1 T-complexes

1.1 Canonical Finite Sets

Definition 1.1.1 (Canonical finite sets)**.** For each $n \in \mathbb{N}$, let $[n] := \{m \in \mathbb{N} \mid m < n\}$ be the *n-th canonical finite set.*

Definition 1.1.2 (Category of nonempty canonical finite sets)**.** Let Fin_+ denote the full subcategory of Set generated by all nonempty canonical finite sets, i.e. all $[n]$ with $n > 0$.

Lemma 1.1.1. *For $m, n \in \mathbb{N}$, the function*

$$p_{m,n} : [m] \times [n] \to [m \cdot n], \quad (a, b) \mapsto an + b$$

is bijective, as well as

$$q_{m,n} : [m] \uplus [n] \to [m + n], \quad \begin{cases} (0, i) & \mapsto i \\ (1, j) & \mapsto m + j \end{cases}$$

The inverse of $p_{m,n}$ is simply the Euclidean division by n, while the inverse of $q_{m,n}$ is the following:

$$r_{m,n} : [m + n] \to [m] \uplus [n], \quad k \mapsto \begin{cases} (0, k) & k < m \\ (1, k - m) & k \geq m \end{cases}$$

Proof. Exercise for reader. $\square$

Corollary 1.1.1. Fin_+ *as defined in Definition 1.1.2 has finite products (in fact even finitely complete) as well as finite nonempty sums. The nullary product is* $[1]$ *and for each* $[m]$ *and* $[n]$*, the binary product is* $[m \cdot n]$*, while the binary sum is* $[m + n]$*, with the encoding demonstrated by Lemma 1.1.1.*

Proof. Exercise for reader. $\square$

For convenience, we shall write $p_{m,n}(a, b)$ simply as $(a; b)$ whenever m and n are clear from context. It resembles the common notation for ordered pairs, thus reflecting its similar purpose to the set-theoretic cartesian product.

Definition 1.1.3 (Global and local elements)**.** Because Set is cartesian closed with $[1]$ being the terminal object, there is the following natural isomorphism between two endofunctors Set $\to$ Set:

$$\mathfrak{g}_A : A \xrightarrow{\sim} \mathrm{Set}([1], A)$$

For each $x \in A$, it can be explicitly defined as

$$\mathfrak{g}_A(x) : [1] \to A, \quad 0 \mapsto x$$

Which we shall call the *global element* for x. Conversely, for each $f : [1] \to A$, we shall call $f(0) \in A$ its corresponding *local element*.

By abuse of notation, we shall sometimes consider $\mathfrak{g}$ to be a natural isomorphism between functors of type $\mathrm{Fin}_+ \to \mathrm{Set}$ instead, and write

$$\mathfrak{g}_n : [n] \xrightarrow{\sim} \mathrm{Fin}_+([1], [n])$$

for every $n \in \mathbb{N}_+$.

1.2 Abstract Simplicial Complexes

Definition 1.2.1 (Abstract simplicial complex)**.** For each set X, a $C \subset \mathfrak{P}(X)$ is called an *abstract simplicial complex (ASC for short) over* X if it satisfies three conditions

1. It is downward closed, i.e. whenever we have $S \subset T \in C$, we also have $S \in C$.
2. It contains all singletons subsets of X, i.e. $\forall x \in X : \{x\} \in C$.
3. It contains the empty set $\emptyset$ (which ensures that C is not empty).

Let ASC_n denote the set of all ASC over $[n]$.

Example.

1. For every set X, $C_0 := \mathfrak{P}(X)$ is the greatest ASC over X with respect to inclusion, and $C_1 := \{\emptyset\} \cup \{\{x\} \mid x \in X\}$ the smallest.
2. $C_2 := \{S \subset X \mid S \neq X\}$ is an ASC over X if $|X| > 1$.
3. $C_3 := \{S \subset \mathbb{N} \mid S \text{ is finite}\}$ is an ASC over $\mathbb{N}$.

□

Definition 1.2.2 (Center of an ASC)**.** Let C be an ASC over some set X. Let $Z(C) := \{x \in X \mid \forall S \in C : \{x\} \cup S \in C\}$ denote the *center* of C. If $Z(C) \neq \emptyset$, we call C *centralized*.

Example. Here we take a look at the center of each example of Definition 1.2.1

1. $Z(C_0) = Z(\mathfrak{P}(X)) = X$, $Z(C_1) = \emptyset$ if $|X| \neq 1$, otherwise $Z(C_1) = \{x\}$ if $X = \{x\}$.
2. $Z(C_2) = \emptyset$
3. $Z(C_3) = \mathbb{N}$

□

Theorem 1.2.1. *Let $C \in \mathsf{ASC}_n$. Then $Z(C) \in C$.*

Proof. Per definition we have $\emptyset \in C$. $Z(C) \in C$ then follows by induction over $|Z(C)|$. □

Remark 1.2.1. Note that Lemma 1.2.1 fails when we consider ASC over an infinite set. In Example 3 of Definition 1.2.2 we see that $Z(C_3) = \mathbb{N} \notin C_3$.

Definition 1.2.3 (Generated ASC)**.** Let $\mathcal{T} \subset \mathfrak{P}(X)$ be a system of subsets of X. We call the ASC $\langle \mathcal{T} \rangle := \{S \subset X \mid \exists T \in \mathcal{T} : S \subset T\} \cup \{\emptyset\} \cup \{\{x\} \mid x \in X\}$ to be *generated by* $\mathcal{T}$.

Example.

1. $\langle \{X\} \rangle = \mathfrak{P}(X)$ and $\langle \emptyset \rangle = \{\emptyset\} \cup \{\{x\} \mid x \in X\}$
2. $\langle X \setminus \{x\} \mid x \in X \rangle = \{S \subset X \mid S \neq X\}$ if $|X| > 1$.

□

The following lemma justifies the name:

Lemma 1.2.1. *$\langle \mathcal{T} \rangle$ is the smallest ASC that contains $\mathcal{T}$, i.e. for every ASC C with $\mathcal{T} \subset C$, we have $\langle \mathcal{T} \rangle \subset C$.*

Proof. Suppose we have an ASC C with $\mathcal{T} \subset C$, and let $S \in \langle \mathcal{T} \rangle$. Then $S = \emptyset$ or $S = \{x\}$ for some $x \in X$, in which case clearly $S \in C$; or there exists some $T \in \mathcal{T}$ s.t. $S \subset T$. Hence we have $S \subset T \in \mathcal{T} \subset C$, therefore $S \in C$. □

Definition 1.2.4. Let $f : X \to Y$ be a function and C an ASC over Y. We call $C^f := \{S \subset X \mid f[S] \in C\}$ the ASC of preimages.

