Flying Calligrapher: Contact-Aware Motion and Force Planning and Control for Aerial Manipulation

I. Newton Euler Equation

We define the position of the end-effector as $\boldsymbol{p}_{ee}$ and the position of the vehicle center of gravity (cg) as $\boldsymbol{p}_{cg}$. Their velocity are denoted as $\boldsymbol{v}_{ee}$ and $\boldsymbol{v}_{cg}$, respectively. We denote $\boldsymbol{\tau}_a = [\boldsymbol{f}_a, \boldsymbol{m}_a]$ as the control wrench and $\boldsymbol{\tau}_c = [\boldsymbol{f}_c, \boldsymbol{m}_c]$ as the contact wrench, where $\boldsymbol{f}$ and $\boldsymbol{m}$ denote the force component and torque component correspondingly. Denote m as the vehicle mass; J as the moment of inertia at the vehicle center of mass; $\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}$ as the relative translation between the vehicle body and the tip of the end-effector. All other definitions follow the main manuscripts.

Given the structure of the UAM, we have

$$ {}_{\mathcal{B}}\boldsymbol{p}_{cg} = {}_{\mathcal{B}}\,\boldsymbol{p}_{ee} - \boldsymbol{t} \tag{1}$$

$$ {}_{\mathcal{W}}\boldsymbol{p}_{cg} = {}_{\mathcal{W}}\,\boldsymbol{p}_{ee} - \boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t} \tag{2}$$

$$ {}_{\mathcal{W}}\boldsymbol{v}_{cg} = {}_{\mathcal{W}}\,\boldsymbol{v}_{ee} - \boldsymbol{\Omega}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t} \tag{3}$$

$$ {}_{\mathcal{W}}\dot{\boldsymbol{v}}_{cg} = {}_{\mathcal{W}}\,\dot{\boldsymbol{v}}_{ee} - (\dot{\boldsymbol{\Omega}}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t} + \boldsymbol{\Omega}\boldsymbol{\Omega}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}) \tag{4}$$

Here $\boldsymbol{\Omega}$ is the skew matrix corresponding to $\boldsymbol{w}$ in the body frame.

For the force component, under the world frame, we have

$$ {}_{\mathcal{W}}\boldsymbol{f}_a + {}_{\mathcal{W}}\,\boldsymbol{f}_c - {}_{\mathcal{W}}\,\boldsymbol{g} = m\dot{\boldsymbol{v}}_{cg} \tag{5}$$

Combining (4) and (5), we have:

$$ \begin{aligned} {}_{\mathcal{W}}\boldsymbol{f}_a + {}_{\mathcal{W}}\,\boldsymbol{f}_c - {}_{\mathcal{W}}\,\boldsymbol{g} &= m({}_{\mathcal{W}}\dot{\boldsymbol{v}}_{ee} - (\dot{\boldsymbol{\Omega}}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t} + \boldsymbol{\Omega}\boldsymbol{\Omega}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t})) \\ &= m{}_{\mathcal{W}}\dot{\boldsymbol{v}}_{ee} + m[\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}]_\times\dot{\boldsymbol{\omega}} + m[\boldsymbol{\Omega}\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}]_\times\boldsymbol{\omega} \end{aligned} \tag{6}$$

Transforming back to the body frame (multiply $\boldsymbol{R}_{\mathcal{B}}^{\mathcal{W}}$) at both sides, we get the first half of the final equation.

$$ \begin{aligned} {}_{\mathcal{E}}\boldsymbol{f}_a + {}_{\mathcal{E}}\,\boldsymbol{f}_c = & m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}\,{}_{\mathcal{W}}\dot{\boldsymbol{v}}_{ee} \\ & + m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}[\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}]_\times\dot{\boldsymbol{\omega}} \\ & + m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}[\boldsymbol{\omega}_\times\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}]_\times\boldsymbol{\omega} + {}_{\mathcal{E}}\,\boldsymbol{g} \end{aligned} \tag{7}$$

In terms of the torque component, we have:

$$ \begin{aligned} {}_{\mathcal{B}}\boldsymbol{m}_a + {}_{\mathcal{B}}\,\boldsymbol{m}_c - {}_{\mathcal{B}}\,\boldsymbol{g} &= \boldsymbol{J}_{cg}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\boldsymbol{J}_{cg}\boldsymbol{\omega}) \\ &= \boldsymbol{J}_{cg}\dot{\boldsymbol{\omega}} - [\boldsymbol{J}_{cg}\boldsymbol{\omega}]_\times\boldsymbol{\omega} \end{aligned} \tag{8}$$

Here $\boldsymbol{J}_{cg} = \boldsymbol{J}$ is the vehicle moment of inertia with respect to the cg in the body frame and is a constant value. Denote $\boldsymbol{J}_{ee}$ as the moment of inertia with respect to the ee in the body frame, we have:

$$ \begin{aligned} \boldsymbol{J}_{ee} &= \boldsymbol{J}_{cg} + m\boldsymbol{t}_\times\boldsymbol{t}_\times^\top \\ &= \boldsymbol{J}_{cg} - m\boldsymbol{t}_\times\boldsymbol{t}_\times \end{aligned} \tag{9}$$

Focusing on the ee, we have:

$$ {}_{\mathcal{E}}\boldsymbol{m}_a + {}_{\mathcal{E}}\,\boldsymbol{m}_c - {}_{\mathcal{E}}\,\boldsymbol{g} = \boldsymbol{J}_{ee}\dot{\boldsymbol{\omega}} - [\boldsymbol{J}_{ee}\boldsymbol{\omega}]_\times\boldsymbol{\omega} \tag{10}$$

Combining (9) and (10), we have

$$ \begin{aligned} &(\boldsymbol{J}_{cg} - m\boldsymbol{t}_\times\boldsymbol{t}_\times)\dot{\boldsymbol{\omega}} - [(\boldsymbol{J}_{cg} - m\boldsymbol{t}_\times\boldsymbol{t}_\times)\boldsymbol{\omega}]_\times\boldsymbol{\omega} \\ &= {}_{\mathcal{E}}\,\boldsymbol{m}_a + {}_{\mathcal{E}}\,\boldsymbol{m}_c - {}_{\mathcal{E}}\,\boldsymbol{g} \end{aligned} \tag{11}$$

Combining (6) and (11), we have the final dynamic equation:

$$ \boldsymbol{M}\dot{\boldsymbol{v}} + \boldsymbol{C}\boldsymbol{v} + \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{B}}\,\mathcal{B}}\boldsymbol{g} = \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{B}}\,\mathcal{B}}\boldsymbol{\tau}_a + \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{C}}\,\mathcal{C}}\boldsymbol{\tau}_c \tag{12}$$

where

$$ \boldsymbol{M} = \begin{bmatrix} m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}} & m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}[\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times \\ \boldsymbol{0}_{3\times3} & \boldsymbol{J} - m[\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times[\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times \end{bmatrix} \tag{13}$$

$$ \boldsymbol{C} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}[\boldsymbol{\omega}_\times\boldsymbol{R}_{\mathcal{W}}^{\mathcal{B}}\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times \\ \boldsymbol{0}_{3\times3} & -[(\boldsymbol{J} - m[\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times[\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_\times)\boldsymbol{\omega}]_\times \end{bmatrix} \tag{14}$$