The virial theorem and the dynamics of clusters of galaxies in the brane world models

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Abstract

A version of the virial theorem, which takes into account the effects of the non-compact extra-dimensions, is derived in the framework of the brane world models. In the braneworld scenario, the four dimensional effective Einstein equation has some extra terms, called dark radiation and dark pressure, respectively, which arise from the embedding of the 3-brane in the bulk. To derive the generalized virial theorem we use a method based on the collisionless Boltzmann equation. The dark radiation term generates an equivalent mass term (the dark mass), which gives an effective contribution to the gravitational energy. This term may account for the well-known virial theorem mass discrepancy in actual clusters of galaxies. An approximate solution of the vacuum field equations on the brane, corresponding to weak gravitational fields, is also obtained, and the expressions for the dark radiation and dark mass are derived. The qualitative behavior of the dark mass is similar to that of the observed virial mass in clusters of galaxies. We compare our model with the observational data for galaxy clusters, and we express all the physical parameters of the model in terms of observable quantities. In particular, we predict that the dark mass must extend far beyond the presently considered virial radius. The behavior of the galaxy cluster velocity dispersion in brane world models is also considered. Therefore the study of the matter distribution and velocity dispersion at the extragalactic scales could provide an efficient method for testing the multi-dimensional physical models.

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I. INTRODUCTION

Several recent astrophysical observations [1] have provided the astonishing result that around $95-96\%$ of the content of the universe is in the form of dark matter + dark energy, with only about $4-5\%$ being represented by baryonic matter. More intriguing, around 70% of the energy-density may be in the form of what is called the dark energy, which provides $\Omega_{DE} \sim 0.7$, and may be responsible for the acceleration of the distant type Ia supernovae. The best candidate for the dark energy is the cosmological constant Λ, which is usually interpreted physically as a vacuum energy, with energy density ρ_Λ and pressure p_Λ satisfying the unusual equation of state $\rho_\Lambda = -p_\Lambda/c^2 = \Lambda/8\pi G$. Its size is of the order $\Lambda \approx 3 \times 10^{-56}$ cm^{-2} [2]. The analysis of the cosmic microwave radiation background data from the WMAP experiment indicated that most of the 22% dark matter content of the Universe must be of non-baryonic nature. More specifically, the analysis of the power spectrum indicates that a theory of gravity based essentially on the properties of baryonic matter would produce a lower third peak [3].

The problem of the dark matter is a long standing problem in modern astrophysics. Two important observational issues, the behavior of the galactic rotation curves and the mass discrepancy in clusters of galaxies led to the necessity of considering the existence of the dark matter at a galactic and extra-galactic scale.

The rotation curves of spiral galaxies [4] are one of the best evidences showing the problems Newtonian mechanics and/or standard general relativity has to face on the galactic/intergalactic scale. In these galaxies neutral hydrogen clouds are observed at large distances from the center, much beyond the extent of the luminous matter. Since the clouds move in circular orbits with velocity $v_{tg}(r)$, the orbits are maintained by the balance between the centrifugal acceleration v_{tg}^2/r and the gravitational attraction force $GM(r)/r^2$ of the total mass $M(r)$ contained within the orbit. This allows the expression of the mass profile of the galaxy in the form $M(r) = rv_{tg}^2/G$.

Observations show that the rotational velocities increase near the center of the galaxy and then remain nearly constant at a value of $v_{tg\infty} \sim 200$ km/s [4]. This leads to a mass profile $M(r) = rv_{tg\infty}^2/G$. Consequently, the mass within a distance r from the center of the galaxy increases linearly with r, even at large distances where very little luminous matter can be detected.

The second astrophysical evidence for dark matter comes from the study of the clusters of galaxies. The total mass of a cluster can be estimated in two ways. Knowing the motions of its member galaxies, the virial theorem gives one estimate, M_V, say. The second is obtained by separately estimating the mass of each individual members, and summing these masses, to give a total baryonic mass M. Almost without exception it is found that M_V is considerably greater than M, $M_V > M$, typical values of M_V/M being about 20-30 [4].

These behaviors at a galactic/extra-galactic scale are usually explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weekly interacting massive particles (WIMP). Their interaction cross section with normal baryonic matter, while extremely small, are expected to be non-zero and we may expect to detect them directly. It has also been suggested that the dark matter in the Universe might be composed of superheavy particles, with mass $\geq 10^{10}$ GeV. But observational results show the dark matter can be composed of superheavy particles only if these interact weakly with normal matter or if their mass is above 10^{15} GeV [5].

Dark matter may also be in the form of Warm Dark Matter particles, consisting either of sterile neutrinos, with masses of several keV, or from early decoupled thermal relics [6]. However, constraints based on X-ray fluxes from the Andromeda galaxy have shown that dark matter particles cannot be sterile neutrinos, unless they are produced by a nonstandard mechanism (resonant oscillations, coupling with the inflaton field), or they have been diluted by some large entropy release. The X-rays produced by the decays of these relic sterile neutrinos can boost the production of molecular hydrogen, which can speed up the cooling of gas and the early star formation [6].

Hence, despite more than 20 years of intense experimental and observational effort, up to now no *non-gravitational* evidence for dark matter has ever been found: no direct evidence of it and no annihilation radiation from it. Moreover, accelerator and reactor experiments do not support the physics (beyond the standard model) on which the dark matter hypothesis is based upon.

Therefore, it seems that the possibility that Einstein's (and the Newtonian) gravity breaks down at the scale of galaxies cannot be excluded *a priori*. Several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves [7].

The proposal by Randall and Sundrum [8] that our four-dimensional space-time is a three-brane, embedded in a five-dimensional space-time (the bulk), had attracted a considerable interest lately. According to the brane-world scenario, the physical fields (electromagnetic, Yang-Mills etc.) in our four-dimensional Universe are confined to the three brane. These fields are assumed to arise as fluctuations of branes in string theories. Only gravity can freely propagate in both the brane and bulk space-times, with the gravitational self-couplings not significantly modified. Even if the fifth dimension is uncompactified, standard $4D$ gravity is reproduced on the brane. Hence this model allows the presence of large, or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time. For a review of dynamics and geometry of brane Universes see [9].

Due to the correction terms coming from the extra dimensions, significant deviations from the Einstein theory occur in brane world models at very high energies [10]. Gravity is largely modified at the electro-weak scale 1 TeV. The cosmological and astrophysical implications of the brane world theories have been extensively investigated in the physical literature [11].

The vacuum field equations on the brane have been reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms (the projections of the Weyl curvature of the bulk, generating non-local brane stresses) in [12]. Several classes of vacuum solutions of the static gravitational field equations in the brane world scenario describing conformally symmetric gravitational fields in the vacuum have been obtained in [13, 14]. As a possible physical application of these solutions the behavior of the angular velocity v_{tg} of the test particles in stable circular orbits on the brane has been considered. In the obtained models in the limit of large radial distances and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies [4], and is usually explained by postulating the existence of the dark matter. Thus, the rotational galactic curves can be naturally explained in brane world models without introducing any additional hypothesis. The galaxy is embedded in a modified, spherically symmetric geometry, generated by the non-zero contribution of the Weyl tensor from the bulk. The extra-terms, which can be described in terms of the dark radiation term U and the dark pressure term P, act as a "matter" distribution outside the galaxy [13, 14, 15, 16].

The existence of the dark radiation term generates an equivalent mass term M_U, which is linearly increasing with the distance, and proportional to the baryonic mass of the galaxy M_B, $M_U(r) \approx M_B(r/r_0)$. The particles moving in this geometry feel the gravitational effects of U, which can be expressed in terms of an equivalent mass. In the framework of this model all the relevant physical parameters (metric tensor components, dark radiation and dark pressure terms) can be obtained as functions of the tangential velocity, and hence they can be determined observationally. Using the smallness of the rotational velocity, a perturbative scheme for reconstructing the metric in a galactic halo in the braneworld models with induced gravity was developed in [17]. In the leading order of expansion, at sufficiently large distances, the obtained result reproduce that obtained in the Randall-Sundrum braneworld model [13, 14, 15, 16].

Similar interpretations of the dark matter as bulk effects have been also considered in [18].

The exact galactic metric, the dark radiation and the dark pressure in the flat rotation curves region in the brane world scenario has been obtained in [15]. The deflection of photons has also been considered and the bending angle of light is computed. The bending angle predicted by the brane world models is much larger than that predicted by standard general relativistic and dark matter models. Therefore the study of the light deflection by galaxies and the gravitational lensing could discriminate between the different dynamical laws proposed to model the motion of particles at the galactic level and the standard dark matter models.

