3、Liner discrimination. → Fisher Classification

Assumption

① $X = (x_1, x_2, x_3, \cdots, x_{N_1})^T$

② $Y = (y_1, y_2, y_3, \cdots y_{N_2})^T$

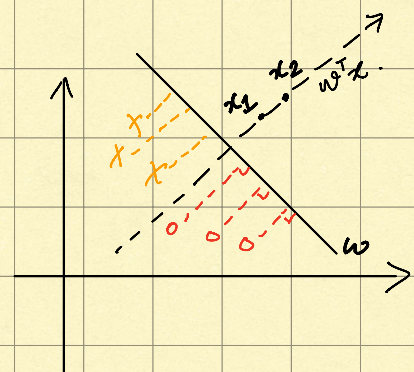
③ $y_i \in \{+1, -1\}$ $(+1 \to \text{class } C_1 ; -1 \to \text{class } C_2)$

④ $|x_{C_1}| + |x_{C_2}| = N_1 + N_2 = N$

Motivation

a.类内:紧凑 $\Rightarrow$

b.类间:分散



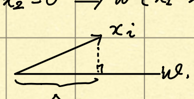
$\|w\| = 1.$

Fig. 1.

相关系数推导

(1) for fig.1: x_1, x_2 位于 $w^T x$: $w^T x_1 - w^T x_2 = 0 \longrightarrow w^T(x_1 - x_2) = 0$

for x_i: x_i 在w上的投影: $w^T x_i$



$\Delta = |x_i| \cdot \cos\theta$ —— ①

$x_i w = |x_i| |w| \cos\theta$ —— ②

$\|w\| = 1$ —— ③

①,②,③: $\Delta = \vec{x_i} \vec{w}$ (向量内积)

(2). 均值 Z

$z_i = w^T x_i \longrightarrow \bar{z}_i = \frac{1}{N}\sum_{i=1}^{N} z_i \longrightarrow \bar{z}_i = \frac{1}{N}\sum_{i=1}^{N} w^T x_i$

$$Var[z_i] = \frac{1}{N}\sum_{i=1}^{N}(z_i - \bar{z}_i)(z_i - \bar{z}_i)^T$$

$$= \frac{1}{N}\sum_{i=1}^{N}(w^T x_i - \bar{z}_i)(w^T x_i - \bar{z}_i)^T \longrightarrow Var[z_i] = S_z.$$

$\therefore C_1: \bar{z}_1 = \frac{1}{N}\sum_{i=1}^{N} w^T x_i ;\quad S_1 = \frac{1}{N}\sum_{i=1}^{N}(w^T x_i - \bar{z}_1)(w^T x_i - \bar{z}_1)^T$

$C_2: \bar{z}_2 = \frac{1}{N}\sum_{i=1}^{N} w^T x_i ;\quad S_2 = \frac{1}{N}\sum_{i=1}^{N}(w^T x_i - \bar{z}_2)(w^T x_i - \bar{z}_2)^T$

(3).类内类间的表示

①.2个样本投影到z轴,用样本点均值作差的平方 ⟶ 类间距.

②类内:各自 C_1 与 C_2 的方差加和,表示它们的 ⟶ 类内

$$\Rightarrow \frac{(\bar{z}_1 - \bar{z}_2)^2}{S_1 + S_2}$$

目标函数

$$J(W)=\frac{(\bar{z}_1-\bar{z}_2)^2}{S_1+S_2} \quad \text{———} \quad ⊛$$

⊛：分子：$(\bar{z}_1-\bar{z}_2)^2=\left(\frac{1}{N_1}\sum_{i=1}^{N_1}W^Tx_i-\frac{1}{N_2}\sum_{i=1}^{N_2}W^Tx_i\right)=\left(W^T\left(\frac{1}{N_1}\sum_{i=1}^{N_1}x_i-\frac{1}{N_2}\sum_{i=1}^{N_2}x_i\right)\right)=\left(W^T(x_{c_1}-x_{c_2})\right)^2=W^T(x_{c_1}-\bar{x}_{c_2})(x_{c_1}-x_{c_2})W$.

