Math Visualizations
Billiard Trajectories

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Contents

1 Billiard Trajectories Background

1.1 Introduction

Suppose we have a squared billiard board and at each corner there is a pocket with shape a quarter of a circle (with radius ϵ). If we shoot a tiny ball/particle (just a point and does not have size) from one fixed corner at a certain angle θ and suppose the ball do not lose any speed until it exit from a pocket, then there are some natural questions one can ask: Is the particle guaranteed to exit? From which pocket will it exit? How long does it take to exit? To study it's trajectory, one could consider allowing the particle pass through the board boundaries and we study the continuing trajectory on the flipped board by the boundary its passing through. Repeating this process, one would see all the pockets lying on the lattice points in $\mathbb{R}^2$ and the trajectory will become a line segment from the origin to one lattice points.

1.2 Main Result

If we should the particle from the point "s" in following graph with angle $\theta = \pi/2$:

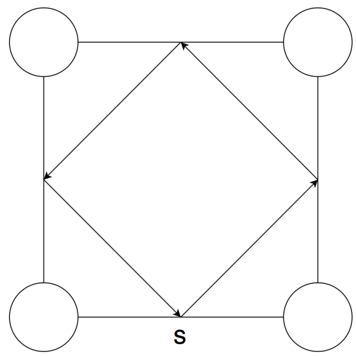

Figure 1: shoot from s with $\theta = \pi/2$

then clearly the particle will never exit.

How ever, when we following the setting of the problem and shoot from a corner:

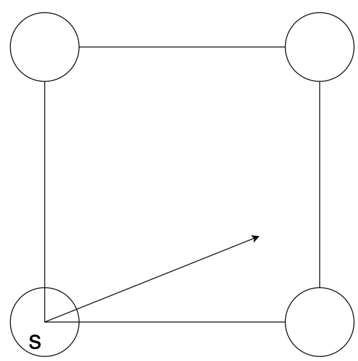

Figure 2: shoot from s with some $\theta \in [0, \frac{\pi}{2}]$

then, in fact, it is guaranteed the ball will fall into one of the other three pockets (distinct from the one at the corner where it was shot from) after a finite time. Moreover, **each hole will have probability $\frac{1}{3}$ to capture the particle**. The length of the trajectory is worth studying and Prof.F. P. Boca, Prof.R. N. Gologan, and Prof. A. Zaharescu have papers containing results about the average length of the trajectories (over all possible angle $\theta \in [0, \frac{\pi}{2}]$). Below is the main result:

Theorem 1. If we define the length of the trajectory as $l(\theta, \epsilon)$ (as a function of θ and ϵ) and r is the power of the length ($r \in \mathbb{N}$), then

$$\lim_{\epsilon \to 0} \epsilon^r \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (l(\theta, \epsilon))^r d\theta = C_r$$

where $C_r \in \mathbb{R}$ and $C_r > 0$.

To prove this, one need to use Farey fractions and Kloosterman sums. For more details, please refer to paper in the reference.

2 Computer Visualizations

2.1 Motivation to Visualizations

- **Covering Space $\mathbb{R}^2$**

  As mentioned in the introduction, we could consider the covering space $\mathbb{R}^2$ to unfold the trajectory and make it a line segment. In this case, the four pockets at corners will become circular disks lying on lattice points on $\mathbb{R}^2$.

- **Torus**

  Because the reflection has order 2, on $\mathbb{R}^2$, we will have the 2×2 block repeating itself (please see the visualization on the webpage, where four boundaries of the board is colored by four different colors.). Then, we could glue the opposite sides of this block to get a torus. By the property of reflections, studying the trajectory on the squared board is the same as studying the trajectory on the corresponding torus. Moreover, the trajectory on the torus will be a continuous curve and it does not "bounce back" at all. After gluing the block, we will have four pockets on the torus, one corresponding to the center of the block and the other corresponding to the corners of the block, and the other two corresponding to the middles of the four sides of the block.

2.2 Visualizations

- **Start**

  One could use to input the value of θ (between 0 and $\pi/2$) or use the mouse cursor to select the direction to shoot. If the mouse mode is chosen, a red dashed line will appear and it moves with the mouse cursor, in order to let the user see their chosen direction.

