Borromean Rings Are Non-Splittable

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Knots and Links

Definition

A knot is a closed curve embedded in $\mathbb{R}^3$ does not decompose the space, but it can be tangled up in inescapable ways.

Definition

A link is a collection of two or more disjoint knots.

A link L is splittable if $\exists$ a 3-ball B, an ordering of components of the link $K_1, K_2, \ldots, K_m$ and a integer $0 < k < m$ such that $K_j \subseteq B$ for $j \leq k$ and $K_i \subseteq S^3 - B$ for $i > k$.

Borromean Rings

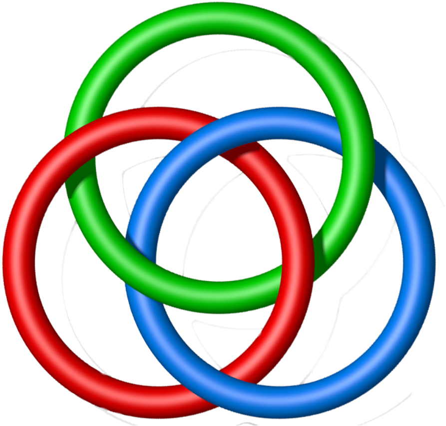
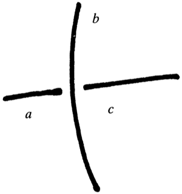


Figure: Borromean Rings

Colorable

A knot or a link K is said to be colorable mod $n \geq 3$ if K has a diagram D in which it is possible to assign an integer to each arc of D which does not contain an undercrossing D such that:
(1) At each crossing we have $a + c = 2b$ (mod n) where b is the integer assigned to the overcrossing and a and c are the integers assigned to the other two arcs
(2) At least 2 distinct integers mod n are used in the diagram.



Colorable

Theorem

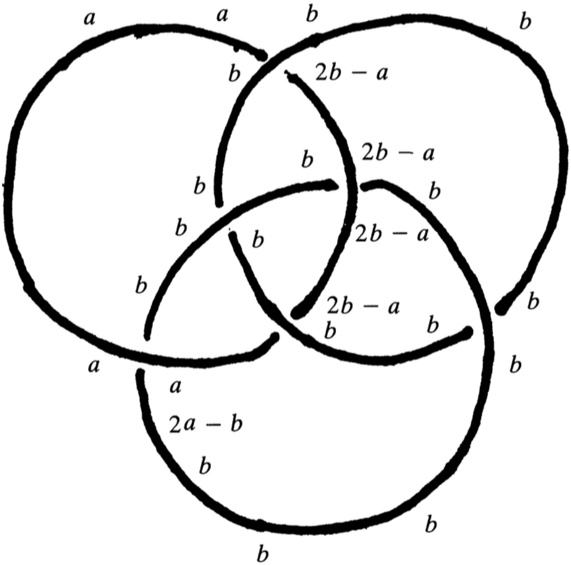
If a link L is splittable, then L is colorable mod 3.

Proof: If L is splittable with a ball B, then there exists a diagram for L in which the images of $L \cap B$ are separated from the images of $L \cap S^3 - B$ by a circle C.

Given the components of the diagram of $L \cap B$ the monochrome coloring by assigning the integer 0 to each arc. Similarly, assign the arcs of the rest of the diagram the integer 1.

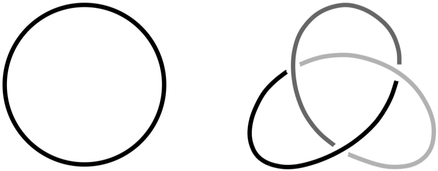
Borromean ring is not colorable mod n for any $n > 1$, and hence, is not splittable.

Colorable



Application of Colorability

The trefoil knot is not equivalent to the unknot. The former is colorable mod 3 whereas the latter is not.



