Math 496 Final Report

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1 Overview

Our research topic was decided due to our shared interests in the problems related to lattice points and convex geometry. We use Python programming to obtain results and then analyze them mathematically. Below are the two topics we mainly worked on.

1.1 Properties of Numbers of Interior Lattice Points and Boundary Lattice Points on a Lattice Triangle of the Circles in the Gauss Circle Problem

We look at the lattice triangles composed by connecting lattice points on the circles in the Gauss Circle Problem, where a circle is centered at the origin with integer radius. This topic is inspired by the study of Pick's Theorem[1] as well as the study by P.R. Scott[2] about the relationship between the number of interior and boundary lattice points on convex lattice polygons.

For a triangle $\triangle$, define B as the number of its boundary lattice points, C as the number of its interior lattice points. Let A denote area. Define:

$$f(\triangle) = B - 2 \cdot C.$$

Since Pick's Theorem gives the method to relate B and C with A for any lattice polygon and P.R. Scott showed that there exists an upper bound of f for lattice polygons with at least one interior lattice point, we want to study how such numerical relationships behave when we put the vertices of lattice polygons on Gauss' Circles. Specifically, we are interested in the triangles circumscribed by Gauss' Circles and how their f-values change with their area.

1.2 An Area Inequality in Planar, Convex, Compact Geometry

The motivation of this section is by Poh Wah Awyong [3] who proves an inequality between circumradius, inradius, and width. It is done by finding the maximum case.
Indeed, some inequalities regarding the area fail because the area can be small while the sides and out-radius are increasing. The following is an area inequality that is bounded on both sides:

$$2Rr < A < 4Rr. \tag{1}$$

This result is first proved by M. Hend and G.A. Tsintsifas [4]; however, I have found a different approach that is inspired by Awyong.

2 Properties of Numbers of Interior Lattice Points and Boundary Lattice Points on a Lattice Triangle in the Gauss Circle Problem

2.1 Notations and Conventions

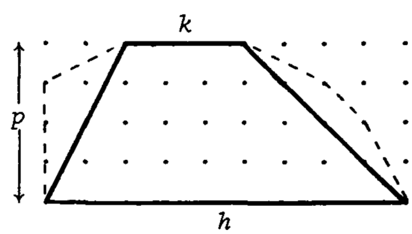
1. Define T_1 as the triangles composed by connect three points of the set $\{(0,R),\ (0,-R),\ (R,0),\ (-R,0)\}$. WLOG, in the discussion below, when we mention T_1, we always use the triangle of $(0,R),\ (R,0),\ (-R,0)$ as the example. Similarly, during the discussion of other kinds of triangles, we only use the triangle above *xaxis* as the example, but the discussion is obviously applicable for other triangles which can be obtained by rotating the example triangle along the origin.

2. Every time we mention the circular lattice triangles of a circle, we mean the triangles obtained by connecting three lattice points on the circle.

3. When we say "maximize the f of a circle", we mean "to get the max value of all possible f of circular lattice triangles of the circle."

4. Since in the Gauss Circle Problem, the radius is an integer, a circle always has at least four lattice points on them: $(R,0),\ (-R,0),\ (0,R),\ (0,-R)$. Because we use them frequently in the following discussion, we call them the "basic four" lattice points of a circle.

5. We would like to mention one idea we borrowed from P.R. Scott[2] and used in our proof:
For all convex polygon Π containing at least one interior lattice point, let Π meet two supporting lines $y=0$ and $y=p$ in segments of length h and k (possibly zero) respectively (see figure_Scott).



(figure_Scott)

We have:

$$B(\Pi) \leq h+k+2p.$$

2.2 Algorithms Involved in Problem Solving

To count the number of boundary lattice points on a lattice triangle, we used the following formula we came up with:

Given two points $(x_1,\ y_1)$, $(x_2,\ y_2)$, the number of lattice points on the line segment connecting these two points, including endpoints, is $gcd(x_2-x_1,\ y_2-y_1)+1$. (gcd: greatest common divisor)
Thus:

$$\begin{aligned}B &= gcd(x_2-x_1,\ y_2-y_1)+1+gcd(x_2-x_3,\ y_2-y_3)+1+gcd(x_3-x_1,\ y_3-y_1)+1-3\\ &= gcd(x_2-x_1,\ y_2-y_1)+gcd(x_2-x_3,\ y_2-y_3)+gcd(x_3-x_1,\ y_3-y_1).\end{aligned}$$

To count the number of interior lattice points inside a lattice triangle, we use the following algorithm (which used the same idea of counting interior points in a circle by Hilbert and Cohn-Vossen[5]: Algorithm Logic: Our algorithm first finds the maximum and minimum of x coordinates of the triangle's vertices. Then for all integral x_i in that range, the algorithm finds the two points(x_i, y_{i-high}), (x_i, y_{i-low}) which lie on the edges of that triangle and have x coordinates x_i, as long as these two points do not lie on the same edge, the number of interior points between these two points are $I_i = \lceil y_{i-high} \rceil - \lfloor y_{i-low} \rfloor - 1$. Otherwise, $I_i = 0$.

