Tutorial for Bayesian PSD Identification

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In this tutorial, we want to give a brief introduction on how to infer flutter derivatives in turbulent flow. The theoretical background is briefly explained. Details can be seen in Chu et al. 2022. The code will be open to everyone for non-commercial use.

Theoretical Background

Consider a stationary vector process $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_d(t)]^T$ and a finite number of discrete data $\mathbf{X}_N = \{\boldsymbol{x}(m), m = 0,1, \dots, N-1\}$. Based on X_N, the following discrete estimator of the spectral density matrix of $x(t)$ is introduced:

$$\mathbf{S}_{\boldsymbol{x},N}(\omega_k) = \mathcal{X}_N(\omega_k)\mathcal{X}_N^{\star}(\omega_k) \tag{1}$$

$$\mathcal{X}_N(\omega_k) = \sqrt{\frac{\Delta t}{2\pi N}} \sum_{m=0}^{N-1} \boldsymbol{x}(m) \exp(-\mathrm{i}\omega_k m \Delta t) \tag{2}$$

For a stationary process, we can obtain the average spectral density estimator (by dividing the time series equally into M pieces) as:

$$\mathbf{S}_N^M(\omega_k) = \frac{1}{M}\sum_{m}^{M} \mathbf{S}_N^{(m)}(\omega_k) \tag{3}$$

$\mathbf{S}_N^M(\omega_k)$ follows a central complex Wishart distribution of dimension d with M sets and the mean value $\mathbb{E}[\mathbf{S}_N^M(\omega_k)|\boldsymbol{\theta}] = \mathbb{E}[\mathbf{S}_N(\omega_k)|\boldsymbol{\theta}]$ ($\boldsymbol{\theta}$ is the parameter vector that needs to be inferred):

$$p[\mathbf{S}_N^M(\omega_k)|\boldsymbol{\theta}] = \frac{\pi^{-\frac{d(d-1)}{2}} M^{M-d+d^2} \left|\mathbf{S}_N^M(\omega_k)\right|^{M-d}}{\left[\prod_{p=1}^{d}(M-p)!\right] |\mathbb{E}[\mathbf{S}_N(\omega_k)|\boldsymbol{\theta}]|^M} \times \exp(-M \cdot \mathrm{tr}\{\mathbb{E}[\mathbf{S}_N(\omega_k)|\boldsymbol{\theta}]^{-1}\mathbf{S}_N^M(\omega_k)\}) \tag{4}$$

Where $\mathbf{S}_N^M(\omega_k)$ is the measured PSD; $\mathbb{E}[\mathbf{S}_N(\omega_k)|\boldsymbol{\theta}]$ is the theoretical PSD.

In a nutshell, what we need to do is find a theoretical physical model to establish $\mathbb{E}[\mathbf{S}_N(\omega_k)|\boldsymbol{\theta}]$. $p[\mathbf{S}_N^M(\omega_i)|\boldsymbol{\theta}]$ and $p\left[\mathbf{S}_N^M(\omega_j)|\boldsymbol{\theta}\right]$ $(i \neq j)$ will be independent if the sampling duration is long enough. Then we have the likelihood function:

$$p\left(\mathbf{S}_N^{M,\mathcal{K}}|\boldsymbol{\theta}\right) = \prod_{k\in\mathcal{K}} p[\mathbf{S}_N^M(k)|\boldsymbol{\theta}] \tag{5}$$

Where $\mathcal{K}$ is the preselected frequency set.

In a Bayesian context, the posterior PDF can be obtained if given some prior PDF. For the practice of flutter derivatives, it is hard to know the values ahead of time. So we use uniform PDF as the prior. Actually, we maximize the likelihood function. Or equivalently (more practical), we minimize the negative log-likelihood function (NLLF). Then the most probable value (MPV) of $\boldsymbol{\theta}$ is $\widehat{\boldsymbol{\theta}}$:

$$\hat{\boldsymbol{\theta}} = \arg\min\{\ln p(\mathbf{S}_N^{M,\mathcal{K}}|\boldsymbol{\theta})\} = \arg\max\{p(\mathbf{S}_N^{M,\mathcal{K}}|\boldsymbol{\theta})\} \tag{6}$$

Code

We give a example of buffeting displacement response measured in a wind tunnel test (demo.mat file).

