge \beta | \gamma)$$

so. $Pr(\alpha | \beta \wedge \gamma) = Pr(\alpha | \gamma)$

is equivalent to $Pr(\alpha \wedge \beta | \gamma) = Pr(\alpha | \gamma) Pr(\beta | \gamma)$.

P3. Let F denote fail in the test.

C denote cheating on partner.

then $P(F|C) = 0.98$ $P(F|\neg C) = 0.02$

with the new information, $P(C) = 1/10000 = 0.0001$

$$P(C|F) = \frac{P(F|C)P(C)}{P(F|C)P(C) + P(F|\neg C)P(\neg C)}$$

$$= \frac{0.98 \times 0.0001}{0.98 \times 0.0001 + 0.02 \times (1 - 0.0001)} \approx 0.49\%$$

So ~~th~~ with the new information. the probability of cheating given that failed the test is too small. I can prove that I am not cheating on my partner.

If I fail two out of three tests. then

$$P(C|F \wedge F \wedge \neg F) = \frac{P(F \wedge F \wedge \neg F|C)P(C)}{P(F \wedge F \wedge \neg F|C)P(C) + P(F \wedge F \wedge \neg F|\neg C)P(\neg C)}$$

$$= \frac{P(F|C)P(F|C)P(\neg F|C)P(C)}{P(F|C)P(F|C)P(\neg F|C)P(C) + P(F|\neg C)P(F|\neg C)P(\neg F|\neg C)P(\neg C)}$$

$\approx 0.49\%$. so the ~~prob~~ probability is still small.

So I can prove that I am not cheating on my partner.

4. Since $P(\alpha|\beta,r) = \dfrac{P(\alpha,\beta,r)}{P(\beta,r)}$

$$= \frac{P(\beta,r|\alpha)P(\alpha)}{P(\beta,r)}$$

$$= \frac{P(r|\alpha,\beta)P(\beta|\alpha)P(\alpha)}{P(\beta|r)P(r)}$$

So. 2 is sufficient.

1 is not sufficient we can not get $P(r|\alpha,\beta)$.

3 is not sufficient. we can not get $P(\beta,r)$.

If β and r are conditionally independent given α.

we have $P(\beta,r|\alpha) = P(\beta|\alpha)P(r|\alpha)$

~~Then 1 is sufficient now~~. since $\dfrac{P(\beta,r|\alpha)P(\alpha)}{P(\beta,r)}$

Then 1.3 are still not sufficient $= \dfrac{P(\beta|\alpha)P(r|\alpha)P(\alpha)}{P(\beta,r)}$.

based on 1 or 3, we just know $P(\alpha), P(\alpha,\beta), P(\alpha,r), P(\beta|\alpha)$
$P(r|\alpha)$. $P(\beta,r|\alpha)$ $P(\alpha,\beta,r)$. $P(\beta|\alpha,r)$ $P(r|\alpha,\beta)$

5. Assume that $I(X, Z, Y \cup W)$ holds. then

$$Pr(X, Y, W \mid Z) = Pr(X \mid Z) \times Pr(Y, W \mid Z)$$

$$Pr(X, Y \mid W, Z) = \frac{Pr(X, Y, W \mid Z)}{Pr(W \mid Z)}$$

$$= \frac{Pr(X \mid Z) \times Pr(Y, W \mid Z)}{Pr(W \mid Z)}$$

$$= Pr(X \mid Z) \times Pr(Y \mid W, Z)$$

Since $Pr(X, Y \mid W, Z) = Pr(X \mid W, Z) \times Pr(Y \mid W, Z)$

We need proof $Pr(X \mid Z) = Pr(X \mid W, Z)$

Since $Pr(X \mid Z)$ is the expectation over Y and W of $P(X \mid Y, W, Z)$, and $P(X \mid Z, W)$ is the expectation over Y of the same quantity. But since we assumed X is conditionally independent of Y and W, taking the expectation over Y or W or both does not change the value.