$$I = \sum I_i.$$

2.3 Lemmas, Theorems and Their Proofs

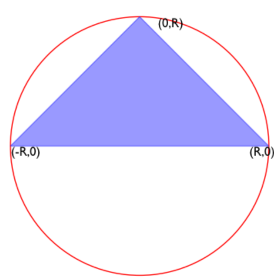
Pick's Theorem states that: For a polygon whose vertices are lattice points, $A = \frac{B}{2} + C - 1$. Thus f can be alternatively expressed as:

$$\begin{aligned} f &= B - 2C \\ &= 2A - 4C + 2 \\ &= 2B - 2A - 2. \end{aligned}$$

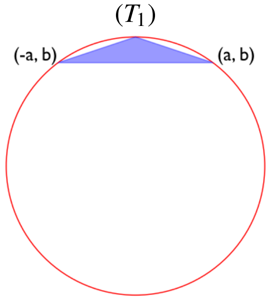
Lemma 1. *In the Gauss Circle Problem (the circle is centered at the origin with integer radius R), if there exist a,b $\in \mathbb{N}$ such that $R^2 = a^2 + b^2$, then the lattice triangle T_1 with vertices $(0,R)$, $(R,0)$, $(-R,0)$ cannot be the smallest circular lattice triangle that maximizes $f = B - 2C$ (i.e. if we let $f(T_1)$ be the f of $T1$, then there must exist another smaller circular lattice triangle with $f(T_2)$ such that $f(T_2) > f(T_1)$).*

Proof. Since $R^2 = a^2 + b^2$, the circle has at least one lattice point (a,b) in the first quadrant. If $a \neq b$, then we have two points (a,b) and (b,a). WLOG, assume $b \geq a$. If $b = a$, then $a = b = \frac{r}{\sqrt{2}}$ and $(a,\ b)$ cannot be a lattice point. Thus, we have $b \geq a + 1$ (i.e. $b > a$).

Claim: The lattice triangle T_2 with vertices $(0,R)$, (a,b), $(-a,b)$ has $f(T_2) > f(T_1)$, where $f(T_1)$ is of lattice triangle $T_1$with vertices $(0,R)$, $(R,0)$, $(-R,0)$.



(T_1)



$$(T_2)$$

For T_2,

$$\begin{aligned} f(T_2) &= 2B_2 - 2A_2 - 1 \\ f(T_2) &= 2B_2 - 2a(R-b) - 1. \end{aligned}$$

For T_1,

$$\begin{aligned} B_1 &= 4R \\ C_1 &= (R-1)^2 \\ f(T_2) &= B_1 - 2C_1 \\ &= -2R^2 + 8R - 2 \\ &= -2(R^2 - 4R + 1). \end{aligned}$$

$$\begin{aligned} f(T_2) - f(T_1) &= 2B_2 - 2aR + 2ab - 1 + 2(R^2 - 4R + 1) \\ &= 2B_2 - 2a\sqrt{a^2+b^2} + 2ab - 1 + 2(a^2 + b^2 - 4\sqrt{a^2+b^2} + 1) \\ &= 2B_2 + 2a^2 + 2b^2 + 2ab - (2a+8)\sqrt{a^2+b^2} + 1. \end{aligned}$$

Since the base of T_2 has length $2a$, so it has at least $2a+1$ lattice points on the base. Plus the one point at the vertex above, we can be sure that T_2 has at least $2a+2$ lattice points. Thus, $B_2 \geq 2a+2$.

$$f(T_2) - f(T_1) \geq 4a + 4 + 2a^2 + 2b^2 + 2ab - 2(a+4)\sqrt{a^2+b^2} + 1.$$

Let the right side of the inequality be H, then:

$$\begin{aligned} \frac{\partial H}{\partial a} &= 4a + 4 + 2b - 2[\sqrt{a^2+b^2} + (a+4)\cdot\frac{2a}{2\sqrt{a^2+b^2}}] \\ &= 4a + 4 + 2b - \frac{2(a^2+b^2) + (2a^2+8a)}{\sqrt{a^2+b^2}} \\ &= 4a + 4 + 2b - \frac{4a^2 + 8a + 2b^2}{\sqrt{a^2+b^2}} \\ &= \frac{4a\sqrt{a^2+b^2} + 4\sqrt{a^2+b^2} + 2b\sqrt{a^2+b^2} - 4a^2 - 8a - 2b^2}{\sqrt{a^2+b^2}}. \end{aligned}$$

We have,

$$\begin{aligned} 4a\sqrt{a^2+b^2} &> 4a\sqrt{a^2+a^2} = 4\sqrt{2}a^2 && (b > a). \\ 4\sqrt{a^2+b^2} &> 4\sqrt{2}a && (b > a). \\ 2b\sqrt{a^2+b^2} &> 2b \cdot b = 2b^2 && (\sqrt{b^2+a^2} > b). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial H}{\partial a} &= \frac{(4a\sqrt{a^2+b^2} - 4a^2) + (4\sqrt{a^2+b^2} - 8a) + (2b\sqrt{a^2+b^2} - 2b^2)}{\sqrt{a^2+b^2}} \\ &> \frac{4(\sqrt{2}-1)a^2 + 4(\sqrt{2}-2)a + 0}{\sqrt{a^2+b^2}}. \end{aligned}$$

