# Almost Beatty Partitions and Optimal Scheduling Problems


Xiaomin Li, A.J. Hildebrand (Faculty Advisor)

ILLINOIS Illinois Geometry Lab



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## Beatty Sequences

**Definition: Beatty Sequence**

Given $\alpha > 0$, define the **Beatty sequence** $B_\alpha$ as

$$B_\alpha := \left\{ \lfloor \frac{n}{\alpha} \rfloor, n = 1, 2, \dots \right\},$$

where $\lfloor x \rfloor$ denotes the floor function.

**Remark:** When $\alpha \leq 1$, then the elements of $B_\alpha$ are distinct and have density $\alpha$ in $\mathbb{N}$. When $\alpha > 1$, $B_\alpha$ has repeated elements.

**Definition: Non-homogeneous Beatty Sequence**

Given $\alpha > 0$ and $\delta \in \mathbb{R}$, define the **non-homogeneous Beatty sequence** $B^*_{\alpha,\delta}$ as:

$$B^*_{\alpha,\delta} := \{ \lfloor \frac{n}{\alpha} + \delta \rfloor, n = 1, 2, \dots \}.$$

## Beatty Partitions

**Beatty's Theorem**

Let $\alpha$ and $\beta$ be two positive irrational numbers such that $\alpha + \beta = 1$. Then $B_\alpha$ and $B_\beta$ form a partition of the positive integers.

**Example: Beatty Partition**

Let $\phi$ be the golden ratio defined by $\phi = (\sqrt{5} + 1)/2$. Let $\alpha = 1/\phi, \beta = 1/\phi^2$. Note that $\alpha + \beta = 1$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_{1/\phi}$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 | 17 | 19 | 21 | 22 | 24 | 25 |
| $B_{1/\phi^2}$ | 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | 26 | 28 | 31 | 34 | 36 | 39 | 41 |

Rearranging the two sequences shows that the Beatty sequences $B_{1/\phi}$ and $B_{1/\phi^2}$ **partition** $\mathbb{N}$:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_{1/\phi}$ | 1 | | 3 | 4 | | 6 | | 8 | 9 | | 11 | 12 | | 14 | | 16 |
| $B_{1/\phi^2}$ | | 2 | | | 5 | | 7 | | | 10 | | | 13 | | 15 | |

## Beatty Partitions into more than two parts?

**Question**

*Does Beatty's Theorem generalize to partitions into 3 parts? That is, given three positive irrational numbers $\alpha$, $\beta$ and $\gamma$ which sum up to 1, do $B_\alpha$, $B_\beta$ and $B_\gamma$ partition the positive integers?*

**Answer: No!**

**Uspensky's Theorem**

Beatty's Theorem does not hold for three (or more) sequences. That is, if $\alpha$, $\beta$ and $\gamma$ are arbitrary positive numbers, then $B_\alpha$, $B_\beta$ and $B_\gamma$ **never** partition the positive integers.

## How close can a 3-part partition be to three Beatty Sequences?

**Definition: Almost Beatty Sequence**

Consider a Beatty sequence with density $\alpha \in (0, 1)$:

$$B_\alpha = (a(n))_{n\in\mathbb{N}}, \text{ where } a(n) = \lfloor \frac{n}{\alpha} \rfloor.$$

We call a sequence

$$\widetilde{B_\alpha} = (\tilde{a}(n))_{n\in\mathbb{N}}$$

an **almost Beatty sequence** with density $\alpha$ if $\|\tilde{a} - a\| < \infty$, where $\|\tilde{a} - a\| = \sup_n |\tilde{a}(n) - a(n)|$.

