I. LIPSCHITZ CONTINUOUS ANALYSIS OF MÖBIUS ADDITION

We present the lipschitz continuous property analysis of the Möbius addition $\boldsymbol{x} \oplus_c \boldsymbol{y}$ with respect to curvatures c, right input $\boldsymbol{y}$, and left input $\boldsymbol{x}$ in Theorems 1, 2, and 3, respectively.

Theorem 1: Given two points on hyperbolic spaces $\boldsymbol{x}$ and $\boldsymbol{y}$, the addition $\boldsymbol{x} \oplus \boldsymbol{y}$ is Lipschitz continuous with curvatures c, *i.e.*,

$$\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\| \leq L_{\oplus_c} |c_1 - c_2|, \tag{1}$$

where $L_{\oplus_c}$ is computed as

$$L_{\oplus_c} \triangleq \frac{2\|\boldsymbol{x}\|\|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2\|\boldsymbol{y}\| + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\|^2 + (c_1+c_2)\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^3 + (c_1+c_2)\|\boldsymbol{x}\|^3\|\boldsymbol{y}\|^2 + 3c_1c_2\|\boldsymbol{x}\|^4\|\boldsymbol{y}\|^3 + c_1c_2\|\boldsymbol{x}\|^3\|\boldsymbol{y}\|^4}{|(1 + 2c_1\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c_1^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2)||(1 + 2c_2\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c_2^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2)|}, \tag{2}$$

We also derive that

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y} - (\boldsymbol{x} + \boldsymbol{y})\| \leq L_{\oplus_{c0}}, \tag{3}$$

where

$$L_{\oplus_{c0}} \triangleq |c| \frac{2\|\boldsymbol{x}\|\|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2\|\boldsymbol{y}\| + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\|^2 + c\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^3 + c\|\boldsymbol{x}\|^3\|\boldsymbol{y}\|^2}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2)|}, \tag{4}$$

Moreover, $L_{\oplus_{c0}}$ satisfies that

$$\lim_{c\to 0} L_{\oplus_{c0}} = 0. \tag{5}$$

Denote the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ as θ. Suppose that θ satisfy $\cos(\theta) \geq \cos\tilde{\theta}$. By utilizing the hyperbolic constraint of $\boldsymbol{x}$ and $\boldsymbol{y}$, $L_{\oplus_x}$ can be further modeled as

$$L_{\oplus_c} \triangleq \frac{\left(\frac{6}{c^3} + \frac{2}{c^2} + \frac{4c_1c_2}{c^7} + \frac{2(c_1+c_2)}{c^5}\right)}{(1-\cos(\tilde{\theta})^2)^2} |c_1 - c_2|, \tag{6}$$

and $L_{\oplus_{c0}}$ can be modeled as

$$L_{\oplus_{c0}} \triangleq \frac{\frac{8}{c^2} + \frac{2}{c}}{(1-\cos(\tilde{\theta})^2)^2}. \tag{7}$$

Proof: Recall that the addition $\oplus$ in hyperbolic spaces is computed as

$$\boldsymbol{x} \oplus_{c_i} \boldsymbol{y} = \frac{(1 + 2c_i\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c_i\|\boldsymbol{y}\|^2)\boldsymbol{x} + (1 - c_i\|\boldsymbol{x}\|^2)\boldsymbol{y}}{1 + 2c_i\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c_i^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}, \quad i = 1, 2. \tag{8}$$

For simplicity, we denote that

$$\begin{aligned} \boldsymbol{A}_i &= (1 + 2c_i\langle \boldsymbol{x}, y\rangle + c_i\|y\|^2)x + (1 - c_i\|\boldsymbol{x}\|^2)y, \\ \boldsymbol{D}_i &= 1 + 2c_i\langle \boldsymbol{x}, y\rangle + c_i^2\|\boldsymbol{x}\|^2\|y\|^2, i = 1, 2. \end{aligned} \tag{9}$$

Thus,

$$\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} = \frac{\boldsymbol{A}_1}{\boldsymbol{D}_1}, \boldsymbol{x} \oplus_{c_2} \boldsymbol{y} \quad = \frac{\boldsymbol{A}_2}{\boldsymbol{D}_2} \tag{10}$$

Then, we can obtain that

$$\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y} = \frac{\boldsymbol{A}_1\boldsymbol{D}_2 - \boldsymbol{A}_2\boldsymbol{D}_1}{\boldsymbol{D}_1\boldsymbol{D}_2}. \tag{11}$$

$\boldsymbol{A}_1\boldsymbol{D}_2$ is expanded as

$$\begin{aligned}\boldsymbol{A}_1\boldsymbol{D}_2 &= \left[(1+2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_1\|\boldsymbol{y}\|^2)\boldsymbol{x} + (1-c_1\|\boldsymbol{x}\|^2)\boldsymbol{y}\right]\cdot\left(1+2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_2^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\right)\\ &= \underbrace{(1+2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_1\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot 1}_{T_1} + \underbrace{(1+2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_1\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot(2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle)}_{T_2}\\ &\quad+\underbrace{(1+2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_1\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot c_2^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}_{T_3}\\ &\quad+\underbrace{(1-c_1\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot 1}_{T_4} + \underbrace{(1-c_1\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot(2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle)}_{T_5}\\ &\quad+\underbrace{(1-c_1\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot c_2^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}_{T_6},\end{aligned} \tag{12}$$

and $\boldsymbol{A}_2\boldsymbol{D}_1$ is expanded as

$$\begin{aligned}\boldsymbol{A}_2\boldsymbol{D}_1 &= \left[(1+2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_2\|\boldsymbol{y}\|^2)\boldsymbol{x} + (1-c_2\|\boldsymbol{x}\|^2)\boldsymbol{y}\right]\cdot\left(1+2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_1^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\right)\\ &= \underbrace{(1+2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_2\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot 1}_{T_1'} + \underbrace{(1+2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_2\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot(2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle)}_{T_2'}\\ &\quad+\underbrace{(1+2c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle + c_2\|\boldsymbol{y}\|^2)\boldsymbol{x}\cdot c_1^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}_{T_3'}\\ &\quad+\underbrace{(1-c_2\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot 1}_{T_4'} + \underbrace{(1-c_2\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot(2c_1\langle\boldsymbol{x},\boldsymbol{y}\rangle)}_{T_5'}\\ &\quad+\underbrace{(1-c_2\|\boldsymbol{x}\|^2)\boldsymbol{y}\cdot c_1^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}_{T_6'}.\end{aligned} \tag{13}$$

The numerator is computed as

$$\boldsymbol{A}_1\boldsymbol{D}_2 - \boldsymbol{A}_2\boldsymbol{D}_1 = \sum_{i=1}^{6} T_i - T_i', \tag{14}$$

where $T_1 - T_1'$ is computed as

$$T_1 - T_1' = (c_2-c_1)(2\langle\boldsymbol{x},\boldsymbol{y}\rangle + \|\boldsymbol{y}\|^2)\boldsymbol{x}, \tag{15}$$

$T_2 - T_2'$ is computed as

$$T_2 - T_2' = (c_2-c_1)2\langle\boldsymbol{x},\boldsymbol{y}\rangle, \tag{16}$$

$T_3 - T_3'$ is computed as

$$T_3 - T_3' = (c_2-c_1)\left((c_1+c_2)\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\boldsymbol{x} + 2c_1c_2\langle\boldsymbol{x},\boldsymbol{y}\rangle\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\boldsymbol{x} + c_1c_2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^4\boldsymbol{x}\right), \tag{17}$$

$T_4 - T_4'$ is computed as

$$T_4 - T_4' = (c_2-c_1)\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|, \tag{18}$$

$T_5 - T_5'$ is computed as

$$T_5 - T_5' = 2(c_2-c_1)\langle\boldsymbol{x},\boldsymbol{y}\rangle\boldsymbol{y}, \tag{19}$$

and $T_6 - T_6'$ is computed as

$$T_6 - T_6' = (c_2-c_1)\left((c_1+c_2)\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\boldsymbol{y} - c_1c_2\|\boldsymbol{x}\|^4\|\boldsymbol{y}\|^2\boldsymbol{y}\right). \tag{20}$$

From the Cauchy-Schwarz inequality, we can derive that

$$\|T_1 - T_1'\| \le |c_1-c_2|\|(2\langle\boldsymbol{x},\boldsymbol{y}\rangle + \|\boldsymbol{y}\|^2)\boldsymbol{x}\| \le |c_1-c_2|(2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\| + \|\boldsymbol{x}\|\|\boldsymbol{y}\|^2), \tag{21}$$

$$\|T_2 - T_2'\| \le |c_1-c_2|2\|\boldsymbol{x}\|\|\boldsymbol{y}\|, \tag{22}$$

$$\|T_3 - T_3'\| \le |c_1-c_2|\left((c_1+c_2)\|\boldsymbol{x}\|^3\|\boldsymbol{y}\|^2 + 2c_1c_2\|\boldsymbol{x}\|^4\|\boldsymbol{y}\|^3 + c_1c_2\|\boldsymbol{x}\|^3\|\boldsymbol{y}\|^4\right), \tag{23}$$

$$\|T_4 - T_4^{'}\| \leq |c_2 - c_1| \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|, \tag{24}$$

$$\|T_5 - T_5^{'}\| \leq |c_2 - c_1| 2\|\boldsymbol{x}\| \|\boldsymbol{y}\|^2, \tag{25}$$

$$\|T_6 - T_6^{'}\| \leq |c_2 - c_1| \left((c_1 + c_2)\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^3 + c_1 c_2 \|\boldsymbol{x}\|^4 \|\boldsymbol{y}\|^3 \right). \tag{26}$$

From the above proof, the numerator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$\begin{aligned} \|\boldsymbol{A}_1 \boldsymbol{D}_2 - \boldsymbol{A}_2 \boldsymbol{D}_1\| \leq |c_1 - c_2| (2\|\boldsymbol{x}\| \|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\| + 3\|\boldsymbol{x}\| \|\boldsymbol{y}\|^2 + (c_1 + c_2)\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^3 \\ + (c_1 + c_2)\|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^2 + 3c_1 c_2 \|\boldsymbol{x}\|^4 \|\boldsymbol{y}\|^3 + c_1 c_2 \|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^4) \end{aligned} \tag{27}$$

The denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$|\boldsymbol{D}_1 \boldsymbol{D}_2| = |(1 + 2c_1 \langle \boldsymbol{x}, \boldsymbol{y} \rangle + c_1^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)| |(1 + 2c_2 \langle \boldsymbol{x}, \boldsymbol{y} \rangle + c_2^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)| \tag{28}$$

By denoting

$$L_{\oplus_c} \triangleq \frac{2\|\boldsymbol{x}\| \|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\| + 2\|\boldsymbol{x}\| \|\boldsymbol{y}\|^2 + (c_1 + c_2)\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^3 + (c_1 + c_2)\|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^2 + 3c_1 c_2 \|\boldsymbol{x}\|^4 \|\boldsymbol{y}\|^3 + c_1 c_2 \|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^4}{|(1 + 2c_1 \langle \boldsymbol{x}, \boldsymbol{y} \rangle + c_1^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)| |(1 + 2c_2 \langle \boldsymbol{x}, \boldsymbol{y} \rangle + c_2^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)|}, \tag{29}$$

we have proved that

$$\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\| \leq L_{\oplus_c} |c_1 - c_2| \tag{30}$$

By assigning $c_1 = c, c_2 = 0$, we can derive that

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y} - \boldsymbol{x} \oplus_0 \boldsymbol{y}\| \leq |c| \frac{2\|\boldsymbol{x}\| \|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\| + 2\|\boldsymbol{x}\| \|\boldsymbol{y}\|^2 + c\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^3 + c\|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^2}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y} \rangle + c^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)|}. \tag{31}$$

By denoting

$$L_{\oplus_{c0}} \triangleq |c| \frac{2\|\boldsymbol{x}\| \|\boldsymbol{y}\| + 3\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\| + 2\|\boldsymbol{x}\| \|\boldsymbol{y}\|^2 + c\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^3 + c\|\boldsymbol{x}\|^3 \|\boldsymbol{y}\|^2}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y} \rangle + c^2 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2)|}, \tag{32}$$

we have proved that

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y} - (\boldsymbol{x} + \boldsymbol{y})\| \leq L_{\oplus_{c0}}. \tag{33}$$

By substituting $c = 0$ here, we have that

$$\|\boldsymbol{x} \oplus_0 \boldsymbol{y} - \boldsymbol{x} \oplus_0 \boldsymbol{y}\| = 0, \tag{34}$$

and thus

$$\lim_{c \to 0} L_{\oplus_{c0}} = 0, \tag{35}$$

which is consistent with common sense.

