Almost Beatty Partitions

Matthew Cho, Xiaomin Li, Yun Xie, Junxian Li (Graduate Student), A.J. Hildebrand (Faculty Advisor)

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Beatty Sequences

Definition

Let γ be an irrational number. The Beatty sequence B_γ is defined as

$$B_\gamma = \{\lfloor n\gamma \rfloor, n = 1, 2 \ldots, \},$$

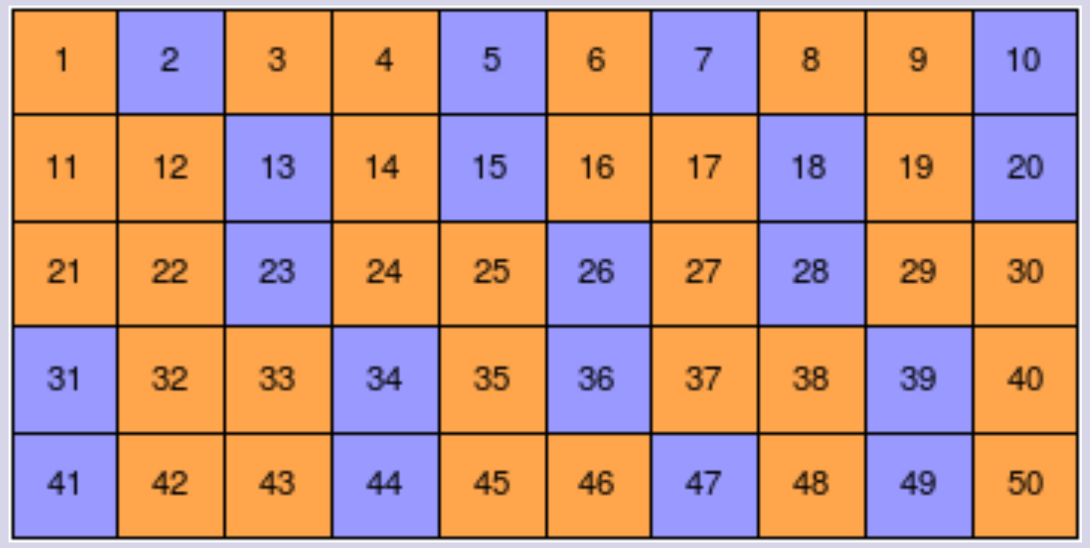
where $\lfloor x \rfloor$ is the floor function.

Example: Partition of Integers

Let ϕ be the golden ratio.

n	1	2	3	4	5	6	7	8
$n\phi$	1.62	3.24	4.85	6.47	8.09	9.70	11.33	12.94
$\lfloor n\phi \rfloor$	1	3	4	6	8	9	11	12
$n\phi^2$	2.61	5.24	7.85	10.47	13.09	15.71	18.33	20.94
$\lfloor n\phi^2 \rfloor$	2	5	7	10	13	15	18	20

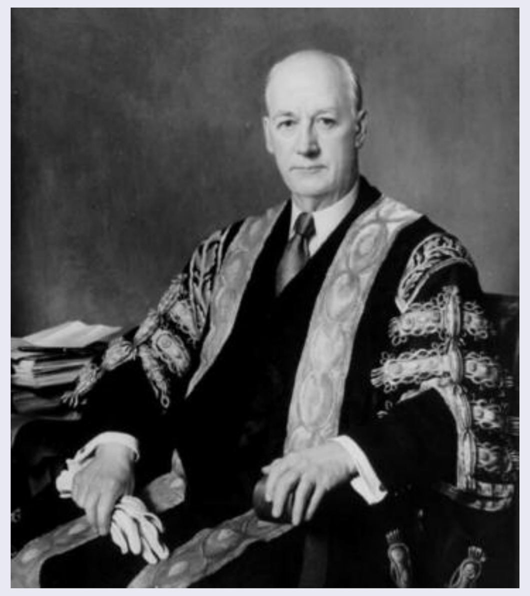
Beatty sequences of B_ϕ and B_{ϕ^2}.



Beatty Partition with B_ϕ and B_{ϕ^2}.