So $Pr(X \mid Z) = Pr(X \mid W, Z)$.

At all, ~~Pr(X,Y~~ $I(X, Z, Y \cup W) \Rightarrow I(X, Z \cup W, Y)$

5. ~~Contract~~ contraction:

Since $P(X,Y,W|Z) = P(X|Y,W,Z)P(Y,W|Z)$

and $P(X,Y,W|Z) = P(X|Y,Z)P(Y,W|Z)$ with $I(X, Z\cup Y, W)$

and $P(X,Y,W|Z) = P(X|Z)P(Y,W|Z)$ with $I(X, Z, Y)$

Hence, $I(X, Z, Y\cup W)$ holds.

Counterexample of intersection property.

For $I_{pr}(X, Z\cup W, Y)$ and $I_{pr}(X, Z\cup Y, W)$ only if $I_{pr}(X, Z, Y\cup W)$.

Assuming we play a coin game. we need to flip two coins (A&B)

The result X is the number of the coins face up.

Z denotes the result of coin A.

W denotes the coin B face up, Y denotes the coin B face down.

Then the intersection property can not hold in this situation.

6. 1. Markovian assumptions.

$I(A, \phi, \{B, E\})$ $\quad$ $I(E, B, \{A, C, F, G\})$

$I(B, \phi, \{A, C\})$ $\quad$ $I(F, \{C, D\}, \{A, B, E\})$

$I(C, A, \{B, D, E\})$ $\quad$ $I(G, F, \{A, B, C, D, E, H\})$

$I(D, \{A, B\}, \{C, E\})$ $\quad$ $I(H, \{E, F\}, \{A, B, C, D, G\})$

2. $Pr(a, b, c, d, e, f, g, h)$

$= Pr(A)\,Pr(B)\,Pr(C|A)\,Pr(D|A,B)\,Pr(E|B)\,Pr(F|C,D)$
$Pr(G|F)\,Pr(H|E,F)$.

3. Since A and B are independent,

$Pr(A=0, B=0) = Pr(A=0)\,Pr(B=0) = 0.8 \times 0.3 = 0.24$

Since $I(A, \phi, \{B, E\})$ and $I(E, B, \{A, C, F, G\})$,

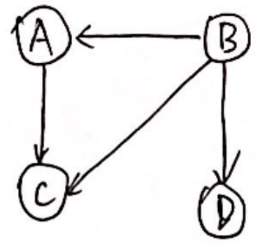
$Pr(E=1 | A=1) = Pr(E=1) = 0.7 \times 0.1 + 0.3 \times 0.9 = 0.34$.

4. (a) True, divergent B is in $\{B, H\}$. ~~convergent H is in $\{B, H\}$~~ sequential $A \to C \to F$ and $A \to D \to F$ are closed so, B and F are closed. Hence, dsep(A, BH, E).

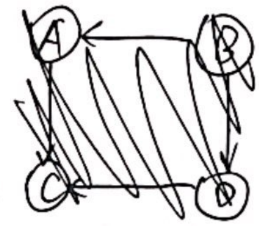
(b) ~~False~~ True. divergent F is open and convergent H is ~~open~~ closed.

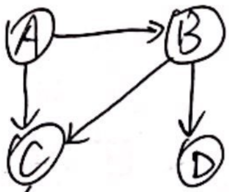
(c) False. sequential $D \to F \to G$ is closed. sequential E is not in $\{F\}$ so E is ~~closed~~ open. Hence we have path $B \to E \to H$.

7.



dsep (A, BC, D)





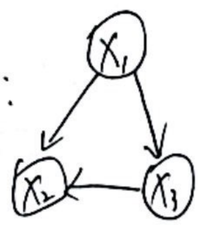
dsep (A, BC, D)

~~dsep (A, BC, D)~~

~~This is a perfect map which is both I-map and D-map.~~

~~$I(X_1, X_2X_3, X_4) \Longleftrightarrow dsep(X_1, X_2X_3, X_4)$~~

8.