From the hyperbolic constraint of $\boldsymbol{x}$ and $\boldsymbol{y}$, *i.e.*, $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \leq \frac{1}{c}$, we can derive that

$$\|T_1 - T_1^{'}\| \leq |c_1 - c_2| \frac{3}{c^3}, \tag{36}$$

$$\|T_2 - T_2^{'}\| |c_1 - c_2| \frac{2}{c^2}, \tag{37}$$

$$\|T_3 - T_3^{'}\| \leq |c_1 - c_2| \left(\frac{3c_1 c_2}{c^7} + \frac{(c_1 + c_2)}{c^5} \right), \tag{38}$$

$$\|T_4 - T_4^{'}\| \leq \frac{|c_1 - c_2|}{c^3}, \tag{39}$$

$$\|T_5 - T_5^{'}\| \leq \frac{2|c_1 - c_2|}{c^3}, \tag{40}$$

$$\|T_6 - T_6^{'}\| \leq |c_1 - c_2| \left(\frac{c_1 + c_2}{c^5} + \frac{c_1 c_2}{c^7} \right). \tag{41}$$

Overall, $\boldsymbol{A}_1\boldsymbol{D}_2 - \boldsymbol{A}_2\boldsymbol{D}_1$ satisfies that

$$\|\boldsymbol{A}_1\boldsymbol{D}_2 - \boldsymbol{A}_2\boldsymbol{D}_1\| \leq \sum_{i=1}^{6}\|T_i - T_i^{'}\| \leq |c_2 - c_1|\left(\frac{6}{c^3} + \frac{2}{c^2} + \frac{4c_1c_2}{c^7} + \frac{2(c_1+c_2)}{c^5}\right). \tag{42}$$

As to the denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$, it holds that

$$\begin{aligned}|\boldsymbol{D}_1| &= |(1 + 2c_1\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c_1^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2)| = 1 + 2c\|\boldsymbol{x}\|\boldsymbol{y}\|\cos(\theta) + c^2\|\boldsymbol{x}\|^2\boldsymbol{y}\|^2 \\ &= (c\|\boldsymbol{x}\|\boldsymbol{y}\| + \cos(\theta))^2 + (1 - \cos(\theta)^2) \geq (1 - \cos(\theta)^2),\end{aligned} \tag{43}$$

where θ denotes the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$. Suppose that the angle satisfies that $\cos(\theta) \geq \cos(\tilde{\theta})$, the denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$|\boldsymbol{D}_1\boldsymbol{D}_2| \geq (1 - \cos(\tilde{\theta})^2)^2. \tag{44}$$

Combing the numerator and denominator, we can derive that

$$\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\| \leq |c_2 - c_1|\frac{\left(\frac{6}{c^3} + \frac{2}{c^2} + \frac{4c_1c_2}{c^7} + \frac{2(c_1+c_2)}{c^5}\right)}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{45}$$

By assigning $c_1 = c$ and $c_2 = 0$, we can obtain that

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y} - (\boldsymbol{x} + \boldsymbol{y})\| \leq \frac{\frac{8}{c^2} + \frac{2}{c}}{(1 - \cos(\tilde{\theta})^2)^2}, \tag{46}$$

due to $\lim\limits_{c\to 0}\boldsymbol{x} \oplus_c \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{y}$. ■

Theorem 2: Given points on hyperbolic spaces $\boldsymbol{x}$, $\boldsymbol{y}$, the addition $\boldsymbol{x} \oplus \boldsymbol{y}$ is Lipschitz continuous with the right input $\boldsymbol{y}$, *i.e.*,

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\| \leq L_{\oplus_y}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{47}$$

where $L_{\oplus_y}$ is computed as

$$\begin{aligned}L_{\oplus_y} \triangleq\ & \frac{1 + c\left(5\|\boldsymbol{x}\|^2 + 5\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|\|\boldsymbol{y}_2\|\right) + c^2\left(13\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| + 6\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|} \\ &+ \frac{c^3\left(6\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|},\end{aligned} \tag{48}$$

Moreover, $L_{\oplus_y}$ satisfies that

$$\lim_{c\to 0}L_{\oplus_y} = 1. \tag{49}$$

Denote the angle between $\boldsymbol{x}$ and $\boldsymbol{y}_1$, and $\boldsymbol{x}$ and $\boldsymbol{y}_2$ as θ_1 and θ_2, respectively. Suppose that θ_1 and θ_2 satisfy $\cos(\theta_1), \cos(\theta_2) \geq \cos\tilde{\theta}$. By utilizing the hyperbolic constraint of $\boldsymbol{x}$ and $\boldsymbol{y}$, $L_{\oplus_y}$ can be further modeled as

$$L_{\oplus_y} = \frac{1 + \frac{13}{c^3} + \frac{23}{c^2} + \frac{11}{c}}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{50}$$

Proof: Recall that the addition is computed as

$$\boldsymbol{x} \oplus_c \boldsymbol{y}_i = \frac{(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c\|\boldsymbol{y}_i\|^2)\boldsymbol{x} + (1 - c\|\boldsymbol{x}\|^2)\boldsymbol{y}_i}{1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_i\|^2}, \quad i = 1, 2. \tag{51}$$

We denote that

$$A_i = (1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c\|\boldsymbol{y}_i\|^2)\boldsymbol{x} + (1 - c\|\boldsymbol{x}\|^2)\boldsymbol{y}_i, \quad D_i = 1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_i\|^2. \tag{52}$$

Therefore,

$$\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2 = \frac{A_1}{D_1} - \frac{A_2}{D_2} = \frac{A_1D_2 - A_2D_1}{D_1D_2}. \tag{53}$$

The numerator $A_1D_2 - A_2D_1$ can be expanded as

$$\begin{aligned}
&\left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)\boldsymbol{x}\right] D_2 - \left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_2\rangle + c\|\boldsymbol{y}_2\|^2)\boldsymbol{x}\right] D_1 + (1-c\|\boldsymbol{x}\|^2)(\boldsymbol{y}_1 D_2 - \boldsymbol{y}_2 D_1) \\
&= \underbrace{\left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)D_2 - (1+2c\langle \boldsymbol{x},\boldsymbol{y}_2\rangle + c\|\boldsymbol{y}_2\|^2)D_1\right]\boldsymbol{x}}_{\text{in terms of } \boldsymbol{x}\text{, denoted as } T_1} + \underbrace{(1-c\|\boldsymbol{x}\|^2)(\boldsymbol{y}_1 D_2 - \boldsymbol{y}_2 D_1)}_{\text{in terms of } \boldsymbol{y}\text{, denoted as } T_2}.
\end{aligned} \tag{54}$$

T_1 can be computed as

$$\begin{aligned}
T_1 &= \left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)D_2 - (1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)D_1\right. \\
&\quad \left.+(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)D_1 - (1+2c\langle \boldsymbol{x},\boldsymbol{y}_2\rangle + c\|\boldsymbol{y}_2\|^2)D_1\right]\boldsymbol{x} \\
&= (1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)(D_2 - D_1)\boldsymbol{x} + \left(2c\langle \boldsymbol{x},\boldsymbol{y}_1-\boldsymbol{y}_2\rangle + c(\|\boldsymbol{y}_1\|^2 - \|\boldsymbol{y}_2\|^2)\right) D_1\boldsymbol{x} \\
&= (1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)(D_2 - D_1)\boldsymbol{x} + \left(2\langle \boldsymbol{x},\boldsymbol{y}_1-\boldsymbol{y}_2\rangle + \langle \boldsymbol{y}_1+\boldsymbol{y}_2,\boldsymbol{y}_1-\boldsymbol{y}_2\rangle\right) cD_1\boldsymbol{x},
\end{aligned} \tag{55}$$

where

$$D_2 - D_1 = -2c\langle \boldsymbol{x},\boldsymbol{y}_1-\boldsymbol{y}_2\rangle - c^2\|\boldsymbol{x}\|^2\langle \boldsymbol{y}_1+\boldsymbol{y}_2,\boldsymbol{y}_1-\boldsymbol{y}_2\rangle. \tag{56}$$

In this way, the norm of T_1 satisfies that

$$\begin{aligned}
\|T_1\| &\le \|\boldsymbol{x}\|\left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)|D_2-D_1| + \left(2\langle \boldsymbol{x},\boldsymbol{y}_1-\boldsymbol{y}_2\rangle + \langle \boldsymbol{y}_1+\boldsymbol{y}_2,\boldsymbol{y}_1-\boldsymbol{y}_2\rangle\right) cD_1\right] \\
&\le \|\boldsymbol{x}\|\left[(1+2c\langle \boldsymbol{x},\boldsymbol{y}_1\rangle + c\|\boldsymbol{y}_1\|^2)|D_2-D_1| + \left(2\|\boldsymbol{x}\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\| + \|\boldsymbol{y}_1+\boldsymbol{y}_2\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\|\right) cD_1\right],
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
|D_2 - D_1| &\le 2c\langle \boldsymbol{x},\boldsymbol{y}_1-\boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\langle \boldsymbol{y}_1+\boldsymbol{y}_2,\boldsymbol{y}_1-\boldsymbol{y}_2\rangle \\
&\le 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1+\boldsymbol{y}_2\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\|.
\end{aligned} \tag{58}$$

Therefore, it holds that

$$\|T_1\| \le \|\boldsymbol{y}_1-\boldsymbol{y}_2\|\|\boldsymbol{x}\|\left[(1+2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{y}_1\|^2)\cdot c\left(2\|\boldsymbol{x}\| + c\|\boldsymbol{x}\|^2(\|\boldsymbol{y}_1\|+\|\boldsymbol{y}_2\|)\right) + c|D_1|\left(2\|\boldsymbol{x}\| + \|\boldsymbol{y}_1\| + \|\boldsymbol{y}_2\|\right)\right], \tag{59}$$

where $|D_1|$ satisfies that

$$|D_1| \le 1 + 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2. \tag{60}$$