Samuel Beatty

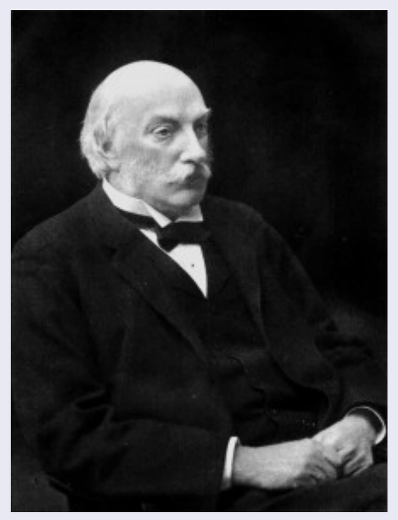
- Published Beatty's Theorem as a problem in the American Mathematical Monthly in 1926.
- First person receiving a Ph.D degree in mathematics from a Canadian university.
- One of the founders and the first president of the Canadian Mathematical Congress.



Samuel Beatty (1881-1970)

John W. Strutt (3rd Baron Rayleigh)

- Stated Beatty's Theorem even earlier in his book "The Theory of Sound" in 1894.
- Received the Nobel Prize in Physics in 1904 for discovering Argon.



John William Strutt (1842-1919)

Partitions of the Integers with Beatty Sequences

Theorem (Beatty's Theorem)

Let α and β be two positive irrational numbers. B_α and B_β form a partition of the integers if and only if

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Theorem (Uspensky's Theorem)

*Beatty's Theorem does not hold for three (or more) sequences. That is, if α, β and γ are arbitrary positive numbers, then B_α, B_β and B_γ do **not** partition the positive integers.*

How close can a 3-part partition be to three Beatty Sequences?

Theorem (3-part Almost Beatty Construction 1)

Let α and β be two irrational numbers such that B_α and B_β are disjoint. Let γ be the irrational number such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Denote

$$B^*_\gamma = \mathbb{N} \backslash (B_\alpha \cup B_\beta).$$

Let $B_\gamma(n)$ be the n-th term of B_γ. Then

$$\max_n (B^*_\gamma(n) - B_\gamma(n)) = \max \left(\left\lfloor \frac{1}{\alpha - 1} \right\rfloor, \left\lfloor \frac{1}{\beta - 1} \right\rfloor \right) + 2.$$

$\alpha = \phi^2$, (red) $\beta = \phi^3$, (blue) $B^*_\gamma(n) - B_\gamma(n) \in \{1, 2\}$.

Theorem (3-part Almost Beatty Construction 2)

Given $\alpha > 2$, let

$$B^*_\alpha = B_\alpha - 1.$$

Let γ be the irrational number such that

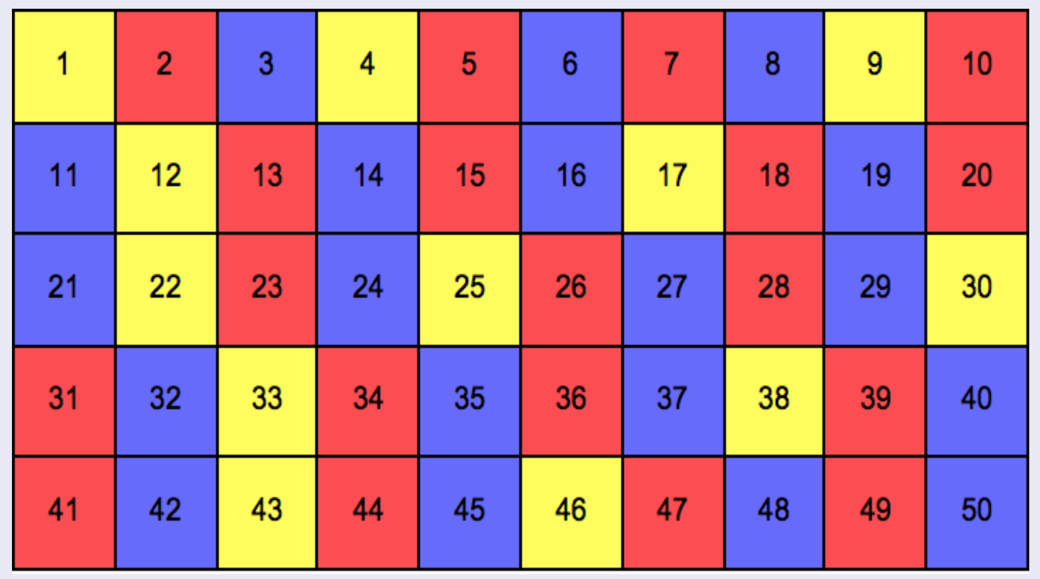
$$\frac{1}{\alpha} + \frac{1}{\alpha} + \frac{1}{\gamma} = 1.$$