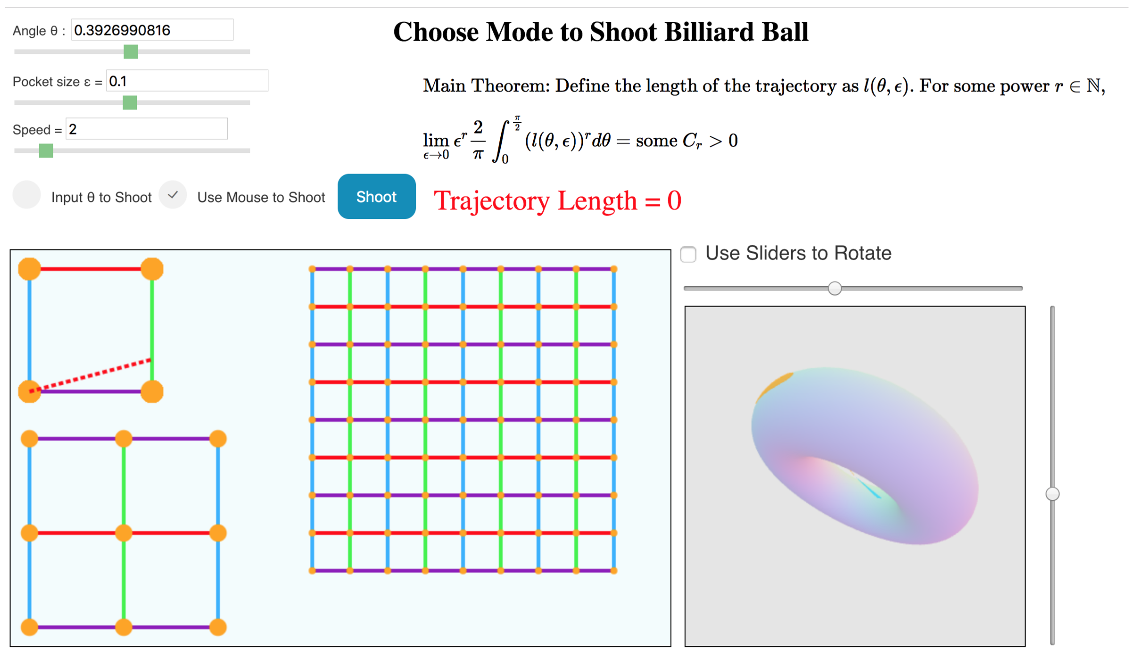

Figure 3: start

- **Change Variables Values**

  One can also use the sliders to set the value for θ, change pocket size ϵ, and set the drawing speed of the trajectory. Moreover, the torus is rotating constantly and one could use the cursor to control the rotation of the torus (the torus will also face the cursor). On the other hand, checking "Use Sliders to Rotate" will stop the torus from facing the mouse cursor. Then, the user can drag the two sliders (one vertical and one horizontal) to adjust the rotation of the torus.

- **Examples**

  Clearly, a large ϵ, makes it easier for the particle to exit and a small ϵ makes it harder to exit.

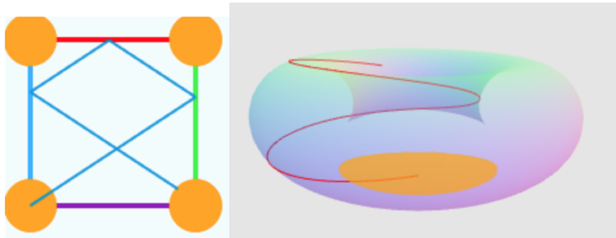

Figure 4: Example 1: large ϵ

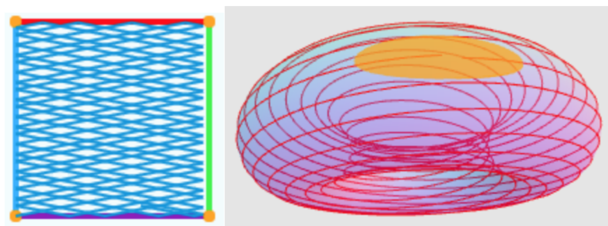

Figure 5: Example 2: small ϵ

- **Analyze Trajectory Length**

  The trajectory length is displayed in red for the user to study the length. The value after *"Trajectory Length ="* is changing dynamically while the trajectories are being drawn.