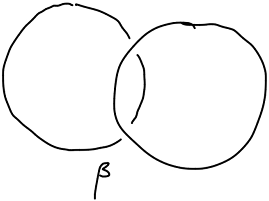
Application of Colorability

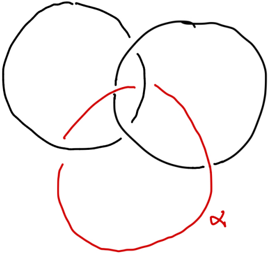
Theorem

If a knot or link K is colorable mod n, then one can obtain a homomorphism from $\pi_1(\mathbb{R}^3 - K)$ onto the dihedral group D_n where $Dn = \{s, t | s^2 = 1 = t^n, sts = t^{n-1}\}$. This homomorphism is determined by the particular choice of coloring.

Fundamental Group

Let β be two knots of the Borromean ring, and the rest one is α. Consider $\mathbb{R}^3 - \beta$ and its fundamental group.

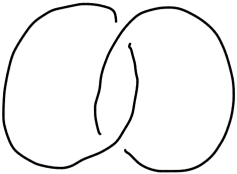




Notice that β is homeomorphic to two unknots, thus $\pi_1(\mathbb{R}^3 - \beta) \cong \mathbb{Z} * \mathbb{Z}$, which is not abelian. While α as a loop in $\mathbb{R}^3 - \beta$, its equivalent class $ab^{-1}a^{-1}b$ is not a unit.

Fundamental Group

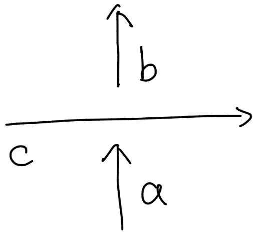
However, if β is not splittable after removing α, just as the figure shows,



then $\mathbb{R}^3 - \beta$ is homeomorphic to a torus whose fundamental group is $\mathbb{Z}^2$, an abelian group. In this case, $\alpha \in ab^{-1}a^{-1}b = 1$ is a trivial loop.
With this idea, let's introduce Wirtinger presentation.

Meridian

Locally consider a crossing, let a, b, c be generators of these three arcs respectively in the fundamental group. Then we have $b = cac^{-1}$, where c separates a and b.



For a link or kont K, one may collect all generators and all crossing relations, to get $\pi_1(\mathbb{R}^3 - K) \cong \langle a_1, a_2, \ldots, a_n | (CrossingRelations) \rangle$.

Longitude

Let K be a knot, which is homeomorphic to S^1. The tubular neighborhood of K is defiend as $N(K) \cong S^1 \times D^2$. A longitude $\lambda \in \pi_1(\partial N(K))$ is a simple closed curve satisfying:
(1) It is parallel to the knot.
(2) It has linking number 0 with the knot, which ensures it does not wrap around the meridian direction.

Informally, one may consider the meridian as a samll loop going around the strand once, and the longitude as a loop going along the strand once staying parallel to it but never linking it.

Longitude

In Borromean ring, let A, B, C be the three knots, and a, b, c be the meridians respectively, and $\lambda_A, \lambda_B, \lambda_C$ be the longitudes respectively. Then we have the following equallities:

$\lambda_A = cb^{-1}c^{-1}b = [c, b^{-1}]$
$\lambda_B = ac^{-1}a^{-1}c = [a, c^{-1}]$
$\lambda_C = ba^{-1}b^{-1}a = [b, a^{-1}]$

This comes from the same idea with the second proof.

Combination of Meridian and Longitude

Recall that at each crossing locally, $b = cac^{-1}$, where c separates a and b. Thus we let a meridian a go along the corresponding longitude λ_A, until it returns its original place, then we get $a = \lambda_A a \lambda_A^{-1}$.
Similarly, we also have $b = \lambda_B b \lambda_B^{-1}$ and $c = \lambda_C c \lambda_C^{-1}$.

Collecting all these generators and relations, we get the fundamental group $\pi_1(\mathbb{R}^3 - Bor) \cong \langle a, b, c | a = \lambda_A a \lambda_A^{-1}, b = \lambda_B b \lambda_B^{-1} \rangle$ after simplifying, where *Bor* is the Borromean ring.

Result

Let γ and τ be two links or knots.

Theorem

$\gamma \cong \tau \Rightarrow \pi_1(\mathbb{R}^3 - \gamma) \cong \pi_1(\mathbb{R}^3 - \tau)$

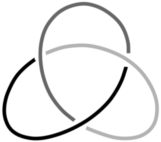
Now let γ be Borromean ring and τ be three unknots. Compare their fundamental group.

