Decomposition Theorems for Spectra

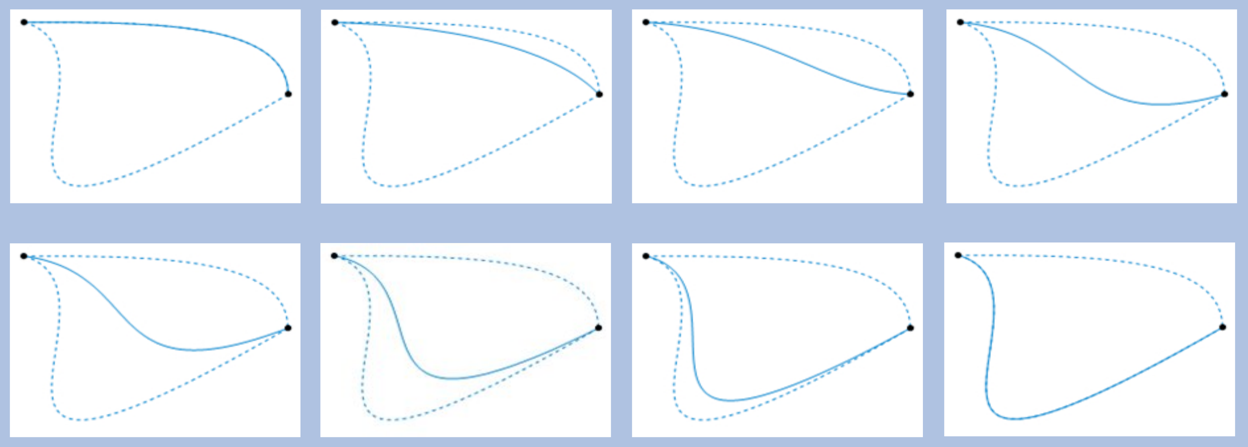
Yichen Ding, Xiaomin Li & Chengyi Liu
Brian Shin (Team Leader)
Dominic Culver (Faculty mentor)

Introduction

In this project, we will develop computer code to find the obstructions to the existence of maps between certain topological objects through an algebraic apparatus known as cohomology. Bob Bruner has developed a wonderful program to compute certain groups, known as Ext groups, and these are where the obstructions live. We will try to determine how many obstructions exist and then, if possible, figure out a way to eliminate these obstructions.

Homotopy

In topology, two continuous functions from one topological space to another are called **homotopic** if one can be "continuously deformed" into the other.
A **homotopy** between two functions f and g from a space X to a space Y is a continuous map H from $X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.



Cohomology

- **Chain Complex:** a **chain complex** is a sequence of maps
$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$
satisfying $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$, where the spaces C_n may be Abelian groups or modules.
- **Homology:** the **n-dimensional homology is defined by:**
$$H_n(C_*;\partial_*) := \frac{Ker\partial_n}{Im\partial_{n+1}}$$
- **Cochain Complex:** roughly speaking, cochain complex is a dual notion of chain complex with the arrows being flipped.
- **Cohomology:** similarly, cohomology is a dual notion of homology.

Exact Sequences & Short Exact Sequences

- **Exact Sequence:** in the context of group theory, a sequence of groups and group homomorphisms
$$\dots \xrightarrow{f_{n+2}} G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} \dots$$
is called **exact** if $Imf_{n+1} = Kerf_n$ for all $n \in \mathbb{Z}$.
- **Short Exact Sequence**: an exact sequence of the form
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
is called a **short exact sequence**.
- *Remark*: in the short exact sequence above, it's easy to see that f must be a monomorphism and g must be an epimorphism.

