WEYL CLOSURE OF HYPERGEOMETRIC SYSTEMS

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ABSTRACT. We show that A-hypergeometric systems and Horn hypergeometric systems are Weyl closed for very generic parameters.

1. INTRODUCTION

Let $D = D_n$ be the (complex) Weyl algebra, that is, the ring of linear partial differential operators with polynomial coefficients in variables $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$, where ∂_i stands for $\frac{\partial}{\partial x_i}$. Write $R = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} D$ for the ring of operators with rational function coefficients. If I is a left D-ideal, then the *Weyl closure* of I is

$$RI \cap D.$$

If I equals its Weyl closure, then I is said to be *Weyl closed*.

The operation of Weyl closure is an analog of the radical operation in the polynomial ring, as the Weyl closure of I is the differential annihilator of the space of (local) holomorphic solutions of I (see Proposition 2.19 in [Tsa00a]). The notion of Weyl closure was introduced by Harrison Tsai in [Tsa00a]. This work contains an algorithm to compute the Weyl closure of a left D-ideal, which has been implemented by Anton Leykin and Harrison Tsai in the computer algebra system Macaulay2 [M2]. Other references are [Tsa00b, Tsa02].

The goal of this note is to show that A-hypergeometric systems and Horn hypergeometric systems are Weyl closed when the parameters are generic enough. We illustrate this in an example. Given $a, a' \in \mathbb{C}\backslash\mathbb{Z}$, consider the series

$$G(s,t) = \sum_{(m,n)\in\mathbb{N}^2} c_{m,n} s^m t^n = \sum_{(m,n)\in\mathbb{N}^2} \frac{(a)_{m-2n}}{(a')_{n-2m}} \frac{s^m t^n}{m!n!},$$

where $(a)_k = \prod_{l=0}^{k-1}(a+l)$ is the Pochhammer symbol for the ascending factorial. This series converges in a neighborhood of the origin, and it is a hypergeometric series, since its coefficients satisfy the following special recurrence relations:

$$\frac{c_{m+1,n}}{c_{m,n}} = \frac{(-2m+n+a'-1)(-2m+n+a'-2)}{(m+1)(m-2m+a)}$$

$$\frac{c_{m,n+1}}{c_{m,n}} = \frac{(m-2n+a-1)(m-2n+a-2)}{(n+1)(-2m+n+a')}$$

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which translate into the following system of differential equations for G:

$$
\begin{aligned}
&\big[\tfrac{1}{s}\theta_s(\theta_s - 2\theta_t + a - 1) - (-2\theta_s + \theta_t + a' - 1)(-2\theta_s + \theta_t + a' - 2)\big]G(s,t) = 0 \\
&\big[\tfrac{1}{t}\theta_t(-2\theta_s + \theta_t + a' - 1) - (\theta_s - 2\theta_t + a - 1)(\theta_s - 2\theta_t + a - 2)\big]G(s,t) = 0
\end{aligned} \tag{1.1}
$$

where $\theta_s = s\frac{\partial}{\partial s}$ and $\theta_t = t\frac{\partial}{\partial t}$.

Question: Is the above system the *differential annihilator* of G? This would mean that any differential equation for G can be obtained by taking combinations (with coefficients in the Weyl algebra in s, t, $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$) of the equations (1.1).

It turns out that it is simpler to study G, and in particular, determine its differential annihilator, if we make a change of variables, as follows.

Define matrices

$$B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The rows of B tell us the factors that appear in the differential equations (1.1); these factors are obtained by adding appropriate parameters to dot products of rows of B with the vector (θ_s, θ_t). The matrix A is chosen so that the columns of B form a basis for its kernel.

Let

$$F(x_1, x_2, x_3, x_4) = x_2^{a'-1}x_3^{a-1}G(\frac{x_1x_3}{x_2^2}, \frac{x_2x_4}{x_3^2}) = x_2^{a'-1}x_3^{a-1}G(x^B).$$

Then F satisfies the following system of differential equations

$$[3\theta_1 + 2\theta_2 + \theta_3 - (2a' + a - 3)]F(x) = 0; \quad [\theta_2 + 2\theta_3 + 3\theta_4 - (2a + a' - 3)]F(x) = 0;$$
$$[\partial_1\partial_3 - \partial_2^2]F(x) = 0; \quad [\partial_2\partial_4 - \partial_3^2]F(x) = 0,$$

where ∂_i stands for $\frac{\partial}{\partial x_i}$ and $\theta_i = x_i\partial_i$. The first two differential equations reflect the change of variables we applied to G. The last two correspond to the differential equations (1.1). We call this system a *Horn system* (Definition 3.1).