Because of its generality and wide range of applications, the virial theorem plays an important role in astrophysics. Assuming steady state, one of the important results which can be obtained with the use of the virial theorem is to deduce the mean density of astrophysical objects such as galaxies, clusters and super clusters, by observing the velocities of test particles rotating around them. Hence the virial theorem can be used to predict the total mass of the clusters of galaxies. The virial theorem is also a powerful tool for stability studies. In a general relativistic framework several versions of the virial theorem have been proposed [19]. Versions of the virial theorem including the effect of a cosmological constat have been derived in [20] and [21], respectively.

It is the purpose of the present paper to consider the virial theorem in the framework of the brane world models. By taking into account the effects of the non-compact extra-dimensions we derive, by using the collisionless Boltzmann equation, a generalized virial equality, which explicitly takes into account the presence of the bulk effects. In the case of static spherically symmetric systems the transmitted projections of the bulk Weyl tensor can be described in terms of two quantities, called the dark radiation and the dark pressure, respectively. The dark radiation term gives an effective contribution to the gravitational energy, the total virial mass being proportional to the mass associated to the effective mass of the dark radiation. This term may account for the well-known virial theorem mass discrepancy in actual clusters of galaxies. The generalized virial theorem in the brane world models can be an efficient tool in observationally testing the brane world models.

An approximate solution of the vacuum field equations on the brane, corresponding to weak gravitational fields, is also obtained, and the expressions for the dark radiation and dark mass are derived. The qualitative behavior of the dark mass is similar to that of the observed virial mass in clusters of galaxies.

We consider several astrophysical applications of the general results obtained in the present paper. The expressions of the dark radiation and of the dark mass can be expressed in terms of observable astrophysical quantities, like the temperature and the gas density profile, which can be obtained from the X-ray observations of the gas in the cluster. From the obtained expression of the dark radius, which describes the relevant length scale for brane world effects, we predict that the dark mass must extend far beyond the presently considered virial radius. The behavior of the galaxy cluster velocity dispersion in brane world models is also considered in detail. Therefore the study of the matter distribution and velocity dispersion at the extragalactic scales could provide an efficient method for testing the multi-dimensional physical models.

The present paper is organized as follows. The basic equations for static spherically symmetric gravitational fields on the brane are presented in Section II. The generalized virial theorem, including the effects of the non-compact extra dimensions is derived in Section III. In Section IV we derive the general expressions of the cluster metric, of the dark radiation and of the dark mass for the clusters of galaxies. Some astrophysical applications are presented in Section V. We discuss and conclude our results in Section VI.

II. STATIC SPHERICALLY SYMMETRIC GRAVITATIONAL FIELDS ON THE BRANE

On the 5-dimensional space-time (the bulk), with the negative vacuum energy Λ_5 and brane energy-momentum as source of the gravitational field, the Einstein field equations are given by

$$G_{IJ} = k_5^2 T_{IJ}, \qquad T_{IJ} = -\Lambda_5 g_{IJ} + \delta(Y) \left[-\lambda_b g_{IJ} + T_{IJ}^{\text{matter}} \right], \tag{1}$$

with λ_b the vacuum energy on the brane and $k_5^2 = 8\pi G_5$. In this space-time a brane is a fixed point of the Z_2 symmetry. In the following capital Latin indices run in the range $0, ..., 4$, while Greek indices take the values $0, ..., 3$.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the $\chi =$constant hypersurfaces and g_{IJ} the induced metric on $\chi =$constant hypersurfaces, the effective four-dimensional gravitational equations on the brane take the form [10]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \tag{2}$$

where $S_{\mu\nu}$ is the local quadratic energy-momentum correction

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu{}^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} \left(3 T^{\alpha\beta} T_{\alpha\beta} - T^2 \right), \tag{3}$$

and $E_{\mu\nu}$ is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor C_{IAJB}, $E_{IJ} = C_{IAJB} n^A n^B$, with the property $E_{IJ} \to E_{\mu\nu} \delta_I^\mu \delta_J^\nu$ as $\chi \to 0$. We have also denoted $k_4^2 = 8\pi G$, with G the usual four-dimensional gravitational constant. Eq. (2) is also known as the effective Einstein equation. It has been shown that for a large class of generalized Randall-Sundrum type models, the characterization of brane-gravity sector by the effective Einstein equation, Codazzi equation and the twice-contracted Gauss equation is equivalent with the bulk five-dimensional Einstein equation [22].

The four-dimensional cosmological constant, Λ, and the four-dimensional coupling constant, k_4, are given by $\Lambda = k_5^2 \left(\Lambda_5 + k_5^2 \lambda_b^2 / 6 \right) / 2$ and $k_4^2 = k_5^4 \lambda_b / 6$, respectively. The four-dimensional gravitational constant G is given by $G = k_5^4 \lambda_b / 48\pi$ [10].

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, $D_\nu T_\mu{}^\nu = 0$, where D_ν denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor should obey the constraint $D_\nu E_\mu{}^\nu = k_5^4 D_\nu S_\mu{}^\nu$ [9].

In the limit $\lambda_b^{-1} \to 0$ we recover standard general relativity. An alternative possibility in recovering the four-dimensional Einstein equations is to take the limit $k_5 \to 0$, while keeping the Newtonian gravitational constant G finite [10]. As one can see from its definition, the existence of the four-dimensional gravitational constant relies on the existence of the vacuum energy λ_b. It is impossible to define an the gravitational constant in an era when the distinction between vacuum energy and matter is ambiguous. Moreover, the positivity of G also requires the positivity of λ_b [10].

The symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^μ as

$$E_{\mu\nu} = -\tilde{k}^4 \left[U \left(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}\right) + 2Q_{(\mu}u_{\nu)} + P_{\mu\nu}\right], \tag{4}$$

where $\tilde{k} = k_5/k_4$, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to u^μ, the "dark radiation" term $U = -\tilde{k}^{-4}E_{\mu\nu}u^\mu u^\nu$ is a scalar, $Q_\mu = \tilde{k}^{-4}h_\mu^\alpha E_{\alpha\beta}u^\beta$ a spatial vector and $P_{\mu\nu} = -\tilde{k}^{-4}\left[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\right]E_{\alpha\beta}$ a spatial, symmetric and trace-free tensor [9].

We assume that the matter on the brane consists of an anisotropic fluid, characterized by an effective energy-density $\rho_{eff} \neq 0$, a radial pressure $p_{eff}^{(r)}$ and a tangential pressure $p_{eff}^{(\perp)}$, respectively. Generally $p_{eff}^{(r)} \neq p_{eff}^{(\perp)}$, but for isotropic systems $p_{eff}^{(r)} = p_{eff}^{(\perp)}$. Hence $T_{\mu\nu} \neq 0$, and consequently also $S_{\mu\nu} \neq 0$. $E_{\mu\nu}$ satisfies the constraint $D_\nu E_\mu{}^\nu = k_5^4 D^\mu S_{\mu\nu}$, where D_μ is the projection (orthogonal to u^μ) of the covariant derivative [10]. In an inertial frame at any point on the brane we have $u^\mu = \delta_0^\mu$ and $h_{\mu\nu} =$diag$(0,1,1,1)$. In the static spherically symmetric case $Q_\mu = 0$ and the constraint for $E_{\mu\nu}$ takes the form [23]

$$\frac{1}{3}D_\mu U + \frac{4}{3}UA_\mu + D^\nu P_{\mu\nu} + A^\nu P_{\mu\nu} = -\left(4\pi G\right)^2 \left(2\rho_{eff} + p_{eff}^{(r)} + p_{eff}^{(\perp)}\right) D_\mu \rho_{eff}, \tag{5}$$

where $A_\mu = u^\nu D_\nu u_\mu$ is the 4-acceleration.

In the static spherically symmetric case we may choose $A_\mu = A(r)r_\mu$ and $P_{\mu\nu} = P(r)\left(r_\mu r_\nu - \frac{1}{3}h_{\mu\nu}\right)$, where $A(r)$ and $P(r)$ (the "dark pressure") are some scalar functions of the radial distance r, and r_μ is a unit radial vector [24].