$\pi_1(\mathbb{R}^3 - \tau) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$\pi_1(\mathbb{R}^3 - \gamma) \cong \langle a, b, c | a = \lambda_A a \lambda_A^{-1}, b = \lambda_B b \lambda_B^{-1} \rangle$, where $\lambda_A = [c, b^{-1}]$, $\lambda_B = [a, c^{-1}]$

Therefore, $\pi_1(\mathbb{R}^3 - \gamma) \ncong \pi_1(\mathbb{R}^3 - \tau) \Rightarrow \gamma \ncong \tau$, which reveals Borromean rings are not splittable.

Let γ be a trefoil knot.



At these three crossings, we can obtain $a = b^{-1}cb, b = c^{-1}ac, c = a^{-1}ba$ respectively. After being simplified, the fundamental group is $\pi_1(\mathbb{R}^3 - \gamma) \cong \langle a, b | aba = bab \rangle$

Application of Wirtinger Presentation

Notice that one may calculate $\pi_1(\mathbb{R}^3 - \gamma)$ by considering the trefoil knot as a $(2, 3)$-torus knot and get $\pi_1(\mathbb{R}^3 - \gamma) = \langle x, y | x^2 = y^3 \rangle$ In fact, it is isomorphic to what he had got.

Construct $\phi : \langle a, b | aba = bab \rangle \to \pi_1(\mathbb{R}^3 - \gamma) = \langle x, y | x^2 = y^3 \rangle$ with $x \mapsto aba$ and $y \mapsto ab$. ϕ is an isomorphism.

Dihedral Group

Recall that if a knot or link K is colorable mod n, then one can obtain a homomorphism from $\pi_1(\mathbb{R}^3 - K)$ onto the dihedral group D_n. Here the trefoil knot is colorable mod 3.

We can construct a homomorphism
$\phi : \pi_1(\mathbb{R}^3 - \gamma) = \langle a, b | aba = bab \rangle \to D_3 = S_3$ by
$\phi(a) = (12), \phi(b) = (23)$, where S_3 is the symmetry group. It's easy to check that ϕ is a surjective homomorphism.

There is also a natural surjective homomorphism from $\langle x, y | x^2 = y^3 \rangle$ to the Dihedral group D_3.

Perspective in Singular Homology

Let $K \subseteq S^3$ be a knot. Take a small tubular neighborhood:$N(K) \cong S^1 \times D^2$ whose boundary is a torus $\partial N(K) = T \cong S^1 \times S^1$. And the knot complement is defined by $X_K = S^3 - int(N(K))$. The meridian μ and the longitude λ form a natural basis of $H_1(T) = \langle [\mu], [\lambda] \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

Consider two inclusion maps $\phi_1 : H_1(T) \to H_1(N(K))$ and $\phi_2 : H_1(T) \to H_1(X_K)$.
Under ϕ_1, $[\mu] \mapsto 0$ and $[\lambda] \mapsto$ generator of $H_1(N(K)) \cong \mathbb{Z}$;
while under ϕ_2, $[\lambda] \mapsto 0$ and $[\mu] \mapsto$ generator of $H_1(X_K) \cong \mathbb{Z}$

Mayer-Vietoris Sequence

$\cdots \to H_2(S^3) \to H_1(T) \to H_1(N(K)) \oplus H_1(X_K) \to H_1(S^3) \to 0$
Since $H_2(S^3) \cong 0$ and $H_1(S^3) \cong 0$, we have the following exact sequence:

$0 \to H_1(T) \to H_1(N(K)) \oplus H_1(X_K) \to 0$

with concrete identifications $H_1(T) = \mathbb{Z}\langle[\mu]\rangle \oplus \mathbb{Z}\langle[\lambda]\rangle$, $H_1(N(K)) = \mathbb{Z}\langle g_N\rangle$, $H_1(X_K) = \mathbb{Z}\langle g_X\rangle$, and $(\phi_1, -\phi_2)([\lambda]) = (g_N, 0)$, $(\phi_1, -\phi_2)([\mu]) = (0, g_X)$.

This is the foundation of Dehn surgery theory.