It turns out (by Theorem 2.8) that, in order to get the differential annihilator of F, we need to add another equation, namely

$$[\partial_1\partial_4 - \partial_2\partial_3]F(x) = 0.$$

When we do this, we obtain an *A-hypergeometric system* (Definition 2.1).

The A-hypergeometric system is strictly larger than the Horn system. To see this, note that the function $x_2^{(a'-1)}x_3^{a-1}$ is a solution of the Horn system, but not of the A-hypergeometric system. On the other hand, by Corollary 3.7, the Horn system itself is also Weyl closed.

This is interesting information. It tells us, for instance, that if we want to understand the monodromy of the function F, the differential equations we should study are the A-hypergeometric system, and not the smaller Horn system.

The plan for this article is as follows. In Section 2, we define A-hypergeometric systems, and show that they are Weyl closed for very generic parameters (Corollary 2.9). A key ingredient is the existence of *fully supported* (Definition 2.6) convergent power series solutions of A-hypergeometric systems [GKZ89, SST00, OT07]. In Section 3 we introduce Horn systems, and again, prove that they are Weyl closed for generic parameters. The proofs in this section rely heavily on results from [DMM06].

2. A-HYPERGEOMETRIC SYSTEMS

We will work in the Weyl algebra $D = D_n$ in $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, and denote $\theta_j = x_j\partial_j$.

Let $A = (a_{ij})$ be a $d \times n$ integer matrix of full rank d, satisfying two conditions on its columns. The first is that they $\mathbb{Z}$-span $\mathbb{Z}^d$, and the second is that they all lie in an open half space of $\mathbb{R}^d$. In particular, A is not allowed to have a zero column.

Definition 2.1. Given A as above, set

$$E_i = \sum_{j=1}^{n} a_{ij}\theta_j : \quad i = 1, \dots, d,$$

and define the *toric ideal* to be

$$I_A = \langle \partial^u - \partial^v : Au = Av \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n].$$

For $\beta \in \mathbb{C}^d$ the A-*hypergeometric system with parameter* β is the left D-ideal

$$H_A(\beta) = I_A + \langle E - \beta \rangle \subseteq D,$$

where $\langle E - \beta \rangle$ is shorthand for $\langle E_i - \beta_i : i = 1, \dots, d \rangle$.

Note that although $\langle E - \beta \rangle$ depends on the matrix A, this is not reflected in the notation.

A-hypergeometric systems were introduced in the work of Gelfand, Graev, Kapranov and Zelevinsky [GGZ87, GKZ89]. The text [SST00] emphasizes computational aspects in the theory of A-hypergeometric equations, and is highly recommended.

We wish to show that $H_A(\beta)$ is Weyl closed for very generic β. Here, *very generic* will mean "outside a countable collection of algebraic varieties". The following definition gives us a countable family of the hyperplanes that we will need to avoid.

Definition 2.2. A *facet* of A is a subset of its columns that is maximal among those minimizing nonzero linear functionals on $\mathbb{Z}^d$. Denote the columns of A by $a_1, \dots, a_n$. For a facet σ of A let ν_σ be its *primitive support function*, the unique rational linear form satisfying

(1) $\nu_\sigma(\mathbb{Z}A) = \mathbb{Z}$,
(2) $\nu_\sigma(a_j) \geq 0$ for all $j \in J$,
(3) $\nu_\sigma(a_j) = 0$ for all $a_j \in \sigma$.

A parameter vector $\beta \in \mathbb{C}^d$ is A-*nonresonant* (or simply *nonresonant*, when it causes no confusion) if $\nu_\sigma(\beta) \notin \mathbb{Z}$ for all facets σ of A. Note that if β is A-nonresonant, then so is $\beta + A(\gamma)$ for any $\gamma \in \mathbb{Z}^n$.

Nonresonant parameters have nice properties, as is illustrated below.

Lemma 2.3. *Fix a nonresonant parameter β. If $P\partial_i \in H_A(\beta)$, then $P \in H_A(\beta + Ae_i)$.*

This is an immediate consequence of the following well known fact, a concise proof of which can be found in [DMM06][Lemma 6.10].

Theorem 2.4. *If β is nonresonant, the map $D/H_A(\beta) \to D/H_A(\beta - Ae_i)$ given by right multiplication by ∂_i is an isomorphism.*

Here is another useful feature of nonresonant parameters.