Since,

$$4(\sqrt{2}-1)a^2+4(\sqrt{2}-2)a = 4a[(\sqrt{2}-1)a+\sqrt{2}-2],$$

the roots are $a_1 = 0,\ a_2 = \frac{2-\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}$.
Since the smallest R such that R^2 can be written as a^2+b^2 $(a,b \in \mathbb{N})$ happens when $R = 5, (a = 3,\ b = 4)$ we have $a \geq 3$ and $b \geq 4$.
Then

$$4a[(\sqrt{2}-1)a+(\sqrt{2}-2)] > 0$$
$$\frac{\partial H}{\partial a} > 0.$$

For $\frac{\partial H}{\partial b}$,

$$\begin{aligned}\frac{\partial H}{\partial b} &= 4b+2a-2(a+4)\frac{2b}{2\sqrt{(a^2+b^2)}}\\ &= 2\cdot[2b+a-\frac{(a+4)b}{\sqrt{a^2+b^2}}]\\ &= 2\cdot\frac{2b\sqrt{a^2+b^2}+a\sqrt{a^2+b^2}-ab-4b}{\sqrt{a^2+b^2}}.\end{aligned}$$

Similarly as before, we have:

$$2b\sqrt{a^2+b^2} > 2b\cdot b = 2b^2$$
$$a\sqrt{a^2+b^2} > a\cdot b = ab.$$

Thus,

$$\begin{aligned}2b\sqrt{a^2+b^2}+a\sqrt{a^2+b^2}-ab-4b &> 2b^2+ab-ab-4b\\ &= 2b^2-4b\\ &= 2b(b-4).\end{aligned}$$

As $b \geq 4$,

$$\frac{\partial H}{\partial b} > 0.$$

When $a = 3,\ b = 4$,

$$H = 4\cdot 3+4+2\cdot 3^2+2\cdot 4^2+2\cdot 3\cdot 4-2\cdot 7\cdot 5+1 = 21 > 0.$$

Also, $\frac{\partial H}{\partial a} > 0$, $\frac{\partial H}{\partial b} > 0$, so we have

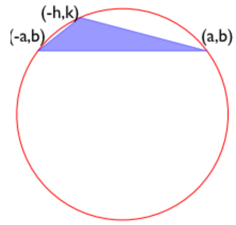
$$H > 0.$$

Thus,

$$f(T_2)-f(T_1) > 0.$$

□

Lemma 2. *In the Gauss Circle Problem, if there exist distinct $(a,\ b)$ and $(h,\ k)$ in $\mathbb{N}^2$ such that $R^2 = a^2+b^2 = h^2+k^2$. The lattice triangle T_3 with vertices $(a,b),(-a,b),(-h,k)$ with $f(T_3) = B-2C$ also satisfies $f(T_3) > f(T_1)$.*



(T_3)

Proof. With lemma 1, we have T_2 such that:

$$\begin{aligned} f(T_2) &= 2B_2 - 2A_2 - 1 \\ &\geq 2(2a+2) - 2a(R-b) - 1 \\ &> f(T_1). \end{aligned}$$

For T_3,

$$\begin{aligned} B_3 &\geq 2a+2 \\ A_3 &< A_2 \\ -2A_3 &> -2A_2 \\ f(T_3) = 2B_3 - 2A_3 - 1 > 2(2a+2) - 2a(r-b) - 1 &> f(T_1). \end{aligned}$$

□

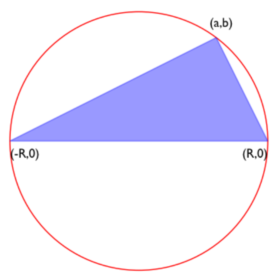
Lemma 3. *In the Gauss Circle Problem, if there exist a, b in $\mathbb{N}$ such that $R^2 = a^2 + b^2$, then any triangle with an area bigger than T_1 cannot be the smallest triangle that maximizes f.*

Proof. When the lattice triangle has some "basic four" points as vertices, the idea of the proof of the most general case (when there are no "basic four" vertices) still apply to them, but we still have to be careful about these special cases, because they might cause some minor changes on the details of our proof. Thus we divide into several cases depending on the situation of having "basic four" as vertices, and prove each of them individually.

Case 1:
If we have 3 triangle vertices from the "basic four", then the triangle is equivalent to T_1, the area is not bigger, which violates the assumption.

Case 2:
If we have 2 triangle vertices from the "basic four", we would have two different situations:

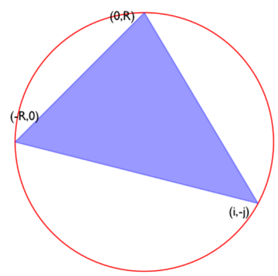
Case 2.1:
When these two vertices lie on the same axis (see T_{1-2}).



(T_{1-2})

The area cannot be bigger than that of T_1, which violates our assumption.

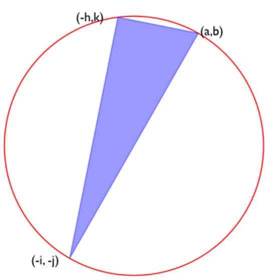
Case 2.2:
If instead, these two vertices lie on different axis (See T_{1-3}). Now the area can be larger than that of T_1.



(T_{1-3})

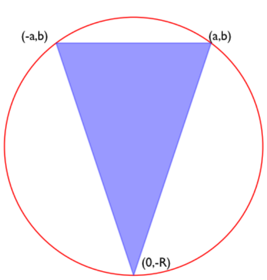
But let us discuss a more general situation first:

Case 3:
If the triangle has 0 or 1 "basic four" as vertices. That is, we can view the triangle such that the vertices are located in three different quadrants or it has one vertex on one axis and other vertices lie in two different quadrants. We can always rotate such a triangle so that it has two vertices (we will further refer them as "the upper two vertices") above the x-axis and one below it (See T_4).