We first get the measured PSD by fft_transfer function. We choose two frequency sets: f(k1:k2), f(k3:k4).

```
[psd_y_M,f,psd_plot] = fft_transfer(df,y,M,0); % y is the measured buffeting
% signal; df is the sampling frequency; fft_transfer is a function used to
% get the PSD of y.
k1 = 405;
k2 = 510;
k3 = 675;
k4 = 790;
disp(['The PSD information in ',num2str(f(k1)),'-',num2str(f(k2)),...
    'Hz and ',num2str(f(k3)),'-',num2str(f(k4)),'Hz will be used.']);
```

With initial guess, lower bound, and higher bound, we can get the MPV of flutter derivatives by simulated annealing (via minimizing the NLLF). Simulated annealing is time-consuming (about 10-30 min, depending on your computer) but very robust. People can also use MCMC, which is very efficient (about 100-300 sec, depending on your computer). But MCMC is less robust and often gets a non-convergent result. We suggest people use simulated annealing.

```
options = optimoptions('simulannealbnd','PlotFcns',...
    {@saplotbestx,@saplotbestf,@saplotx,@saplotf},'MaxFunctionEvaluations', 100000);
[theta_opt,fval,exitFlag,output] = simulannealbnd(@(theta) calculate_NLLF_noncross(k1,
k2, k3, k4, M, w_h, w_a, zeta_h, zeta_a, theta, psd_y_M, f),x0,lb,ub,options);
```

Please notice that the initial guess, lower bound, and higher bound should be adjusted in different wind tunnel tests. In an engineering practice, we do not know the theoretical flutter derivatives. We can reconstruct the buffeting displacement PSD by the identified MPV. Then compare the measured PSD with the reconstructed PSD (as shown in Figure 1), using λ indicator (Chu et al. 2022).

$$\lambda = \frac{\left(n\sum_k\{\mathbb{E}[\mathbf{S}_N(k)|\hat{\boldsymbol{\theta}}]\mathbf{S}_N^M(k)\} - \sum_k \mathbb{E}[\mathbf{S}_N(k)|\hat{\boldsymbol{\theta}}]\cdot\sum_k \mathbf{S}_N^M(k)\right)^2}{\left(n\sum_k\left(\mathbb{E}[\mathbf{S}_N(k)|\hat{\boldsymbol{\theta}}]\right)^2 - \left(\sum_k \mathbb{E}[\mathbf{S}_N(k)|\hat{\boldsymbol{\theta}}]\right)^2\right)\left(n\sum_k\left(\mathbf{S}_N^M(k)\right)^2 - (\sum_k \mathbf{S}_N^M(k))^2\right)} \tag{7}$$

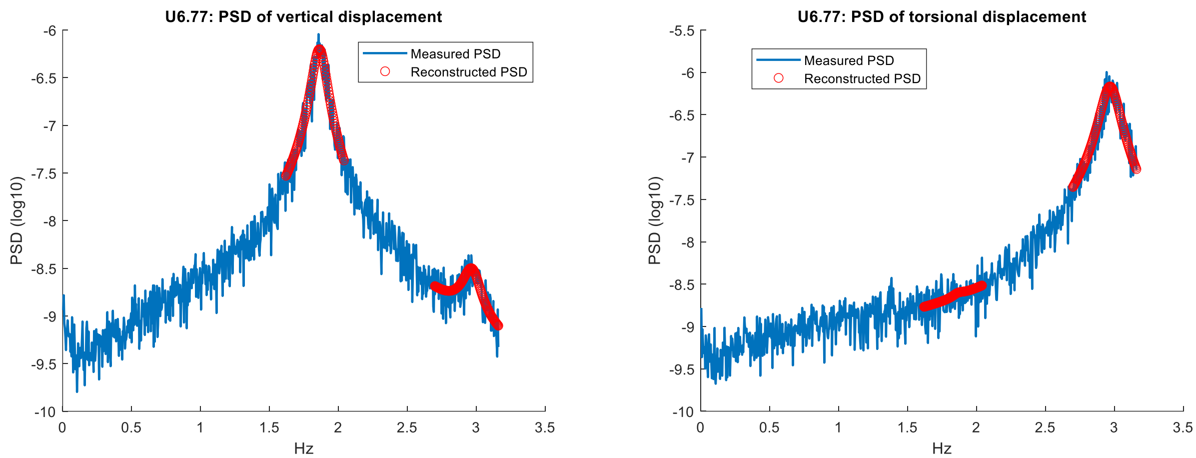

Figure 1 Comparison between measured PSD and reconstructed PSD

Actually, $\hat{\theta}$ can also be obtained by:

$$\hat{\theta} = \arg\min\{\ln p\left(\mathbf{S}_N^{M,\mathcal{K}}|\boldsymbol{\theta}\right) + (1-\lambda)\} \tag{8}$$

Chu, X., Cui, W., Liu, P., Zhao, L. and Ge, Y., 2022. Bayesian spectral density approach for identification of bridge section's flutter derivatives operated in turbulent flow. *Mechanical Systems and Signal Processing*, *170*, p.108782.