Example. Again, we take a look at the examples introduced after Definition 1.2.1

1. $\mathfrak{P}(X)^f = \mathfrak{P}(Y)$ and $(\{\emptyset\} \cup \{\{x\} \mid x \in X\})^f = \{\emptyset\} \cup \{\{y\} \mid y \in Y\}$ for every $f : Y \to X$.
2. If $f : Y \to X$ is not surjective, then $C_2^f = \mathfrak{P}(Y)$.
3. For every $f : Y \to \mathbb{N}$, $C_3^f = \{S \subset Y \mid S \text{ is finite}\}$. In particular, if Y is finite, then $C_3^f = \mathfrak{P}(Y)$; if $Y = \mathbb{N}$, then $C_3^f = C_3$.

□

Again, the following lemma justifies the name:

Lemma 1.2.2. *For $f : X \to Y$ and C an ASC over Y, $C^f = \langle f^{-1}[S] \mid S \in C \rangle$.*

Proof. According to Lemma 1.2.2, we can simply prove that C^f is the smallest ASC that contains all preimages $f^{-1}[S]$ with $S \in C$.

Firstly, it is clear that C^f itself contains all such preimages, since $f\left[f^{-1}[S]\right] \subset S \in C$.

Secondly, for every ASC D over X that contains all such preimages $f^{-1}[S]$ with $S \in C$, and for every $T \in C^f$, we have $f[T] \in C$, hence $T \subset f^{-1}[f[T]] \in D$, therefore $T \in D$. □

Theorem 1.2.2. *Let $f : X \twoheadrightarrow Y$ be a surjection and let C be an ASC over Y. Then $Z\left(C^f\right) = f^{-1}[Z(C)]$. In particular, if C is centralized, then so is C^f.*

Proof. Let $x \in Z\left(C^f\right)$, and let $T \in C$. Since f is surjective, we have $f\left[f^{-1}[T]\right] = T$. Furthermore, $\{x\} \cup f^{-1}[T] \in C^f$, which means

$$f\left[\{x\} \cup f^{-1}[T]\right] = f[\{x\}] \cup f\left[f^{-1}[T]\right] = \{f(x)\} \cup T \in C$$

Since T was chosen arbitrarily, this shows that $f(x) \in Z(C)$, or $x \in f^{-1}[Z(C)]$. Since x was also chosen arbitrarily, this shows that $Z\left(C^f\right) \subset f^{-1}[Z(C)]$.

Conversely, whenever we have $x \in f^{-1}[Z(C)]$, i.e. $f(x) \in Z(C)$, and $S \in C^f$, we have

$$f[\{x\} \cup S] = \{f(x)\} \cup f[S] \in C$$

Hence $\{x\} \cup S \in C^f$. Again, because x was chosen arbitrarily, we have $f^{-1}[Z(C)] \subset Z\left(C^f\right)$. □

Corollary 1.2.1. *Let X, Y be lets, X nonempty, C an ASC over Y. Let $\pi^2 : X \times Y \twoheadrightarrow Y$ denote the projection from the cartesian product. Then $Z\left(C^{\pi^2}\right) = X \times Z(C)$.*

Proof. π^2 is surjective if X is nonempty. Theorem 1.2.2 then delivers the specified equality. □

1.3 (Concrete) Simplicial Complexes

From here on, we will drop the brackets and simply write $[n]$ as n whenever a natural number is used where a set is expected.

Definition 1.3.1. Let PSF be the category of presheaves over Fin_+, i.e. $\mathrm{Set}^{\mathrm{Fin}_+^{\mathrm{op}}}$, and let $\mathcal{Y} : \mathrm{Fin}_+ \hookrightarrow \mathrm{PSF}$ denote the Yoneda-embedding of Fin_+. Furthermore, for every $n \in \mathbb{N}$ and $C \in \mathsf{ASC}_n$, let $\mathcal{X}(C)$ denote the subpresheaf of $\mathcal{Y}(n)$ consisting of only functions $f : m \to n$ with $\mathrm{im}(f) \in C$. Let the inclusion $\mathcal{X}(C) \subset \mathcal{Y}(n)$ be denoted by $\iota(C) : \mathcal{X}(C) \hookrightarrow \mathcal{Y}(n)$. Clearly, $\mathcal{Y}(n) = \mathcal{X}(\mathfrak{P}(n))$.

Definition 1.3.2 (Simplexes and (concrete) simplicial complexes)**.** Let $n \in \mathbb{N}_+$ and $C \in \mathsf{ASC}_n$, and $P : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf over Fin_+. A natural transformation $s : \mathcal{Y}(n) \to P$ is called an *n-simplex* in P, while a $c : \mathcal{X}(C) \to P$ is called a *(concrete) simplicial complex of shape C* in P.

Remark 1.3.1. Yoneda lemma asserts $\mathrm{Nat}(\mathcal{Y}(n), P) \cong P(n)$. Hence one could alternatively define an n-simplex in P simply as a $s \in P(n)$.

Definition 1.3.3. For $m, n \in \mathbb{N}$ and $f : m \to n$, $\mathcal{Y}(f) : \mathcal{Y}(m) \to \mathcal{Y}(n)$ is a natural transformation. Analogously, let $\mathcal{X}(f) : \mathcal{X}(C^f) \to \mathcal{X}(C)$ also denote the natural transformation defined by composition with f, i.e.

$$\mathcal{X}(f) : \mathcal{X}(C^f) \to \mathcal{X}(C), \quad g \mapsto f \circ g$$

Lemma 1.3.1. *The following square is a pullback:*

$$\begin{tikzcd}
\mathcal{X}(C^f) \arrow[r, "\mathcal{X}(f)"] \arrow[d, hook, "\iota(C^f)"'] & \mathcal{X}(C) \arrow[d, hook, "\iota(C)"] \\
\mathcal{Y}(m) \arrow[r, "\mathcal{Y}(f)"'] & \mathcal{Y}(n)
\end{tikzcd}$$