(T_4)

If we have one vertex that is one of the "basic four", we rotate the graph so that the "basic four" vertex lie on $(0,-R)$ (See T_{4-1}).

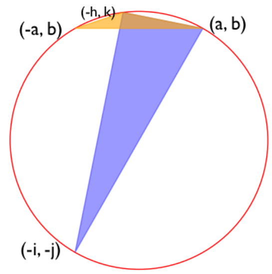


(T_{4-1})

We will prove that one can always build a smaller triangle which gives a larger $B-2C$.

Case 3.1 If the upper two vertices do not have the same y-value:
This is actually the most general case among all cases mentioned in the proof of this lemma.

Using the upper two vertices, determine the one with smaller y-value and call it V_{low}. We can build a T_3 type triangle by adding an extra vertex that is symmetric about the y-axis with V_{low} (See $T_{3,4}$).



$(T_{3,4})$

For consistency, we still denote the three vertices by (a,b), $(-a,b)$, $(-h,k)$, just as in lemma 2.

Proof. Since $f = 2B - 2A - 1$ and we have $A_4 > A_1 > A_2 > A_3$,

$$\begin{aligned} A_4 &\geq a^2 + b^2 = A_1 \\ A_3 &\leq a(R-b) \\ A_4 - A_3 &\geq a^2 + b^2 - a(R-b). \end{aligned}$$

For $B_3 - B_4$, with the use of the method described in notation 5, we have:

$$\begin{aligned} B_3 &\geq 2a + 2 \\ B_4 &\leq (h+a) + 2k + 2k \\ &\leq (h+a) + 2R + 2R \\ &\leq 2a + 4R \\ B_3 - B_4 &\geq 2a + 2 - (2a + 4R) \\ &\geq 2 - 4R. \end{aligned}$$

Thus,

$$\begin{aligned} (B_3 - B_4) + (A_4 - A_3) &\geq a^2 + b^2 - a(\sqrt{a^2+b^2} - b) + 2 - 4\sqrt{a^2+b^2} \\ &= a^2 + b^2 + ab + 2 - (a+4)\sqrt{a^2+b^2}. \end{aligned}$$

$$\begin{aligned} \frac{\partial H}{\partial a} &= 2a + b - (\sqrt{a^2+b^2} + (a+4) \cdot \frac{a}{\sqrt{a^2+b^2}}) \\ &= 2a + b - \sqrt{a^2+b^2} - \frac{a^2+4a}{\sqrt{a^2+b^2}} \\ &= \frac{2a\sqrt{a^2+b^2} + b\sqrt{a^2+b^2} - 2a^2 - b^2 - 4a}{\sqrt{a^2+b^2}} \\ &= \frac{(2a\sqrt{a^2+b^2} - 2a^2 - 4a) + (b\sqrt{a^2+b^2} - b^2)}{\sqrt{a^2+b^2}}. \end{aligned}$$

As proved before,

$$b\sqrt{a^2+b^2} - b^2 > 0.$$

For $2a\sqrt{a^2+b^2}-2a^2-4a$. Since $a \geq 3$,

$$(a-3)(a+1) \geq 0$$
$$(a-3)(a+1) \geq 0$$
$$a^2-2a-3 \geq 0$$
$$a^2 = 2a = 1-4a-4 \geq 0$$
$$(a+1)^2 \geq 4a+4.$$

Since $b \geq a+1$,

$$b^2 \geq 4a+4$$
$$a^2+b^2 \geq a^2+4a+4$$
$$\sqrt{a^2+b^2} \geq a+2$$
$$\sqrt{a^2+b^2}-a-2 \geq 0$$
$$2a\sqrt{a^2+b^2}-2a^2-4a \geq 0.$$

Thus:

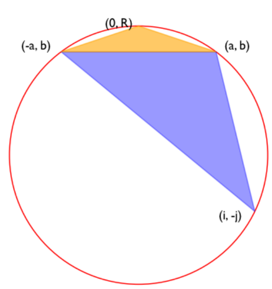
$$\frac{\partial H}{\partial a} > 0.$$

For $\frac{\partial H}{\partial b}$, name the H in this proof H_{now} and the H we referred to in the proof of lemma 1 H_{before}. Since $H_{new} = H_{before} + (2a+\frac{1}{2})$, $\frac{\partial H}{\partial b}$ does not change and $\frac{\partial H}{\partial b} > 0$. Again, plug in $(a,b) = (3,4)$, we get $H > 0$. Also, the fact that H is increasing shows $H > 0$ is always true. Therefore:

$$f(T_3) > f(T_4).$$

□

Case 3.2: If the upper two vertices have the same y-value:
We build a smaller triangle with the upper two vertices and the point $(0,R)$. That is a type 2 triangle (see T_{2-4}).



(T_{2-4})

We still denote the coordinates of these vertices in the same fashion as we did in lemma 2, only $(-h,k) = (0,R)$ this time.

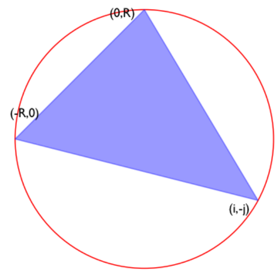
Proof. Since the area and number of boundary lattice points of so created T_2 still satisfy that:

$$A_2 \leq a(R-b)$$
$$B_3 \geq 2a+2.$$

Therefore, the proving process for Case 3-1 is still valid. And we can get $f(T_2) > f(T_4)$.