Proposition 2.5. *For any nonresonant β, if $x_iP \in H_A(\beta)$, then $P \in H_A(\beta)$.*

Proof. By contradiction, assume that the set

$$\{(\beta, P) \in \mathbb{C}^d \times D : \beta \text{ is nonresonant}, P \notin H_A(\beta) \text{ but } \; x_iP \in H_A(\beta)\} \tag{2.1}$$

is nonempty. Choose a pair in this set with P of lowest possible order.

Since $x_iP \in H_A(\beta)$, we can find binomials $b_1, \dots, b_r \in I_A$ and operators $F_1 + x_iF_1^o, \dots, F_r + x_iF_r^o, G_1 + x_iG_1^o, \dots, G_d + x_iG_d^o$ such that

$$x_iP = \sum_j (F_j + x_iF_j^o)b_j + \sum_l (G_l + x_iG_l^o)(E_l - \beta_l),$$

and the operators F_j, G_l contain no terms with x_i. Then,

$$x_i(P - \sum_j F_j^o b_j - \sum_l G_l^o(E_l - \beta_l)) = \sum_j F_jb_j + \sum_l G_l(E_l - \beta_l). \tag{2.2}$$

Since $(x^\mu\partial^\nu)x_i = x^{\mu+e_i}\partial^\nu + \nu_i x^\mu \partial^{\nu - e_i}$, the only part of the right hand side of (2.2) that contains x_i is

$$x_i(\sum_l a_{li}G_l)\partial_i, \tag{2.3}$$

and therefore, since the terms without x_i must all cancel, (2.3) must equal the right hand side of (2.2).

Now, the left hand side of (2.2) belongs to $H_A(\beta)$, so $x_i(\sum_l a_{li}G_l)\partial_i \in H_A(\beta)$. Since β is nonresonant, we have $x_i(\sum_l a_{li}G_l) \in H_A(\beta + Ae_i)$ by Lemma 2.3. If $\sum_l a_{li}G_l \notin H_A(\beta)$ then $(\beta + Ae_i, \sum_l a_{li}G_l)$ belongs to the set (2.1), as $\beta + Ae_i$ is nonresonant. This is a contradiction, since $\sum_l a_{li}G_l$ has lower order than P.

If $\sum_l a_{li}G_l \in H_A(\beta)$, then we can conclude that $P \in H_A(\beta)$, another contradiction. □

We want to show that an A-hypergeometric system is the differential annihilator of a special kind of function, that we define below.

Definition 2.6. A formal power series φ is *supported on a translate of a lattice* $L \subseteq \mathbb{Z}^n$ if it is of the form $x^v \sum_{u \in L} \lambda_u x^u$. The set $\{v + u : \lambda_u \neq 0\}$ is called the *support* of φ. If the support of φ is Zariski dense in the Zariski closure of $v + L$, then φ is *fully supported*.

We can guarantee the existence of fully supported solutions of $H_A(\beta)$ if we require that the parameters be generic.

Theorem 2.7. *If β is generic, then $H_A(\beta)$ has a holomorphic solution that can be represented as a fully supported power series on a translate of the lattice $\ker_{\mathbb{Z}}(A)$.*

Proof. This follows from [SST00][Proposition 3.4.4, Lemma 3.4.6] in the case that the toric ideal I_A is homogeneous in the usual grading of the polynomial ring $\mathbb{C}[\partial]$. Another proof can be found in [GKZ89]. When I_A is not homogeneous, we use [OT07][Theorem 2]. $\square$

We are now ready to prove the main result in this section.

Theorem 2.8. *If $P \in D$ annihilates a fully supported solution f of $H_A(\beta)$, and β is nonresonant, then $P \in H_A(\beta)$.*

Proof. Let f be a fully supported solution of $H_A(\beta)$, and let $P \in D$ such that $Pf = 0$.

The Weyl algebra is $\mathbb{Z}^d$-graded via $\deg(x^\mu\partial^\nu) = A(\mu-\nu)$. If $x^\mu\partial^\nu$ and $x^{\mu'}\partial^{\nu'}$ have different A-degrees, then $x^\mu\partial^\nu f$ and $x^{\mu'}\partial^{\nu'} f$ have disjoint supports, since f is supported on a translate of the lattice $\ker_{\mathbb{Z}}(A)$. Thus, we may assume that P is an A-homogeneous differential operator. Moreover, if $x^{\mu^o}\partial^{\nu^o}$ is a monomial with nonzero coefficient in P, then $x^{\nu^o}P\partial^{\mu^o}$ is homogeneous of degree $0 \in \mathbb{Z}^d$. By Theorem 2.4, we can find a solution g of $H_A(\beta - A\mu^o)$ such that $\partial^{\mu^o} g = f$. Since f is fully supported, so is g. Finally $\beta - A\mu^o$ is nonresonant.