Therefore, $\|T_1\|$ satisfies that

$$\begin{aligned}
\|T_1\| &\le \|\boldsymbol{y}_1-\boldsymbol{y}_2\|\|\boldsymbol{x}\|\left[(1+2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{y}_1\|^2)\cdot c\left(2\|\boldsymbol{x}\| + c\|\boldsymbol{x}\|^2(\|\boldsymbol{y}_1\|+\|\boldsymbol{y}_2\|)\right)\right. \\
&\quad \left.+c(1+2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)\left(2\|\boldsymbol{x}\| + \|\boldsymbol{y}_1\| + \|\boldsymbol{y}_2\|\right)\right] \\
&= \|\boldsymbol{y}_1-\boldsymbol{y}_2\|\left(4c\|\boldsymbol{x}\|^2 + c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{x}\|\|\boldsymbol{y}_2\| + 9c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| + 4c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + 2c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right. \\
&\quad \left.+4c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 2c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\right).
\end{aligned} \tag{61}$$

T_2 and its norm satisfy that

$$\begin{aligned}
\|T_2\| \triangleq \|(1+c\|\boldsymbol{x}\|^2)(\boldsymbol{y}_1 D_2 - \boldsymbol{y}_2 D_1)\| &= \|(1+c\|\boldsymbol{x}\|^2)(\boldsymbol{y}_1 D_2 - \boldsymbol{y}_1 D_1 + \boldsymbol{y}_1 D_1 - \boldsymbol{y}_2 D_1)\| \\
&\le (1+c\|\boldsymbol{x}\|^2)\left[\|\boldsymbol{y}_1 D_2 - \boldsymbol{y}_1 D_1\| + \|\boldsymbol{y}_1 D_1 - \boldsymbol{y}_2 D_1\|\right] \\
&\le (1+c\|\boldsymbol{x}\|^2)\left[\|\boldsymbol{y}_1\||D_2 - D_1| + \|\boldsymbol{y}_1-\boldsymbol{y}_2\||D_1|\right].
\end{aligned} \tag{62}$$

By utilizing Eq. (58), it holds that

$$\begin{aligned}
\|T_2\| &\le (1+c\|\boldsymbol{x}\|^2)\left[\|\boldsymbol{y}_1\|\left(2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1+\boldsymbol{y}_2\|\|\boldsymbol{y}_1-\boldsymbol{y}_2\|\right) + \|\boldsymbol{y}_1-\boldsymbol{y}_2\||D_1|\right] \\
&= (1+c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}_1-\boldsymbol{y}_2\|\left[\|\boldsymbol{y}_1\|\left(2c\|\boldsymbol{x}\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1+\boldsymbol{y}_2\|\right) + |D_1|\right],
\end{aligned} \tag{63}$$

where $|D_1|$ satisfies that

$$|D_1| \le 1 + 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2. \tag{64}$$

Thus, $\|T_2\|$ can be modeled as

$$\begin{aligned}
\|T_2\| &\le (1+c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}_1-\boldsymbol{y}_2\|\left[\|\boldsymbol{y}_1\|\left(2c\|\boldsymbol{x}\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1+\boldsymbol{y}_2\|\right) + 1 + 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2\right] \\
&= \|\boldsymbol{y}_1-\boldsymbol{y}_2\|(1+c\|\boldsymbol{x}\|^2)\left(1 + 4c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + 2c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right) \\
&= \|\boldsymbol{y}_1-\boldsymbol{y}_2\|\left(1 + 4c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + 2c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + c\|\boldsymbol{x}\|^2\right. \\
&\quad \left.+4c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + 2c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right).
\end{aligned} \tag{65}$$

The numerator $\|A_1D_2 - A_2D_1\|$ satisfies that

$$\begin{aligned}
&\|A_1D_2 - A_2D_1\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|\|\boldsymbol{x}\| \left[(1 + 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{y}_1\|^2) \cdot c \left(2\|\boldsymbol{x}\| + c\|\boldsymbol{x}\|^2(\|\boldsymbol{y}_1\| + \|\boldsymbol{y}_2\|)\right)\right. \\
&\left.+c(1 + 2c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)\left(2\|\boldsymbol{x}\| + \|\boldsymbol{y}_1\| + \|\boldsymbol{y}_2\|\right)\right] \\
&+ \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|(1 + c\|\boldsymbol{x}\|^2)\left(1 + 4c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + 2c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right) \\
&= \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|\left(1 + 5c\|\boldsymbol{x}\|^2 + 5c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{x}\|\|\boldsymbol{y}_2\| + 13c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| + 6c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2\right. \\
&\left.+3c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 6c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 3c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\right).
\end{aligned} \tag{66}$$

The denominator of $\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2$ satisfies that

$$|\boldsymbol{D}_1\boldsymbol{D}_2| = |(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)| \tag{67}$$

Overall, $\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\|$ satisfies that

$$\begin{aligned}
&\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\| \\
&\leq \left(\frac{1 + c\left(5\|\boldsymbol{x}\|^2 + 5\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|\|\boldsymbol{y}_2\|\right) + c^2\left(13\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| + 6\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|}\right. \\
&\left.+\frac{c^3\left(6\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|}\right)\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|.
\end{aligned} \tag{68}$$

By denoting

$$\begin{aligned}
L_{\oplus_y} &\triangleq \frac{1 + c\left(5\|\boldsymbol{x}\|^2 + 5\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|\|\boldsymbol{y}_2\|\right) + c^2\left(13\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + \|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| + 6\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|} \\
&+ \frac{c^3\left(6\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\right)}{|(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)||(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)|},
\end{aligned} \tag{69}$$

we have proved

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\| \leq L_{\oplus_y}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \tag{70}$$

In this way, the addition $\oplus_c(\cdot, \cdot)$ on hyperbolic space is $L_{\oplus_y}-$Lipschitz continuous with respect to the right input.

Note that $\lim_{c\to 0}\boldsymbol{x} \oplus_c \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{y}$. Therefore, it holds that

$$\lim_{c\to 0}\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\| = \|(\boldsymbol{x} + \boldsymbol{y}_1) - (\boldsymbol{x} + \boldsymbol{y}_2)\| = 1 \times \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{71}$$

When $c \to 0$, we can observe that

$$\lim_{c\to 0}L_{\oplus_y} = 1, \tag{72}$$

which is consistent with the Eq. (71), further validating the correctness of our proof.

From the hyperbolic constraint of $\boldsymbol{x}, \boldsymbol{y}$, *i.e.*, $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \leq \frac{1}{c}$, the numerator of $\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\|$ satisfies that

$$\|A_1D_2 - A_2D_1\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|\left[1 + \frac{13}{c^3} + \frac{23}{c^2} + \frac{11}{c}\right]. \tag{73}$$

As to $\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2$, its denominator holds that

$$\begin{aligned}
|\boldsymbol{D}_1| &= |(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_1\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2)| = 1 + 2c\|\boldsymbol{x}\|\boldsymbol{y}_1\|\cos(\theta_1) + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 \\
&= (c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + \cos(\theta_1))^2 + (1 - \cos(\theta_1)^2) \geq (1 - \cos(\theta_1)^2),
\end{aligned} \tag{74}$$

where θ_1 denotes the angle between $\boldsymbol{x}$ and $\boldsymbol{y}_1$. Similarly, it also holds that

$$\begin{aligned}
|\boldsymbol{D}_2| &= |(1 + 2c\langle \boldsymbol{x}, \boldsymbol{y}_2\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2)| = 1 + 2c\|\boldsymbol{x}\|\boldsymbol{y}_2\|\cos(\theta_2) + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_2\|^2 \\
&= (c\|\boldsymbol{x}\|\|\boldsymbol{y}_2\| + \cos(\theta_2))^2 + (1 - \cos(\theta_2)^2) \geq (1 - \cos(\theta_2)^2),
\end{aligned} \tag{75}$$

where θ_2 denotes the angle between $\boldsymbol{x}$ and $\boldsymbol{y}_2$. Suppose that the angle θ_1, θ_2 satisfies that $\cos(\theta_1), \cos(\theta_2) \geq \cos(\tilde{\theta})$, the denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$|\boldsymbol{D}_1 \boldsymbol{D}_2| \geq (1 - \cos(\tilde{\theta})^2)^2. \tag{76}$$

Combing the numerator and denominator, we can derive that

$$\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \frac{1 + \frac{13}{c^3} + \frac{23}{c^2} + \frac{11}{c}}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{77}$$

In this way, the Lipschitz continuous constant $L_{\oplus_y}$ can be further modeled as

$$L_{\oplus_y} = \frac{1 + \frac{13}{c^3} + \frac{23}{c^2} + \frac{11}{c}}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{78}$$

■

Corollary 1: Supposing that $\|\boldsymbol{x}\| \leq \frac{1}{c}, \|\boldsymbol{y}\| \leq \frac{1}{\sqrt{c}}$, the addition $\boldsymbol{x} \oplus_c \boldsymbol{y}$ is $L_{\oplus_y}$−Lipschitz continuous with respect to $\boldsymbol{y}$, *i.e.*,

$$\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\| \leq L_{\oplus_y} \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{79}$$

and $L_{\oplus_y}$ can be re-formulated as

$$L_{\oplus_y} \triangleq \frac{\left(\frac{9}{c^2} + \frac{18}{c^{\frac{3}{2}}} + \frac{14}{c} + \frac{6}{\sqrt{c}} + 1\right)}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{80}$$

Proof: Recall that the numerator in $\|\boldsymbol{x} \oplus_c \boldsymbol{y}_1 - \boldsymbol{x} \oplus_c \boldsymbol{y}_2\|$ can be expanded as The numerator $\|A_1 D_2 - A_2 D_1\|$ satisfies that

$$\begin{aligned} \|A_1 D_2 - A_2 D_1\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \big(&1 + 5c\|\boldsymbol{x}\|^2 + 5c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\| + c\|\boldsymbol{x}\|\|\boldsymbol{y}_2\| + 13c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\| + c^2\|\boldsymbol{x}\|^3\|\boldsymbol{y}_2\| \\ &+6c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|^2 + 3c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 6c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|^2 + 3c^3\|\boldsymbol{x}\|^4\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\| + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^3 + 2c^3\|\boldsymbol{x}\|^3\|\boldsymbol{y}_1\|^2\|\boldsymbol{y}_2\|\big). \end{aligned} \tag{81}$$

From the assumptions, the numerator satisfies that

$$\|A_1 D_2 - A_2 D_1\| \leq \frac{9}{c^2} + \frac{18}{c^{\frac{3}{2}}} + \frac{14}{c} + \frac{6}{\sqrt{c}} + 1, \tag{82}$$

and the denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$|\boldsymbol{D}_1 \boldsymbol{D}_2| \geq (1 - \cos(\tilde{\theta})^2)^2. \tag{83}$$

Combing the numerator and denominator, we can derive that

$$\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \frac{\left(\frac{9}{c^2} + \frac{18}{c^{\frac{3}{2}}} + \frac{14}{c} + \frac{6}{\sqrt{c}} + 1\right)}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{84}$$

In this way, the $L_{\oplus_y}$ is given by

$$L_{\oplus_y} \triangleq \frac{\left(\frac{9}{c^2} + \frac{18}{c^{\frac{3}{2}}} + \frac{14}{c} + \frac{6}{\sqrt{c}} + 1\right)}{(1 - \cos(\tilde{\theta})^2)^2}. \tag{85}$$