**Theorem (Partition into 2 Exact and 1 Almost Beatty Sequence)**

Let $\alpha$, $\beta$, and $\gamma$ be positive irrational numbers such that $\alpha + \beta + \gamma = 1$. Let $B_\alpha = (a(n))_{n\in\mathbb{N}}$, $B_\beta = (b(n))_{n\in\mathbb{N}}$ and $B_\gamma = (c(n))_{n\in\mathbb{N}}$ be the corresponding Beaty sequences. Define

$$\widetilde{B_\gamma} = \mathbb{N} \backslash (B_\alpha \cup B_\beta) = (\tilde{c}(n))_{n\in\mathbb{N}}.$$

Then:

(i) $B_\alpha$, $B_\beta$, $\widetilde{B_\gamma}$ **form a partition of** $\mathbb{N}$ if and only if $r\alpha + s\beta = 1$ for some $r, s \in \mathbb{N}$.

(ii) If this condition is satisfied, then $\widetilde{B_\gamma}$ **is an almost Beatty sequence** with perturbation errors satisfying $|c(n) - \tilde{c}(n)| \leq \max(\lfloor \frac{2-\alpha}{1-\alpha} \rfloor, \lfloor \frac{2-\beta}{1-\beta} \rfloor)$.

**Theorem (Partition into 1 Exact and 2 Almost Beatty Sequences)**

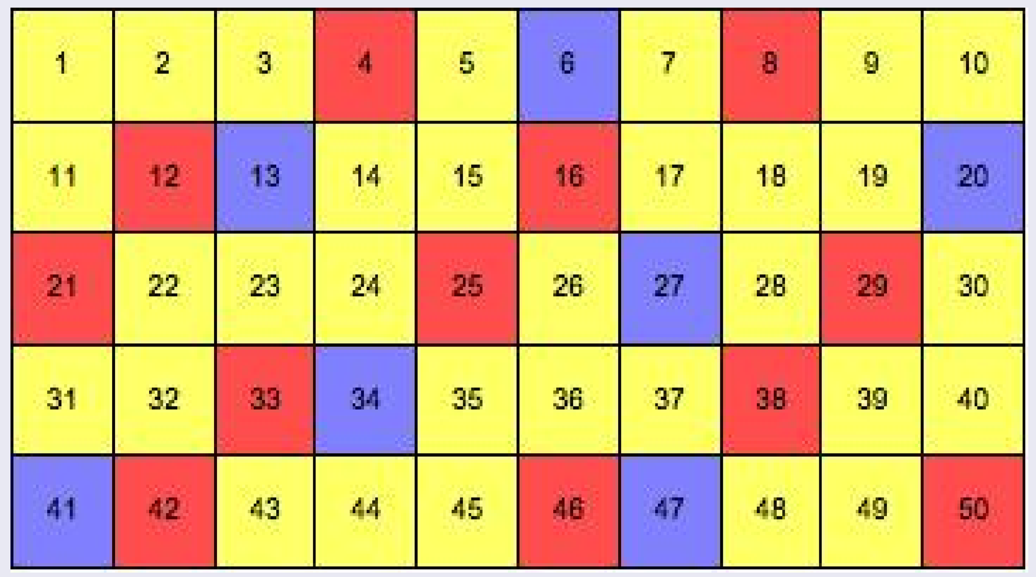
Let $\alpha$, $\beta$, and $\gamma$ be positive irrational numbers such that $\alpha + \beta + \gamma = 1$ and $\max(\alpha, \beta) < \gamma$. Define $\widetilde{B_\beta} = (\tilde{b}(n))_{n\in\mathbb{N}}$ as

$$\tilde{b}(n) = \begin{cases} b(n), & \text{if } b(n) \notin B_\alpha \\ b(n) - 1, & \text{if } b(n) \in B_\alpha \end{cases}$$

Denote

$$\widetilde{B_\gamma} = \mathbb{N} \backslash (B_\alpha \cup \widetilde{B_\beta}).$$

Then $B_\alpha$, $\widetilde{B_\beta}$, and $\widetilde{B_\gamma}$ partition all positive integers and $b(n) - \tilde{b}(n) \in \{0, 1\}$, $c(n) - \tilde{c}(n) \in \{0, 1, 2\}$




Almost Beatty Partition with $B_{1/\phi^3}$, $\widetilde{B}_{1/\phi^4}$, and $\widetilde{B}_{1/\phi}$.