Two short exact sequences are said to be **isomorphic** if there is a commutative diagram of module homomorphisms as the following:

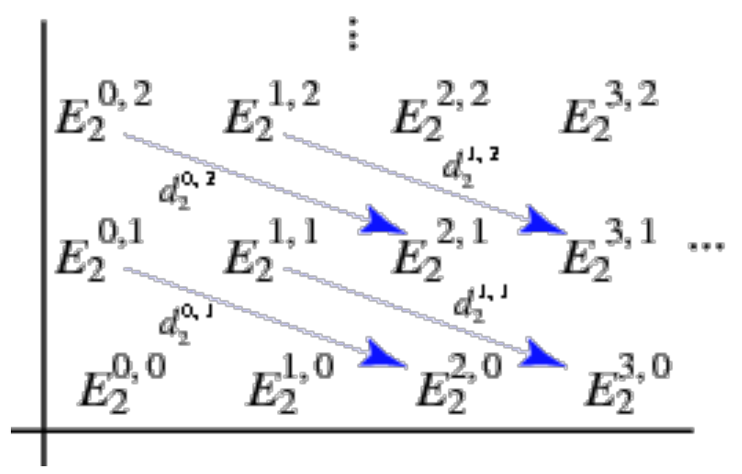
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Spectral Sequences

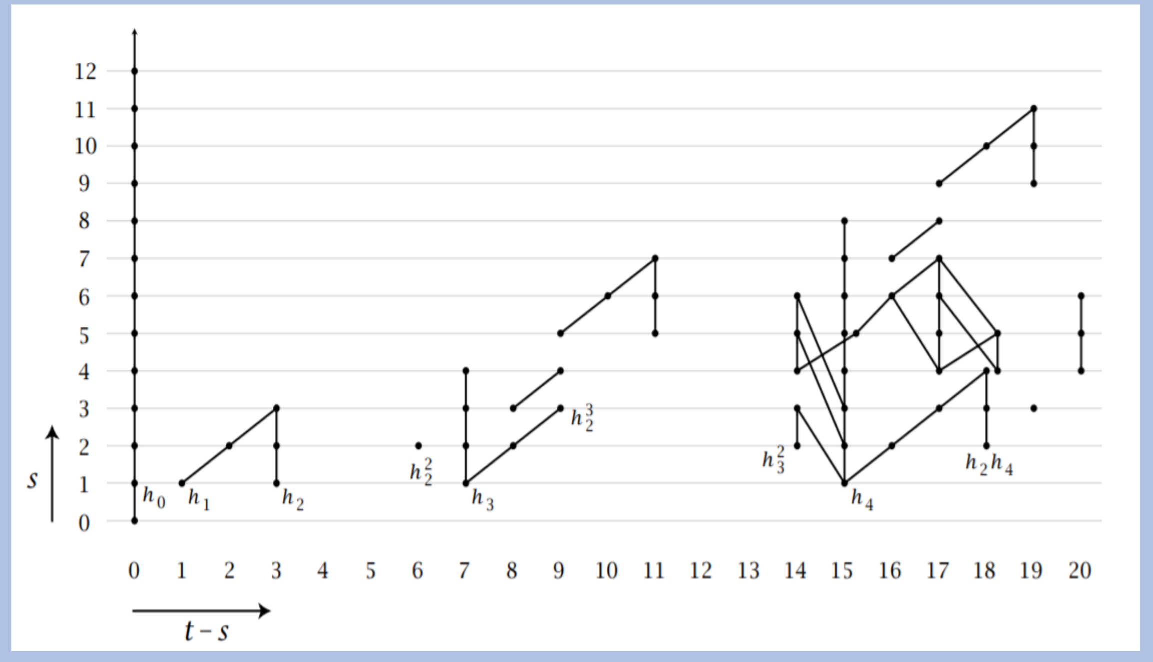
- **Spectral Sequence**: fix an abelian category, such as a category of modules over a ring. A **spectral sequence** is a choice of a nonnegative integer r_0 and a collection of three sequences:

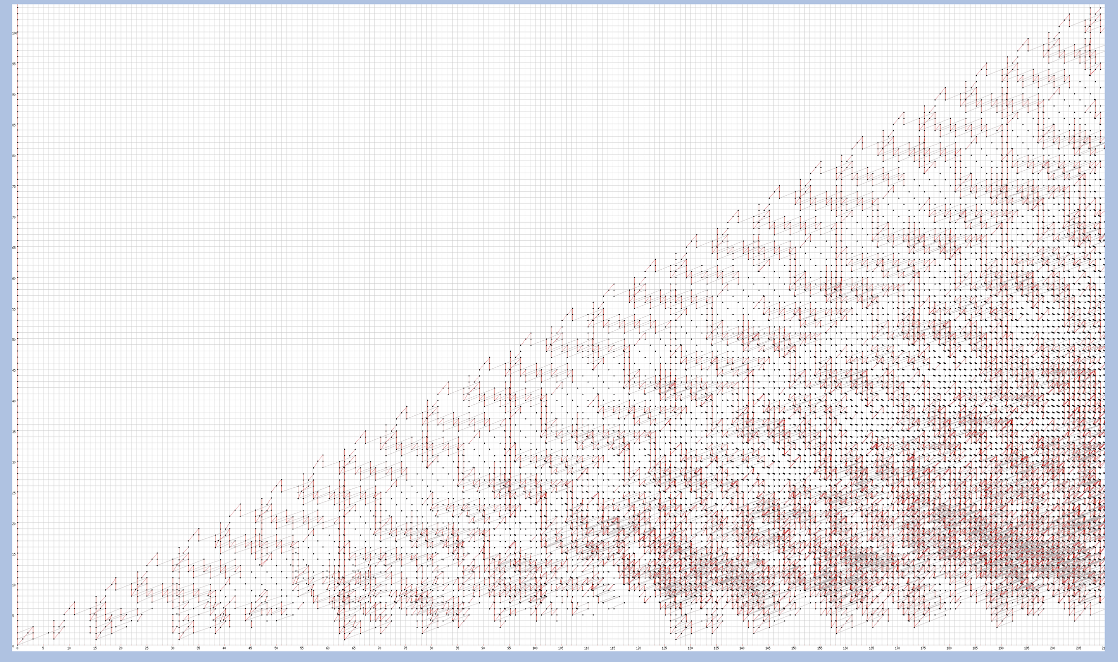
1. For all integers $r \geq r_0$, an object E_r, called a sheet.
2. Endomorphisms d_r: $E_r \to E_r$ satisfying $d_r \circ d_r = 0$, called boundary maps or differentials.
3. Isomorphisms of E_{r+1}, with $H(E_r)$, the homology of E_r, with respect to d_r.

- **Ext:** $Ext_R^{*,*}(H^*(Y), H^*(X))$ is a "derived" version of $HomR$ and a "computable" approximation of $[X, Y]$.
There is a correspondence between some elements in $Ext_R^{*,*}$ and equivalent classes in $[X, Y]$. Other elements are "noises".



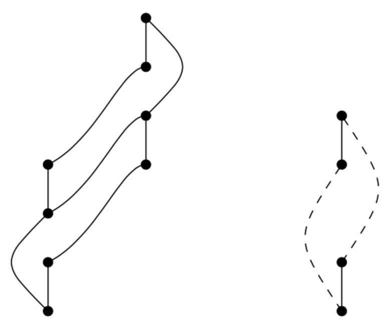
Examples





Steenrod Algebra

- For a given prime number p, the Steenrod algebra $\mathcal{A}_p$ is the graded Hopf algebra over the field F_p of order p, consisting of all stable cohomology operations for mod p cohomology. It is generated by the **Steenrod squares** introduced by Norman Steenrod (1947) for $p = 2$, and by the **Steenrod reduced pth powers** introduced in Steenrod (1953) and the Bockstein homomorphism for $p > 2$.
- Example: $\mathcal{A}_1$ (left) and its subalgebra $\mathcal{E}_1$ (right). The dashed lines represent the action of $Q_1 = Sq_1Sq_2 + Sq_2Sq_1$:



Future Work

- **The Ultimate Goal:**

The group $Ext_R^{*,*}(H^*(Y), H^*(X))$ approximates the set of continuous maps from X to Y. We have identified a certain element of this group and we want to know whether it actually represents a continuous map from X to Y.

- **Some Intermediate Goals:**

Utilize software to help calculate groups $Ext_R^{*,*}(H^*(Y), H^*(X))$, identifying which elements are "noises" and which represent actual continuous maps.

- **Immediate Goals:**

Study some basics of homological algebra and learn the background needed to work with $Ext_R^{*,*}$ and spectral sequences.

References

- Hatcher, A. (n.d.). The Adams spectral sequence. In Spectral Sequences in Algebraic Topology.
- Hungerford, T. W. (2011). Algebra. Place of publication not identified: Springer.