■

Theorem 3: Given points on hyperbolic spaces $\boldsymbol{x}$, $\boldsymbol{y}$, the addition $\boldsymbol{x} \oplus_c \boldsymbol{y}$ is Lipschitz continuous with $\boldsymbol{x}$, *i.e.*,

$$\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq L_{\oplus_x} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \tag{86}$$

where $L_{\oplus_x}$ is computed as

$$\begin{aligned} L_{\oplus_x} \triangleq\; &\frac{1 + 3c\|\boldsymbol{y}\|^2 + 3c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\| + 7c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\| + 7c^2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^2 + c^2\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|^3 + 5c^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3}{|1 + 2c\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1 + 2c\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|} \\ &+ \frac{6c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2 + 4c^3\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3 + 4c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^3 + 2c^3\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^4 + c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^4}{|1 + 2c\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1 + 2c\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|}. \end{aligned} \tag{87}$$

Moreover, $L_{\oplus_x}$ satisfies that

$$\lim_{c\to 0} L_{\oplus_x} = 1. \tag{88}$$

Denote the angle between $\boldsymbol{x}_1$ and $\boldsymbol{y}$, and $\boldsymbol{x}_2$ and $\boldsymbol{y}$ as θ_1 and θ_2, respectively. Suppose that θ_1 and θ_2 satisfy $\cos(\theta_1), \cos(\theta_2) \geq \cos\tilde{\theta}$. By utilizing the hyperbolic constraint of $\boldsymbol{x}$ and $\boldsymbol{y}$, $L_{\oplus_x}$ can be further modeled as

$$L_{\oplus_x} \triangleq \frac{(1+\frac{13}{c}+\frac{19}{c^2}+\frac{11}{c^3})}{(1-\cos(\tilde{\theta})^2)^2}. \tag{89}$$

Proof: Recall that the addition is computed as

$$\boldsymbol{x} \oplus_c \boldsymbol{y}_i = \frac{(1+2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c\|\boldsymbol{y}_i\|^2)\boldsymbol{x} + (1-c\|\boldsymbol{x}\|^2)\boldsymbol{y}_i}{1+2c\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle + c^2\|\boldsymbol{x}\|^2\|\boldsymbol{y}_i\|^2}, \quad i = 1, 2. \tag{90}$$

We denote that

$$\begin{aligned} &N(\boldsymbol{x}_i) = (1+2c\langle \boldsymbol{x}_i, \boldsymbol{v}\rangle + c\|\boldsymbol{v}\|^2)\boldsymbol{x}_i + (1-c\|\boldsymbol{x}_i\|^2)\boldsymbol{v}, \\ &D(\boldsymbol{x}_i) = 1+2c\langle \boldsymbol{x}_i, \boldsymbol{v}\rangle + c^2\|\boldsymbol{x}_i\|^2\|\boldsymbol{v}\|^2, \\ &\text{where} \quad i = 1, 2. \end{aligned} \tag{91}$$

Then $\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}$ can be presented by

$$\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y} = \frac{N(\boldsymbol{x}_1)D(\boldsymbol{x}_2) - N(\boldsymbol{x}_2)D(\boldsymbol{x}_1)}{D(\boldsymbol{x}_1)D(\boldsymbol{x}_2)}. \tag{92}$$

Its numerator equals to

$$\begin{aligned} &N(\boldsymbol{x}_1)D(\boldsymbol{x}_2) - N(\boldsymbol{x}_2)D(\boldsymbol{x}_1) \\ &= (1+c\|\boldsymbol{y}\|^2 + 4c^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)(\boldsymbol{x}_1 - \boldsymbol{x}_2) + 2c(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_2) \\ &+ (2c^2\|\boldsymbol{y}\|^2 + 2c)(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_2) + (c^2\|\boldsymbol{y}\|^2 + c^3\|\boldsymbol{y}\|^4)(\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1 - \|\boldsymbol{x}_1\|^2\boldsymbol{x}_2) \\ &+ 2c^3\|\boldsymbol{y}\|^2(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2) \\ &+ \left[c(\|\boldsymbol{x}_2\|^2 - \|\boldsymbol{x}_1\|^2) + 2c(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle) + 2c^2(\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle - \|\boldsymbol{x}_1\|^2\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle) + c^2\|\boldsymbol{y}\|^2(\|\boldsymbol{x}_2\|^2 - \|\boldsymbol{x}_1\|^2)\right]\boldsymbol{y} \end{aligned}. \tag{93}$$

Due to the property of norm, $\|N(\boldsymbol{x}_1)D(\boldsymbol{x}_2) - N(\boldsymbol{x}_2)D(\boldsymbol{x}_1)\|$ satisfies that

$$\begin{aligned} &\|N(\boldsymbol{x}_1)D(\boldsymbol{x}_2) - N(\boldsymbol{x}_2)D(\boldsymbol{x}_1)\| \\ &\leq \underbrace{\|(1+c\|\boldsymbol{y}\|^2 + 4c^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)(\boldsymbol{x}_1 - \boldsymbol{x}_2)\|}_{(1)} + \underbrace{\|2c(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_2)\|}_{(2)} \\ &+ \underbrace{\|(2c^2\|\boldsymbol{y}\|^2 + 2c)(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_2)\|}_{(3)} + \underbrace{\|(c^2\|\boldsymbol{y}\|^2 + c^3\|\boldsymbol{y}\|^4)(\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1 - \|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\|}_{(4)} \\ &+ \underbrace{\|2c^3\|\boldsymbol{y}\|^2(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\|}_{(5)} + \underbrace{\|(c + c^2\|\boldsymbol{y}\|^2)(\|\boldsymbol{x}_2\|^2 - \|\boldsymbol{x}_1\|^2)\boldsymbol{y}\|}_{(6)} \\ &+ \underbrace{\|2c(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle)\boldsymbol{y}\|}_{(7)} + \underbrace{\|2c^2(\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle - \|\boldsymbol{x}_1\|^2\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)\boldsymbol{y}\|}_{(8)} \end{aligned}. \tag{94}$$

Let's analyze each part of Eq. (94) separately.

Part (1):

$$\|(1+c\|\boldsymbol{y}\|^2 + 4c^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)(\boldsymbol{x}_1 - \boldsymbol{x}_2)\| \leq \left(1+c\|\boldsymbol{y}\|^2 + 4c^2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^2\right)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{95}$$

Part (2):

$$\begin{aligned} \|2c(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_2)\| &= 2c\| - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_2 + \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_1 - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \boldsymbol{x}_1 + \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \boldsymbol{x}_1\| \\ &\leq 2c\left(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| + \langle \boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{y}\rangle\|\boldsymbol{x}_1\|\right) \\ &\leq 2c(\|\boldsymbol{x}_2\|\|\boldsymbol{y}\| + \|\boldsymbol{x}_1\|\|\boldsymbol{y}\|)\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| \end{aligned} \tag{96}$$

.

Part (3):

$$\begin{aligned}\|(2c^2\|\boldsymbol{y}\|^2+2c)(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle \boldsymbol{x}_1-\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle \boldsymbol{x}_2)\| &= |(2c^2\|\boldsymbol{y}\|^2+2c)|\|\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle \boldsymbol{x}_1-\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle \boldsymbol{x}_2+\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle \boldsymbol{x}_2-\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle \boldsymbol{x}_2\| \\ &\le |(2c^2\|\boldsymbol{y}\|^2+2c)|\left(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1-\boldsymbol{x}_2\|+\langle \boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|\right) \\ &\le |(4c^2\|\boldsymbol{y}\|^2+4c)|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|\|\boldsymbol{x}_1-\boldsymbol{x}_2\| \\ &= (4c^2\|\boldsymbol{y}\|^2+4c)\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|\|\boldsymbol{x}_1-\boldsymbol{x}_2\|.\end{aligned} \tag{97}$$

Part (4):

$$\begin{aligned}\|(c^2\|\boldsymbol{y}\|^2+c^3\|\boldsymbol{y}\|^4)(\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1-\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\| &= |(c^2\|\boldsymbol{y}\|^2-c^3\|\boldsymbol{y}\|^4)|\|\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1-\|\boldsymbol{x}_2\|^2\boldsymbol{x}_2+\|\boldsymbol{x}_2\|^2\boldsymbol{x}_2-\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2\| \\ &\le |(c^2\|\boldsymbol{y}\|^2+c^3\|\boldsymbol{y}\|^4)|\left(\|\boldsymbol{x}_2\|^2\|\boldsymbol{x}_1-\boldsymbol{x}_2\|+(\|\boldsymbol{x}_2\|^2-\|\boldsymbol{x}_1\|^2)\|\boldsymbol{x}_2\|\right) \\ &\le |(c^2\|\boldsymbol{y}\|^2+c^3\|\boldsymbol{y}\|^4)|\left(\|\boldsymbol{x}_2\|^2+\|\boldsymbol{x}_2\|\|\boldsymbol{x}_2+\boldsymbol{x}_1\|\right)\|\boldsymbol{x}_2-\boldsymbol{x}_1\| \\ &= (c^2\|\boldsymbol{y}\|^2+c^3\|\boldsymbol{y}\|^4)(\|2\boldsymbol{x}_2\|^2+\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|)\|\boldsymbol{x}_2+\boldsymbol{x}_1\|.\end{aligned} \tag{98}$$

Part (5):

$$\begin{aligned}&\|2c^3\|\boldsymbol{y}\|^2(\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1-\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\| \\ &= 2c^3\|\boldsymbol{y}\|^2\|-\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2+\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_1-\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_1+\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1\| \\ &\le 2c^3\|\boldsymbol{y}\|^2\left(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2-\boldsymbol{x}_1\|\right)+2c^3\|\boldsymbol{y}\|^2\|\boldsymbol{x}_1\|\left(\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2-\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\right) \\ &\le 2c^3\|\boldsymbol{y}\|^2\left(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2-\boldsymbol{x}_1\|\right)+2c^3\|\boldsymbol{y}\|^2\|\boldsymbol{x}_1\|\left(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle(\|\boldsymbol{x}_2\|^2-\|\boldsymbol{x}_1\|^2)+\langle \boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\right) \\ &\le 2c^3\|\boldsymbol{y}\|^3\left(\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|+\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2+\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{x}_1+\boldsymbol{x}_2\|\right)\|\boldsymbol{x}_1-\boldsymbol{x}_2\| \\ &= 2c^3\left(2\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|+2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2\right)\|\boldsymbol{y}\|^3\|\boldsymbol{x}_1-\boldsymbol{x}_2\|.\end{aligned} \tag{99}$$

Part (6):

$$\|(c+c^2\|\boldsymbol{y}\|^2)(\|\boldsymbol{x}_2\|^2-\|\boldsymbol{x}_1\|^2)\boldsymbol{y}\| \le (c\|\boldsymbol{y}\|+c^2\|\boldsymbol{y}\|^3)(\|\boldsymbol{x}_1\|+\|\boldsymbol{x}_2\|)\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{100}$$

Part (7):

$$\|2c(\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle-\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle)\boldsymbol{y}\| \le 2c\|\boldsymbol{y}\|^2\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{101}$$