We choose the static spherically symmetric metric on the brane in the form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{6}$$

Then the gravitational field equations and the effective energy-momentum tensor conservation equation for an anisotropic static spherically symmetric system take the form

[12, 13, 14, 15, 23]

$$-e^{-\lambda}\left(\frac{1}{r^{2}}-\frac{\lambda'}{r}\right)+\frac{1}{r^{2}}=8\pi G\rho_{eff}\left(1+\frac{\rho_{eff}}{2\lambda_{b}}\right)+\frac{48\pi G}{k_{4}^{4}\lambda_{b}}U+\Lambda, \tag{7}$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^{2}}\right)-\frac{1}{r^{2}}=8\pi Gp_{eff}^{(r)}+\frac{4\pi G}{\lambda_{b}}\rho_{eff}\left(\rho_{eff}+2p_{eff}^{(r)}\right)+\frac{16\pi G}{k_{4}^{4}\lambda_{b}}\left(U+2P\right)-\Lambda, \tag{8}$$

$$e^{-\lambda}\left(\nu''+\frac{\nu'^{2}}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right)=16\pi Gp_{eff}^{(\perp))}+\frac{8\pi G}{\lambda_{b}}\rho_{eff}\left(\rho_{eff}+2p_{eff}^{(\perp)}\right)+\frac{32\pi G}{k_{4}^{4}\lambda_{b}}\left(U-P\right)-2\Lambda, \tag{9}$$

$$\nu'=-\frac{U'+2P'}{2U+P}-\frac{6P}{r\left(2U+P\right)}-\left(4\pi G\right)^{2}\frac{2\rho_{eff}+p_{eff}^{(r)}+p_{eff}^{(\perp)}}{2U+P}\rho_{eff}'. \tag{10}$$

In the following we shall denote $\alpha=16\pi G/k_{4}^{4}\lambda_{b}=1/4\pi G\lambda_{b}$.

III. THE VIRIAL THEOREM IN THE BRANE WORLD MODELS

We consider an isolated, spherically symmetric cluster, situated in a space with metric given by Eq. (6). We describe the galaxies, which are treated as identical, collisionless point particles, by a distribution function which obeys the general relativistic Boltzmann equation.

Consider a time-oriented Lorentzian four-dimensional space-time manifold M, with metric g of signature $(-,+,+,+)$. The tangent bundle $T(M)$ is a real vector bundle whose fibers at a point $x\in M$ is given by the tangent space $T_{x}\left(M\right)$. In the space-time M the instantaneous state of a particle with mass m_{0} is given by a four-momentum $p\in T_{x}\left(M\right)$ at an event $x\in M$. The one-particle phase space P_{phase} is a subset of the tangent bundle given by [25]

$$P_{phase}:=\left\{(x,p)\mid x\in M,p\in T_{x}\left(M\right),p^{2}=-m_{0}^{2}\right\}. \tag{11}$$

A state of a multi-particle system is described by a continuous, non-negative distribution function $f\left(x,p\right)$, defined on P_{phase}, and which gives the number dN of the particles of the system which cross a certain space-like volume dV at x, and whose 4-momenta p lie within a corresponding three-surface element $d\vec{p}$ in the momentum space. The mean value of f gives the average number of occupied particle states (x,p). Macroscopic, observable quantities can be defined as moments of f [25].

Let $\{x^{\alpha}\}$, $\alpha=0,1,2,3$ be a local coordinate system in M, defined in some open set $U\subset M$. The coordinates are chosen so that ∂_{t} is time-like future directed and ∂_{a}, $a=1,2,3$, are spacelike. Then $\{\partial/\partial x^\alpha\}$ is the corresponding natural basis for tangent vectors. We express each tangent vector p in U in terms of this basis as $p = p^\alpha \partial/\partial x^\alpha$ and define a system of local coordinates $\left\{z^A\right\}$, $A = 0, ..., 7$ in $T_U(M)$ as $z^\alpha = x^\alpha$, $z^{\alpha+4} = p^\alpha$. This defines a natural basis in the tangent space given by $\left\{\partial/\partial z^A\right\} = \{\partial/\partial x^\alpha, \partial/\partial p^\alpha\}$ [25].

A vertical vector field over TM is given by $\pi = p^\alpha \partial/\partial p^\alpha$. The geodesic flow field σ, which can be constructed over the tangent bundle, is defined as $\sigma = p^\alpha \partial/\partial x^\alpha - p^\alpha p^\gamma \Gamma^\beta_{\alpha\gamma} \partial/\partial p^\beta = p^\alpha D_\alpha$, where $D_\alpha = \partial/\partial x^\alpha - p^\gamma \Gamma^\beta_{\alpha\gamma} \partial/\partial p^\beta$, and $\Gamma^\beta_{\alpha\gamma}$ are the connection coefficients. Physically, σ describes the phase flow for a stream of particles whose motion through space-time is geodesic [25].

Therefore the transport equation for the propagation of a particle in a curved arbitrary Riemannian space-time is given by the Boltzmann equation [25]

$$\left(p^\alpha \frac{\partial}{\partial x^\alpha} - p^\alpha p^\beta \Gamma^i_{\alpha\beta} \frac{\partial}{\partial p^i}\right) f = 0. \tag{12}$$

For many applications it is convenient to introduce an appropriate orthonormal frame or tetrad $e^{(a)}_\mu(x)$, $a = 0, 1, 2, 3$, which varies smoothly over some coordinates neighborhood U and satisfies the condition $e^{(a)}_\mu(x)\, e^{(b)}_\mu(x) = \eta^{ab}$ for all $x \in U$ [20, 25]. Any tangent vector p^μ at x can be expressed as $p^\mu = p^a e^\mu_{(a)}$, which defines the tetrad components p^a.

In the case of the spherically symmetric line element given by Eq. (6) we introduce the following frame of orthonormal vectors [20, 25]:

$$e^{(0)}_\mu = e^{\nu/2}\delta^0_\mu, e^{(1)}_\mu = e^{\lambda/2}\delta^1_\mu, e^{(2)}_\mu = r\delta^2_\mu, e^{(3)}_\mu = r\sin\theta\delta^3_\mu. \tag{13}$$

Let u^μ be the four-velocity of a typical galaxy, satisfying the condition $u^\mu u_\mu = -1$, with tetrad components $u^{(a)} = u^\mu e^{(a)}_\mu$. The relativistic Boltzmann equation in tetrad components is

$$u^{(a)} e^\mu_{(a)} \frac{\partial f}{\partial x^\mu} + \gamma^{(a)}_{(b)(c)} u^{(b)} u^{(c)} \frac{\partial f}{\partial u^{(a)}} = 0, \tag{14}$$

where the distribution function $f = f\left(x^\mu, u^{(a)}\right)$ and $\gamma^{(a)}_{(b)(c)} = e^{(a)}_{\mu;\nu} e^\mu_{(b)} e^\nu_{(c)}$ are the Ricci rotation coefficients [20, 25]. By denoting

$$u^{(0)} = u_t, u^{(1)} = u_r, u^{(2)} = u_\theta, u^{(3)} = u_\phi, \tag{15}$$

and by assuming that the only coordinate dependence of the distribution function is upon the radial coordinate r, Eq. (14) becomes [20]

$$u_r \frac{\partial f}{\partial r} - \left(\frac{1}{2} u_t^2 \frac{\partial \nu}{\partial r} - \frac{u_\theta^2 + u_\phi^2}{r}\right) \frac{\partial f}{\partial u_r} - \frac{1}{r} u_r \left(u_\theta \frac{\partial f}{\partial u_\theta} + u_\phi \frac{\partial f}{\partial u_\phi}\right) -$$

$$\frac{1}{r}e^{\lambda/2}u_{\phi}\cot\theta\left(u_{\theta}\frac{\partial f}{\partial u_{\phi}}-u_{\phi}\frac{\partial f}{\partial u_{\theta}}\right)=0. \tag{16}$$

The spherical symmetry requires that the coefficient of $\cot\theta$ be zero, which implies that f is a function of r, u_r and $u_\theta^2+u_\phi^2$ only. We multiply now Eq. (16) by $mu_r du$, where m is the mass of each galaxy and $du = du_r du_\theta du_\phi/u_t$ is the invariant volume element of the velocity space. By integrating over the velocity space and by assuming that f vanishes sufficiently rapidly as the velocities tend to $\pm\infty$, we obtain

$$r\frac{\partial}{\partial r}\left[\rho\left\langle u_r^2\right\rangle\right]+\frac{1}{2}\rho\left[\left\langle u_t^2\right\rangle+\left\langle u_r^2\right\rangle\right]r\frac{\partial\nu}{\partial r}-\rho\left[\left\langle u_\theta^2\right\rangle+\left\langle u_\phi^2\right\rangle-2\left\langle u_r^2\right\rangle\right]=0, \tag{17}$$

where, at each point, $\langle u_r^2\rangle$ is the average value of u_r^2 etc. and ρ is the mass density.