$\square$

Now let us go back to Case 2.2, T_{1-3},



(T_{1-3})

With the discussions above, here we could simply use $B_4 \leq 2R+2R = 4R < 2a+4R$, $f(T_{1-3}) < f(T_4) < f(T_3)$ to prove this case.

Combining the above three cases gives the proof of lemma 3. $\square$

Theorem 4. *In the Gauss Circle Problem, the ratio between the circular lattice triangle's area which maximizes $f = B-2C$ and the circle's area has an upper bound: $\frac{Area(\triangle)}{Area(\bigcirc)} \leq \frac{1}{\pi}$. (i.e. $Area(\triangle) \leq R^2$)*

Proof. The proof is direct by combining lemma 1, 2, and 3. We have proved that any triangle bigger than T_1 cannot maximize $f = B-2C$. Thus the area of the triangle which maximizes f cannot be greater than R^2. Hence $\frac{Area(\triangle)}{Area(\bigcirc)} \leq \frac{1}{\pi}$. $\square$

Theorem 5. *In the Gauss Circle Problem, the ratio between the circular lattice triangle's area which maximizes $f = B-2C$ and the circle's area = $\frac{1}{\pi}$ if and only if $s(R^2) = 1$, where $s(n)$ denotes the number of ways to write n as the sum of two perfect squares, ignoring sign or order.*

Note: the standard "sum of squares function" $r_k(n)$ denotes the number of ways to write integer n as the sum of k perfect squares allowing zero, signs and order. When we omit the k, by default $k=2$. Instead, since the sign and order are not important in our consideration, here we use our own defined "sum of squares function" $s(n)$, which does not allow negative sign and different order but does allow zero (it is obvious that $r(n^2) \geq 1$, $\forall n \in \mathbb{Z}$).

One way to calculate $s(n)$ for a given $n \in \mathbb{Z}$ is provided by Albert H. Beiler[6] as the following:
Factor n as:

$$n = 2^{a_0} p_1^{2a_1} ... p_r^{2a_r} q_1^{b_1} ... q_s^{b_s},$$

where p_i are primes of the form $4k+3$ and q_i are primes of the form $4k+1$. Define

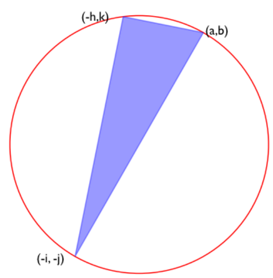
$$B \equiv (b_1+1)(b_2+1)...(b_s+1).$$

The number of representations of n as two squares ignoring orders or signs, namely $s(n)$, is given by:

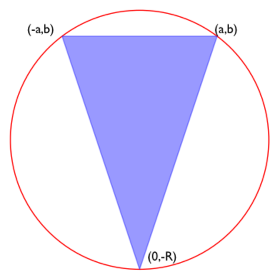
$$s(n) = \begin{cases} 0, & \text{if any } a_i \text{ is a half integer.} \\ \frac{B}{2}, & \text{if all } a_i \text{ are integers and } B \text{ is even.} \\ \frac{1}{2}(B-(-1)^{a_0}), & \text{if all } a_i \text{ are integers and } B \text{ is odd.} \end{cases}$$

Proof. $\Leftarrow$:
If $r(R^2) = 1$, then there exists only four lattice points on the circle, which is indeed the "basic four". So one can only draw one type of lattice triangle, T1, on that circle. It automatically becomes the one that gives the greatest f value and also has area R^2.

$\Rightarrow$: Given that $\frac{Area(\triangle)}{Area(\bigcirc)} = \frac{1}{\pi}$, there are 3 different cases in total.
Case 1: The lattice triangle maximizing f has 0 or 1 vertex being the "basic four".
Since the area of the triangle is R^2, such triangle must be a type 4 or a type 4-1 triangle. Otherwise, the area would not be large enough.



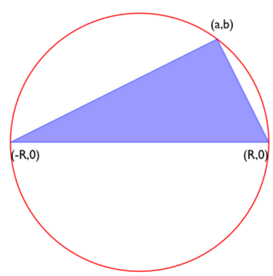
(T_4)



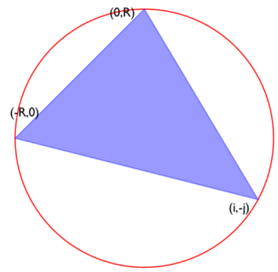
(T_{4-1})

Now assume $s(R^2) > 1$, $\exists\, a, b \in \mathbb{N}$ such that $R^2 = a^2 + b^2$.
As we proved in lemma 3, no triangle of type 4 or type 4-1 can maximize $f - B - 2C$ under such condition. Meanwhile, if $s(R^2) = 1$, a type 4 or 4-1 triangle can never be created.
Case 1 is then excluded.

Case 2: The lattice triangle maximizing f has 2 vertices being the "basic four".
This is a type 1-2 or type 1-3 triangle, and $A_{1-3or1-2} \neq A_1$ since they share the same base but have different corresponding height.