Part (8):

$$\begin{aligned}&\|2c^2(\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle-\|\boldsymbol{x}_1\|^2\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle)\boldsymbol{y}\| \\ &= 2c^2\|\boldsymbol{y}\|\|\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle-\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle+\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle-\|\boldsymbol{x}_1\|^2\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\| \\ &\le 2c^2\|\boldsymbol{y}\|\left(\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{y}\rangle+(\|\boldsymbol{x}_2\|^2-\|\boldsymbol{x}_1\|^2)\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle\right) \\ &\le 2c^2\|\boldsymbol{y}\|^2\left(\|\boldsymbol{x}_2\|^2+\|\boldsymbol{x}_2\|\|\boldsymbol{x}_2+\boldsymbol{x}_1\|\right)\|\boldsymbol{x}_2-\boldsymbol{x}_1\| \\ &= 2c^2\|\boldsymbol{y}\|^2(2\|\boldsymbol{x}_2\|^2+\|\boldsymbol{x}_2\|\|\boldsymbol{x}_1\|)\|\boldsymbol{x}_2-\boldsymbol{x}_1\|.\end{aligned} \tag{102}$$

Overall, $\|N(\boldsymbol{x}_1)D(\boldsymbol{x}_2)-N(\boldsymbol{x}_2)D(\boldsymbol{x}_1)\|$ satisfies that

$$\begin{aligned}&\|N(\boldsymbol{x}_1)D(\boldsymbol{x}_2)-N(\boldsymbol{x}_2)D(\boldsymbol{x}_1)\| \\ &\le \big(1+3c\|\boldsymbol{y}\|^2+3c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|+7c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|+7c^2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^2+c^2\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|^3+5c^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3 \\ &\quad +6c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2+4c^3\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3+4c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^3+2c^3\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^4+c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^4\big)\|\boldsymbol{x}_2-\boldsymbol{x}_1\|.\end{aligned} \tag{103}$$

The denominator holds that

$$|D(\boldsymbol{x}_1)D(\boldsymbol{x}_2)| = |1+2c\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1+2c\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|. \tag{104}$$

In this way, the $\|\boldsymbol{x}_1\oplus_c\boldsymbol{y}-\boldsymbol{x}_2\oplus_c\boldsymbol{y}\|$ satisfies that

$$\begin{aligned}&\|\boldsymbol{x}_1\oplus_c\boldsymbol{y}-\boldsymbol{x}_2\oplus_c\boldsymbol{y}\| \le \|\boldsymbol{x}_2-\boldsymbol{x}_1\|\times \\ &\Bigg(\frac{1+3c\|\boldsymbol{y}\|^2+3c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|+7c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|+7c^2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^2+c^2\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|^3+5c^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3}{|1+2c\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1+2c\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|} \\ &+\frac{6c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2+4c^3\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3+4c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^3+2c^3\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^4+c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^4}{|1+2c\langle \boldsymbol{x}_1,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1+2c\langle \boldsymbol{x}_2,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|}\Bigg).\end{aligned} \tag{105}$$

By denoting that

$$L_{\oplus_x} \triangleq \frac{1+3c\|\boldsymbol{y}\|^2+3c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|+7c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|+7c^2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^2+c^2\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|^3+5c^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3}{|1+2c\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1+2c\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|} + \frac{6c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2+4c^3\|\boldsymbol{x}_1\|^2\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^3+4c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^3+2c^3\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^4+c^3\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|^4}{|1+2c\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2||1+2c\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle+c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2|}, \tag{106}$$

we have proved that

$$\|\boldsymbol{x}_1\oplus_c\boldsymbol{y}-\boldsymbol{x}_2\oplus_c\boldsymbol{y}\| \leq L_{\oplus_x}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{107}$$

Note that $\lim_{c\to 0}\boldsymbol{x}\oplus_c\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{y}$. Therefore, it holds that

$$\lim_{c\to 0}\|\boldsymbol{x}_1\oplus_c\boldsymbol{y}-\boldsymbol{x}_2\oplus_c\boldsymbol{y}\| = \|(\boldsymbol{x}_1+\boldsymbol{y})-(\boldsymbol{x}_2+\boldsymbol{y})\| = 1\times\|\boldsymbol{x}_1-\boldsymbol{x}_2\|, \tag{108}$$

When $c\to 0$, we can observe that

$$\lim_{c\to 0}L_{\oplus_x}=1, \tag{109}$$

which is consistent with the Eq. (71), further validating the correctness of our proof.

We again utilize

$$\|\boldsymbol{x}_1,\boldsymbol{x}_2\| \leq \frac{1}{c}, \|\boldsymbol{y}\| \leq \frac{1}{c}, \tag{110}$$

and thus

$$\|\boldsymbol{x}_1+\boldsymbol{x}_2\| \leq \frac{2}{c}. \tag{111}$$

Then, as to each part, we can obtain the simplified results.

Part (1):

$$\|(1-c\|\boldsymbol{y}\|^2+4c^2\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle)(\boldsymbol{x}_1-\boldsymbol{x}_2)\| \leq (1+\frac{1}{c}+\frac{4}{c^2})\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{112}$$

Part (2):

$$\|2c(\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle\boldsymbol{x}_2-\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle\boldsymbol{x}_1)\| \leq \frac{4}{c}\|\boldsymbol{x}_2-\boldsymbol{x}_1\|. \tag{113}$$

Part (3):

$$\|(2c^2\|\boldsymbol{y}\|^2-2c)(\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle\boldsymbol{x}_1-\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle\boldsymbol{x}_2)\| \leq \frac{4}{c^2}(1+c)\|\boldsymbol{x}_1-\boldsymbol{x}_2\| \quad . \tag{114}$$

Part (4):

$$\|(c^2\|\boldsymbol{y}\|^2-c^3\|\boldsymbol{y}\|^4)(\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1-\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\| \leq \frac{3}{c^3}(1+c)\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{115}$$

Part (5):

$$\|2c^3\|\boldsymbol{y}\|^2(\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle\|\boldsymbol{x}_1\|^2\boldsymbol{x}_2-\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1)\| \leq \frac{8}{c^3}\|\boldsymbol{x}_1-\boldsymbol{x}_2\| \quad . \tag{116}$$

Part (6):

$$\|(c+c^2\|\boldsymbol{y}\|^2)(\|\boldsymbol{x}_1\|^2-\|\boldsymbol{x}_2\|^2)\boldsymbol{y}\| \leq \frac{2}{c^2}(c+1)\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{117}$$

Part (7):

$$\|2c(\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle-\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle)\boldsymbol{y}\| \leq \frac{2}{c}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|. \tag{118}$$

Part (8):

$$\|2c^2(\|\boldsymbol{x}_2\|^2\langle\boldsymbol{x}_1,\boldsymbol{y}\rangle-\|\boldsymbol{x}_1\|^2\langle\boldsymbol{x}_2,\boldsymbol{y}\rangle)\boldsymbol{y}\| \leq \frac{6}{c^2}\|\boldsymbol{x}_2-\boldsymbol{x}_1\| \quad . \tag{119}$$

Overall,

$$\|N_1D_2-N_2D_1\| \leq (1+\frac{13}{c}+\frac{19}{c^2}+\frac{11}{c^3})\|\boldsymbol{x}_2-\boldsymbol{x}_1\|. \tag{120}$$

As to $\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}$, its denominator holds that

$$\begin{aligned} |\boldsymbol{D}_1| &= |(1 + 2c\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2)| = 1 + 2c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\|\cos(\theta_1) + c^2\|\boldsymbol{x}_1\|^2\|\boldsymbol{y}\|^2 \\ &= (c\|\boldsymbol{x}_1\|\|\boldsymbol{y}\| + \cos(\theta_1))^2 + (1 - \cos(\theta_1)^2) \geq (1 - \cos(\theta_1)^2), \end{aligned} \tag{121}$$

where θ_1 denotes the angle between $\boldsymbol{x}_1$ and $\boldsymbol{y}$. Similarly, it also holds that

$$\begin{aligned} |\boldsymbol{D}_2| &= |(1 + 2c\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle + c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2)| = 1 + 2c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\|\cos(\theta_2) + c^2\|\boldsymbol{x}_2\|^2\|\boldsymbol{y}\|^2 \\ &= (c\|\boldsymbol{x}_2\|\|\boldsymbol{y}\| + \cos(\theta_2))^2 + (1 - \cos(\theta_2)^2) \geq (1 - \cos(\theta_2)^2), \end{aligned} \tag{122}$$

where θ_2 denotes the angle between $\boldsymbol{x}_2$ and $\boldsymbol{y}$. Suppose that the angle θ_1, θ_2 satisfies that $\cos(\theta_1), \cos(\theta_2) \geq \cos(\tilde{\theta})$, the denominator of $\|\boldsymbol{x} \oplus_{c_1} \boldsymbol{y} - \boldsymbol{x} \oplus_{c_2} \boldsymbol{y}\|$ satisfies that

$$|\boldsymbol{D}_1\boldsymbol{D}_2| \geq (1 - \cos(\tilde{\theta})^2)^2. \tag{123}$$

Combing the numerator and denominator, we can derive that

$$\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq \frac{(1 + \frac{13}{c} + \frac{19}{c^2} + \frac{11}{c^3})}{(1 - \cos(\tilde{\theta})^2)^2}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|. \tag{124}$$

In this way, the Lipschitz continuous constant $L_{\oplus_x}$ can be further modeled as

$$L_{\oplus_x} = \frac{(1 + \frac{13}{c} + \frac{19}{c^2} + \frac{11}{c^3})}{(1 - \cos(\tilde{\theta})^2)^2}, \tag{125}$$

and have proved that $\|\boldsymbol{x}_1 \oplus_c \boldsymbol{x} - \boldsymbol{x}_2 \oplus_c \boldsymbol{x}\| \leq L_{\oplus_x}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$.

■

Corollary 2: Supposing that $\|\boldsymbol{x}\| \leq \frac{1}{c}, \|\boldsymbol{y}\| \leq \frac{1}{\sqrt{c}}$, the Möbius addition $\boldsymbol{x} \oplus_c \boldsymbol{y}$ is Lipschitz continuous with respect to the left input $\boldsymbol{x}$, *i.e.*,

$$\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq L_{\oplus_x}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \tag{126}$$

and $L_{\oplus_x}$ is re-formulated as

$$L_{\oplus_x} = \frac{(4 + \frac{16}{c} + \frac{16}{\sqrt{c}} + \frac{8}{c^{\frac{3}{2}}})}{(1 - \cos(\tilde{\theta})^2)^2}, \tag{127}$$

Proof: We utilize the assumption, *i.e.*,

$$\|\boldsymbol{x}_1, \boldsymbol{x}_2\| \leq \frac{1}{c}, \|\boldsymbol{y}\| \leq \frac{1}{\sqrt{c}}, \tag{128}$$

and thus

$$\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \frac{2}{c}. \tag{129}$$

We separately analyze the each part of numerator with respect to $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$.