By multiplying Eq. (17) by $4\pi r^2$ and integrating over the cluster we obtain [20]

$$-\int_0^R 4\pi\rho\left[\left\langle u_r^2\right\rangle+\left\langle u_\theta^2\right\rangle+\left\langle u_\phi^2\right\rangle\right]r^2dr+\frac{1}{2}\int_0^R 4\pi r^3\rho\left[\left\langle u_t^2\right\rangle+\left\langle u_r^2\right\rangle\right]\frac{\partial\nu}{\partial r}dr=0. \tag{18}$$

In terms of the distribution function the energy-momentum tensor of the matter can be written as [25]

$$T_{\mu\nu}=\int f m u_\mu u_\nu du, \tag{19}$$

which gives

$$\rho_{eff}=\rho\left\langle u_t^2\right\rangle, p_{eff}^{(r)}=\rho\left\langle u_r^2\right\rangle, p_{eff}^{(\perp)}=\rho\left\langle u_\theta^2\right\rangle=\rho\left\langle u_\phi^2\right\rangle. \tag{20}$$

Then, with the use of this form of the energy-momentum tensor, by adding the gravitational field equations Eqs. (7)-(9), we immediately obtain the following relation:

$$\begin{aligned} e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{r}-\frac{\nu'\lambda'}{4}\right) &= 3\alpha U-\Lambda+4\pi G\rho\left[\left\langle u_t^2\right\rangle+\left\langle u_r^2\right\rangle+\left\langle u_\theta^2\right\rangle+\left\langle u_\phi^2\right\rangle\right]+ \\ &\quad \frac{4\pi G}{\lambda_b}\rho^2\left[\left\langle u_t^2\right\rangle^2+\left\langle u_r^2\right\rangle^2+\left\langle u_\theta^2\right\rangle^2+\left\langle u_\phi^2\right\rangle^2\right]. \end{aligned} \tag{21}$$

It is convenient to introduce some approximations at this moment. First of all, we assume that ν and λ are small, so that in Eq. (21) the quadratic terms can be neglected. Secondly, we assume that the galaxies have velocities much smaller than the velocity of the light, so that $\langle u_r^2\rangle, \langle u_\theta^2\rangle, \left\langle u_\phi^2\right\rangle << \langle u_t^2\rangle \approx 1$. Since for clusters of galaxies the ratio of the matter density and of the brane tension is much smaller than 1, $\rho/\lambda_b << 1$, we can neglect in Eq. (21) the quadratic term in the matter density. These conditions certainly apply to test particles in stable circular motion around galaxies, and to the galactic clusters.

Therefore Eqs. (18) and (21) become

$$-2K+\frac{1}{2}\int_0^R 4\pi r^3\rho\frac{\partial\nu}{\partial r}dr=0, \tag{22}$$

$$4\pi G\rho=\frac{1}{2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\nu}{\partial r}\right)+\Lambda-3\alpha U, \tag{23}$$

respectively, where

$$K=\int_0^R 2\pi\rho\left[\left\langle u_r^2\right\rangle+\left\langle u_\theta^2\right\rangle+\left\langle u_\phi^2\right\rangle\right]r^2dr, \tag{24}$$

is the total kinetic energy of the galaxies. The gravitational potential energy Ω of the system is defined by

$$\Omega=-\int_0^R\frac{GM(r)}{r}dM(r), \tag{25}$$

where $M(r)$ is the mass out to radius r, so that $dM(r)=4\pi\rho r^2dr$. The total mass of the system is given by $M=\int_0^R dM(r)=4\pi\int_0^R\rho r^2dr$. The main contribution to M is represented by the baryonic mass of the intra-cluster gas and of the stars, but other particles, like, for example, massive neutrinos, may also give a significant contribution to M.

Multiplying Eq. (23) by r^2 and integrating from 0 to r we obtain

$$GM(r)=\frac{1}{2}r^2\frac{\partial\nu}{\partial r}+\frac{1}{3}\Lambda r^3-3\alpha\int_0^r U(r)r^2dr. \tag{26}$$

In the following we denote

$$GM_U(r)=3\alpha\int_0^r U(r)r^2dr. \tag{27}$$

This quantity may be called as the dark mass. By multiplying Eq. (26) with $dM(r)$, integrating from 0 to R, by introducing the moment of inertia of the system as $I=\int_0^R r^2dM(r)$, and by denoting

$$\Omega_U=\int_0^R\frac{GM_U(r)dM(r)}{r}, \tag{28}$$

we obtain

$$-\Omega=\frac{1}{2}\int_0^R 4\pi r^3\rho\frac{\partial\nu}{\partial r}dr+\frac{1}{3}\Lambda I-\Omega_U. \tag{29}$$

Finally, with the use of the Eq. (22), we obtain the generalization of the virial theorem in the brane world models in the form

$$2K+\Omega+\frac{1}{3}\Lambda I-\Omega_U=0. \tag{30}$$

The generalized virial theorem, given by Eq. (30), can be written in a simpler form if we introduce the radii R_V, R_I and R_U defined by

$$R_V = M^2 / \int_0^R \frac{M(r)}{r} dM(r), \tag{31}$$

$$R_I = \left[\left(\int_0^R r^2 dM(r) \right) / M(r) \right]^{1/2}, \tag{32}$$

$$R_U = M_U^2 / \int_0^R \frac{M_U(r) dM(r)}{r}, \tag{33}$$

so that

$$\Omega = -\frac{GM^2}{R_V}, \tag{34}$$

$$I = MR_I^2, \tag{35}$$

$$\Omega_U = \frac{GM_U^2}{R_U}. \tag{36}$$

R_U may be called the dark radius of the cluster of galaxies. The virial mass M_V is defined as [20]

$$2K = \frac{GM_V^2}{R_V}. \tag{37}$$

If this expression is substituted into the virial theorem, given by Eq. (30), we obtain

$$\frac{M_V}{M} = \left(1 + \frac{M_U^2 R_V}{M^2 R_U} - \frac{\Lambda}{4\pi G \bar{\rho}} \right)^{1/2}, \tag{38}$$

where $\bar{\rho} = 3M/4\pi R_V R_I^2$.

For $U = 0$, $M_U = 0$ and we reobtain the virial theorem in the presence of a cosmological constant [20].

If $M_V/M > 3$, a condition which is true for most of the clusters, the term unity can be neglected in the bracket in Eq. (38) with little loss of accuracy. Moreover, the contribution of the cosmological constant to the mass energy of the galaxy can also be ignored as being several orders of magnitude smaller than the observed masses. Therefore the virial mass in the brane world models is given by

$$M_V \approx M_U \sqrt{\frac{R_V}{R_U}}. \tag{39}$$

From observational point of view the virial mass M_V is determined from the study of the velocity dispersion σ_r^2 of the stars and of the galaxies in the clusters. According to our interpretation, most of the mass in a cluster with mass M_{tot} should be in the form of the dark mass M_U, so that $M_U \approx M_{tot}$. A possibility of detecting the presence of the dark mass and of the astrophysical effects of the extra dimensions is through gravitational lensing, which can provide direct evidence of the mass distribution and of the gravitational effects even at distances extending far beyond of the virial radius of the cluster.

IV. DARK MASS, DARK RADIATION AND METRIC FOR CLUSTERS OF GALAXIES ON THE BRANE

The total mass of galaxy clusters ranges from $10^{13}\ M_\odot$ for groups up to a few $10^{15}\ M_\odot$ for very rich systems. The cluster morphology is usually dominated by a regular centrally peaked main component [27, 28].

As clusters are "dark matter" dominated objects, their formation and evolution is driven by gravity. The mass function of the clusters is determined by the initial conditions of the mass distribution set in the early universe. The evolution of the large scale matter distribution on scales comparable to the size of the clusters is linear. The overall process of the gravitational growth of the density fluctuations and the development of gravitational instabilities leading to cluster formation has been extensively studied by using both analytical and numerical methods [29].

An other important component is the intra-cluster gas, which is assumed to be isothermal, and in hydrostatic equilibrium. Most of the baryonic mass in the cluster is contained in the gas. The properties of the gas are important observational quantities, since the total gravitational mass of the cluster M_{tot} is determined by using the radial gas distribution [27]. By assuming that the density and pressure of the gas are ρ_g and p_g, respectively, and by neglecting the quadratic terms in the field equations, the basic equations describing the structure of a galactic cluster in the brane world model are given by

$$-e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2}=8\pi G\rho_g+3\alpha U+\Lambda, \tag{40}$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=8\pi G p_g+\alpha\left(U+2P\right)-\Lambda, \tag{41}$$

$$e^{-\lambda}\frac{1}{2}\left(\nu''+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right)=8\pi G p_g+\alpha\left(U-P\right)-\Lambda, \tag{42}$$

$$\nu'=-\frac{U'+2P'}{2U+P}-\frac{6P}{r\left(2U+P\right)}-2\left(4\pi G\right)\frac{\rho_g+p_g}{2U+P}\rho_g^{'}. \tag{43}$$

Eq. (40) can immediately be integrated to give

$$e^{-\lambda} = 1 - \frac{C}{r} - \frac{2GM_g}{r} - \frac{GM_U(r)}{r} - \frac{\Lambda}{3}r^2, \tag{44}$$

where C is an arbitrary constant of integration, M_U is defined according to Eq. (27) and M_g is the total mass of the gas.