G: 

$I_{pr}(G) = \{\phi\}$ G is a I-map.

G: 

$I_{pr}(G) = (X_1, X_2, X_3)$ G is a D-map.

9. Let Y denotes the whole set of the variables.

Then W_1 denotes $Y - B_1 - X$. W_2 denotes $Y - B_2 - X$.

W_3 denotes $Y - (B_1 \cap B_2) - X$. B_0 denotes $B_1 \cap B_2$.

Since $I_{pr}(X, B_1, W_1)$ and $I_{pr}(X, B_2, W_2)$ holds,

$I_{pr}(X, B_0 \cup W_2, W_1)$ and $I_{pr}(X, B_0 \cup W_1, W_2)$

Hence, based on intersection axiom,

$I_{pr}(X, B_0, W_1 \cup W_2)$

So, $I_{pr}(X, B_1 \cap B_2, Y - (B_1 \cap B_2) - X)$

$B_1 \cap B_2$ is also a Markov blanket for X.

10. dsep (B, ∅, {E, T, N})

dsep (E, ∅, {B, T, N})

dsep (T, ∅, {B, E, N, M})

dsep (N, ∅, {B, E, T, J})

dsep (J, ∅, N)

dsep (M, ∅, T)

dsep (B, E, {T, N})

dsep (B, J, N)

dsep (B, T, {E, N})

dsep (B, M, {T})

dsep (B, N, {E, T})

dsep (B, {E, J}, N)

dsep (B, {E, T}, N)

dsep (B, {E, M}, T)

dsep (B, {E, N}, T)

dsep (B, {J, T}, N)

dsep (B, {J, M}, ∅)

dsep (B, {J, N}, ∅)

dsep (B, {T, M}, ∅)

dsep (B, {T, N}, ∅)

dsep (B, {M, N}, T)

dsep (B, {T, M, N}, ∅)

dsep (B, {J, M, N}, ∅)

dsep (B, {J, T, N} ∅)

dsep (B, {J, T, M} ∅)

dsep (B, {E, M, N} T)

dsep, (B, {E, T, N} ∅)

dsep (B, {E, T, M} ∅)

dsep (B, {E, J, N} ∅)

dsep (B, {E, J, M} ∅)

dsep (B, {E, J, T}, N)

dsep (B, {J, T, M, N} ∅)

dsep (B {E, T, M, N} ∅)

dsep (B, {E, J, M, N} ∅)

dsep (B, {E, J, T, N} ∅)

dsep (B, {E, J, T, M}, ∅)

dsep(J, B, N)

(J, E, N)

(J, T, N)

(J, N, ∅)

(J, M, ∅)

(J, {B, E}, N)

(J, {B, T}, N)

(J, {E, T}, N)

(J, {B, E, T}, N)

(T, B, {E, N, M})

(T, E, {B, N, M})

(T, J, N)

(T, M, {B, E, N})

(T, N, {B, E, M})

(T, {B, E}, {M, N})

(T, {B, J}, N)

(T, {B, M}, {E, N})

(T, {B, N}, {E, M})

(T, {E, J}, N)

(T, {E, M}, {B, N})

(T, {E, N}, {B, M})

(T, {J, M}, N)

(T, {J, N}, ∅)

(T, {M, N}, {B, E})

(T, {J, M, N}, ∅)

(T, {E, M, N}, B)

(T, {E, J, N}, ∅)

(T, {E, J, M}, N)

(T, {B, M, N}, E)

(T, {B, J, N}, ∅)

(T, {B, J, M}, N)

(T, {B, E, N}, M)

(T, {B, E, M}, N)

(T, {B, E, J}, N)

(T, {E, J, M, N}, ∅)

(T, {B, J, M, N}, ∅)

(T, {B, E, M, N}, ∅)

(T, {B, E, J, N}, ∅)

(T, {B, E, J, M}, N)

B → E