Part (1):

$$\|(1 - c\|\boldsymbol{y}\|^2 + 4c^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)(\boldsymbol{x}_1 - \boldsymbol{x}_2)\| \leq (2 + \frac{4}{c})\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{130}$$

Part (2):

$$\|2c(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle\boldsymbol{x}_2 - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\boldsymbol{x}_1)\| \leq \frac{4}{\sqrt{c}}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|. \tag{131}$$

Part (3):

$$\|(2c^2\|\boldsymbol{y}\|^2 - 2c)(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle\boldsymbol{x}_1 - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle\boldsymbol{x}_2)\| \leq \frac{8}{\sqrt{c}}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{132}$$

Part (4):

$$\|(c^2\|\boldsymbol{y}\|^2 - c^3\|\boldsymbol{y}\|^4)(\|\boldsymbol{x}_2\|^2\boldsymbol{x}_1 - \|\boldsymbol{x}_1\|^2\boldsymbol{x}_2)\| \leq \frac{6}{c}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{133}$$

Part (5):

$$\|2c^3\|\boldsymbol{y}\|^2(\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \|\boldsymbol{x}_1\|^2\boldsymbol{x}_2 - \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle \|\boldsymbol{x}_2\|^2\boldsymbol{x}_1)\| \leq \frac{8}{c^{\frac{3}{2}}}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{134}$$

Part (6):

$$\|(c + c^2\|\boldsymbol{y}\|^2)(\|\boldsymbol{x}_1\|^2 - \|\boldsymbol{x}_2\|^2)\boldsymbol{y}\| \leq \frac{4}{\sqrt{c}}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{135}$$

Part (7):

$$\|2c(\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle - \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)\boldsymbol{y}\| \leq 2\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{136}$$

Part (8):

$$\|2c^2(\|\boldsymbol{x}_2\|^2\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle - \|\boldsymbol{x}_1\|^2\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle)\boldsymbol{y}\| \leq \frac{6}{c}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|. \tag{137}$$

Overall,

$$\|N_1D_2 - N_2D_1\| \leq (4 + \frac{16}{c} + \frac{16}{\sqrt{c}} + \frac{8}{c^{\frac{3}{2}}})\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|. \tag{138}$$

The denominator $\boldsymbol{x}_1 \oplus_c \boldsymbol{y}_2 - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}$ satisfies that

$$|\boldsymbol{D}_1\boldsymbol{D}_2| \geq (1 - \cos(\tilde{\theta})^2)^2. \tag{139}$$

Therefore, $\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\|$ satisfies that

$$\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq \frac{(4 + \frac{16}{c} + \frac{16}{\sqrt{c}} + \frac{8}{c^{\frac{3}{2}}})}{(1 - \cos(\tilde{\theta})^2)^2}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|. \tag{140}$$

By denoting

$$L_{\oplus_x} = \frac{(4 + \frac{16}{c} + \frac{16}{\sqrt{c}} + \frac{8}{c^{\frac{3}{2}}})}{(1 - \cos(\tilde{\theta})^2)^2}, \tag{141}$$

we have proved $\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq L_{\oplus_x}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$.

■

II. Lipschitz Continuous Analysis of Exponential Map

In this section, we present the Lipschitz continuous analysis of exponential map $\mathrm{expm}^c_{\boldsymbol{y}}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ in Theorems 4 and 5.

Theorem 4: The exponential map $\mathrm{expm}^c_{\boldsymbol{y}}(\boldsymbol{x})$ is Lipschitz continuous with the constant L_{expm_y} with respect to $\boldsymbol{y}$. Mathematically, for any $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, we have that

$$\|\mathrm{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq L_{\mathrm{expm}_y}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \tag{142}$$

where L_{expm_y} is computed as

$$L_{\mathrm{expm}_y} \triangleq L_{\oplus_y} c\|\boldsymbol{x}\|(\|\boldsymbol{y}_2\| + \|\boldsymbol{y}_1\|) + L_{\oplus_x}. \tag{143}$$

Moreover, L_{expm_y} satisfies that

$$\lim_{c \to 0} L_{\mathrm{expm}_y} = 1. \tag{144}$$

By utilizing the hyperbolic constraint of $\boldsymbol{y}$, L_{expm_y} can be further modeled as

$$L_{\mathrm{expm}_y} \triangleq 2L_{\oplus_y}\|\boldsymbol{x}\| + L_{\oplus_x} \tag{145}$$

Proof: Recall that the exponential map is computed as

$$\mathrm{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{y} \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|} \right), \tag{146}$$

where $\lambda^c_{\boldsymbol{y}} = 2/(1+c\|\boldsymbol{y}\|^2)$ is the conformal factor. Thus we have that

$$\|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| = \Big\|\boldsymbol{y}_1 \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}_1}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|}\right) - \boldsymbol{y}_2 \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}_2}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|}\right).\Big\| \tag{147}$$

For the scale of simplicity, we denote

$$\begin{aligned} \boldsymbol{v}_1 &= \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}_1}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|}\right), \\ \boldsymbol{v}_2 &= \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}_2}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|}\right) \end{aligned} \tag{148}$$

In this way, Eq. (171) can be simplified as

$$\|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| = \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_1 - \boldsymbol{y}_2 \oplus_c \boldsymbol{v}_2,\| \tag{149}$$

and further can be transformed as

$$\begin{aligned} \|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| &= \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_1 - \boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2 + \boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2 - \boldsymbol{y}_2 \oplus_c \boldsymbol{v}_2,\| \\ &\le \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_1 - \boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2\| + \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2 - \boldsymbol{y}_2 \oplus_c \boldsymbol{v}_2.\| \end{aligned} \tag{150}$$

In the above theorems, we have proved the addition $\oplus_c$ also satisfies the Lipschitz continuity. Because $\boldsymbol{y}$ is on hyperbolic spaces, and the property of $\boldsymbol{v}_1, \boldsymbol{v}_2$, we have that

$$\|\boldsymbol{y}\| \le \frac{1}{c}, \|\boldsymbol{v}_1\| \le \frac{1}{\sqrt{c}}, \|\boldsymbol{v}_2\| \le \frac{1}{\sqrt{c}}. \tag{151}$$

From Corollary 1 and Corollary 2, it holds that

$$\begin{aligned} \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_1 - \boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2\| &\le L_{\oplus_y}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \\ \|\boldsymbol{y}_1 \oplus_c \boldsymbol{v}_2 - \boldsymbol{y}_2 \oplus_c \boldsymbol{v}_2\| &\le L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \end{aligned} \tag{152}$$

By substituting Eq. (175) into Eq. (171), we can obtain that

$$\|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \le L_{\oplus_y}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| + L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \tag{153}$$

Next, we provide the upper bound of $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|$. $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are computed as

$$\boldsymbol{v}_1 = \frac{\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}\|}{1+c\|\boldsymbol{y}_1\|^2}\right)}{\sqrt{c}}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \boldsymbol{v}_2 = \frac{\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}\|}{1+c\|\boldsymbol{y}_2\|^2}\right)}{\sqrt{c}}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}. \tag{154}$$

For the scale of simplicity, we denote

$$\begin{aligned} \boldsymbol{v}_1 &= \frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \quad \text{where} \quad \boldsymbol{a}_1 = \frac{\sqrt{c}\|\boldsymbol{x}\|}{1+c\|\boldsymbol{y}_1\|^2} \\ \boldsymbol{v}_2 &= \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \quad \text{where} \quad \boldsymbol{a}_2 = \frac{\sqrt{c}\|\boldsymbol{x}\|}{1+c\|\boldsymbol{y}_2\|^2}. \end{aligned} \tag{155}$$

Therefore,

$$\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| = \left\|\frac{\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right\| = \frac{|\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)|}{\sqrt{c}} \tag{156}$$

Because $\tanh(\cdot)$ is $1-$Lipschitz continuous, *i.e.*, for any $x, y \in \mathbb{R}$, it holds that

$$|\tanh(x) - \tanh(y)| \le |x - y|. \tag{157}$$

Thus, $|\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)|$ satisfies that

$$\begin{aligned} |\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)| \le |\boldsymbol{a}_1 - \boldsymbol{a}_2| &= \sqrt{c}\|\boldsymbol{x}\|\left|\frac{1}{1+c\|\boldsymbol{y}_1\|^2} - \frac{1}{1+c\|\boldsymbol{y}_2\|^2}\right| \\ &= \frac{c\sqrt{c}\|\boldsymbol{x}\|\|\boldsymbol{y}_2\|^2 - \|\boldsymbol{y}_1\|^2|}{(1+c\|\boldsymbol{y}_1\|^2)(1+c\|\boldsymbol{y}_2\|^2)}. \end{aligned} \tag{158}$$

Because $\boldsymbol{y}_1, \boldsymbol{y}_2$ locate on the hyperbolic spaces, $\boldsymbol{y}_1, \boldsymbol{y}_2 \leq \frac{1}{c}$. Then, $(1+c\|\boldsymbol{y}_1\|^2)(1+c\|\boldsymbol{y}_2\|^2) \geq 1$. In this way, $|\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)|$ further satisfies that

$$\begin{aligned} |\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)| &\leq c\sqrt{c}\|\boldsymbol{x}\| \|\boldsymbol{y}_2\|^2 - \|\boldsymbol{y}_1\|^2| = c\sqrt{c}\|\boldsymbol{x}\| |\langle \boldsymbol{y}_2 + \boldsymbol{y}_1, \boldsymbol{y}_2 - \boldsymbol{y}_1 \rangle| \\ &\leq c\sqrt{c}\|\boldsymbol{x}\|(\|\boldsymbol{y}_2\| + \|\boldsymbol{y}_1\|)\|\boldsymbol{y}_2 - \boldsymbol{y}_1\|. \end{aligned} \tag{159}$$

By substituting Eq. (159) into Eq. (181), it holds that

$$\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \leq c\|\boldsymbol{x}\|(\|\boldsymbol{y}_2\| + \|\boldsymbol{y}_1\|)\|\boldsymbol{y}_2 - \boldsymbol{y}_1\|. \tag{160}$$

In this way, we can derive that

$$\begin{aligned} \|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| &\leq L_{\oplus_y}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| + L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \\ &\leq \left(L_{\oplus_y} c\|\boldsymbol{x}\|(\|\boldsymbol{y}_2\| + \|\boldsymbol{y}_1\|) + L_{\oplus_x}\right)\|\boldsymbol{y}_2 - \boldsymbol{y}_1\|. \end{aligned} \tag{161}$$

By denoting

$$L_{\text{expm}_y} \triangleq L_{\oplus_y} c\|\boldsymbol{x}\|(\|\boldsymbol{y}_2\| + \|\boldsymbol{y}_1\|) + L_{\oplus_x}, \tag{162}$$

we have proved that $\|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq L_{\text{expm}_y}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|$. Moreover, we observe that

$$\lim_{c\to 0} L_{\text{expm}_y} = 1, \tag{163}$$

since

$$\lim_{c\to 0} L_{\text{expm}_y} = \lim_{c\to 0} L_{\oplus_x} = 1. \tag{164}$$

We further utilize the constraint of $\boldsymbol{y}$, *i.e.*, $\|\boldsymbol{y}\| \leq \frac{1}{c}$, we can further derive that

$$\|\text{expm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \text{expm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq \left(2L_{\oplus_y}\|\boldsymbol{x}\| + L_{\oplus_x}\right)\|\boldsymbol{y}_2 - \boldsymbol{y}_1\|. \tag{165}$$

By denoting $L_{\text{expm}_y} = 2L_{\oplus_y}\|\boldsymbol{x}\| + L_{\oplus_x}$, we have proved the Lipschitz continuity of the exponential map.