By substituting ν' given by Eq. (43) into Eq. (41) and with the use of Eq. (44) we obtain the following system of differential equations satisfied by the gas density, pressure and the dark radiation term U, the dark pressure P and the dark mass M_U, respectively, describing the gravitational field of a cluster of galaxies in the brane world model:

$$\frac{dM_g}{dr} = 4\pi\rho_g r^2. \tag{45}$$

$$\frac{dM_U}{dr} = \frac{3\alpha}{G} r^2 U. \tag{46}$$

$$\frac{dU}{dr} = -\frac{(2U+P)\left\{C + 2GM_g + GM_U + \left[\alpha\left(U+2P\right) + 8\pi p_g\right] r^3\right\} - \frac{2}{3}\Lambda r^3}{r^2\left(1 - \frac{C}{r} - \frac{2GM_g}{r} - \frac{GM_U(r)}{r} - \frac{\Lambda}{3}r^2\right)} - 2\frac{dP}{dr} - \frac{6P}{r} + 2\left(4\pi G\right)\frac{\rho_g + p_g}{2U+P}\rho_g'. \tag{47}$$

Since the most important contribution to the structure and dynamics of the clusters of galaxies comes from the "dark matter", which we interpret as a multi-dimensional effect, described by the contribution of the Weyl tensor from the bulk, in discussing the properties of the clusters we can neglect, in the first approximation, the effect of the ordinary matter, by taking $\rho_g \approx 0$ and $p_g \approx 0$, respectively.

Therefore the metric components $\nu(r)$ and $\lambda(r)$ inside the spherically symmetric static cluster are related to the dark radiation U and to the dark pressure P via the Einstein gravitational field equations and the effective energy-momentum tensor conservation equation, which in the vacuum take the form [12, 13]

$$\frac{dM_U}{dr} = \frac{3\alpha}{G} r^2 U. \tag{48}$$

$$\frac{dU}{dr} = -\frac{(2U+P)\left\{C + GM_U + \left[\alpha\left(U+2P\right)\right] r^3\right\} - \frac{2}{3}\Lambda r^3}{r^2\left(1 - \frac{C}{r} - \frac{GM_U(r)}{r} - \frac{\Lambda}{3}r^2\right)} - 2\frac{dP}{dr} - \frac{6P}{r}. \tag{49}$$

The system of equations (48) and (49) can be transformed to an autonomous system of differential equations by means of the transformations

$$q = \frac{C}{r} + \frac{GM_U}{r} + \frac{\Lambda}{3}r^2, \quad \mu = 3\alpha r^2 U + 3r^2\Lambda, \quad p = 3\alpha r^2 P - 3r^2\Lambda, \quad \theta = \ln r. \tag{50}$$

We shall call μ and p the "reduced" dark radiation and pressure, respectively.

With the use of the new variables given by Eqs. (50), Eqs. (48) and (49) become

$$\frac{dq}{d\theta} = \mu - q, \tag{51}$$

$$\frac{d\mu}{d\theta} = -\frac{(2\mu + p)\left[q + \frac{1}{3}(\mu + 2p)\right]}{1-q} - 2\frac{dp}{d\theta} + 2\mu - 2p, \tag{52}$$

In order to close the system of equations Eqs. (51) and (52), we need to specify the "equation of state" of the dark pressure. As a possible equation of state relating p and μ we will assume a general linear relation of the form

$$p(\mu) = (\Gamma - 2)\,\mu + B, \tag{53}$$

with Γ and B arbitrary constants. Therefore Eq. (52) takes the form

$$\frac{d}{d\theta}\left[(2\Gamma - 3)\,\mu\right] = -\frac{(\Gamma\mu + B)\left[q + (2\Gamma - 3)\,\mu/3 + 2B/3\right]}{1-q} + 2\,(3-\Gamma)\,\mu - 2B, \tag{54}$$

where we have neglected the possible effect of the cosmological constant on the structure of the cluster.

For clusters of galaxies with virial masses of the order of $M_V \approx 10^{14} M_\odot$ and virial radii of the order of $R_V \approx 2$ Mpc, the quantity GM/R is of the order of 2×10^{-6}. Since observations show that inside the cluster the mass is a linearly increasing function of the radius r, the value of this ratio is roughly the same at all points in the cluster. Therefore from its definition it follows that generally $q << 1$, and $1-q \approx 1$. Moreover, the quantities q^2 and $qdq/d\theta$ are also very small as compared to q. Eq. (51) gives $\mu = q + dq/d\theta$, $d\mu/d\theta = dq/d\theta + d^2q/d\theta^2$. Hence, by neglecting the second order terms, we obtain for q the following differential equation:

$$\frac{d^2q}{d\theta^2} + m\frac{dq}{d\theta} + nq = b, \tag{55}$$

where we denoted

$$m = 1 + \frac{B}{3} + \frac{2\Gamma\,(B+3) - 18}{3\,(2\Gamma - 3)}, \tag{56}$$

$$n = \frac{B}{3} + \frac{2\,(B+3)\,\Gamma + 3\,(B-6)}{3\,(2\Gamma - 3)}, \tag{57}$$

and

$$b = \frac{2B(B+3)}{3\,(3-2\Gamma)}, \tag{58}$$

respectively. The general solution of Eq. (55) is given by

$$q(\theta) = q_0 + C_1 e^{l_1\theta} + C_2 e^{l_2\theta}, \tag{59}$$

where C_1 and C_2 are arbitrary constants of integration, and we denoted

$$q_0 = \frac{b}{n}, \tag{60}$$

and

$$l_{1,2} = \frac{-m \pm \sqrt{m^2 - 4n^2}}{2}, \tag{61}$$

respectively. The reduced dark radiation term is given by

$$\mu(\theta) = q_0 + C_1 (1 + l_1) e^{l_1\theta} + C_2 (1 + l_2) e^{l_2\theta}. \tag{62}$$

For a positive m from Eq. (61) it follows that both l_1 and l_2 are negative numbers, $l_1 < 0$ and $l_2 < 0$, respectively. In the original radial variable r we obtain for the dark radiation and the mass distribution inside the cluster the expressions

$$3\alpha U(r) = \frac{q_0}{r^2} + C_1 (1 + l_1) r^{l_1 - 2} + C_2 (1 + l_2) r^{l_2 - 2}, \tag{63}$$

and

$$GM_U(r) = q_0 r \left(1 + \frac{C_1}{q_0} r^{l_1} + \frac{C_2}{q_0} r^{l_2} - \frac{C}{q_0 r}\right), \tag{64}$$

respectively, where we have neglected the possible effect of the cosmological constant.

The metric coefficient ν can be calculated from the equation

$$\nu'(\theta) = q(\theta) + \frac{2\Gamma - 3}{3}\mu(\theta) + \frac{2B}{3}, \tag{65}$$

giving

$$e^{\nu(r)} = C_3 r^{2(q_0\Gamma + B)/3} \exp\left[C_1 \frac{3 + (2\Gamma - 3)(1 + l_1)}{3l_1} r^{l_1} + C_2 \frac{3 + (2\Gamma - 3)(1 + l_2)}{3l_2} r^{l_2}\right], \tag{66}$$

where C_3 is an arbitrary integration constant. In the limit of large distances $e^{\nu(r)}$ behaves like $e^{\nu(r)} \approx C_3 r^{2(q_0\Gamma + B)/3}$. Inside the cluster we can approximate $e^{-\lambda} \approx 1 - C/r - GM_U(r)/r$.

The first term in the mass profile of the dark mass given by Eq. (64) is linearly increasing with r, thus having a similar behavior to the dark matter in clusters of galaxies. In order to obtain a model consistent with observations, the second and third terms in Eq. (64) must be decreasing with increasing r. Therefore the constants l_1 and l_2 must also satisfy the conditions $l_1 + 1 < 0$ and $l_2 + 1 < 0$, respectively. By assuming that m and n are positive numbers, the conditions $l_1 < 0$ and $l_2 < 0$ are realized automatically.

In the particular case $B = 0$, we have $q_0 \equiv 0$, and the solution is defined only for values of Γ so that $\Gamma \geq 21/8$. This shows that a radiation like equation of state of the form $P = U/3$ is not allowed in the present model. Moreover, the obtained mass profile is not consistent with the observations, and for such an equation of state the dark radiation cannot play the role of the dark matter. For the case $\Gamma = 4$, corresponding to an equation of state $P = 2U$, the dark mass is given by $GM_U(r) = C_1/r^{1.97} + C_2/r^{0.82} - C$.