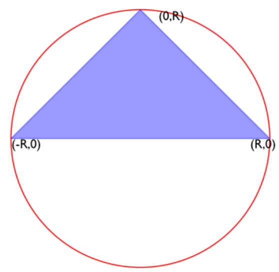
(T_{1-2})



(T_{1-3})

Case 2 is then excluded.

Case 3: The lattice triangle maximizing f has 3 vertices being the "basic four".
This is a type 1 triangle.

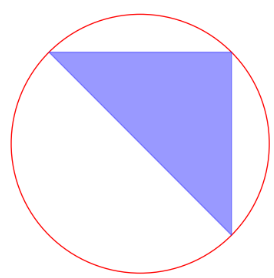


(T_1)

Assume $\exists\, a,b \in \mathbb{N}$ such that $R^2 = a^2 + b^2$, namely $s(R^2) > 1$. By Lemma 1, T_1 cannot maximize f under such assumption. We arrive at a contradiction. Thus $s(R^2) = 1$. $\square$

2.4 Discovery

If we extend the problem set: instead of requiring $R \in \mathbb{N}$, we only require $R^2 \in \mathbb{N}$, then we have found out that Theorem 1 and Theorem 2 still hold. But now the desired triangles whose areas equal to R^2 can have and only have two situations: either be a type 1 triangle with radius such that $s(R^2) = 1$. or a type 5 triangle (See T_5 below). A type 5 triangle is constructed by vertices lie on a circle centered at origin that has radius $R' = 2R, s(R^2) = 1$.



(T_5)

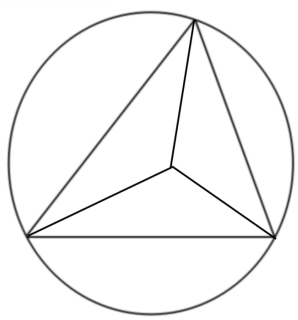
R^2 for T1	R^2 for T5
4	8=4*2
9	18=9*2
16	32=16*2
36	72=36*2
49	98=49*2
64	128=64*2
81	162=81*2
...	...

3 An Area Inequality in Planar, Convex, Compact Geometry

There are some notations to be clarified before starting to state the theorem. Let K be, a planar, compact, convex set, then $D = D(K)$ denotes the greatest distance between two points in the set; $r = r(K)$ denotes the radius of the largest circle inscribed in K; $R = R(K)$ denotes the radius of the smallest circle containing K; $w = w(K)$ denotes the smallest distance between two parallel lines which are tangent to K; $A = A(K)$ denotes the area of K; $C = C(K)$ denotes the circumference of K. We first introduce two lemmas:

Lemma 6. *Consider a planar, convex, compact, set K. Let r be the inradius of K, p be the conference of K, and A be the area of K, then:*

$$rC \leq A.$$



Proof. First, consider the case when K is a convex polygon with vertices $B_1, B_2, ..., B_n$, where n is the number of vertices. Let O be the center of the largest circle inscribed by K. If the side B_iB_{i+1} is tangent to the largest circle inscribed by K, then $B_{\triangle B_iB_{i+1}O} = r|B_iB_{i+1}|$, where $B_{\triangle A_iA_{i+1}O}$ denotes the area of the triangle $B_iB_{i+1}O$. If the side is not tangent to the largest circle inscribed by K, then $A_{\triangle B_iB_{i+1}O} = h_{\triangle B_iB_{i+1}O}|B_iB_{i+1}| \geq r|B_iB_{i+1}|$, where $h_{\triangle B_iB_{i+1}O}$ is the height of $\triangle B_iB_{i+1}O$ relating the side B_iB_{i+1}. The area of K satisfies:

$$A = \sum A_{\triangle B_iB_{i+1}O} \geq \sum r|B_iB_{i+1}| = rC, \tag{2}$$

which proves the case for planar, compact, convex polygon K. Since any convex set K is approached by a series of polygon K_n such that as $K_n \to K$ pointwisely, we have $A_n \to A, r_n \to r, R_n \to R$.
The area of A is thus proved to have a lower bound, but it remains to show the boundary for p. ☐

Theorem 7.

$$4R < C \leq 2\pi R$$

.

Proof. To prove the left part of the inequality, there are two cases to consider:
Case 1: there exist exactly two points in K that are on the smallest circle containing K. Let these two points be B and P, and U be a neighbor of B satisfying: $\forall E \in$, $|DB| < \sigma$, where σ is a positive number. Then Choose two points G, H on different sides of line BP. The quadrilateral BGPH has a smaller circumference than K because of the convexity of K, and the smallest circle containing $BGPH$ is the same as the smallest circle containing K. By the triangle inequality, $C_{BGPH} = |BG| + |GP| + |PH| + |HB| > 2|BC|$. It follows that:

$$2D = 2|BC| < C_{BGCH} < C.$$

Case 2: there exist more than three points in K that are on the smallest circle containing K. Let these three points be E, F, and G such that EFG is an acute triangle. Then by the convexity of K, $C_{EFG} < C$. In addition,