■

Theorem 5: The exponential map $\text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x})$ is Lipschitz continuous with the constant L_{expm_x} with respect to $\boldsymbol{x}$. Mathematically, for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we have that

$$\|\text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}_1) - \text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}_2)\| \leq L_{\text{expm}_x}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \tag{166}$$

where L_{expm_x} is computed as

$$L_{\text{expm}_x} \triangleq L_{\oplus_y}\left(\frac{1}{1+c\|\boldsymbol{y}\|^2} + \frac{2\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}\|\boldsymbol{x}_2\|}\right), \tag{167}$$

and L_{expm_x} can satisfy that

$$\lim_{c\to 0} L_{\text{expm}_x} = 1. \tag{168}$$

From the hyperbolic constraint of $\boldsymbol{y}$, we can further model L_{expm_x} as

$$L_{\text{expm}_x} = \left(L_{\oplus_y} + 2L_{\oplus_y}\frac{\tanh(\|\boldsymbol{x}_2\|)}{\sqrt{c}\|\boldsymbol{x}_2\|}\right). \tag{169}$$

Proof: Recall that the exponential map is computed as

$$\text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{y} \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}}\|\boldsymbol{x}\|}{2})\frac{\boldsymbol{x}}{\sqrt{|c|}\|\boldsymbol{x}\|}\right), \tag{170}$$

where $\lambda^c_{\boldsymbol{y}} = 2/(1+c\|\boldsymbol{y}\|^2)$ is the conformal factor. Thus we have that

$$\|\text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}_1) - \text{expm}^c_{\boldsymbol{y}}(\boldsymbol{x}_2)\| = \left\|\boldsymbol{y} \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}}\|\boldsymbol{x}_1\|}{2})\frac{\boldsymbol{x}_1}{\sqrt{|c|}\|\boldsymbol{x}_1\|}\right) - \boldsymbol{y} \oplus_c \left(\tanh(\sqrt{|c|}\frac{\lambda^c_{\boldsymbol{y}}\|\boldsymbol{x}_2\|}{2})\frac{\boldsymbol{x}_2}{\sqrt{|c|}\|\boldsymbol{x}_2\|}\right).\right\| \tag{171}$$

For the scale of simplicity, we denote

$$\begin{aligned}\boldsymbol{v}_1 &= \left(\tanh(\sqrt{|c|}\frac{\lambda_{\boldsymbol{y}}^c\|\boldsymbol{x}_1\|}{2})\frac{\boldsymbol{x}_1}{\sqrt{|c|}\|\boldsymbol{x}_1\|}\right),\\ \boldsymbol{v}_2 &= \left(\tanh(\sqrt{|c|}\frac{\lambda_{\boldsymbol{y}}^c\|\boldsymbol{x}_2\|}{2})\frac{\boldsymbol{x}_2}{\sqrt{|c|}\|\boldsymbol{x}_2\|}\right)\end{aligned} \tag{172}$$

In this way, Eq. (171) can be simplified as

$$\|\text{expm}_{\boldsymbol{y}_1}^c(\boldsymbol{x}) - \text{expm}_{\boldsymbol{y}_2}^c(\boldsymbol{x})\| = \|\boldsymbol{y}\oplus_c \boldsymbol{v}_1 - \boldsymbol{y}\oplus_c \boldsymbol{v}_2\|. \tag{173}$$

In the above theorem and corollary, we have proved the addition $\oplus_c$ also satisfies the Lipschitz continuity. Because $\boldsymbol{y}$ is on hyperbolic spaces, and the property of $\boldsymbol{v}_1, \boldsymbol{v}_2$, we have that

$$\|\boldsymbol{y}\| \leq \frac{1}{c}, \|\boldsymbol{v}_1\| \leq \frac{1}{\sqrt{c}}, \|\boldsymbol{v}_2\| \leq \frac{1}{\sqrt{c}}. \tag{174}$$

From Corollary 1, it holds that

$$\|\text{expm}_{\boldsymbol{y}_1}^c(\boldsymbol{x}) - \text{expm}_{\boldsymbol{y}_2}^c(\boldsymbol{x})\| = \|\boldsymbol{y}_1\oplus_c \boldsymbol{v}_1 - \boldsymbol{y}_1\oplus_c \boldsymbol{v}_2\| \leq L_{\oplus_y}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|. \tag{175}$$

Next, we provide the upper bound of $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|$. $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are computed as

$$\boldsymbol{v}_1 = \frac{\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}_1\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}, \boldsymbol{v}_2 = \frac{\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}}\frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}. \tag{176}$$

For the scale of simplicity, we denote

$$\begin{aligned}\boldsymbol{v}_1 &= \frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}, \quad \text{where} \quad \boldsymbol{a}_1 = \frac{\sqrt{c}\|\boldsymbol{x}_1\|}{1+c\|\boldsymbol{y}\|^2}\\ \boldsymbol{v}_2 &= \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}, \quad \text{where} \quad \boldsymbol{a}_2 = \frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}.\end{aligned} \tag{177}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| &= \left\|\frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} - \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}\right\|\\ &\leq \left\|\frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} - \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}\right\| + \left\|\frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} - \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}\right\|\\ &\leq \left\|\frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}} - \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\right\| + \left|\frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\right| \left\|\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} - \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}\right\|,\end{aligned} \tag{178}$$

where

$$\begin{aligned}\left\|\frac{\tanh(\boldsymbol{a}_1)}{\sqrt{c}} - \frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\right\| &\leq \frac{1}{\sqrt{c}}\Big|\tanh(\boldsymbol{a}_1) - \tanh(\boldsymbol{a}_2)\Big| \leq \frac{1}{\sqrt{c}}\Big|\boldsymbol{a}_1 - \boldsymbol{a}_2\Big|\\ &= \Big|\frac{\sqrt{c}\|\boldsymbol{x}_1\|}{1+c\|\boldsymbol{y}\|^2} - \frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\Big|\frac{1}{\sqrt{c}}\\ &\leq \frac{1}{1+c\|\boldsymbol{y}\|^2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,\end{aligned} \tag{179}$$

and

$$\Big|\frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\Big|\left\|\frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} - \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|}\right\| \leq \Big|\frac{\tanh(\boldsymbol{a}_2)}{\sqrt{c}}\Big|\frac{2\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|}{\|\boldsymbol{x}_2\|}. \tag{180}$$

Then, we can obtain that

$$\begin{aligned}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| &\leq \frac{1}{1+c\|\boldsymbol{y}\|^2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + \frac{2\tanh(\boldsymbol{a}_2)}{\sqrt{c}\|\boldsymbol{x}_2\|}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|\\ &= \frac{1}{1+c\|\boldsymbol{y}\|^2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + \frac{2\tanh\left(\frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}\|\boldsymbol{x}_2\|}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|.\end{aligned} \tag{181}$$

In this way, we can derive that

$$\|\text{expm}_{\boldsymbol{y}}^c(\boldsymbol{x}_1) - \text{expm}_{\boldsymbol{y}}^c(\boldsymbol{x}_2)\| \leq L_{\oplus_y}\left(\frac{1}{1+c\|\boldsymbol{y}\|^2}\|\boldsymbol{x}_1-\boldsymbol{x}_2\| + \frac{2\text{tanh}\left(\frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}\|\boldsymbol{x}_2\|}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|\right). \tag{182}$$

By denoting

$$L_{\text{expm}_x} \triangleq L_{\oplus_y}\left(\frac{1}{1+c\|\boldsymbol{y}\|^2} + \frac{2\text{tanh}\left(\frac{\sqrt{c}\|\boldsymbol{x}_2\|}{1+c\|\boldsymbol{y}\|^2}\right)}{\sqrt{c}\|\boldsymbol{x}_2\|}\right), \tag{183}$$

we have proved that

$$\|\text{expm}_{\boldsymbol{y}}^c(\boldsymbol{x}_1) - \text{expm}_{\boldsymbol{y}}^c(\boldsymbol{x}_2)\| \leq L_{\text{expm}_x}\|\boldsymbol{x}_2-\boldsymbol{x}_1\|. \tag{184}$$

Moreover, we observe that

$$\lim_{c\to 0} L_{\text{expm}_x} = 1, \tag{185}$$

by limiting the increasing scale of $\|\boldsymbol{y}\|$ is less than $\frac{1}{\sqrt{c}}$, and the increasing scale of $\|\boldsymbol{x}_2\|$ is more than $\frac{1}{\sqrt{c}}$. Mathematically, $\|\boldsymbol{y}\|$ and $\|\boldsymbol{x}_2\|$ satisfy that

$$\|\boldsymbol{y}\| = o(\frac{1}{\sqrt{c}}), \|\boldsymbol{x}_2\| = \omega o(\frac{1}{\sqrt{c}}), \tag{186}$$

where $\omega > 1$. From the hyperbolic constraint of $\boldsymbol{y}$, *i.e.*, $\|\boldsymbol{y}\| \leq \frac{1}{c}$, we can further model L_{expm_x} as

$$L_{\text{expm}_x} = \left(L_{\oplus_y} + 2L_{\oplus_y}\frac{\tanh(\|\boldsymbol{x}_2\|)}{\sqrt{c}\|\boldsymbol{x}_2\|}\right). \tag{187}$$

■

III. Lipschitz Continuous Analysis of Logarithmic Map

In Theorem 6, we present the Lipschitz continuous analysis of logarithmic map $\text{logm}_{\boldsymbol{y}}^c(\boldsymbol{x})$ with respect to $\boldsymbol{y}$.

Theorem 6: The logarithmic map $\text{logm}_{\boldsymbol{y}}^c(\boldsymbol{x})$ is Lipschitz continuous with the constant L_{logm_y} with respect to $\boldsymbol{y}$. Mathematically, for any $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, we have that

$$\|\text{logm}_{\boldsymbol{y}_1}^c(\boldsymbol{x}) - \text{logm}_{\boldsymbol{y}_2}^c(\boldsymbol{x})\| \leq L_{\text{logm}_y}\|\boldsymbol{y}_1-\boldsymbol{y}_2\|, \tag{188}$$

where L_{logm_y} is computed as

$$L_{\text{logm}_y} \triangleq L_{\oplus_x}(1+c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\|)^2 + \frac{\sqrt{c}}{8}\|\boldsymbol{y}_1+\boldsymbol{y}_2\| + \sqrt{|c|}L_{\oplus_x}, \tag{189}$$

and L_{logm_y} satisfies that

$$\lim_{c\to 0} L_{\text{logm}_y} = 1. \tag{190}$$

From the hyperbolic constraint of $\boldsymbol{x}, \boldsymbol{y}$, L_{logm_y} can be modeled as

$$L_{\text{logm}_y} = (1+\frac{1}{c})^2 L_{\oplus_x} + \frac{1}{4\sqrt{c}} + \sqrt{|c|}L_{\oplus_x}. \tag{191}$$

Proof: Recall that the Logarithmic map $\text{logm}_{\boldsymbol{y}}^c(\boldsymbol{x})$ is computed as

$$\text{logm}_{\boldsymbol{y}}^c(\boldsymbol{x}) = \frac{2}{\sqrt{|c|}\lambda_{\boldsymbol{y}}^c}\text{tanh}^{-1}(\sqrt{|c|}\|-\boldsymbol{y}\oplus_c\boldsymbol{x}\|)\frac{-\boldsymbol{y}\oplus_c\boldsymbol{x}}{\|-\boldsymbol{y}\oplus_c\boldsymbol{x}\|}. \tag{192}$$

Then, $\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})$ is computed as

$$\begin{aligned}\|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| = \Bigg\| &\frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_1}}\tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|)\frac{-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}}{\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} \\ &-\frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_2}}\tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|)\frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|}\Bigg\|.\end{aligned} \tag{193}$$

For simplicity, we denote

$$\begin{aligned}Ap_1 &\triangleq \frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_1}}, Bp_1 \triangleq \tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|), Cp_1 \triangleq \frac{-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}}{\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|}, \\ Ap_2 &\triangleq \frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_2}}, Bp_2 \triangleq \tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|), Cp_2 \triangleq \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|}.\end{aligned} \tag{194}$$

In this way, $\|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\|$ is given by

$$\begin{aligned}\|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| &= \|Ap_1Bp_1Cp_1 - Ap_2Bp_2Cp_2\| \\ &= \|Ap_1Bp_1Cp_1 - Ap_1Bp_2Cp_2 + Ap_1Bp_2Cp_2 - Ap_2Bp_2Cp_2\| \\ &= \|Ap_1(Bp_1Cp_1 - Bp_2Cp_2) + (Ap_1 - Ap_2)Bp_2Cp_2\| \\ &\leq \|Ap_1(Bp_1Cp_1 - Bp_2Cp_2)\| + \|(Ap_1 - Ap_2)Bp_2Cp_2\| \\ &\leq \|Ap_1\|\|(Bp_1Cp_1 - Bp_2Cp_2)\| + \|(Ap_1 - Ap_2)\|\|Bp_2Cp_2\| \\ &\leq \|Ap_1\|\left(\|Bp_1\|\|(Cp_1 - Cp_2)\| + \|(Bp_1 - Bp_2)\|\|Cp_2\|\right) + \|(Ap_1 - Ap_2)\|\|Bp_2Cp_2\|.\end{aligned} \tag{195}$$

Then, we separately analyze the each part in Eq. (222).