Since according to our physical interpretation M_U is an effective, geometry induced mass, it must satisfy the condition $M_U \geq 0$ for all r. Therefore the solution obtained in the present Section is valid only for values of the coordinate radius r so that $q_0 r + C_1 r^{l_1+1} + C_2 r^{l_2+1} - C \geq 0$, where we assume that all integration constants are non-zero. In the limit of small r, taking into account that $l_1 + 1 < 0$ and $l_2 + 1 < 0$, and by assuming that the constant C is small and can be neglected near the origin of the cluster, we obtain $C_1 r^{l_1} + C_2 r^{l_2} \approx 0$. If both C_1 and C_2 are strictly positive, then the dark mass diverges at the center of the cluster, $\lim_{r\to 0} M_U(r) = \infty$. On the other hand, if C_1 and C_2 have different signs, there is a single point r_0, given by $r_0 \approx (C_2/C_1)^{1/(l_1 - l_2)}$, so that $M_U(r_0) \approx 0$ and $M_U(r_0) < 0$, for $r < r_0$. In this case our solution has physically acceptable properties only in the region $r \geq r_0$.

The solution of the brane world model Einstein field equations, given by Eqs. (64), (63) and 66) depends on five, equation of state related or integration constants, q_0, C and C_i, $i = 1, 2, 3$, respectively. These constants must be determined by using some appropriate boundary conditions, or by using some other observable quantities. Without any loss in generality we can take the integration constant $C_3 = 1$, since the metric tensor component $\exp(\nu)$ can be arbitrarily re-parameterized by a transformation of the time coordinate.

An important observational quantity is the radial velocity dispersion σ_r^2, which is related to the total mass in the cluster by the relation $GM_V = \sigma_r^2 R_V$. [26]. By assuming that $R_U \approx R_V$ gives, with the use of the generalized virial theorem, $M_V(R_V) \approx M_U(R_V)$, respectively. For $r = R_V$ Eq. (64) becomes $M_U(R_V) \approx q_0 R_V$, thus giving

$$q_0 \approx \sigma_r^2. \tag{67}$$

Observationally, the mass profiles of the clusters of galaxies are obtained from the equation

$$GM_{tot}\left(r\right)=-\sigma_r^2\left(r\right)r\left[\frac{d\ln\sigma_r^2(r)}{d\ln r}+\frac{d\ln n(r)}{d\ln r}+2\beta_\sigma\right], \tag{68}$$

where $n(r)$ is the spatial galaxy number density and β_σ is the velocity anisotropy parameter [26]. This equation has a very similar mathematical structure with Eq. (64), with the total observed mass of the cluster interpreted as the dark mass. Therefore the other constants in the model can be, at least in principle, obtained by fitting the dark radiation $U(r)$ and dark mass $M_U(r)$ with the observed density and mass profiles of the dark matter in clusters of galaxies.

V. ASTROPHYSICAL APPLICATIONS

Astrophysical observations together with cosmological simulations have shown that the virialized part of the cluster corresponds roughly to a fixed density contrast $\delta\sim 200$ as compared to the critical density of the universe, $\rho_c\left(z\right)$, at the considered redshift, so that $\rho_V=3M_V/4\pi R_V^3=\delta\rho_c\left(z\right)$, where ρ_V is the virial density, M_V and R_V are the virial mass and radius, $\rho_c\left(z\right)=h^2(z)3H_0^2/8\pi G$, and $h(z)$ is the Hubble parameter normalized to its local value: $h^2(z)=\Omega_m\left(1+z\right)^3+\Omega_\Lambda$, where Ω_m is the mass density parameter and Ω_Λ is the dark energy density parameter [28].

Once the integrated mass as a function of radius is determined for galaxy clusters, a physically meaningful fiducial radius for the mass measurement has to be defined. The radii commonly used are either r_{200} or r_{500}. These radii are the radii within the mean gravitational mass density of the matter $\langle\rho_{tot}\rangle=200\rho_c$ or $500\rho_c$. A pragmatic approach to the virial mass is to use r_{200} as the outer boundary [27].

The numerical values of the radius r_{200} are in the range $r_{200}=0.85$ Mpc (for the cluster NGC 4636) and $r_{200}=4.49$ Mpc (for the cluster A2163). A typical value for r_{200} is 2 Mpc. The masses corresponding to r_{200} and r_{500} are denoted by M_{200} and M_{500}, respectively. Usually it is assumed that $M_V=M_{200}$ and $R_V=r_{200}$ [27].

A. Dark radiation, dark mass and dark radius from galactic cluster observations

In clusters of galaxies most of the baryonic mass is in the form of the intra-cluster gas. The gas mass density ρ_g distribution can be fitted with the observational data by using the following expression for the radial baryonic mass (gas) distribution [27]

$$\rho_g(r) = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-3\beta/2}, \tag{69}$$

where r_c is the core radius, and ρ_0 and β are (cluster-dependent) constants.

A static spherical system in collisionless equilibrium can be described by the Jeans equation, which is given by [4]

$$\frac{d}{dr}\left[\rho_g \sigma_r^2\right] + \frac{2\rho_g(r)}{r}\left(\sigma_r^2 - \sigma_{\theta,\phi}^2\right) = -\rho_g(r)\frac{d\Phi}{dr}, \tag{70}$$

where $\Phi(r)$ is the gravitational potential, σ_r and $\sigma_{\theta,\phi}$ are the mass-weighted velocity dispersions in the radial and tangential directions, respectively. We assume that the gas is distributed isotropically inside the clusters and therefore we take $\sigma_r = \sigma_{\theta,\phi}$. The pressure profile P_g is related to the velocity dispersion by $P_g = \rho_g \sigma_r^2$.

Therefore the equation describing the isotropic equilibrium of the gas is

$$\frac{dP_g(r)}{dr} = -\rho_g(r)\frac{d\Phi(r)}{dr} = -\frac{GM_{tot}(r)}{r^2}\rho_g(r), \tag{71}$$

where $M_{tot}(r)$ is the total mass inside radius r. In obtaining Eq. (71) we have assumed that the gravitational field is weak, and that the gravitational potential satisfies the usual Poisson equation $\Delta\Phi \approx 4\pi\rho_{tot}$, with the total energy density $\rho_{tot} \approx \rho_g + \rho_m + 3\alpha U/8\pi G$ also including the energy density ρ_m of other forms of matter, different from gas, like, for example, luminous matter, massive neutrinos etc., and the bulk (five-dimensional) effects. In the following we will systematically neglect the quadratic contribution to the energy-momentum tensor. The energy density of the gas is much larger than the pressure, $\rho_g >> P_g$.

The observed X-ray emission from the hot, ionized intra-cluster gas is usually interpreted by assuming that the gas is in isothermal equilibrium. Therefore, we may assume that the pressure P_g of the gas satisfies the equation of state $P_g(r) = (k_B T_g/\mu m_p)\rho_g(r)$, where k_B is Boltzmann's constant, T_g is the gas temperature, $\mu \approx 0.61$ [27] is the mean atomic weight of the particles in the cluster gas, and m_p is the proton mass. Then Eq. (71) gives

$$M_{tot}(r) = -\frac{k_B T_g}{\mu m_p G} r^2 \frac{d}{dr}\ln\rho_g. \tag{72}$$

By using the density profile of the gas given by Eq. (69) we obtain for the mass profile inside the cluster the relation [27]

$$M_{tot}(r) = \frac{3k_B\beta T_g}{\mu m_p G}\frac{r^3}{r_c^2 + r^2}. \tag{73}$$

On the other hand, from the Einstein field equation Eq. (7) it follows that the total mass inside the radius r satisfies the following mass continuity equation,

$$\frac{dM_{tot}\left(r\right)}{dr}=4\pi r^{2}\rho_{g}\left(r\right)+3\alpha r^{2}U\left(r\right), \tag{74}$$

where we have neglected again the quadratic contributions in the gas density and pressure to the total mass. Since the gas density and the total mass profile inside the cluster are given by Eqs. (69) and (73), respectively, we can obtain immediately the expression of the dark radiation term inside the cluster as

$$3\alpha GU(r)=\left[\frac{3k_{B}\beta T_{g}r^{2}\left(r^{2}+3r_{c}^{2}\right)}{\mu m_{p}\left(r_{c}^{2}+r^{2}\right)^{2}}-\frac{4\pi G\rho_{0}r^{2}}{\left(1+r^{2}/r_{c}^{2}\right)^{3\beta/2}}\right]\frac{1}{r^{2}}. \tag{75}$$

In the limit $r>>r_c$ we obtain for the dark radiation the simple relation

$$3\alpha GU(r)=\left[\frac{3k_{B}\beta T_{g}}{\mu m_{p}}-4\pi G\rho_{0}r_{c}^{3\beta}r^{2-3\beta}\right]\frac{1}{r^{2}}. \tag{76}$$