Proof. $\mathrm{im}(g) \in C^f$ if and only if $\mathrm{im}(f \circ g) \in C$, therefore $\mathcal{X}(C^f) \subset \mathcal{Y}(m)$ is essentially the preimage of $\mathcal{X}(C) \subset \mathcal{Y}(n)$ under $\mathcal{Y}(f)$. $\square$

Theorem 1.3.1 (Simplicial complexes in $\mathcal{Y}(n)$)**.** *Every simplicial complex in $\mathcal{Y}(n)$, i.e. every $c : \mathcal{X}(C) \to \mathcal{Y}(n)$ with $C \in \mathsf{ASC}_m$, can be lifted into a simplex $s : \mathcal{Y}(m) \to \mathcal{Y}(n)$ in a unique way, i.e. there is a unique $s : \mathcal{Y}(m) \to \mathcal{Y}(n)$ s.t. the following triangle commutes:*

$$\begin{tikzcd} \mathcal{X}(C) \arrow[r, "c"] \arrow[d, "\iota_C"'] & \mathcal{Y}(n) \\ \mathcal{Y}(m) \arrow[ur, "s"'] & \end{tikzcd} \tag{1}$$

Proof. Suppose such an lifting simplex exists, then per Yoneda-lemma it must arise from some $f : m \to n$ with $s = \mathcal{Y}(f)$. For each $i \in [m]$, $\mathrm{im}(\mathfrak{g}_m(i)) = \{i\} \in C$, so $\mathfrak{g}_m(i) \in \mathcal{X}(C)(m)$. So since $c = s \circ \iota_C$, we must have

$$s_1(\mathfrak{g}_m(i)) = s_1(\iota_{C1}(\mathfrak{g}_m(i))) = c_1(\mathfrak{g}_m(i))$$

Under the Yoneda-correspondence, $s_l(h) = f \circ h$ for every $h : l \to m$, so the above equation becomes

$$f \circ \mathfrak{g}_m(i) = c_1(\mathfrak{g}_m(i))$$

Both sides are in fact functions of type $1 \to n$, so applying both sides to $0 \in [1]$, we get

$$f(\mathfrak{g}_m(i)(0)) = c_1(\mathfrak{g}_m(i))(0)$$

But $\mathfrak{g}_m(i)(0)$ is per definition simply i, so the left-hand side just becomes $f(i)$:

$$f(i) = c_1(\mathfrak{g}_m(i))(0) \tag{2}$$

So we've shown that if the lifting simplex $s : \mathcal{Y}(m) \to \mathcal{Y}(n)$ exists, it must arise from some $f : m \to n$ satisfying equation 2. On the other hand, equation 2 also uniquely defines f, and one can easily verify that $s = \mathcal{Y}(f)$ is indeed lifts c. $\square$

Remark 1.3.2. Theorem 1.3.1 unfortunately does not generalize to simplicial complexes in $\mathcal{X}(C)$ for arbitrary $C \in \mathsf{ASC}_n$, even if C is centralized. This can be seen by the fact that the identity transformation $\mathrm{id}_{\mathcal{X}(C)} : \mathcal{X}(C) \to \mathcal{X}(C)$ is a simplicial complex in $\mathcal{X}(C)$, however it has no lifting simplex $\mathcal{Y}(n) \to \mathcal{X}(C)$ if $C \neq \mathfrak{P}(n)$, since any such lifting simplex must send $\mathrm{id}_n \in \mathcal{Y}(n)(n)$ to itself, which is not in $\mathcal{X}(C)(n)$.

Definition 1.3.4. For a simplicial complex $c : \mathcal{X}(C) \to \mathcal{Y}(n)$, let the unique lifting simplex delivered by Theorem 1.3.1 be denoted by $\mathfrak{F}(c) : \mathcal{Y}(m) \to \mathcal{Y}(n)$. Using equation 2, it can be explicitly defined by

$$\mathfrak{F}(c) := \mathcal{Y}(\mathfrak{g}_n^{-1} \circ c_1 \circ \mathfrak{g}_m)$$

It is trivial to show that if $c : \mathcal{X}(C) \twoheadrightarrow \mathcal{Y}(n)$ is an epimorphism, then so is $\mathfrak{F}(c) : \mathcal{Y}(m) \twoheadrightarrow \mathcal{Y}(n)$, simply because $c = \mathfrak{F}(c) \circ \iota_C$. This fact will probably never be used in proofs, but we will nonetheless reference it in our diagrams.

Definition 1.3.5 (T-complexes)**.** A *T-complex* consists of an ordered pair (P, ϕ) where:

1. $P : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf over Fin_+;
2. ϕ is a family of simplexes, with $\phi^s : \mathcal{Y}(n) \to P$ for each centralized $C \in \mathsf{ASC}_n$ and each simplicial complex $s : \mathcal{X}(C) \to P$, that lifts s i.e. $s = \phi^s \circ \iota(C)$, called the *filler* for s. In diagram:

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[r, "s"] \arrow[d, hook, "\iota(C)"'] & P \\
\mathcal{Y}(n) \arrow[ur, "\phi^s"'] &
\end{tikzcd}$$

The fillers are subject to the following two axioms, for every $C \in \mathsf{ASC}_n$ centralized:

1. For every $f : m \to n$ s.t. C^f is also centralized, the following diagram, in particular the lower triangle, commutes:

$$\begin{tikzcd}
\mathcal{X}(C^f) \arrow[rr, "\mathcal{X}(f)"] \arrow[dd, hook, "\iota(C^f)"'] & & \mathcal{X}(C) \arrow[dl, hook', "\iota(C)"'] \arrow[dd, "s"] \\
& \mathcal{Y}(n) \arrow[dr, "\phi^s"] & \\
\mathcal{Y}(m) \arrow[ur, "\mathcal{Y}(f)"] \arrow[rr, "\phi^{s \circ \mathcal{X}(f)}"'] & & P
\end{tikzcd} \tag{3}$$

   Summarized in an equation, this just says $\phi^{s \circ \mathcal{X}(f)} = \phi^s \circ \mathcal{Y}(f)$.