(1) As to $\|(Ap_1 - Ap_2)\|$, it holds that

$$\begin{aligned}\|(Ap_1 - Ap_2)\| &= \left\|\frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_1}} - \frac{2}{\sqrt{|c|}\lambda^c_{\boldsymbol{y}_2}}\right\| = \frac{2}{\sqrt{c}}\left\|\frac{1+c\|\boldsymbol{y}_1\|^2}{2} - \frac{1+c\|\boldsymbol{y}_2\|^2}{2}\right\| \\ &= \frac{\sqrt{c}}{2}(\|\boldsymbol{y}_1 + \boldsymbol{y}_2\|\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|)\end{aligned} \tag{196}$$

(2) As to $\|(Bp_1 - Bp_2)\|$, it holds that

$$\begin{aligned}\|(Bp_1 - Bp_2)\| &= \|\tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|) - \tanh^{-1}(\sqrt{|c|}\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|)\| \\ &\leq \frac{1}{\omega}\left|\sqrt{|c|}\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\| - \sqrt{|c|}\|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|\right| \\ &= \frac{\sqrt{|c|}}{\omega}\left|\|-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\| - \|-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|\right| \\ &\leq \frac{\sqrt{|c|}}{\omega}\left|(-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}) - (-\boldsymbol{y}_2 \oplus_c \boldsymbol{x})\right|.\end{aligned} \tag{197}$$

The first inequality holds because we present an additional assumption, *i.e.*, the $\tanh^{-1}$ is $\frac{1}{\omega}-$Lipschitz continuous. The last inequality holds because the norm satisfies $1-$Lipschitz continuous. From Theorem 3, we can obtain that

$$\|\boldsymbol{x}_1 \oplus_c \boldsymbol{y} - \boldsymbol{x}_2 \oplus_c \boldsymbol{y}\| \leq L_{\oplus_x}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|. \tag{198}$$

Therefore, $\|(Bp_1 - Bp_2)\|$ satisfies that

$$\|(Bp_1 - Bp_2)\| \leq \frac{\sqrt{|c|}}{\omega}L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \tag{199}$$

(3) As to $\|(Cp_1 - Cp_2)\|$, it holds that

$$\begin{aligned}
\|(Cp_1 - Cp_2)\| &= \left\| \frac{-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} - \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|} \right\| \\
&= \left\| \frac{-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} - \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} + \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} - \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|} \right\| \\
&\leq \left\| \frac{-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} - \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} \right\| + \left\| \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} - \frac{-\boldsymbol{y}_2 \oplus_c \boldsymbol{x}}{\| -\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\|} \right\| \\
&\leq \frac{\|(-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}) - (-\boldsymbol{y}_2 \oplus_c \boldsymbol{x})\|}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} + \frac{\|\| -\boldsymbol{y}_2 \oplus_c \boldsymbol{x}\| - \| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|\|}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|} \\
&\leq \frac{2\,\|(-\boldsymbol{y}_1 \oplus_c \boldsymbol{x}) - (-\boldsymbol{y}_2 \oplus_c \boldsymbol{x})\|}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|}.
\end{aligned} \tag{200}$$

From Theorem 3, we can derive that

$$\|(Cp_1 - Cp_2)\| \leq \frac{2L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|}{\| -\boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|}. \tag{201}$$

Then, we further analyze the lower bound of $\| -\boldsymbol{y} \oplus_c \boldsymbol{x}\|$. Its numerator satisfies that

$$\| -(1 - 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y} + (1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\| \geq \max\{|a_1 - b_2|, |a_2 - b_1|\}, \tag{202}$$

and

$$\| -(1 - 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y} + (1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\| \leq a_2 + b_2, \tag{203}$$

where a_1, a_2 are lower bound and upper bound of $\|(1-2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y}\|$, and b_1, b_2 are lower bound and upper bound of $\|(1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\|$, respectively. In terms of $\|(1 - 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y}\|$, we can observe that

$$\|(1 - 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y}\| \leq (1 + 2c\|\boldsymbol{y}\|\|\boldsymbol{x}\| + c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}\|, \tag{204}$$

and

$$\|(1 - 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c\|\boldsymbol{x}\|^2)\boldsymbol{y}\| \geq \max\{(1 + c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}\|+, |2c\|\boldsymbol{y}\|\|\boldsymbol{x}\| - 1|\|\boldsymbol{y}\|\}. \tag{205}$$

For $\|(1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\|$, we can derive that

$$\|(1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\| \leq \|(1 + c\|\boldsymbol{y}\|^2)\|\boldsymbol{x}\|, \tag{206}$$

and

$$\|(1 - c\|\boldsymbol{y}\|^2)\boldsymbol{x}\| \geq |1 - c\|\boldsymbol{y}\|^2|\|\boldsymbol{x}\|. \tag{207}$$

Then, a_1, a_2, b_1, b_2 are computed as

$$\begin{aligned}
&a_1 = \max\{(1 + c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}\|+, |2c\|\boldsymbol{y}\|\|\boldsymbol{x}\| - 1|\|\boldsymbol{y}\|\}, a_2 = (1 + 2c\|\boldsymbol{y}\|\|\boldsymbol{x}\| + c\|\boldsymbol{x}\|^2)\|\boldsymbol{y}\| \\
&b_1 = |1 - c\|\boldsymbol{y}\|^2|\|\boldsymbol{x}\|, b_2 = \|(1 + c\|\boldsymbol{y}\|^2)\|\boldsymbol{x}\|.
\end{aligned} \tag{208}$$

The denominator of $\| -\boldsymbol{y} \oplus_c \boldsymbol{x}\|$ satisfies that

$$\|1 + 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c^2\|\boldsymbol{y}\|^2\|\boldsymbol{x}\|^2\| \leq (1 + c\|\boldsymbol{y}\|\|\boldsymbol{x}\|)^2, \tag{209}$$

and

$$\|1 + 2c\langle \boldsymbol{y}, \boldsymbol{x}\rangle + c^2\|\boldsymbol{y}\|^2\|\boldsymbol{x}\|^2\| \geq 1. \tag{210}$$

Therefore,

$$\| -\boldsymbol{y} \oplus_c \boldsymbol{x}\| \geq \frac{\max\{|a_1 - b_2|, |a_2 - b_1|\}}{(1 + c\|\boldsymbol{y}\|\|\boldsymbol{x}\|)^2}. \tag{211}$$

Then, it holds that

$$\|(Cp_1 - Cp_2)\| \leq 2L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|\frac{(1 + c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\|)^2}{\max\{|a_1 - b_2|, |a_2 - b_1|\}}, \tag{212}$$

where

$$\max\{|a_1 - b_2|, |a_2 - b_1|\} \geq 1. \tag{213}$$

Then Eq. (212) can be transformed as

$$\|(Cp_1 - Cp_2)\| \leq 2L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|(1 + c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\|)^2. \tag{214}$$

(4) As to Ap_1, Ap_2, it holds that

$$\|Ap_1\|, \|Ap_2\| \leq 2. \tag{215}$$

(5) As to Bp_1, Bp_2, the domain of definition of function $\tanh^{-1}$ is $(-1, 1)$, one has to constraint the input of $\tanh^{-1}$. In implementation of hyperbolic neural networks and hyperbolic image embedding, researchers set the 'torch.clamp(-1+1e-5, 1-1e-5)' to avoid overstep the bounds of definition. Here, for analysis, we assume that $\tanh^{-1}(\sqrt{c}\| - \boldsymbol{y}_1 \oplus_c \boldsymbol{x}\|) \leq \frac{1}{4}$. In this way, we can obtain that

$$\|Bp_1\|, \|Bp_2\| \leq \frac{1}{4}. \tag{216}$$

(6) As to Cp_1, Cp_2, it holds that

$$\|Cp_1\|, \|Cp_2\| = 1. \tag{217}$$

For simplicity, we assume that $\omega = 2$, and then $\|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\|$ satisfies that

$$\begin{aligned} \|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq{}& L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|(1 + c\|\boldsymbol{y}\|\|\boldsymbol{x}\|)^2 \\ &+ \frac{\sqrt{c}}{8}(\|\boldsymbol{y}_1 + \boldsymbol{y}_2\|\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|) + \sqrt{|c|}L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \end{aligned} \tag{218}$$

By denoting

$$L_{\mathrm{logm}_y} \triangleq L_{\oplus_x}(1 + c\|\boldsymbol{x}\|\|\boldsymbol{y}_1\|)^2 + \frac{\sqrt{c}}{8}\|\boldsymbol{y}_1 + \boldsymbol{y}_2\| + \sqrt{|c|}L_{\oplus_x}, \tag{219}$$

we have proved that

$$\|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq L_{\mathrm{logm}_y}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \tag{220}$$

Moreover, we observe that

$$\lim_{c\to 0} L_{\mathrm{logm}_y} = \lim_{c\to 0} L_{\oplus_x} = 1. \tag{221}$$

By utilizing the the hyperbolic property of $\boldsymbol{y}, \boldsymbol{x}$, we can derive that

$$\begin{aligned} \|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| \leq{}& L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|(1 + \frac{1}{c})^2 \\ &+ \frac{1}{4\sqrt{c}}(\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|) + \sqrt{|c|}L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \end{aligned} \tag{222}$$

Therefore, it can be derived as

$$\begin{aligned} \|\mathrm{logm}^c_{\boldsymbol{y}_1}(\boldsymbol{x}) - \mathrm{logm}^c_{\boldsymbol{y}_2}(\boldsymbol{x})\| &\leq L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|(1 + \frac{1}{c})^2 + \frac{1}{4\sqrt{c}}(\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|) + \sqrt{|c|}L_{\oplus_x}\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|, \\ &= \left((1 + \frac{1}{c})^2 L_{\oplus_x} + \frac{1}{4\sqrt{c}} + \sqrt{|c|}L_{\oplus_x}\right)\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|. \end{aligned} \tag{223}$$

In this way, L_{logm_y} can be modeled as

$$L_{\mathrm{logm}_y} = (1 + \frac{1}{c})^2 L_{\oplus_x} + \frac{1}{4\sqrt{c}} + \sqrt{|c|}L_{\oplus_x}. \tag{224}$$

■