The dark mass can be obtained generally as

$$GM_{U}(r)=3\alpha\int_{0}^{r}r^{2}U(r)dr=\left[\frac{3k_{B}\beta T_{g}}{\mu m_{p}}\frac{r}{1+r_{c}^{2}/r^{2}}-4\pi G\rho_{0}\int_{0}^{r}\frac{r^{2}dr}{\left(1+r^{2}/r_{c}^{2}\right)^{3\beta/2}}\right]. \tag{77}$$

In the limit $r>>r_c$ the dark mass M_U is given as a function of r by the expression

$$GM_{U}\left(r\right)\approx\left[\frac{3k_{B}\beta T_{g}}{\mu m_{p}}-\frac{4\pi G\rho_{0}r_{c}^{3\beta}r^{2-3\beta}}{3\left(1-\beta\right)}\right]r. \tag{78}$$

Let's assume first that we can neglect the contribution of the gas to the dark radiation and dark mass, respectively. Then we obtain

$$3\alpha GU(r)\approx\left(\frac{3k_{B}\beta T_{g}}{\mu m_{p}}\right)r^{-2}, \tag{79}$$

and

$$GM_{U}\left(r\right)\approx\left(\frac{3k_{B}\beta T_{g}}{\mu m_{p}}\right)r, \tag{80}$$

respectively.

There is a simple way to estimate an upper bound for the cutoff of the dark mass. The idea is to consider the point at which the decaying density profile of the dark radiation associated to the galaxy cluster becomes smaller than the average energy density of the Universe. Let the value of the coordinate radius at the point where the two densities are equal to be $R_U^{(cr)}$. Then at this point $3\alpha GU\left(R_U^{(cr)}\right)=\rho_{univ}$, where ρ_{univ} is the mean energy density of the universe. By assuming $\rho_{univ} = \rho_c = 3H^2/8\pi G = 4.6975 \times 10^{-30} h_{50}^2$ g/cm^{-3}, where $H = 50h_{50}$km/Mpc/s [27], we obtain

$$R_U^{(cr)} = \left(\frac{3k_B\beta T_g}{\mu m_p G\rho_c}\right)^{1/2} = 91.33\sqrt{\beta}\left(\frac{k_B T_g}{5\text{keV}}\right)^{1/2} h_{50}^{-1}\text{Mpc}. \tag{81}$$

The total dark mass corresponding to this value is

$$M_U^{(cr)} = M_U\left(R_U^{(cr)}\right) = 4.83 \times 10^{16}\beta^{3/2}\left(\frac{k_B T_g}{5\text{keV}}\right)^{3/2} h_{50}^{-1} M_{\odot}. \tag{82}$$

This value of the mass is consistent with the observations of the mass distribution in the clusters of galaxies. However, according to the brane world scenario, we predict that the dark mass and its effects extends beyond the virial radius of the clusters, which is of the order of only a few Mpc.

Let's now briefly consider the effect of the gas on the dark mass distribution. If $\beta < 2/3$, the dark mass has a maximum value, which occurs at a radius R_1, which can be obtained from the condition $dM_U(r)/dr = 0$, and is given by

$$R_1 = \left(\frac{3k_B\beta T_g}{4\pi G\rho_0\mu m_p r_c^{3\beta}}\right)^{1/(2-3\beta)}. \tag{83}$$

The maximum value of the dark mass can be found as

$$GM_U^{\max}\left(R_1\right) = \frac{(2-3\beta)}{3\left(1-\beta\right)}\left(\frac{3k_B\beta T_g}{\mu m_p}\right)^{\frac{3(1-\beta)}{2-3\beta}}\left(4\pi G\rho_0 r_c^{3\beta}\right)^{-1/(2-3\beta)}. \tag{84}$$

The central density of the gas ρ_0 can be estimated from the total gas mass M_g as

$$\rho_0 \approx \frac{3\left(1-\beta\right)M_g\left(R_V\right)}{4\pi r_c^3\left(R_V/r_c\right)^{3(1-\beta)}}. \tag{85}$$

Hence we obtain

$$R_1 = \left[\frac{k_B\beta T_g}{\left(1-\beta\right)GM_g\left(R_V\right)\mu m_p}\right]^{1/(2-3\beta)} R_V^{3(1-\beta)/(2-3\beta)}, \tag{86}$$

and

$$GM_U^{\max}\left(R_1\right) = \frac{2-3\beta}{\left[3\left(1-\beta\right)\right]^{3(1-\beta)}}\left(\frac{3k_B\beta T_g}{\mu m_p}\right)^{\frac{3(1-\beta)}{2-3\beta}}\left[GM_g\left(R_V\right)\right]^{-1/(2-3\beta)} R_V^{3(1-\beta)/(2-3\beta)}. \tag{87}$$

Usually the mass of the gas represent $5-10\%$ of the total mass in the cluster [26, 27]. For $\beta = 1/2$, $k_B T_g = 5$ keV, and assuming for the mass of the gas a value of the order

$M_g = 7 \times 10^{13} M_\odot$, we obtain $R_1 \approx 50.92$ Mpc and $M_U^{\max}(R_1) = 6.6 \times 10^{16} M_\odot$, respectively. For $\beta \geq 2/3$ the mass is a linearly increasing function of the radius.

By comparing Eqs. (81) and (86) we predict that, due to the presence of the gas in the cluster, the dark mass, playing the role of the "dark matter", should cut-off earlier, and before it merges into the background.

Astrophysical observations are extended and interpreted in terms of an outer density of the order of $\delta\rho_{univ}$, with a fixed density contrast $\delta = 200$. In order to compare the predictions of the brane world models with the observations it is necessary to estimate the dark radius for an outer density of the dark radiation of the order of $\delta\rho_{cr}$, with the density contrast $\delta = 200$. Hence, the dark radius R_U^{200} corresponding to the density contrast $\delta = 200$ is given as a solution of the non-linear algebraic equation

$$\frac{3k_B\beta T_g}{\mu m_p} - 4\pi G\rho_0 \left(\frac{r_c}{R_U^{200}}\right)^{3\beta} \left(R_U^{200}\right)^2 = \delta G\rho_c \left(R_U^{200}\right)^2. \tag{88}$$

By evaluating the gas density at R_U^{200} we obtain

$$\rho_g\left(R_U^{200}\right) \approx \rho_0 \left(\frac{r_c}{R_U^{200}}\right)^{3\beta} = \delta_{gd}\rho_c, \tag{89}$$

where δ_{gd} gives the density of the amount of the total mass in form of the gas at a distance R_U^{200} from the center of the cluster with respect to the critical density. Then we obtain for the dark radius of the cluster the expression

$$R_U^{200} = \sqrt{\frac{3k_B\beta T_g}{\mu G m_p \rho_c \left(4\pi\delta_{gd} + \delta\right)}} = \frac{R_U^{(cr)}}{\sqrt{4\pi\delta_{gd} + \delta}} = \frac{91.33}{\sqrt{4\pi\delta_{gd} + \delta}} \sqrt{\beta} \left(\frac{k_B T_g}{5\text{keV}}\right)^{1/2} h_{50}^{-1}\text{Mpc}. \tag{90}$$

For $k_B T_g = 5$ keV, $\beta = 2/3$ and $\delta_{gd} = 20$ we obtain $R_U^{200} \approx 3.51$ Mpc. This shows that R_U^{200} is of the same order of magnitude as the virial radius $R_V = r_{200}$, and the presence of the gas considerably decreases the magnitude of the dark radius. Therefore it follows from the virial theorem that the virial mass of the galaxy clusters is mainly determined by the dark mass associated to the transmitted projection of the bulk Weyl tensor.

B. Radial velocity dispersion in galactic clusters

The virial mass can also be expressed in terms of the characteristic velocity dispersion σ_1 as [26]

$$M_V = \frac{3}{G}\sigma_1^2 R_V. \tag{91}$$

By assuming that the velocity distribution in the cluster is isotropic, we have $\langle u^2\rangle = \langle u_r^2\rangle + \langle u_\theta^2\rangle + \left\langle u_\phi^2\right\rangle = 3\langle u_r^2\rangle = 3\sigma_r^2$, with σ_r^2 the radial velocity dispersion. σ_1 and σ_r are related by $3\sigma_1^2 = \sigma_r^2$.