2. For $c : \mathcal{X}(C) \twoheadrightarrow \mathcal{Y}(m)$ an epic centralized simplicial complex in $\mathcal{Y}(m)$ and $s : \mathcal{Y}(m) \to P$ an m-simplex in P, then $\phi^{s \circ c}$ factors through $\mathfrak{F}(c)$. So the following diagram commutes:

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[r, two heads, "c"] \arrow[dr, hook, "\iota_C"'] & \mathcal{Y}(m) \arrow[r, "s"] & P \\
& \mathcal{Y}(n) \arrow[u, two heads, "\mathfrak{F}(c)"] \arrow[ur, "\phi^{s \circ c}"'] &
\end{tikzcd} \tag{4}$$

As is common with algebraic structures, we shall denote a T-complex simply (P, ϕ) simply by its underlying presheaf P, whenever the fillers can be inferred from the context. It should be kept in mind though that the fillers are really the more important part of the structure.

Definition 1.3.6 (Coherent transformations and the category of T-complexes)**.** Let (P, ϕ) and (Q, ψ) be two T-complexes. A natural transformation $\eta : P \to$

Q is called *coherent* if the following diagram, in particular the lower triangle, commutes for every centralized $C \in \mathsf{ASC}_n$ and $s : \mathcal{X}(C) \to P$:

$$\begin{tikzcd}
 & \mathcal{X}(C) \arrow[dl, hook', "\iota(C)"'] \arrow[d, "s"] \\
\mathcal{Y}(n) \arrow[r, "\phi^s"] \arrow[dr, "\psi^{\eta\circ s}"'] & P \arrow[d, "\eta"] \\
 & Q
\end{tikzcd}$$

Summarized in an equation, it just says $\eta \circ \phi^s = \psi^{\eta \circ s}$, i.e. that η preserves fillers.

Clearly, every identity transformation is coherent, as well as the composition of two coherent transformations. Therefore, the T-complexes and coherent transformations between them form a category, which shall be denoted by Tcplx.

1.4 Examples for T-complexes and coherent transformations

Here we give a couple of important examples for T-complexes and coherent transformations that we will encounter again.

First we introduce some related functors:

Definition 1.4.1 (Constant embedding)**.** Let $\mathcal{K} : \mathrm{Set} \hookrightarrow \mathrm{PSF}$ denote the embedding that sends each set A to the constant presheaf which simply sends every object in Fin_+ to A and every morphism to id_A.

It is easy to see that $\mathcal{K}(1) \cong \mathcal{Y}(1)$, and both of which are terminal objects in PSF.

Definition 1.4.2. Let $\mathcal{Q} : \mathrm{PSF} \twoheadrightarrow \mathrm{Set}$ be the functor that sends each presheaf P to the set $P(1)$ and each natural transformation $\eta : P \to Q$ to its first component $\eta_1 : P(1) \to Q(1)$. By abuse of notation, we shall also sometimes take Tcplx to be the domain of $\mathcal{Q}$, i.e. $\mathcal{Q} : \mathrm{Tcplx} \twoheadrightarrow \mathrm{Set}$.

Definition 1.4.3 (Extended Yoneda-embedding)**.** The domain of the Yoneda-embedding $\mathcal{Y} : \mathrm{Fin}_+ \hookrightarrow \mathrm{PSF}$ can indeed be extended to Set, i.e. giving rise to a functor $\mathrm{Set} \hookrightarrow \mathrm{PSF}$, which maps each set A and each $n \in \mathbb{N}_+$ to the set $\mathrm{Set}([n], A)$. By abuse of notation, we shall also denote this functor by $\mathcal{Y} : \mathrm{Set} \hookrightarrow \mathrm{PSF}$, since there is no room for ambiguity. Yoneda-lemma still guarantees that it is fully faithful since Fin_+ is a subcategory of Set, and it is still an embedding since the following composition maps each set A to $\mathrm{Set}([1], A)$, so it is naturally isomorphic to the identity functor through $\mathfrak{g}$:

$$\mathrm{Set} \xrightarrow{\mathcal{Y}} \mathrm{PSF} \xrightarrow{\mathcal{Q}} \mathrm{Set}$$

Theorem 1.4.1. *The functor* $\mathcal{Q} : \mathrm{PSF} \twoheadrightarrow \mathrm{Set}$ *has both a left and a right adjoint. The left adjoint is* $\mathcal{K} : \mathrm{Set} \to \mathrm{PSF}$*, and the right adjoint is* $\mathcal{Y} : \mathrm{Set} \to \mathrm{PSF}$*.*

Proof. To show that K is the left adjoint of $\mathcal{Q}$, we show that the following (obviously) natural transformation is componentwise bijective, hence a natural isomorphism:

$$\begin{aligned} \mathrm{Nat}(\mathcal{K}(A), P) &\to \mathrm{Set}(A, P(1)) \\ \eta &\mapsto \eta_1 \end{aligned}$$

We do it by showing that for each set-theoretic function $f : A \to P(1)$, there is exactly one $\eta : \mathcal{K}(A) \to P$ s.t. $\eta_1 = f$. Suppose that such an η exists. By naturality, the following square must commute for every $n \in \mathbb{N}_+$:

$$\begin{tikzcd} \mathcal{K}(A)(1) \arrow[r, "\mathcal{K}(A)(!_n)"] \arrow[d, "\eta_1"'] & \mathcal{K}(A)(n) \arrow[d, "\eta_n"] \\ P(1) \arrow[r, "P(!_n)"'] & P(n) \end{tikzcd}$$

However, since $\eta_1 = f$ and the top row is simply the identity function id_A on A, it simplifies to the following commutative triangle:

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "\eta_n"] & \\ P(1) \arrow[r, "P(!_n)"'] & P(n) \end{tikzcd}$$

Which reads $\eta_n = P(!_n) \circ f$ as an equation. So if $\eta : \mathcal{K}(A) \to P$ with $\eta_1 = f$ exists, it must be uniquely defined by this equation, which, one can easily verify, indeed yields a valid natural transformation.

