$$\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2 \qquad (1.56)$$

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n \qquad (1.55)$$

Before sdve this question, here are 2 things need to be know.

μ_{ML} is Unbiased estimate

σ^2_{ML} is Biased estimate

Which means $E[\mu_{ML}] = \mu$

$E[\sigma^2_{ML}] \neq \sigma^2 \rightarrow$ actually $\left(\frac{N-1}{N}\sigma^2\right)$

1. How to get $E[\mu_{ML}] = \mu$

2. How to get $E[\sigma^2_{ML}] = \frac{N-1}{N}\sigma^2$

3. Why we could get $E[\sigma^2_{ML}] = \sigma^2$

1.

How to get $E[\mu_{ML}] = \mu$

$$E[\mu_{ML}] = E\left[\frac{1}{N}\sum_{n=1}^{N} x_n\right] = \frac{1}{N}\cdot\sum_{n=1}^{N} E[x_n] = \frac{1}{N}\cdot N\cdot\mu = \mu$$

$\blacksquare$

2. How to get $E[\sigma^2_{ML}] = \frac{N-1}{N}\sigma^2$

Here we will use $\boxed{\mu = \mu_{ML}}$, for $\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2$: (*)

$$E[\sigma^2_{ML}] = E\left[\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2\right]$$

Assume $y = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2$

So, $y = \frac{1}{N}\sum_{n=1}^{N}\left(x_n^2 - 2x_n\mu_{ML} + \mu_{ML}^2\right)$

$$= \frac{1}{N}\sum_{n=1}^{N} x_n^2 - \underbrace{\frac{1}{N}\sum_{n=1}^{N} 2x_n\mu_{ML}}_{1} + \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mu_{ML}^2}_{2}$$

① : Because, $E[\mu_{ML}] = E[\frac{1}{N}\sum_{n=1}^{n} x_n] = \mu = \mu_{ML}$ (See (*))

☆

② : $\frac{1}{N}\sum_{n}^{N} \mu_{ML}^2 = \mu_{ML}^2$

So, ①, ②, (*) $\Rightarrow E[y] = E[\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2$

$= E[\frac{1}{N}\sum_{n=1}^{N} x_n^2 - 2\mu_{ML}\cdot\mu_{ML} + \mu_{ML}^2$

$= E[\frac{1}{N}\sum_{n=1}^{N} x_n^2 - \mu_{ML}^2]$ ——————— ③

From (*): $\mu_{ML} = \mu$, and combine ③ :

/subtract μ and add μ doesn't impact the result /

③ $= E[\frac{1}{N}\sum_{n=1}^{N}(x_n^2 - \mu^2 - (\mu_{ML}^2 - \mu^2))]$

$= E[\frac{1}{N}\sum_{n=1}^{N}(x_n^2 - \mu^2)] - E[\frac{1}{N}\sum_{n=1}^{N}(\mu_{ML}^2 - \mu^2)]$

$= E[\frac{1}{N}\sum_{n=1}^{N}(x_n^2 - \mu^2)] - E[\underbrace{N \times \frac{1}{N}}_{=1}(\mu_{ML}^2 - \mu^2)]$

$= E[\frac{1}{N}\sum_{n=1}^{N}(x_n^2 - \mu^2)] - (E(\mu_{ML}^2] - E[\mu^2])$

$= \underbrace{\frac{1}{N}\cdot\sum_{n=1}^{N}(E[x_n^2] - E[\mu^2])}_{④} - \underbrace{(E(\mu_{ML}^2) - E[\mu^2])}_{⑤}$

④ : $\frac{1}{N}\sum_{n=1}^{N}(E[x_n^2] - E[\mu^2])$

Because, μ is a constant here, $E[\mu^2] = \mu^2$

$= \frac{1}{N}\sum_{n=1}^{N}(E[x_n^2] - \mu^2)$

Because $x_n \sim N(\mu, \sigma^2)$, so $E[x_n] = \mu$ (1.49)

So, ④ $\Rightarrow \frac{1}{N}\sum_{n=1}^{N}(E[x_n^2] - E[x]^2)$

$= \frac{1}{N}\sum_{n=1}^{N} Var[x_n]$

$= \frac{1}{N}\sum_{n=1}^{N} \sigma^2$

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⑤ $E[\mu_{ML}^2] - E[\mu^2]$

$= E[\mu_{ML}^2] - \mu^2$

Because $\mu = E[\mu_{ML}]$ (see (*))

So, ⑤ $\Rightarrow$

$E[\mu_{ML}^2] - E[\mu_{ML}]^2$

$= Var[\mu_{ML}]$

$= Var[\frac{1}{N}\sum_{n=1}^{N} x_n]$ (1.55)

According to rule of Var :

$= (\frac{1}{N})^2 \sum_{n=1}^{N} Var[x_n]$

$= \frac{1}{N^2}\sum_{n=1}^{N} \sigma^2$

④, ⑤ $\Rightarrow$

$$E[\sigma_{ML}^2] = ④ - ⑤$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sigma^2 - \frac{1}{N^2}\sum_{n=1}^{N}\sigma^2$$

$$= \Big(\underbrace{N \times \frac{1}{N}}_{1} - \underbrace{\frac{1}{N^2}\cdot N}_{\frac{1}{N}}\Big)\sigma^2$$

$$= \frac{N-1}{N}\sigma^2$$

$\rightarrow$ Biased estimate

$\blacksquare$

3. Why we could get $E[\sigma_{ML}^2] = \sigma^2$

See problem 1.13.