$$C_{EFG}/2R = (e+f+g)R = sin\angle E + sin\angle F + sin\angle G,$$

which satisfies

$$2 < sin\angle E + sin\angle F + sin\angle G <= 3/2\sqrt{3}.$$

Overall,

$$4R < C_{EFG} < C.$$

Therefore, we have proved that the left half of the inequality holds.
For the right half of the inequality, if there exists a vertex of K that is not on the boundary of the outer circle, let this vertex be B. Let P be a vertex on the circle that is very close to B, Q also be a vertex on the circle that is very close to B from another direction. Stretch a side of K that is split by PQ and contains B perpendicular to PQ until there is a point (not necessary B) that touches the boundary of the circle. Such operation will increase the circumference of K but keep R the same. So for any K with a point located on the boundary of K but not on the boundary of R, there is a planar, compact and convex set K' that has a larger circumference while the radius of the smallest circle that contains K' is the same as the radius of the smallest circle that contains K. The maximum value can be reached only if all points on the boundary of K are on the boundary of the smallest circle that contains K. Therefore, the boundary of K is the same as the boundary of the smallest circle that contains K. And by the convexity of K, set K is the same as the smallest circle that contains K. The maximum value of C that is achieved in this case is $2\pi R$. So it follows that

$$C \leq 2\pi r.$$

□

Combining Lemma 6 with Theorem 7, we have $A > 4rR$. For the maximum value of A/Rr, we first prove that there is an upper bound of this value.

Theorem 8. *For any convex set K, $A < 6Rr$.*

To prove this theorem, we first prove the following lemma.

Lemma 9. *(Blaschke) For any planar, compact, convex set K*

$$w \leq 3r.$$

Proof. For the largest circle inscribed in K, there are at least two points on the circumference of this circle where the boundary of K is tangent to this circle. There are two situations:
Case 1: there are exactly two points. Denote them by P and Q. Then the tangent line passing

through P has to be parallel to the tangent line passing through Q; a central dilatation centered at the intersection of the tangent line passing through P and that passing through Q will create a larger circle that is also contained in K. As a result, $w = 2r$.
Case 2: there are at least three points P, Q and S where the boundary of K is tangent to the largest circle contained in K. If two tangent lines are equivalent, similar to case 1, $w = 2r$. So we can assume that the tangent line passing through P, Q and S can form a triangle. Denote this triangle by $\triangle EFG$. Since K is contained by $\triangle EFG$, the width of K is less than the width of $\triangle EFG$. And the width of $\triangle EFG$ is just the length of the least altitude. So,

$$w \leq w(EFG) \leq \frac{A_{EFG}}{max\{|EF|,|FG|,|GE|\}} \leq \frac{3A_{EFG}}{|EF|+|FG|+|GE|} = r(EFG) = 3r.$$

Also, $D \leq 2R$, and $A \leq wD$ are trivially true. Combining the lemma and these two inequalities gives

$$A \leq wD \leq 6rR.$$

Thus, the theorem is proved. □

However, $3r = w$ requires K to be a regular triangle. On the other hand, the condition for $D = 2R$ is that: K contains a diameter of the smallest circle that contains K. Therefore, the equal sign can not be achieved. Theorem 8 does not give an accurate maximum boundary, but it shows that the upper bound exists.

Since the limit of a convex set is also a convex set, the compactness of the convex set implies the compactness of the family of convex sets. So we can assume there exists a convex set K having the maximum A/Rr value. We have the following lemma.

Lemma 10. *For any planar, convex, compact set K, $O \in K$, where O is the center of the least circle that contains K.*

Proof. If K does not contain O, because of convexity, there is a line l passing through O that does not share any point with K. In addition, since K^c is open, there exists $\sigma > 0$ such that translating line l in any direction will keep the property that line l does not intersect with K. So for some direction, after translation, O is on the different side of the line compared with K. It follows that K is contained in the dome formed by l and the smallest circle containing K. Moreover, there exists a smaller circle that contains the dome (and thus contains K). We arrive at a contradiction. Therefore, K has to contain O. □
Now, we can start proving the left part of the inequality

$$A < 4rR.$$

First, we assume that in a certain circle Cir_{out}, with a fixed largest circle Cir_{in} inscribed in, a convex set K has a maximum value of A/rR. There are two situations:

Case I: $Card(\partial K \bigcap \partial Cir_{in}) = 2$ (card denotes the cardinality). In this case, if the two lines are not parallel to each other, rotate one line centered at the center of Cir_{in}, until it is paralleled to the other. We thus obtain a convex set K'. It's easy to verify that the area of K' is larger than that of K (see Appendix). The area maximizes when two tangent lines are parallel to each other. Now the only change we can make to this convex set is translating along the direction that is perpendicular to the tangent line. It's also easy to verify that the maximum is achieved when the lengths of the two paralleled lines are the same, which means the center of Cir_{out} is in the middle of the two parallel tangent lines.

Case II: $Card(\partial K \bigcap \partial Cir_{in}) \geq 3$. In this case, choose three points from $\partial K \bigcap \partial Cir_{in}$, and denote them as E, F and G. Let $\triangle B$ be the triangle formed by three tangent lines passing through E, F and G that are tangent to Cir_{in}. Let $K' = \triangle B \bigcap Cir_{out}$ which is also convex and contains K, we have $A(K') \geq A(K)$. Now we only consider K'. If two tangent lines l_E and l_F are paralleled to each other, then let K'' be the part of the area formed by l_E and l_f. It has a larger area than K', so this will become the other condition. If no two tangent lines are parallel to each other, in the appendix, case 1,2 explain the inequality between two line segments on a tangent line during the rotation. In fact, the area change of K' in rotation l_E is $\alpha|EE_1|^2 - \alpha|EE_2|^2$ if EE_1 is the longer line segment. Since the inequality $|EE_1| > |EE_2|$ remains true, the area is increasing. The only exception is when Cir_{in} and Cir_{out} have the same center. In this case, the rotation in appendix 1 makes the area non-decreasing, but the rotation in appendix 2 makes it decreasing. The case can therefore be converted to case 1.