To show that $\mathcal{Y}$ is the right adjoint of $\mathcal{Q}$, we show that the following (obviously) natural transformation is also componentwise bijective:

$$\begin{aligned} \mathrm{Set}(P(1), A) &\leftarrow \mathrm{Nat}(P, \mathcal{Y}(A)) \\ \mathfrak{g}_A^{-1} \circ \eta_1 &\leftarrow\!\shortmid \eta \end{aligned}$$

We do it by showing that for each set-theoretic function $f : P(1) \to A$, there is exactly one $\eta : P \to \mathcal{Y}(A)$ s.t. $f = \mathfrak{g}_A^{-1} \circ \eta_1$. Suppose that such an η exists. By naturality, the following square must commute for every $n \in \mathbb{N}_+$ and $k \in [n]$:

$$\begin{tikzcd} P(n) \arrow[r, "P(\mathfrak{g}_n(k))"] \arrow[d, "\eta_n"'] & P(1) \arrow[d, "\eta_1"] \\ \mathrm{Set}(n, A) \arrow[r, "\mathrm{Set}(\mathfrak{g}_n(k), \mathrm{id}_A)"'] & \mathrm{Set}(1, A) \end{tikzcd}$$

Now, take an arbitrary $x \in P(n)$, we can turn the commutative square into the following equation:

$$\eta_n(x) \circ \mathfrak{g}_n(k) = \eta_1(P(\mathfrak{g}_n(k))(x))$$

Both sides are now functions of type $1 \to A$. Applying them to 0 and using the fact $\mathfrak{g}_n(k)(0) = k$, we get

$$\eta_n(x)(k) = \eta_1(P(\mathfrak{g}_n(k))(x))(0) \tag{5}$$

The right-hand side is still a handful to write. However, remember we haven't used the assumption $f = \mathfrak{g}_A^{-1} \circ \eta_1$ yet. Applying both sides of this equation to some arbitrary $y \in P(1)$, we get:

$$f(y) = \mathfrak{g}_A^{-1}(\eta_1(y)) = \eta_1(y)(0)$$

Setting $y = P(\mathfrak{g}_n(k))(x)$ and putting it together with equation 5 above, we get

$$\eta_n(x)(k) = f(P(\mathfrak{g}_n(k))(x))$$

This equality would uniquely determine $\eta : P \to \mathcal{Y}(A)$, and indeed, one may easily check that this gives a well-defined natural transformation. □

The first example has already been foreshadowed by Theorem 1.3.1:

Theorem 1.4.2 ($\mathcal{Y}(A)$ are T-complexes)**.** *For every set A, $\mathcal{Y}(A) : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ can be given a T-complex structure in a unique way, such that any natural transformation from a T-complex P into $\mathcal{Y}(A)$ is also coherent.*

Proof. For the case of $A = [n]$, theorem 1.3.1 guarantees that every simplicial complex (even non-centralized ones) in $\mathcal{Y}(n)$ has a unique lifting simplex, which can be chosen as fillers. Furthermore, since lifting simplex are unique, both axioms in Definition 1.3.5 are automatically satisfied, and for the same reason they are also preserved by every natural transformation into $\mathcal{Y}(n)$.

The general case is completely analogous. □

Lemma 1.4.1 (Centralized simplicial complexes in constant presheaves)**.** *Let $c : \mathcal{X}(C) \to \mathcal{K}(A)$ be a centralized simplicial complex in $\mathcal{K}(A)$. Then there is some global element $\mathfrak{g}_x : 1 \hookrightarrow A$ s.t. c is equal to the following composition:*

$$\mathcal{X}(C) \xrightarrow{!_{\mathcal{X}(C)}} \mathcal{K}(1) \xrightarrow{\mathcal{K}(\mathfrak{g}_x)} \mathcal{K}(A)$$

Proof. Say $C \in \mathsf{ASC}_n$. Because C is centralized, we have some $k \in Z(C) \subset [n]$. Let $x := c_1(\mathfrak{g}_k) \in A$.

We show that $c_m(f) = x$ for every $f \in \mathcal{X}(C)(m)$, i.e. c must in fact be constant, which is the same as saying that c factors through $\mathcal{K}(\mathfrak{g}_x)$. Consider the function $\langle \mathfrak{g}_k; f\rangle : 1+m \to n$, which maps 0 to k and $1+j$ to $f(j)$ for $j \in [m]$. Furthermore, let $i^1 : 1 \to 1+m$ and $i^2 : m \to 1+m$ be injections into the sum $[1+m]$. Obviously per definition of sum/coproduct we have:

$$\langle \mathfrak{g}_k; f\rangle \circ i^1 = \mathfrak{g}_k \qquad \langle \mathfrak{g}_k; f\rangle \circ i^2 = f$$

Furthermore, $\mathrm{im}\langle \mathfrak{g}_k; f\rangle = \mathrm{im}(\mathfrak{g}_k) \cup \mathrm{im}(f) = \{k\} \cup \mathrm{im}(f) \in C$, since $k \in Z(C)$ and $\mathrm{im}(f) \in C$. So $\langle \mathfrak{g}_k; f\rangle \in \mathcal{X}(C)(1+m)$. In addition, regarding i^1, we have the following commutative square:

$$\begin{tikzcd}
\mathcal{X}(C)(1+m) \arrow[r, "\mathcal{X}(C)(i^1)"] \arrow[d, "c_{1+m}"'] & X(C)(1) \arrow[d, "c_1"] \\
\mathcal{K}(A)(1+m) \arrow[r, "\mathcal{K}(A)(i^1)"'] & \mathcal{K}(A)(1)
\end{tikzcd}$$

However, the bottom row is simply the identity function id_A on A. So the diagram simplifies to:

$$\begin{tikzcd}
\mathcal{X}(C)(1+m) \arrow[r, "\mathcal{X}(C)(i^1)"] \arrow[d, "c_{1+m}"'] & \mathcal{X}(C)(1) \arrow[dl, "c_1"] \\
A &
\end{tikzcd}$$

In equation, it says $c_{1+m} = c_1 \circ \mathcal{X}(C)(i^1)$. Specifically for $\langle \mathfrak{g}_k; f\rangle \in \mathcal{X}(C)(1+m)$, it says

$$c_{1+m}\langle \mathfrak{g}_k; f\rangle = c_1(\langle \mathfrak{g}_k; f\rangle \circ i^1) = c_1(\mathfrak{g}_k) = x \tag{6}$$

Similarly, regarding i^2, the following diagram also commutes:

$$\begin{tikzcd}
\mathcal{X}(C)(1+m) \arrow[r, "\mathcal{X}(C)(i^2)"] \arrow[d, "c_{1+m}"'] & \mathcal{X}(C)(m) \arrow[dl, "c_m"] \\
A &
\end{tikzcd}$$