Write $x^{\nu^o}P\partial^{\mu^o} = \sum_{\mu,\nu} c_{\mu,\nu}x^\mu\partial^\nu$. Since this operator is A-homogeneous of degree 0, we have $A\mu = A\nu$ whenever $c_{\mu,\nu} \neq 0$.

Now
$$x^{\nu^o}P\partial^{\mu^o} = \sum c_{\mu,\nu}x^\mu\partial^\nu = \sum c_{\mu,\nu}x^\mu(\partial^\nu - \partial^\mu) + \sum c_{\mu,\nu}x^\mu\partial^\mu.$$
Then $\sum c_{\mu,\nu}x^\mu\partial^\mu$ annihilates g. Moreover,
$$x^\mu\partial^\mu = \prod_{j=1}^{n}\prod_{k=0}^{\mu_j-1}(\theta_j - k),$$
so we can write $\sum c_{\mu,\nu}x^\mu\partial^\mu = p(\theta_1,\dots,\theta_n)$ for some polynomial p. Write
$$g = x^v \sum_{u\in\ker_{\mathbb{Z}}(A)} \lambda_u x^u,$$
where $Av = \beta - A\mu^o$. Then
$$0 = \sum c_{\mu,\nu}x^\mu\partial^\mu g = p(\theta_1,\dots,\theta_n)g = \sum_{u\in\ker_{\mathbb{Z}}(A)} \lambda_u p(v+u)x^{v+u},$$
which means that $p(v+u) = 0$ whenever $\lambda_u \neq 0$. But the fact that g is fully supported implies that the set $\{v+u : \lambda_u \neq 0\}$ is Zariski-dense in $v+\ker(A)$, so p must vanish on all of $v+\ker(A)$. By the Nullstellensatz, this implies that $p(\theta_1,\dots,\theta_n) \in \langle E-(\beta-A\mu^o)\rangle \subseteq \mathbb{C}[\theta_1,\dots,\theta_n]$. Thus $x^{\nu^o}P\partial^{\mu^o} \in H_A(\beta - A\mu^o)$, and by Lemma 2.3 and Proposition 2.5, $P \in H_A(\beta)$. $\square$

Corollary 2.9. *If β is very generic, then $H_A(\beta)$ is Weyl closed.*

Proof. If we choose β generic so that $H_A(\beta)$ has a fully supported series solution and also require β to be nonresonant, we fall into the hypotheses of Theorem 2.8, which implies that $H_A(\beta)$ is a differential annihilator, and therefore Weyl closed. $\square$

3. Horn Systems

In this section, we show that Horn systems are Weyl closed for very generic parameters.

Let B be an $n \times m$ integer matrix of full rank m such that every nonzero element of the $\mathbb{Z}$-column span of B is *mixed*, meaning that each such vector has a strictly positive and a strictly negative entry. In particular, the columns of B are mixed. In this case, we can find a matrix A as in Section 2 with $d = n - m$.

Definition 3.1. Given $u \in \mathbb{Z}^n$, write u_+ for the vector defined by $(u_+)_i = u_i$ if $u_i \leq 0$, and $(u_+)_i = 0$ otherwise. Let $u_- = u_+ - u$. The ideal

$$I(B) = \langle \partial^{u_+} - \partial^{u_-} : u \text{ is a column of } B \rangle \subseteq \mathbb{C}[\partial]$$

is called a *lattice basis ideal* for the lattice $\mathbb{Z}B$ spanned by the columns of B. For any $\beta \in \mathbb{C}^d$ the left D-ideal

$$H(B, \beta) = I(B) + \langle E - \beta \rangle \subseteq D$$

is called a *Horn system with parameter* β.

Remark 3.2. This is the binomial formulation for Horn systems. For the relation with the classical systems of equations introduced by Appell and Horn [App1880, Hor1889], see [DMS05, DMM06].

In order to prove that Horn systems are Weyl closed, we need to describe their solution spaces. This requires information about the lattice basis ideal $I(B)$, namely, its primary decomposition. The main references for primary decomposition of binomial ideals in general, and lattice basis ideals in particular, are [ES96, FS96, HS00, DMM06].

