Almost Beatty Partitions and Optimal Scheduling Problems

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Credits

This talk is based on a joint work with:

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It originated with a project in Spring 2018 at the **Illinois Geometry Lab** at the University of Illinois at Urbana-Champaign.

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- **Graduate student:** Junxian Li
- **Undergraduate students:** Weiru Chen, Matthew Cho, Jared Krandel, Xiaomin Li and Yun Xie

Outline

Beatty Sequences: Definition

Definition: Beatty Sequence

Given $\alpha > 0$, define the **Beatty sequence** B_α as

$$B_\alpha := \left\{ \lfloor \frac{n}{\alpha} \rfloor, n = 1, 2, \dots \right\},$$

where $\lfloor x \rfloor$ denotes the floor function.

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where $\lfloor x \rfloor$ denotes the floor function.

Remarks

- When $\alpha \leq 1$, then the elements of B_α are distinct and have density α in $\mathbb{N}$.
- When $\alpha > 1$, B_α has repeated elements.

Beatty Sequences: Examples

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\alpha = 1/3$	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48

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$\alpha = 1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

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$\alpha = \sqrt{2}$	0	1	2	2	3	4	4	5	6	7	7	8	9	9	10	11

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$\alpha = \pi$	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5

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Examples of Beatty Sequences B_α with $\alpha = 1/3, 1, \sqrt{2}, \pi,$.

Beatty Sequences: Examples II

Let ϕ be the golden ratio defined by $\phi = (\sqrt{5} + 1)/2$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$B_{1/\phi}$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25
B_{1/ϕ^2}	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41

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Rearranging the last two sequences gives:

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Observations:

- The Beatty sequences $B_{1/\phi}$ and B_{1/ϕ^2} **partition** $\mathbb{N}$.

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Observations:

- The Beatty sequences $B_{1/\phi}$ and B_{1/ϕ^2} **partition** $\mathbb{N}$.
- $1/\phi + 1/\phi^2 = 1$ (i.e. the densities of the last two sequences **sum up to 1**).

Beatty's Theorem: Statement

The following theorem shows that the previous observation holds whenever the densities α and β are irrational and sum up to 1.

Theorem (Beatty's Theorem)

Let α and β be two positive irrational numbers such that $\alpha + \beta = 1$. Then B_α and B_β form a partition of the positive integers.

Beatty's Theorem: Proof, Part I

Let α and β be two positive irrational numbers such that $\alpha + \beta = 1$. First, prove by contradiction that the two Beatty sequences are disjoint.

Suppose there exists an integer j such that $j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor$.

Then:

$$j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor$$

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Suppose there exists an integer j such that $j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor$. Then:

$$j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor$$

$$\iff j \leq \frac{n}{\alpha} < j + 1 \text{ and } j \leq \frac{m}{\beta} < j + 1$$

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$$j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor$$

$$\Longleftrightarrow j \leq \frac{n}{\alpha} < j + 1 \text{ and } j \leq \frac{m}{\beta} < j + 1$$

$$\Longleftrightarrow j < \frac{n}{\alpha} < j + 1 \text{ and } j < \frac{m}{\beta} < j + 1 \text{ (since } \alpha, \beta \text{ are irrational)}$$

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$$\iff j\alpha < n < (j+1)\alpha \text{ and } j\beta < m < (j+1)\beta$$

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Then:

$$
\begin{aligned}
& j = \lfloor \frac{n}{\alpha} \rfloor = \lfloor \frac{m}{\beta} \rfloor \\
\iff & j \leq \frac{n}{\alpha} < j + 1 \text{ and } j \leq \frac{m}{\beta} < j + 1 \\
\iff & j < \frac{n}{\alpha} < j + 1 \text{ and } j < \frac{m}{\beta} < j + 1 \text{ (since } \alpha, \beta \text{ are irrational)} \\
\iff & j\alpha < n < (j+1)\alpha \text{ and } j\beta < m < (j+1)\beta \\
\iff & j < n + m < j + 1 \text{ (since } \alpha + \beta = 1)
\end{aligned}
$$

This is a contradiction since n, m, j are integers.

Beatty's Theorem: Proof, Part II

Second, prove by contradiction that any integer belongs to at least one of the sequences.