In equation: $c_{1+m} = c_m \circ \mathcal{X}(C)(i^2)$, and specifically for $\langle \mathfrak{g}_k; f\rangle$:

$$c_{1+m}\langle \mathfrak{g}_k; f\rangle = c_m(\langle \mathfrak{g}_k; f\rangle \circ i^2) = c_m(f)$$

Together with equation (6) above, this gives $c_m(f) = x$, which is what we claimed at the beginning of the paragraph. $\square$

Theorem 1.4.3 (Constant presheaves are T-complexes)**.** *For every set A, $\mathcal{K}(A)$ can be given a T-complex structure in a unique way. Furthermore, every natural transformation $\eta : P \to \mathcal{K}(A)$ from some T-complex P into $\mathcal{K}(A)$, as well as every natural transformation $\varepsilon : \mathcal{K}(A) \to P$ from $\mathcal{K}(A)$ into a T-complex P, is coherent.*

Proof. Consider a simplicial complex $c : \mathcal{X}(C) \to \mathcal{K}(A)$ with $C \in \mathsf{ASC}_n$, $n \in \mathbb{N}_+$ and suppose it has a lifting simplex $s : \mathcal{Y}(n) \to \mathcal{K}(A)$. By the previous Lemma 1.4.1, c must factor through some $\mathcal{K}(\mathfrak{g}_x)$ with $x \in A$, and s must also factor through some $\mathcal{K}(\mathfrak{g}_y)$ with $y \in A$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[rr, "c"] \arrow[dr, two heads, "!_{\mathcal{X}(C)}"] \arrow[ddr, bend right, "\iota_C"'] & & \mathcal{K}(A) \\
& \mathcal{K}(1) \arrow[ur, hook, shift left, "\mathfrak{g}_x"] \arrow[ur, hook, shift right, "\mathfrak{g}_y"'] & \\
& \mathcal{Y}(n) \arrow[u, two heads, "!_{\mathcal{Y}(n)}"] \arrow[uur, bend right, "s"'] &
\end{tikzcd}$$

It's not hard to see that both paths $\mathcal{X}(C) \twoheadrightarrow \mathcal{K}(1) \hookrightarrow \mathcal{K}(A)$ in this diagram are equal, and that the unique natural transformation $!_{\mathcal{X}(C)} : \mathcal{X}(C) \twoheadrightarrow \mathcal{K}(1)$ is componentwise surjective ($\mathcal{X}(C)(m)$ is nonempty for every $m \in \mathbb{N}_+$) and hence indeed an epimorphism. So, it follows that $\mathfrak{g}_x = \mathfrak{g}_y$, thus $s = \mathfrak{g}_x \circ\, !_{\mathcal{Y}(n)}$ is uniquely determined.

So we've shown that every simplicial complex in $\mathcal{K}(A)$ has a unique lifting simplex, and by the same reasoning as Theorem 1.4.2 it follows that $\mathcal{K}(A)$ can be given a T-complex structure in a unique way and that every natural transformation from some T-complex into $\mathcal{K}(A)$ is coherent.

To show that a natural transformation $\varepsilon : \mathcal{K}(A) \to P$ into some T-complex P also preserves fillers, consider some simplicial complex $c : \mathcal{X}(C) \to \mathcal{K}(A)$ with $C \in \mathsf{ASC}_n$. We need to show the following diagram, in particular the right triangle, commutes:

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[r, "c"] \arrow[dr, "\iota_C"'] & \mathcal{K}(A) \arrow[r, "\varepsilon"] & P \\
& \mathcal{Y}(n) \arrow[u] \arrow[ur, "\phi^c"'] &
\end{tikzcd}$$

Again, by invoking Lemma 1.4.1, we decompose c into $\mathcal{X}(C) \twoheadrightarrow \mathcal{K}(1) \xrightarrow{\mathfrak{g}_x} \mathcal{K}(A)$ with some $x \in A$, and by replacing $\mathcal{K}(1)$ with $\mathcal{Y}(1)$, we get the following diagram, which commutes because of (4):

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[r, two heads] \arrow[dr, "\iota_C"'] & \mathcal{Y}(1) \arrow[r, hook] & \mathcal{K}(A) \arrow[d, "\varepsilon"] \\
& \mathcal{Y}(n) \arrow[u, two heads, "\mathcal{Y}(!_n)"] \arrow[r, "\phi^c"'] & P
\end{tikzcd}$$

□

Because of Theorem 1.4.2 and 1.4.3, we shall consider the functors $\mathcal{Y}$ and $\mathcal{K}$ to take values in Tcplx instead of PSF, i.e. $\mathcal{Y}, \mathcal{K} : \mathrm{Set} \hookrightarrow \mathrm{Tcplx}$, since there is hardly any ambiguity.

Theorem 1.4.4. *The restricted version of Q, i.e. $\mathcal{Q} : \mathrm{Tcplx} \twoheadrightarrow \mathrm{Set}$ also has both a left and a right adjoint, with $\mathcal{K} : \mathrm{Set} \hookrightarrow \mathrm{Tcplx}$ being the left adjoint and $\mathcal{Y} : \mathrm{Set} \hookrightarrow \mathrm{Tcplx}$ the right adjoint.*

Proof. We already know from Theorem 1.4.1 of the adjoints of $\mathcal{Q} : \mathrm{PSF} \to \mathrm{Set}$, and of the following two natural isomorphisms:

$$\mathrm{Nat}(\mathcal{K}(A), P) \cong \mathrm{Set}(A, P(1)) \qquad \mathrm{Set}(P(1), A) \cong \mathrm{Nat}(P, \mathcal{Y}(A))$$

We also have the following equality between sets from Theorem 1.4.2:

$$\mathrm{Nat}(P, \mathcal{Y}(A)) = \mathrm{Coh}(P, \mathcal{Y}(A))$$

as well as the following from Theorem 1.4.3:

$$\mathrm{Nat}(\mathcal{K}(A), P) = \mathrm{Coh}(\mathcal{K}(A), P)$$

whenever $P : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ is a T-complex. Putting them all together, we get the following natural isomorphisms:

$$\mathrm{Coh}(\mathcal{K}(A), P) \cong \mathrm{Set}(A, P(1)) \qquad \mathrm{Set}(P(1), A) \cong \mathrm{Coh}(P, \mathcal{Y}(A))$$

which means $\mathcal{Q} : \mathrm{Tcplx} \twoheadrightarrow \mathrm{Set}$ has the same adjunction property as $\mathcal{Q} : \mathrm{PSF} \twoheadrightarrow \mathrm{Set}$. $\square$

Definition 1.4.4 (Global elements in PSF and Tcplx)**.** For $P : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ a presheaf and $x \in P(1)$, let $\kappa^x : \mathcal{K}(1) \to P$ denote the adjunct of the global element $\mathfrak{g}_{P(1)}(x)$. To spell it out, each component $\kappa^x_n : 1 \to P(n)$ is defined as the following composition:

$$1 \xrightarrow{\mathfrak{g}_{P(1)}(x)} P(1) \xrightarrow{P(!_n)} P(n)$$

Furthermore, Theorem 1.4.3 guarantees that κ^x is coherent whenever P is a T-complex, and Theorems 1.4.1 and 1.4.4 assert that every global element in PSF as well as in Tcplx must be of this form.