  As we mentioned before, if we color the boundaries, then one could see the 2×2 copy of the board is repeaing in $\mathbb{R}^2$ and thus if we just look at the trajectory in this 2×2 block, then one could only see parallel lines (when the particle meets the boundary on one side, it reappear on the boundary on the other side, just like the video game "Snake").

  On the torus, we have four pockets, two in orange and two in blue. The outer orange one is the one where the particle is shot from. It could return back to either of the two outer pockets, or the particle could also return to the location behind one of the pockets (the inner orange one and the inner blue one). The latter happens exactly when the x or y projection of the trajectory on $\mathbb{R}^2$ equals to an odd number.

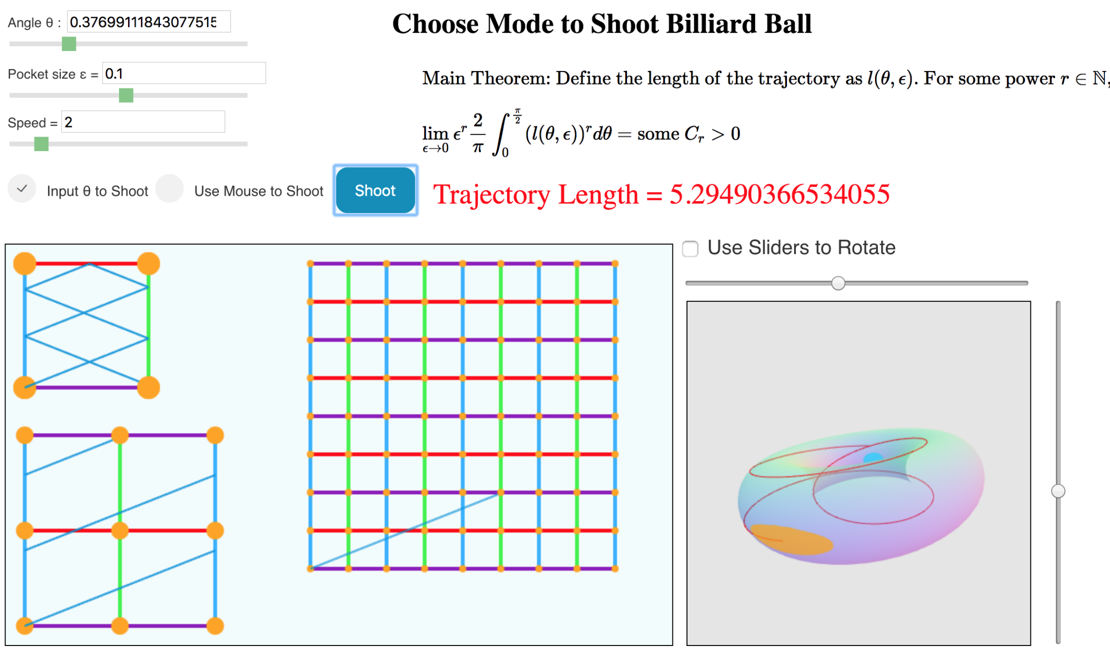

Figure 6: Example 2: small ϵ

2.3 Tools

- Used HTML Canvas and JavaScript to implement the graphics.
- Used the JavaScript library *Three.js* to draw the torus, the pockets on the torus, and the curve on the torus.
- Use JavaScript library “MathJax” to display the formula.

2.4 Future Directions

- Output the average trajectory length $\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(l(\theta,\epsilon))^r d\theta$ with a given r.
- Visualize the geodesics of a torus and compare it with the billiard trajectory on the torus.
- Cut the torus and study the geodesics of a torus on the corresponding square.

3 Acknowledgements

I would like to thank Prof. George Francis for helping me form ideas about possible interesting visualizations of this problem. I appreciate that he always mentions many interesting problems or theories related to this visualization project, from which I have learned a lot. Moreover, I would also like to thank the two wonderful teaching assistants of this research: Karthik Vasu and Daniel Carmody. Particularly, I appreciate Daniel discusssing the underlying mathematics with me and helping me address debugs in my code.

4 References

- Boca, Florin Gologan, Radu Zaharescu, Alexandru. (2001). *The average length of a trajectory in a certain billiard in a flat two-torus*. J. Math.. 9.