Each of the minimal primes of $I(B)$ arises, after row and column permutations, from a block decomposition of B of the form

$$\left[\begin{array}{c|c} N & B_J \\ \hline M & 0 \end{array} \right], \tag{3.1}$$

where M is a mixed submatrix of B of size $q \times p$ for some $0 \leq q \leq p \leq m$ [HS00]. (Matrices with $q = 0$ rows are automatically mixed; matrices with $q = 1$ row are never mixed.) We note that not all such decompositions correspond to minimal primes: the matrix M has to satisfy another condition which Hoşten and Shapiro call irreducibility [HS00, Definition 2.2 and Theorem 2.5]. If $I(B)$ is a complete intersection, then only square matrices M will appear in the block decompositions (3.1), by a result of Fischer and Shapiro [FS96].

Let $\bar{J}$ be the set of indices of the q rows of M (before permuting) and let $J = \{1, \ldots, n\} \backslash \bar{J}$ be the index set of B_J (again, before permuting). Denote by A_J the matrix whose columns are the columns of A indexed by J. Split the variables $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$ into two blocks each:

$$x_J = \{x_j : j \in J\} \quad \text{and} \quad x_{\bar{J}} = \{x_j : j \notin J\}.$$
$$\partial_J = \{\partial_j : j \in J\} \quad \text{and} \quad \partial_{\bar{J}} = \{\partial_j : j \notin J\}.$$

Let $\operatorname{sat}(\mathbb{Z}B_J) = \mathbb{Q}B_J \cap \mathbb{Z}^J$. For each partial character $\rho : \operatorname{sat}(\mathbb{Z}B_J) \to \mathbb{C}^*$ extending the trivial character on $\mathbb{Z}B_J$, the ideal

$$I_{\rho,J} = I_\rho + \langle \partial_j : j \notin J \rangle, \quad \text{where } I_\rho = \langle \partial_J^w - \rho(w - w')\partial_J^{w'} : w, w' \in \mathbb{N}^J, A_J w = A_J w' \rangle,$$

is an associated prime of $I(B)$. Note that the symbol ρ here includes the specification of the sublattice $\operatorname{sat}(\mathbb{Z}B_J) \subseteq \mathbb{Z}^J$.

Definition 3.3. (cf. [DMM06][Definition 3.6, Lemma 6.2, Observation 6.3]) If the matrix M is square and invertible, the prime $I_{\rho,J}$ is called a *toral associated prime of* $I(B)$. The corresponding primary component of $I(B)$, denoted by $C_{\rho,J}$, is called a *toral component of* $I(B)$. Associated primes and primary components that are not toral are called *Andean*.

The primary decomposition of $I(B)$, and in particular, its toral components, are important to the study of Horn systems because of the following fact [DMM06][Theorem 5.8].

$$\frac{D}{H(B,\beta)} \cong \bigoplus_{C_{\rho,J} \text{ toral}} \frac{D}{C_{\rho,J} + \langle E-\beta\rangle} \quad \text{for generic } \beta.$$

This implies that, for generic β, the solution space of $H(B,\beta)$ is the direct sum of the solution spaces of the systems $C_{\rho,J} + \langle E-\beta\rangle$, for toral $C_{\rho,J}$. In order to describe these solution spaces, we need an explicit expression for the toral components $C_{\rho,J}$.

Fix a toral component $C_{\rho,J}$ coming from a decomposition (3.1).

Define a graph Γ whose vertices are the points in $\mathbb{N}^{\bar{J}}$. Two vertices $u, u' \in \mathbb{N}^{\bar{J}}$ are connected by an edge if $u-u'$ or $u'-u$ is a column of the matrix M. The connected components of the graph Γ are called the *M-subgraphs of* $\mathbb{N}^{\bar{J}}$. If $u \in \mathbb{N}^{\bar{J}}$, call $\Gamma(u)$ the M-subgraph that u belongs to. Then, by [DMM06][Example 3.14],

$$C_{\rho,J} = I(B) + I_{\rho,J} + \langle \partial_{\bar{J}}^{u} : \Gamma(u) \text{ is unbounded}\rangle.$$

Let S be a set of representatives of the bounded M-subgraphs of $\mathbb{N}^{\bar{J}}$. By [DMM06][Proposition 6.6], a basis of the space of polynomial solutions of the lattice basis ideal $I(M) \subseteq \mathbb{C}[\partial_{\bar{J}}]$ considered as a system of differential equations, consists of polynomials

$$G_u = x^u \sum_{u+Mv \in \Gamma(u)} c_v x_{\bar{J}}^{Mv}, \quad u \in S,$$

where the all the coefficients c_v are nonzero. Fix a basis $\mathcal{B}_u$ of $I_{\rho,J} + \langle E - (\beta - A_{\bar{J}}u)\rangle$, where $A_{\bar{J}}$ is the matrix whose columns are the columns of A indexed by $\bar{J}$.