2 Tcplx as a topos

2.1 Products and Sums

In this subsection, we show that an arbitrary (small) product or sum of T-complexes is again a T-complex. This means the category Tcplx has all (small) products and sums.

The case for products is quite straightforward:

Theorem 2.1.1 (Product of T-complexes)**.** *Let $(P_i, \phi_i)_{i \in I}$ be a family of T-complexes. The product of there underlying presheaves, i.e. $\prod_i P_i$, can also be made into a T-complex in a unique way s.t. all projections $\pi_i : \prod_i P_i \to P_i$ are coherent.*

Proof. Let $\prod_i \phi_i$ denote the fillers for $\prod_i P_i$ that we shall define. In order for all projections to be coherent, it must satisfy the equation

$$\pi_i \circ \left(\prod_i \phi_i\right)^c = \phi_i^{\pi_i \circ c} \tag{7}$$

for every centralized simplicial complex $c : \mathcal{X}(C) \to \prod_i P_i$ and $i \in I$. This indeed uniquely defines $\prod_i \phi_i$ because of the universal property of cartesian products.

The fact that the fillers defined as such indeed satisfies the axioms in Definition 1.3.5 also follows directly from the universal property of cartesian products, and that all ϕ_i satisfy these axioms. The exact proof shall hence not be spelled out here, and instead left as an exercise for the inquisitive among the readers. $\square$

The same statement for sums is somewhat more complicated to prove and requires a lemma first:

Lemma 2.1.1 (Centralized simplicial complexes in sums of presheaves)**.** *Let $(P_i)_{i\in I}$ be a family of* Fin_+*-presheaves, and $c : \mathcal{X}(C) \to \coprod_i P_i$ be a centralized simplicial complex into their sum. Then there exists some unique $i \in I$ and some $d : \mathcal{X}(C) \to P_i$ s.t. c is equal to the composition of d and the inclusion $P_i \hookrightarrow \coprod_i P_i$.*

Proof. Say $C \in \mathsf{ASC}_n$. Because C is centralized, there is some $k \in Z(C) \subset [n]$. We have $c_1(\mathfrak{g}_k) \in \coprod_i P_i(1)$, so there exists some unique i with $c_1(\mathfrak{g}_k) \in P_i(1)$.

We now show that $c_m(f) \in P_i(m)$ for every $m \in \mathbb{N}_+$ and $f \in \mathcal{X}(C)(m)$, which is same as saying that c factors through some $d : \mathcal{X}(C) \to P_i$. Similar to the proof of Lemma 1.4.1, we consider the function $\langle \mathfrak{g}_k; f\rangle : 1+m \to n$, and the inclusions $i^1 : 1 \to 1+m$ and $i^2 : m \to 1+m$. Per definition of sum/coproduct we have:

$$\langle \mathfrak{g}_k; f\rangle \circ i^1 = \mathfrak{g}_k \quad \langle \mathfrak{g}_k; f\rangle \circ i^2 = f$$

and also $\mathrm{im}\langle \mathfrak{g}_k; f\rangle = \mathrm{im}(\mathfrak{g}_k) \cup \mathrm{im}(f) = \{k\} \cup \mathrm{im}(f) \in C$, as $k \in Z(C)$ and $\mathrm{im}(f) \in C$. By naturality of c, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X}(C)(1+m) \arrow[r, "\mathcal{X}(C)(i^1)"] \arrow[d, "c_{1+m}"'] & \mathcal{X}(C)(1) \arrow[d, "c_1"] \\
(\coprod_i P_i)(1+m) \arrow[r, "(\coprod_i P_i)(i^1)"'] & (\coprod_i P_i)(1)
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
\mathcal{X}(C)(1+m) \arrow[r, "\mathcal{X}(C)(i^2)"] \arrow[d, "c_{1+m}"'] & \mathcal{X}(C)(m) \arrow[d, "c_m"] \\
(\coprod_i P_i)(1+m) \arrow[r, "(\coprod_i P_i)(i^2)"'] & (\coprod_i P_i)(m)
\end{tikzcd}$$

Applying both diagrams to $\langle \mathfrak{g}_k; f \rangle \in \mathcal{X}(C)(1+m)$, we first have

$$\left(\coprod_i P_i\right)(i^1)(c_{1+m}\langle \mathfrak{g}_k; f \rangle) = c_1(\langle \mathfrak{g}_k; f \rangle \circ i^1) = c_1(\mathfrak{g}_k) \tag{8}$$

and then

$$\left(\coprod_i P_i\right)(i^2)(c_{1+m}\langle \mathfrak{g}_k; f \rangle) = c_m(\langle \mathfrak{g}_k; f \rangle \circ i^2) = c_m(f) \tag{9}$$

Now recall $c_1(\mathfrak{g}_k) \in P_i(1)$. Together with equation (8), this gives

$$\left(\coprod_i P_i\right)(i^1)(c_{1+m}\langle \mathfrak{g}_k; f \rangle) \in P_i(1)$$

which in particular means $c_{1+m}\langle \mathfrak{g}_k; f \rangle \in P_i(1+m)$. Plugging this into equation (9), we get

$$c_m(f) = \left(\coprod_i P_i\right)(i^2)(c_{1+m}\langle \mathfrak{g}_k; f \rangle) = P_i(i^2)(c_{1+m}\langle \mathfrak{g}_k; f \rangle) \in P_i(m)$$

which is what we claimed. □

With this, it is now easy to see how we can equip $\coprod_i P_i$ with a T-complex structure:

Theorem 2.1.2. *For every family of T-complexes $(P_i, \phi_i)_{i \in I}$, there is a unique way to equip their sum $\coprod_i P_i$ with a T-complex structure s.t. all inclusions $P_i \hookrightarrow \coprod_i P_i$ are coherent.*

Proof. Again, let $\coprod_i \phi_i$ denote the fillers for $\coprod_i P_i$, and let $c : \mathcal{X}(C) \to \coprod_i P_i$ be an arbitrary centralized simplicial complex in the sum, with $C \in \mathsf{ASC}_n$. As proven in the previous Lemma 2.1.1, it factors through some simplicial complex $d : \mathcal{X}(C) \to P_i$ for some unique $i \in I$. Hence, in order for the inclusion $P_i \hookrightarrow \coprod_i P_i$ to be coherent, $\coprod_i \phi_i$ must be defined in a way s.t. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{X}(C) \arrow[r, "d"] \arrow[dr, "\iota_C"'] & P_i \arrow[r, hook] & \coprod_i P_i \\
& \mathcal{Y}(n) \arrow[u, "\phi_i^d"] \arrow[ur, "(\coprod_i \phi_i)^c"'] &
\end{tikzcd}$$

Indeed, it is again easy to check that this in fact uniquely defines $\coprod_i \phi_i$ and that it satisfies all the required axioms, thus making $\coprod_i P_i$ into a T-complex. □

Remark 2.1.1. For a set A, we can view the constant presheaf $\mathcal{K}(A)$ as the A-indexed sum $\coprod_{x \in A} \mathcal{Y}(1)$. Thus, Lemma 1.4.1 and Theorem 1.4.3 (including their proofs) are simply special cases of Lemma 2.1.1 and 2.1.2

2.2 Exponentials

In order to define exponentials (a.k.a. internal homs) in Tcplx, we start by revisiting the known construction for exponentials in PSF:

Theorem 2.2.1. *For $P, Q : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$, define the exponential $Q^P : \mathrm{Fin}_+^{\mathrm{op}} \to \mathrm{Set}$ as follows*

$$\begin{aligned} Q^P(n) &:= \mathrm{Nat}(\mathcal{Y}(n) \times P, Q) \\ Q^P(f) : Q^P(n) &\to Q^P(m) \quad (\text{for } f : m \to n) \\ \eta &\mapsto \eta \circ (\mathcal{Y}(f) \times \mathrm{id}_P) \end{aligned}$$

This makes PSF *into a cartesian closed category. In particular, we have the following isomorphism that is natural in P, Q, R:*

$$\iota_{P,Q,R} : \mathrm{Nat}(P \times Q, R) \xrightarrow{\sim} \mathrm{Nat}(P, R^Q)$$

Proof. We start by defining ι. Expanding the definition step by step, here's what we need to define

$$\begin{aligned} \iota_{P,Q,R} &: \mathrm{Nat}(P \times Q, R) \to \mathrm{Nat}(P, R^Q) \\ \iota_{P,Q,R}(\eta) &: P \to R^Q \\ \iota_{P,Q,R}(\eta)_n &: P(n) \to R^Q(n) = \mathrm{Nat}(\mathcal{Y}(n) \times Q, R) \\ \iota_{P,Q,R}(\eta)_n(x) &: \mathcal{Y}(n) \times Q \to R \\ \iota_{P,Q,R}(\eta)_n(x)_m &: \mathrm{Fin}_+(m, n) \times Q(m) \to R(m) \end{aligned}$$

With $\eta : P \times Q \to R$, $m, n \in \mathbb{N}_+$ and $x \in P(n)$ arbitrary. We can now give an explicit expression for the last function:

$$(f, y) \mapsto \eta_m(P(f)(x), y)$$

As it is rather tedious and boring to check that this is indeed a well-defined natural transformation, it is left as an exercise for the curious (or simply bored) reader. To show that this is an isomorphism, we give an explicit definition for its inverse, which we shall call κ. To define it we again expand the definition step by step:

$$\begin{aligned} \kappa_{P,Q,R} &: \mathrm{Nat}(P, R^Q) \to \mathrm{Nat}(P \times Q, R) \\ \kappa_{P,Q,R}(\varepsilon) &: P \times Q \to R \\ \kappa_{P,Q,R}(\varepsilon)_n &: P(n) \times Q(n) \to R(n) \end{aligned}$$

With $\varepsilon : P \to Q^R$ and $n \in \mathbb{N}_+$ arbitrary. We can again give an explicit expression for the last function:

$$(x, y) \mapsto \varepsilon_n(x)_n(\mathrm{id}_n, y)$$

Again, it is left as an exercise for the reader to check that this is well-defined. We will however verify here that ι and κ are indeed inverse to each other. For $\eta : P \times Q \to R$, $n \in \mathbb{N}_+$, $x \in P(n)$ and $y \in Q(n)$, we have

$$\begin{aligned}
&\kappa_{P,Q,R}(\iota_{P,Q,R}(\eta))_n(x, y) \\
&= \iota_{P,Q,R}(\eta)_n(x)_n(\mathrm{id}_n, y) \\
&= \eta_n(P(\mathrm{id}_n)(x), y) \\
&= \eta_n(x, y)
\end{aligned}$$

So $\kappa \circ \iota = \mathrm{id}$. For the other composition, take $\varepsilon : P \to R^Q$, $m, n \in \mathbb{N}_+$, $x \in P(n)$ and $y \in Q(m)$:

$$\begin{aligned}
&\iota_{P,Q,R}(\kappa_{P,Q,R}(\varepsilon))_n(x)_m(y) \\
&= \kappa_{P,Q,R}(\varepsilon)_m(P(f)(x), y) \\
&= \varepsilon_m(P(f)(x))_m(\mathrm{id}_m, y) \\
&= R^Q(f)(\varepsilon_n(x))_m(\mathrm{id}_m, y) \\
&= \varepsilon_n(x)_m((\mathcal{Y}(f) \times \mathrm{id}_Q)_m(\mathrm{id}_m, y)) \\
&= \varepsilon_n(x)_m(\mathrm{id}_m \circ f, y) \\
&= \varepsilon_n(x)_m(f, y)
\end{aligned}$$

So $\iota \circ \kappa = \mathrm{id}$ as well. This concludes the proof. $\square$