Denote

$$B^*_\gamma = \mathbb{N} \backslash (B_\alpha \cup B^*_\alpha).$$

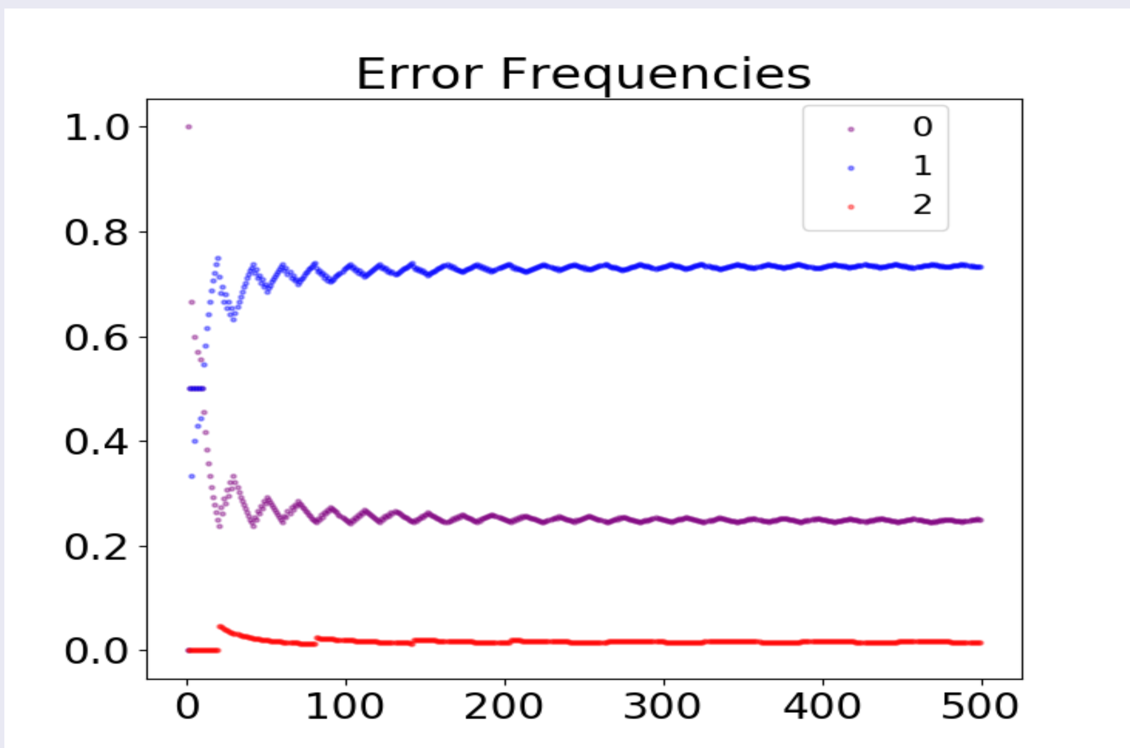
Let $B_\gamma(n)$ be the n-th term of B_γ. Then