## The Chairman Assignment Problem

**Chairman Assignment Problem (Robert Tijdeman, 1980)**

Suppose a set of $k$ $(\geq 2)$ states $S = \{S_1, S_2, \dots, S_k\}$ form a union and a union chair has to be selected every year. Each state $S_i$ has a positive weight $\lambda_i$ with $\sum_{i=1}^k \lambda_i = 1$.

Denote the state designating the chairman in the $n$th year by $\omega_n$. Hence $\omega = \{\omega_n\}_{n\in\mathbb{N}}^\infty$ is a sequence in $S$. Let $A_\omega(i, N)$ denote the number of chairmen representing $S_i$ in the first $N$ years.

The problem asks for the assignment $\omega$ which minimizes the perturbation error

$$D(\omega) = \sup_{i=1,\dots,k} \sup_{N\in\mathbb{N}} |\lambda_i N - A_\omega(i, N)|.$$

**Example: Chairman Assignment for 2 States**

| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| State 1 | 1 | | 3 | 4 | | 6 | | 8 | 9 | | 11 | 12 | | 14 |
| State 2 | | 2 | | | 5 | | 7 | | | 10 | | | 13 | |
| Assignment $\omega_n$ | 1 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 2 | 1 |

This example illustrates the connection between chairman assignments with $k$ states and partitions of $\mathbb{N}$ into $k$ sequences.

**Question:**

*Is the assignment given by a Beatty partition an optimal assignment (i.e. minimizes the error $D(\omega)$ defined by Tijdeman)?*

**Theorem**

Let $\alpha$, $\beta$ be positive irrational numbers such that $\alpha + \beta = 1$. Let $\delta = 1 - \frac{1}{2\alpha}$, $\epsilon = 1 - \frac{1}{2\beta}$.

- The chairman assignment $\omega$ corresponding to the Beatty partition $\mathbb{N} = B_\alpha \cup B_\beta$ satisfies
$$D(\omega) = \max\{\alpha, 1 - \alpha\}.$$
- The chairman assignment $\omega$ corresponding to the non-homogeneous Beatty partition $\mathbb{N} = B^*_{\alpha,\delta} \cup B^*_{\beta,\epsilon}$ satisfies
$$D(\omega) = \frac{1}{2}$$

Tijdeman proved that $D(\omega) = 1/2$ corresponds to an optimal chairman assignment. Thus, the above non-homogeneous Beatty partition solves the Chairman Assignment Problem when there are two states.

## Other Applications

- Frequency Hopping
- Carpool Problem
- Weighted Fair Queueing
- Computer Networks

## Future Work

- Investigate almost Beatty partitions into more than 3 sequences.
- Investigate almost Beatty partitions using non-homogeneous Beatty sequences. Can such sequences give an optimal chairman assignment?
- Consider other perturbation measures in the Chairman Assignment Problem.
- Look for applications of Beatty partitions to other optimal scheduling problems (e.g. frequency hopping).
- Investigate connections between non-homonegeous Beatty sequences and sequences given by Tijdeman's recursive algorithm.

## References


- Beatty, S. (1926). *Problem 3173.* American Mathematical Monthly. 33 (3): 159. doi:10.2307/2300153
- Hildebrand, A. J., Li, J., Li, X., Xie, Y. (2018). Almost Beatty Partitions. arXiv preprint arXiv:1809.08690.
- Lord Rayleigh (1894). *The Theory of Sound.* 1 (Second ed.). Macmillan. p. 123. 10.1016/0012-365X(80)90269-1.
- Skolem, T. (1957). On certain distributions of integers in pairs with given differences, *Math. Scand.* **5** (1957), 57–68.
- Tijdeman, R. (1980). The chairman assignment problem. *Discrete Mathematics.* 32. 323-330.
- Uspensky, J. V. (1927). *On a problem arising out of the theory of a certain game.* Amer. Math. Monthly 34 (1927), pp. 516–521. Canadian Journal of Mathematics, 21, 6-27.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