By [DMM06][Theorem 6.13], if β is very generic, a basis of the local solutions of $C_{\rho,J} + \langle E-\beta\rangle$ is given by the functions

$$F_{u,f} = x_{\bar{J}}^{u} \sum_{u+Mv \in \Gamma(u)} c_v x_{\bar{J}}^{Mv} \partial_J^{-Nv}(f) \ : \ u \in S, \ f \in \mathcal{B}_u. \tag{3.2}$$

To make sense of the notation $\partial_J^{-Nv}(f)$, we need the following result. As Lemma 2.3, the proof of this fact goes along the same lines as that of [DMM06][Lemma 6.10].

Theorem 3.4. *If β is very generic and $\alpha \in \mathbb{N}^J$, the map*

$$\frac{D}{C_{\rho,J} + \langle E - \beta \rangle} \longrightarrow \frac{D}{C_{\rho,J} + \langle E - (\beta + A_J \alpha)\rangle}$$

given by right multiplication by ∂_J^α is an isomorphism. Consequently, if $P\partial_J^\alpha$ belongs to $C_{\rho,J} + \langle E - (\beta + A_J\alpha)\rangle$, then $P \in C_{\rho,J} + \langle E - \beta\rangle$.

The isomorphism from Theorem 3.4 implies that differentiation ∂_J^α is an isomorphism between the solution space of $C_{\rho,J} + \langle E - (\beta + A_J\alpha)\rangle$ and the solution space of $C_{\rho,J} + \langle E - \beta\rangle$, whose inverse we denote by $\partial_J^{-\alpha}$. This, and [DMM06][Remark 6.12] explain the notation $\partial_J^{-Nv} f$ from (3.2).

Theorem 3.5. *Let $C_{\rho,J}$ be a toral component of a lattice basis ideal $I(B)$. If β is very generic, then $C_{\rho,J} + \langle E - \beta\rangle$ is the annihilator of its solution space, and is therefore Weyl closed.*

Proof of Theorem 3.5. First note that if β is very generic, then $I_{\rho,J} + \langle E - \beta\rangle$ is Weyl closed. In fact, we may assume that this system has a solution that can be represented as a fully supported power series on a translate of $\ker_{\mathbb{Z}}(A_J)$, and then $I_{\rho,J} + \langle E - \beta\rangle$ is the differential annihilator of this function.

If $I_\rho = I_{A_J}$, this follows from Theorems 2.7 and 2.8. To adapt those results to more general I_ρ, we note that I_ρ is isomorphic to I_{A_J} by adequately rescaling the variables ∂_j, $j \in J$.

If $q = \#\overline{J} = 0$, then the preceding paragraphs prove Theorem 3.5, so assume $q \neq 0$.

Pick a local basis of solutions of $C_{\rho,J} + \langle E - \beta\rangle$ as in (3.2). We assume that β is generic enough that at least one element of $\mathcal{B}_u$ can be represented as a fully supported series on a translate of $\ker_{\mathbb{Z}}(A_J)$.

Let $P \in D$ that annihilates all the functions (3.2). We want to show that $P \in C_{\rho,J} + \langle E - \beta\rangle$.

Write $P = \sum \lambda_{\mu,\overline{\mu},\nu,\overline{\nu}} x_J^\mu x_{\overline{J}}^{\overline{\mu}} \partial_J^\nu \partial_{\overline{J}}^{\overline{\nu}}$, where all the λs are nonzero complex numbers. We may assume that all the $\overline{\nu}$ appearing in P belong to bounded M-subgraphs.

We introduce a partial order on the set of bounded M-subgraphs as follows: $\Gamma(u) \leq \Gamma(u')$ if and only if there exist elements $v \in \Gamma(u)$ and $v' \in \Gamma(u')$ such that $v \leq v'$ coordinate-wise. Note that if $\Gamma(u) \leq \Gamma(u')$, then for every $v \in \Gamma(u)$ there exists $v' \in \Gamma(u')$ such that $v \leq v'$.

Consider the set $\{\Gamma(\overline{\nu})\}$ of bounded M-subgraphs which have representatives in P, and choose a minimal element in this set, $\Gamma(\overline{\gamma})$, and a corresponding term in P $\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha x_{\overline{J}}^{\overline{\alpha}} \partial_J^\gamma \partial_{\overline{J}}^{\overline{\gamma}}$.