分母：$S_1+S_2=\frac{1}{N_1}\sum_{i=1}^{N_1}(W^Tx_i-\bar{z}_1)(W^Tx_i-\bar{z}_1)^T+\frac{1}{N_2}\sum_{i=1}^{N_2}(W^Tx_i-\bar{z}_2)(W^Tx_i-\bar{z}_2)^T$

$$=\frac{1}{N_1}\sum_{i=1}^{N_1}W^T\left(x_i-\boxed{\frac{1}{N_1}\sum_{j=1}^{N_1}x_j}\right)\left(x_i-\boxed{\frac{1}{N_1}\sum_{j=1}^{N_1}x_j}\right)^T(W^T)^T+\cdots\cdots$$

$$=W^T\left[\frac{1}{N_1}\sum_{i=1}^{N_1}(x_i-\bar{x}_{c_1})(x_i-\bar{x}_{c_1})^T\right]\cdot W+\cdots\cdots \Longrightarrow S_1+S_2=W^TS_{c_1}W+W^TS_{c_2}W=W^T(S_{c_1}+S_{c_2})W$$

(1) $J(w)=\frac{(\bar{z}_1-\bar{z}_2)^2}{S_{z_1}+S_{z_2}}=\frac{w^T(\bar{x}_{c_1}-\bar{x}_{c_2})(x_{c_1}-\bar{x}_{c_2})^Tw}{w^T(S_{c_1}+S_{c_2})w}$

(S_B: between-class 类间方差；S_w: with-class: 类内方差)

$$=\frac{w^TS_Bw}{w^TS_ww}.$$

(2). 最优解.

$$\frac{\partial J(w)}{\partial w}=\frac{2S_Bw}{w^TS_ww}\cdot w^TS_Bw\cdot(-1)(w^TS_ww)^{-2}\cdot 2S_ww=0 \quad \text{———} \quad ①$$

$$\frac{①}{2}=\frac{S_Bw}{w^TS_ww}\cdot w^TS_Bw(-1)(w^TS_ww)^{-2}\cdot S_w\cdot w=0$$

$$\Rightarrow S_Bw\cdot(w^TS_ww)^{-1}\cdot(w^TS_Bw)(-1)(w^TS_ww)^{-2}\cdot S_w\cdot w=0$$

$$\Rightarrow S_Bw\cdot(w^TS_ww)-(w^TS_Bw)\cdot S_w\cdot w$$

$$\Rightarrow (w^TS_ww)S_Bw=(w^TS_Bw)\cdot S_w\cdot w. \quad \text{———} \quad ②$$

②：w^T：$1\times p$维；w：$p\times 1$维；S_B：$p\times p$维. $\Rightarrow w^TS_Bw\in\mathbb{R}^+$(1维).
同理. (w^TS_ww)一维.($\in\mathbb{R}^+$). (对方向无影响)

$$\Rightarrow S_w w = \frac{w^T S_w w}{w^T S_B w} \cdot S_B \cdot w.$$

$$\Rightarrow S_w \cdot w = R^+ \cdot S_B \cdot w$$

$$\Rightarrow w = R^+ \cdot S_B w \, S_w^{-1} \quad (w \propto S_B w \, S_w^{-1}) \text{ ——— ③}$$

for ③: $S_B w = (\bar{x}_{c_1} - \bar{x}_{c_2})\underbrace{\underbrace{(\bar{x}_{c_1} - \bar{x}_{c_2})^T}_{1\times p} \cdot \underbrace{w}_{p\times 1}}_{\text{1维的实数 } R^+\text{，对方向无影响.}}$

$$\Rightarrow w \propto \boxed{S_w^{-1}(\bar{x}_{c_1} - \bar{x}_{c_2})}$$

我们所关注的 w 的方向