The last thing to consider is the size of Cir_{out}. Now fix Cir_{in}, only consider A/R. WLOG, K is bounded by the rectangle with sides $2R$ and $2r$, where the two shorter sides are tangent to Cir_{out}. $A/rR < 4$ is then achieved.

□

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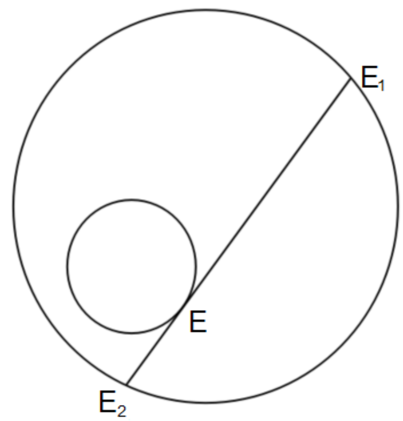
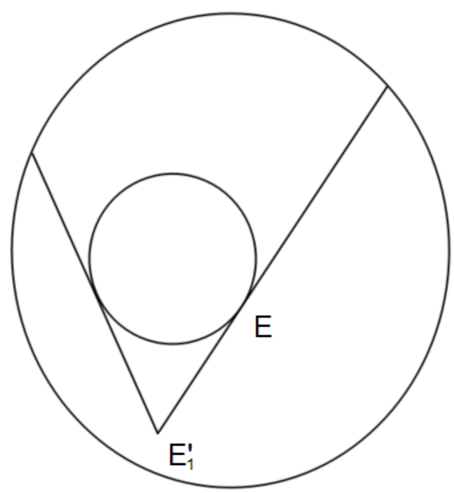
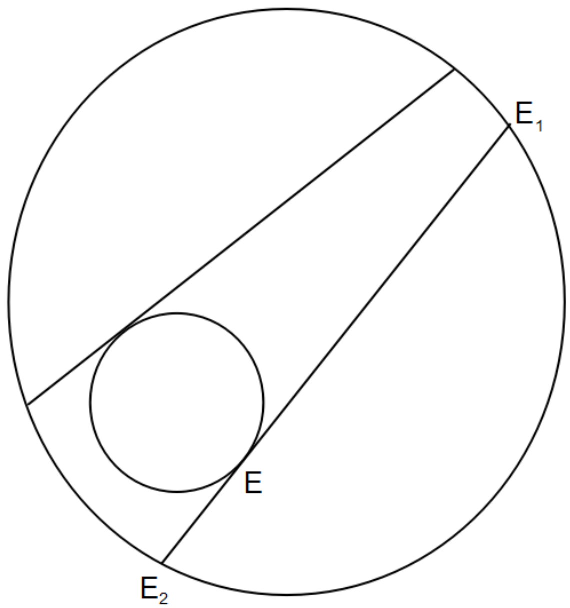
4 Appendix: Graphs For the Last Theorem



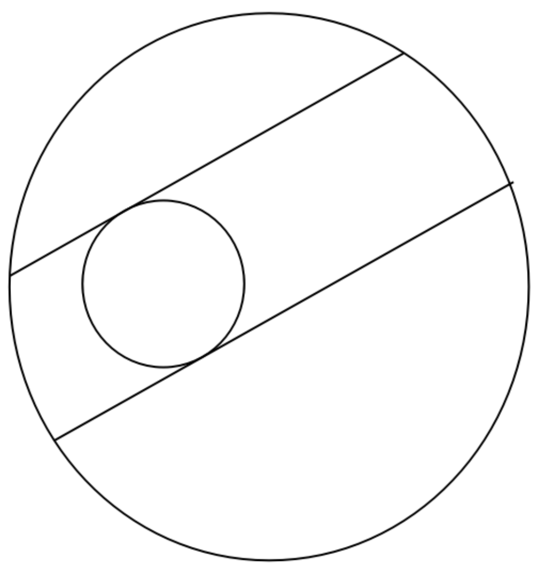
Case 1. Tangent line of E does not intersect with other tangent lines.
Other two tangent lines are not shown on the graph for simplicity.
As $\forall E \in = \partial Cir_{O_1}$, if $O_1 \neq O_2$, then $|E_1E \neq E_2E|$. In the graph above, $|E_1E < E_2E|$. The inequality will hold until it's perpendicular to the line connecting two centers of circles.



Case 2.
One tangent line is not shown on the graph for simplicity.
For the above case, if the tangent line passing through E intersects with another tangent line. Because of the direction of notation, (it's clockwise in the graph above) the shorter one of EE_1, EE_2 is the part that will intersect with another tangent line, and it becomes even shorter, so the inequality in case 1 still holds.
Indeed, at the beginning of rotation the tangent line will intersect with the circle. Therefore, the longer part between EE_1, EE_2 will always intersect with the circle.



When there are only two lines(shown in the graph above). The situation is the same with Case 1. The rotation will end when two lines are parallel to each other (shown in the graph below).



After two lines are parallel, consider translating *K* for maximum. Obviously, it reaches the maximum when the two parallel line segments have the same length (graph below).