Suppose there exists an integer j s.t. $j \neq \lfloor \frac{n}{\alpha} \rfloor$ and $j \neq \lfloor \frac{n}{\beta} \rfloor$ for any n. As j must be between two consecutive elements in B_α, similarly for B_β, so there exists integers k_1 and k_2 such that:

$$\frac{k_1}{\alpha} < j \text{ and } j + 1 \leq \frac{k_1 + 1}{\alpha}, \quad \frac{k_2}{\beta} < j \text{ and } j + 1 \leq \frac{k_2 + 1}{\beta}$$

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$$\iff k_1 < j\alpha \text{ and } (j+1)\alpha < k_1+1, k_2 < j\beta \text{ and } (j+1)\beta < k_2+1$$

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$$\iff k_1 < j\alpha \text{ and } (j+1)\alpha < k_1+1, k_2 < j\beta \text{ and } (j+1)\beta < k_2+1$$

$$\iff k_1+k_2 < j < k_1+k_2+1 \text{ (since } \alpha+\beta=1)$$

This is a contradiction since j, k_1, k_2 are integers. □

Outline

Beatty Partitions into More Than Two Parts?

Question

Does Beatty's Theorem generalize to partitions into 3 parts?

That is, given three positive irrational numbers α, β and γ which sum up to 1, do B_α, B_β and B_γ partition the positive integers?

Beatty Partitions into More Than Two Parts?

Question

Does Beatty's Theorem generalize to partitions into 3 parts?

That is, given three positive irrational numbers α, β and γ which sum up to 1, do B_α, B_β and B_γ partition the positive integers?

Answer: No!

Theorem (Uspensky's Theorem)

Beatty's Theorem does not hold for three (or more) sequences. That is, if α, β and γ are arbitrary positive numbers, then B_α, B_β and B_γ ***never*** *partition the positive integers.*

Almost Beatty Sequences

Question

How close can three Beatty sequences come to a 3-part partition?

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Definition: Almost Beatty Sequence

Consider a Beatty sequence with density $\alpha \in (0, 1)$:

$$B_\alpha = (a(n))_{n\in\mathbb{N}}, \text{ where } a(n) = \lfloor \frac{n}{\alpha} \rfloor.$$

We call a sequence

$$\widetilde{B_\alpha} = (\widetilde{a}(n))_{n\in\mathbb{N}}$$

an **almost Beatty sequence** with density α if $\|\widetilde{a} - a\| < \infty$, where $\|\widetilde{a} - a\| = \sup_n |\widetilde{a}(n) - a(n)|$.

Almost Beatty Partitions: Construction 1

Theorem (Partition into 2 Exact and 1 Almost Beatty Sequence)

Let α, β, and γ be positive irrational numbers such that $\alpha + \beta + \gamma = 1$. Let $B_\alpha = (a(n))_{n\in\mathbb{N}}$, $B_\beta = (b(n))_{n\in\mathbb{N}}$ and $B_\gamma = (c(n))_{n\in\mathbb{N}}$ be the corresponding Beaty sequences. Define

$$\widetilde{B_\gamma} = \mathbb{N}\backslash(B_\alpha \cup B_\beta) = (\widetilde{c}(n))_{n\in\mathbb{N}}.$$

Then:

(i) $B_\alpha, B_\beta, \widetilde{B_\gamma}$ **form a partition of** $\mathbb{N}$ if and only if $r\alpha + s\beta = 1$ for some $r, s \in \mathbb{N}$.

(ii) If this condition is satisfied, then $\widetilde{B_\gamma}$ **is an almost Beatty sequence** with perturbation errors satisfying $|c(n) - \widetilde{c}(n)| \leq \max(\lfloor \frac{2-\alpha}{1-\alpha} \rfloor, \lfloor \frac{2-\beta}{1-\beta} \rfloor)$.

Almost Beatty Partitions: Construction 1

Remarks

- Part (i) of the theorem (i.e. the **partition property**) follows from the following Theorem of Skolem (1957):

  Let α,β be irrational numbers in $(0,1)$. Then the Beatty sequences B_α and B_β are disjoint if and only if there exist positive integers r and s such that $r\alpha + s\beta = 1$.

- In part (ii) (the **almost Beatty sequence property**), the bound on the perturbation errors $|c(n) - \tilde{c}(n)|$ is the best possible. If $\max(\alpha, \beta) < \frac{1}{2}$, then the bound simplifies to $|c(n) - \tilde{c}(n)| \leq 2$.

Construction 1: Example

Example: $\alpha = 1/\phi^3$, $\beta = 1/\phi^4$, $\gamma = 1/\phi$

We have $1/\phi^3 + 1/\phi^4 + 1/\phi = 1$ and $3/\phi^3 + 2/\phi^4 = 1$.
Therefore, the condition $r\alpha + s\beta = 1$ holds for $r = 3$ and $s = 2$.