Now $\overline{\gamma} \in \Gamma(u)$ for some $u \in S$. Consider one of the functions $F_{u,f}$ from (3.2). We know that $PF_{u,f} = 0$. Also,

$$\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha x_{\overline{J}}^{\overline{\alpha}} \partial_J^\gamma \partial_{\overline{J}}^{\overline{\gamma}} F_{u,f} = \overline{\gamma}! \lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha x_{\overline{J}}^{\overline{\alpha}} \partial_J^{\gamma - Nz} f(x_J), \tag{3.3}$$

where $\overline{\gamma}!$ is the product of the factorials of the coordinates of γ, and $\overline{\gamma} = u + Mz$. The reason that only one term of $F_{u,f}$ survives is that the $x_{\overline{J}}$ monomials appearing in $F_{u,f}$ are of the form $\overline{\gamma} + My$. If we could find My whose coordinates are all positive, then there would be no bounded M-subgraphs. Therefore, some coordinate of My must be negative, so that applying $\partial_{\overline{J}}^{\overline{\gamma}}$ to $x_{\overline{J}}^{\overline{\gamma}+My}$ gives zero.

In order to cancel the term from (3.3), P must contain a term $\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}}$ such that $\partial_{\bar{J}}^{\overline{\tau}}$ does not kill all the monomials $x_{\bar{J}}^v : v \in \Gamma(u) = \Gamma(\overline{\gamma})$. If $\overline{\tau} \notin \Gamma(\overline{\gamma})$, we would have that $\Gamma(\overline{\tau}) < \Gamma(\overline{\gamma})$, which contradicts the choice of $\overline{\gamma}$. Thus, $\overline{\tau} = \overline{\gamma} + My$ for some y.

Now, that $\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}} F_{u,f}$ is a multiple of (3.3), means that

$$\overline{\gamma}!\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha x_{\bar{J}}^{\overline{\alpha}} \partial_J^\gamma \partial_J^{-Nz} f(x_J) = c\overline{\tau}!\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^\tau \partial_J^{-N(z+y)} f(x_J) \tag{3.4}$$

for some nonzero c. Therefore $\overline{\sigma} = \overline{\alpha}$.

Assume that $\tau - Ny$ is coordinate-wise non-negative. (The case when $\tau - Ny$ has some strictly negative coordinates is resolved by multiplying P on the right by a suitable monomial in the variables ∂_J, working with a different (albeit still very generic) parameter, and then applying Theorem 3.4).

Then formula (3.4) implies that

$$\Big(\overline{\gamma}!\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha \partial_J^\gamma - c\overline{\tau}!\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma \partial_J^{\tau-Ny}\Big) \partial_J^{-Nz} f(x_J) = 0.$$

Since $\partial_J^{-Nz} f(x_J)$ is a (fully supported) solution of $I_{\rho,J} + \langle E - (\beta - A_{\bar{J}}u - A_J Nz)\rangle$, we conclude that

$$\big(\overline{\gamma}!\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha \partial_J^\gamma - c\overline{\tau}!\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma \partial_J^{\tau-Ny}\big) \in I_{\rho,J} + \langle E - (\beta - A_{\bar{J}}u - A_J Nz)\rangle.$$

Write

$$\big(\overline{\gamma}!\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha \partial_J^\gamma - c\overline{\tau}!\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma \partial_J^{\tau-Ny}\big) = Q + \sum_{j\in\bar{J}} Q_j \partial_{\bar{J}}^{e_j},$$

where $Q \in I_\rho + \langle E - \beta\rangle$. Then

$$\big(\overline{\gamma}!\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha \partial_J^\gamma - c\overline{\tau}!\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma \partial_J^{\tau-Ny}\big)\partial_{\bar{J}}^{\overline{\gamma}} = Q\partial_{\bar{J}}^{\overline{\gamma}} + \sum_{j\in\bar{J}} \partial_{\bar{J}}^{\overline{\gamma}+e_j},$$

where $Q \in I_\rho + \langle E - (\beta - A_{\bar{J}}u - A_J Nz - A_{\bar{J}}\overline{\gamma})\rangle$. As

$$\beta - A_{\bar{J}}u - A_J Nz - A_{\bar{J}}\overline{\gamma} = \beta - A_{\bar{J}}u - A_J Nz - A_{\bar{J}}(u + Mz) = \beta - A(Nz + Mz) = \beta,$$

we have $Q\partial_{\bar{J}}^{\overline{\gamma}} \in I_\rho + \langle E - \beta\rangle$.