$$B^*_\gamma(n) - B_\gamma(n) \in \{0, 1\}$$

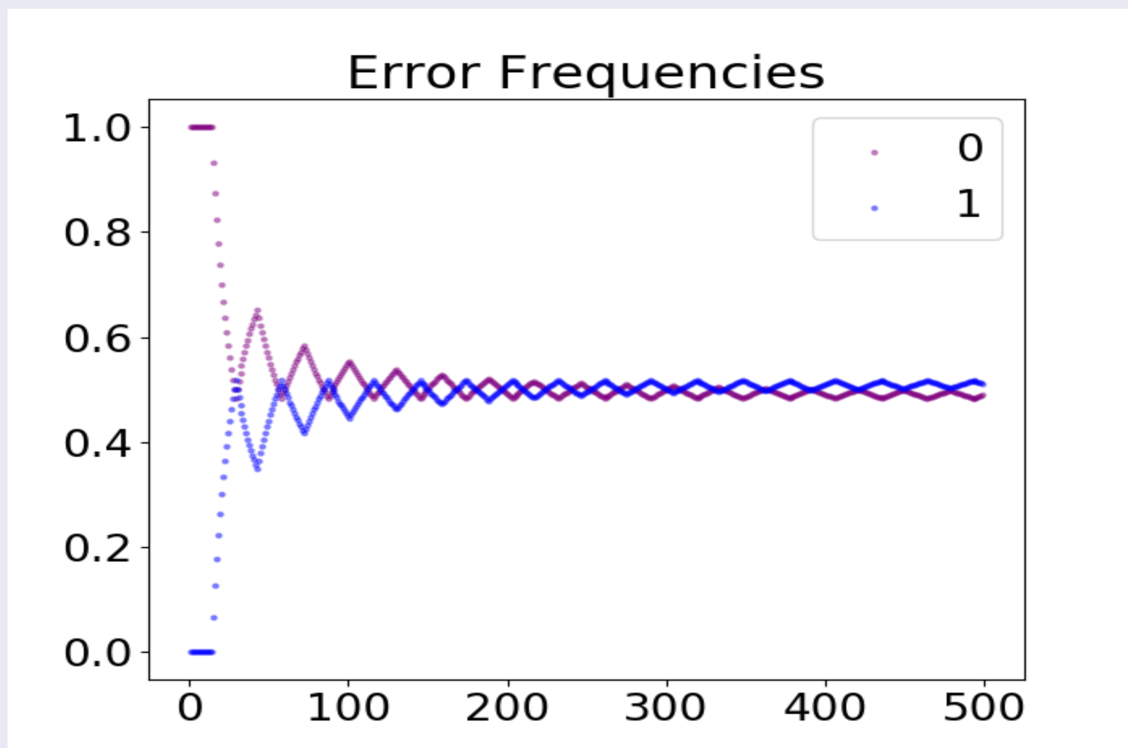


$\alpha = \phi^2$, (red) $B^*_\gamma(n) - B_\gamma(n) \in \{0, 1\}$.

Numerical Data on Distribution of Errors: $B^*_\gamma(n) - B_\gamma(n)$



$\alpha = \pi^3$, $\beta = \frac{3\alpha}{\alpha - 2}$, $B^*_\gamma(n) - B_\gamma(n) \in \{0, 1, 2\}$



$\alpha = \pi^3$, $B^*_\gamma(n) - B_\gamma(n) \in \{0, 1\}$,

Greedy Construction

Let α, β, γ be positive irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Construct sequences $B^*_\alpha, B^*_\beta, B^*_\gamma$ iteratively as follows:

- For each $n = 1, 2, 3, \ldots$ place n into the sequence for which the error

$$B^*_\alpha(n) - B_\alpha(n)|, |B^*_\beta(n) - B_\beta(n)|, |B^*_\gamma(n) - B_\gamma(n)|$$

is smallest.
- By construction, the resulting sequences $B^*_\alpha, B^*_\beta, B^*_\gamma$ form a partition of the positive integers.

Conjecture

The partition generated by the Greedy Construction satisfies

$$|B^*_\alpha(n) - B_\alpha(n)|, |B^*_\beta(n) - B_\beta(n)|, |B^*_\gamma(n) - B_\gamma(n)| \leq 2$$

for all positive integers n.

Numerical Example: Distribution of Errors

$\alpha \approx 2.27$, $\beta \approx 12.53$, $\gamma \approx 2.09$

Future Work

- Determine the possible errors and their frequencies in terms of α and β.
- Classify the cases when the errors are 0.
- Investigate the possible relations between different types of constructions.
- Extend current constructions to partitions with more than three parts.

References

- John William Strutt, 3rd Baron Rayleigh (1894). *The Theory of Sound*. 1 (Second ed.). Macmillan. p. 123.
- Beatty, Samuel (1926). *Problem 3173*. American Mathematical Monthly. 33 (3): 159. doi:10.2307/2300153
- Uspensky, J. V. (1927). *On a problem arising out of the theory of a certain game*. Amer. Math. Monthly 34 (1927), pp. 516–521.