In order to derive the radial velocity dispersion relation for clusters of galaxies in brane world models we start from Eq. (17). Taking into account that the velocity distribution is isotropic, we obtain

$$\frac{d}{dr}\left(\rho\sigma_r^2\right) + \frac{1}{2}\rho\frac{d\nu}{dr} = 0. \tag{92}$$

Since inside the cluster $e^{-\lambda} \approx 1$, by neglecting the cosmological constant the Einstein field equation Eq. (21) becomes

$$\frac{\nu''}{2} + \frac{\nu'}{r} = 3\alpha U + 4\pi G\rho, \tag{93}$$

and can be written in the equivalent form

$$\frac{1}{2r^2}\frac{d}{dr}\left(r^2\nu'\right) = 3\alpha U + 4\pi G\rho, \tag{94}$$

giving upon integration

$$r^2\nu' = 2GM_U(r) + 2GM(r) + 2C_4, \tag{95}$$

where C_4 is an arbitrary constant of integration. Since from Eq. (92) we have $\nu' = -(2/\rho)d\left(\rho\sigma_r^2\right)/dr$, it follows that the radial velocity dispersion of the galactic clusters on the brane satisfies the differential equation

$$\frac{d}{dr}\left(\rho\sigma_r^2\right) = -\frac{GM_U(r)}{r^2}\rho(r) - \frac{GM(r)}{r^2}\rho(r) - \frac{C_4}{r^2}\rho(r), \tag{96}$$

with the general solution given by

$$\sigma_r^2(r) = -\frac{1}{\rho}\int\left[\frac{GM_U(r)}{r^2}\rho(r) + \frac{GM(r)}{r^2}\rho(r) + \frac{C_4}{r^2}\rho(r)\right]dr + \frac{C_5}{\rho}, \tag{97}$$

where C_5 is an integration constant.

As an example of the application of Eq. (97) we consider the case in which the density ρ of the normal matter inside the cluster has a power law distribution, so that

$$\rho(r) = \rho_0 r^{-\gamma}, \tag{98}$$

with ρ_0 and $\gamma \neq 1, 3$ positive constants. The corresponding normal matter mass profile is $M(r) = 4\pi\rho_0 r^{3-\gamma}/(3-\gamma)$. The dark mass $GM_U = q_0 r$ is linearly proportional to r, as has been shown in the previous Section. Therefore after some simple calculations we obtain

$$\sigma_r^2(r) = \frac{q_0}{\gamma} + \frac{2\pi G\rho_0}{(\gamma-1)(3-\gamma)}r^{2-\gamma} + \frac{C_4}{\gamma+1}\frac{1}{r} + \frac{C_5}{\rho_0}r^\gamma, \gamma \neq 1, 3. \tag{99}$$

In the case $\gamma = 1$ we obtain

$$\sigma_r^2(r) = q_0 + \frac{C_4}{2r} - 2\pi G\rho_0 r \ln r + \frac{C_5}{\rho_0} r, \gamma = 1. \tag{100}$$

For $\gamma = 3$ we find

$$\sigma_r^2(r) = \frac{q_0}{3} - \pi G\rho_0 \left(\ln r + \frac{1}{4}\right)\frac{1}{r^4} + \frac{C_4}{4}\frac{1}{r} + \frac{C_5}{\rho_0} r^3, \gamma = 3. \tag{101}$$

Usually the observed data for the velocity dispersion in clusters of galaxies are analyzed by assuming the simple form $\sigma_r^2(r) = B/(r+b)$ for the radial velocity dispersion, with B and b constants. As for the density of the galaxies in the clusters the relation $\rho(r) = A/r\,(r+a)^2$, with A and a constants, is used. The data are then fitted with these functions by using a non-linear fitting procedure [26]. For $r << a$, $\rho(r) \approx A/r$, while for $r >> a$, $\rho(r)$ behaves like $\rho(r) \approx A/r^3$. Therefore the comparison of the observed velocity dispersion profiles of the galaxy clusters and the velocity dispersion profiles predicted by the brane world scenario may give a powerful method to discriminate between the different theoretical models.

VI. DISCUSSIONS AND FINAL REMARKS

The galactic rotation curves and the mass distribution in clusters of galaxies continue to pose a challenge to present day physics. One would like to have a better understanding of some of the intriguing phenomena associated with them, like their universality and the very good correlation between the amount of dark matter and the luminous matter in the galaxy. To explain these observations, the most commonly considered models are based on particle physics in the framework of Newtonian gravity, or of some extensions of general relativity [7].

In the present paper we have considered, and further developed, an alternative view to the dark matter problem [13, 14, 15, 16], namely, the possibility that the galactic rotation curves and the mass discrepancy in clusters of galaxies can naturally be explained in models in which our Universe is a domain wall (a brane) in a multi-dimensional space-time (the bulk). The extra-terms in the gravitational field equations on the brane induce a supplementary gravitational interaction, which can account for the observed behavior of the galactic rotation curves. By using the simple observational fact of the constancy of the galactic rotation curves, the galactic metric and the corresponding Weyl stresses (dark radiation and dark pressure) can be completely reconstructed [15, 16].

In order to estimate the effect of the bulk effects on the extra-galactic scale, at the level of the clusters of galaxies, we have generalized the virial theorem to include the contribution of the non-compact extra dimensions. To derive the generalized virial theorem we have used a method based on the collisionless Boltzmann equation. The generalized virial theorem allows to remove the virial mass discrepancy, by showing that the virial mass M_V is proportional, as one can see from Eq. (39), to the dark mass M_U associated to the dark radiation term U, which is the scalar component of the projected Weyl tensor $E_{\mu\nu}$. The dark mass can be directly related to the observed virial mass, and, for typical clusters of galaxies, it is of the same order of magnitude as the virial mass. This shows that at the extra-galactic scale the dark mass plays the role of what is conventionally called the dark matter.

In the present model the cut-off of the dark mass (which represents the dark matter in the conventional interpretation) is not due to the merging of the dark radiation with the cosmological background, but it is due to the presence of the intra-cluster gas, which is the only form of conventional matter in our theory.

In order to derive our main results we have used a set of approximations and assumptions, which can be classified as a) brane world model related b) small velocity limit approximation and c) weak gravitational field approximation, respectively. In our treatment of the brane world models we have systematically ignored the second order terms in the matter density. This approximation is justified since the ratio ρ/λ_b, where λ_b is the brane tension, is very small in the case of the galaxy clusters. By taking for the mean density of the baryonic matter in the cluster a value of $\rho \approx 10^{-20} - 10^{-23}$ g/cm^3 [27, 28, 29], and for the brane tension a value of the order of $\lambda_b = (100 \text{ GeV})^4 = 1.37 \times 10^{21}$ g/cm^3 [23], it follows that the ratio ρ/λ_b is indeed negligible small at the level of the clusters of galaxies, $\rho/\lambda_b << 1$, and hence the second order effects in the matter density can be safely neglected. Since the dispersion of the velocity of the galaxies in the clusters is of the order $600-1000$ km/s [27, 28, 29], giving for the square of the ratio of the velocity and of the speed of light a value of the order of $(v/c)^2 \approx 4\times 10^{-6}$ - $1.11\times 10^{-5} << 1$, we can neglect the relativistic effects in the Boltzmann equation, and use the small velocity limit of it. The intensity of the gravitational effects can be estimated from the ratio GM/R, which for typical clusters is of the order of $10^{-6} << 1$. Therefore inside galactic clusters the gravitational field is weak, and the linear approximation of the gravitational equations provides an accurate description of the dynamics of the massive test bodies. We have also systematically neglected the possible effects of the cosmological constant. As for the physical structure of the clusters, we have assumed that they are dark radiation dominated spherically symmetric static astrophysical systems, with a normal baryonic matter content representing only $5-10\%$ of the total equivalent mass energy density. Both these assumptions are very well supported by observations and cosmological simulations [27, 28, 29].

In the present model all the relevant physical quantities, including the dark mass and the dark radius R_U, which describe the non-local effects due to the gravitational field of the bulk, are expressed in terms of observable parameters (the temperature and the virial radius of the cluster, and the observed velocity dispersion). Hence it follows that the contributions of the bulk Weyl tensor in the case of the clusters of galaxies on the brane - the dark mass M_U, the dark radius R_U, and the gravitational energy Ω_U - can be determined from the study of the X-ray properties of the gas within the virial radius.

Generally, the virial mass M_V is obtained from the observational study of the velocity dispersions of the stars in the cluster. Thus, it cannot give a reliable estimation of the numerical value of the total mass $M_g + M_U$ in the cluster. A very useful method for the study of the total mass distribution in clusters is the gravitational lensing of the light, which may provide direct evidence for the total mass distribution, and for the gravitational effects at large distances from the cluster.

Therefore, this opens the possibility of testing the brane world models by using astronomical and astrophysical observations at the extra-galactic scale. In this paper we have provided some basic theoretical tools necessary for the in depth comparison of the predictions of the brane world model and of the observational results.

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