Define

$$\begin{aligned}
&P_o = \\
&= \lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha x_{\bar{J}}^{\overline{\alpha}} \partial_J^\gamma \partial_{\bar{J}}^{\overline{\gamma}} + \lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}} - (\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} c\tfrac{\overline{\tau}!}{\overline{\gamma}!} + \lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}}) x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^{\tau-Ny} \partial_{\bar{J}}^{\overline{\gamma}} \\
&= x_{\bar{J}}^{\overline{\alpha}}\big(\lambda_{\alpha,\overline{\alpha},\gamma,\overline{\gamma}} x_J^\alpha \partial_J^\gamma - c\tfrac{\overline{\tau}!}{\overline{\gamma}!}\lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma \partial_J^{\tau-Ny}\big)\partial_{\bar{J}}^{\overline{\gamma}} + \lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} (\partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}} - \partial_J^{\tau-Ny}\partial_{\bar{J}}^{\overline{\tau}-My}) \\
&= x_{\bar{J}}^{\overline{\alpha}} Q\partial_{\bar{J}}^{\overline{\gamma}} + \textstyle\sum_{j\in\bar{J}} x_{\bar{J}}^{\overline{\alpha}} Q_j \partial^{\overline{\gamma}+e_i} + \lambda_{\sigma,\overline{\sigma},\tau,\overline{\tau}} x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} (\partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}} - \partial_J^{\tau-Ny}\partial_{\bar{J}}^{\overline{\tau}-My}) \\
&\equiv \textstyle\sum_{j\in\bar{J}} x_{\bar{J}}^{\overline{\alpha}} Q_j \partial_{\bar{J}}^{\overline{\gamma}+e_j} \quad \mathrm{mod}\,(C_{\rho,J} + \langle E - \beta\rangle),
\end{aligned}$$

since $Q\partial_{\bar{J}}^{\overline{\gamma}} \in I_\rho + \langle E - \beta\rangle$ and $\partial_J^\tau \partial_{\bar{J}}^{\overline{\tau}} - \partial_J^{\tau-Ny}\partial_{\bar{J}}^{\overline{\tau}-My} \in I(B)$.

Now consider the operator $P - P_o - \sum_{j\in\bar{J}} x_{\bar{J}}^{\overline{\alpha}} Q_j \partial_{\bar{J}}^{\overline{\gamma}+e_j}$, which is congruent to P modulo $C_{\rho,J} + \langle E - \beta\rangle$. Note that eliminates this two of the terms in P at the cost of adding terms with strictly higher monomials in $\partial_{\bar{J}}$ than $\partial_{\bar{J}}^{\overline{\gamma}}$, and possibly adding a multiple of $x_J^\sigma x_{\bar{J}}^{\overline{\sigma}} \partial_J^{\tau-Ny} \partial_{\bar{J}}^{\overline{\gamma}}$.

We apply the same treatment to $P - P_o - \sum_{j\in\bar{J}} x_{\bar{J}}^{\overline{\alpha}} Q_j \partial_{\bar{J}}^{\overline{\gamma}+e_j}$ that we did to P, and repeat. Eventually, this procedure will get rid of all the terms that have $\partial_{\bar{J}}^{u}$ with $\Gamma(u)$ bounded.

We conclude that $P \in \mathbb{C}_{\rho,J} + \langle E-\beta\rangle$.

□

We need one more ingredient to prove that Horn systems are Weyl closed.

Proposition 3.6. *If β is generic, and $C_{\rho_1,J_1},\dots,C_{\rho_r,J_r}$ are the toral components of the lattice basis ideal $I(B)$, then*

$$\bigcap_{i=1}^{r} \big(C_{\rho_i,J_i} + \langle E-\beta\rangle\big) = I(B) + \langle E-\beta\rangle = H(B,\beta). \tag{3.5}$$

Proof. The inclusion $\supseteq$ follows from the fact that $\cap_{i=1}^r C_{\rho_i,J_i} \supseteq I(B)$. To prove the reverse inclusion, let P be an element of the left hand side of (3.5).

By the proof of [DMM06][Theorem 5.8], the natural map

$$\frac{D}{H(B,\beta)} \longrightarrow \bigoplus_{i=1}^{r} \frac{D}{C_{\rho_i,J_i} + \langle E-\beta\rangle}$$

is an isomorphism when β is generic. Since P belongs to the left hand side of (3.5), its image under this map is zero. Therefore P must be an element of $H(B,\beta)$. □

Corollary 3.7. *For very generic β, the Horn system $H(B,\beta)$ is Weyl closed.*

Proof. For generic β, Proposition 3.6 says that $H(B,\beta)$ is the intersection of the systems $C_{\rho,J} + \langle E-\beta\rangle$ corresponding to the toral components of $I(B)$. Each of these is Weyl closed for very generic parameters by Theorem 3.5. We finish by noting that the intersection of Weyl closed D-ideals is Weyl closed. □

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