B_α	$a(n)$	4	8	12	16	21	25	29	33	38	42	46	50
B_β	$b(n)$	6	13	20	27	34	41	47	54	61	68	75	82
B_γ	$c(n)$	1	3	4	6	8	9	11	12	14	16	17	19
$\widetilde{B_\gamma}$	$\widetilde{c}(n)$	1	2	3	5	7	9	10	11	14	15	17	18
Error	$c(n) - \widetilde{c}(n)$	0	1	1	1	1	0	1	1	0	1	0	1

Construction 1: Analysis of Perturbation Errors

Theorem (Perturbation Errors in Construction 1)

Assume the conditions of the above theorem are satisfied and suppose $\max(\alpha, \beta) < \frac{1}{2}$. Then the perturbation errors $c(n) - \widetilde{c}(n)$ are 0,1,2 and each value is attained infinitely often. Moreover, the densities of these values are given by:

$$d_0 = \frac{r(s-1)(1-2\gamma) + (2r\gamma - 1 - \lfloor r\gamma \rfloor)\lfloor r\gamma \rfloor}{2r(r-s)\gamma}$$

$$d_1 = \frac{r^2(4\gamma - 1) - r(s - 2 + 4\gamma) - (4r\gamma - 2)\lfloor r\gamma \rfloor + 2\lfloor r\gamma \rfloor^2}{2r(r-s)\gamma}$$

$$d_2 = \frac{(-r + 2r\gamma - \lfloor r\gamma \rfloor)(1 - r + \lfloor r\gamma \rfloor)}{2r(r-s)\gamma}$$

Example: Density of Perturbation Errors

Example: $\alpha = 1/\phi^3$, $\beta = 1/\phi^4$, $\gamma = 1/\phi$

B_α	$a(n)$	4	8	12	16	21	25	29	33	38	42	46	50
B_β	$b(n)$	6	13	20	27	34	41	47	54	61	68	75	82
B_γ	$c(n)$	1	3	4	6	8	9	11	12	14	16	17	19
$\widetilde{B_\gamma}$	$\widetilde{c}(n)$	1	2	3	5	7	9	10	11	14	15	17	18
Error	$c(n) - \widetilde{c}(n)$	0	1	1	1	1	0	1	1	0	1	0	1

The densities of the perturbation errors 0,1,2 are given by:

$$d_0 = (1 + \sqrt{5})/12 = 0.26967\ldots,$$
$$d_1 = (19 - 5\sqrt{5})/12 = 0.65163\ldots,$$
$$d_2 = (\sqrt{5} - 2)/3 = 0.078689\ldots$$

Almost Beatty Partitions: Construction 2

Theorem (Partition into 1 Exact and 2 Almost Beatty Sequences)

Let α, β, and γ be positive irrational numbers such that $\alpha+\beta+\gamma=1$ and $\max(\alpha,\beta)<\gamma$. Define $\widetilde{B_\beta}=(\widetilde{b}(n))_{n\in\mathbb{N}}$ as

$$\widetilde{b}(n)=\begin{cases} b(n), & \text{if } b(n)\notin B_\alpha \\ b(n)-1, & \text{if } b(n)\in B_\alpha \end{cases}$$

Denote

$$\widetilde{B_\gamma}=\mathbb{N}\backslash(B_\alpha\cup\widetilde{B_\beta}).$$

Then B_α, $\widetilde{B_\beta}$, and $\widetilde{B_\gamma}$ partition all positive integers and $b(n)-\widetilde{b}(n)\in\{0,1\}$, $c(n)-\widetilde{c}(n)\in\{0,1,2\}$

Remark: Construction 2 applies to any irrational densities α,β,γ that sum up to 1 and satisfy $\max(\alpha,\beta)<\gamma$.

Is this Construction Best Possible?

Question

Is there a partition of the positive integers into one exact and two almost Beatty sequences with perturbation errors less than or equal to 1?

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Answer: No (in general)!

Theorem (Nonexistence Result)

Let α, β, and γ be positive irrational numbers such that $\alpha+\beta+\gamma=1$. If $\alpha>\frac{1}{3}$ and $1,\alpha,\beta$ are linearly independent over $\mathbb{Q}$, then **there does not exist** an almost Beatty partition $B_\alpha \cup \widetilde{B_\beta} \cup \widetilde{B_\gamma} = \mathbb{N}$ such that $\widetilde{B_\beta} = (\widetilde{b}(n))_{n\in\mathbb{N}}$, $\widetilde{B_\gamma} = (\widetilde{c}(n))_{n\in\mathbb{N}}$ satisfy

$$\sup_n |b(n)-\widetilde{b}(n)| \leq 1, \quad \sup_n |c(n)-\widetilde{c}(n)| \leq 1.$$

Outline

Applications

- Chairman Assignment Problem
- Frequency Hopping
- Carpool Problem
- Weighted Fair Queueing
- Computer Networks

Outline

Future Work

- Investigate almost Beatty partitions into more than 3 sequences.
- Investigate almost Beatty partitions using non-homogeneous Beatty sequences. Can such sequences give an optimal chairman assignment?
- Look for applications of Beatty partitions to other optimal scheduling problems (e.g. frequency hopping).

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