CS 229, Fall 2018
Problem Set #0 Solutions: Linear Algebra and Multivariable Calculus

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1. (a) **Answer:**

$$\begin{aligned}
f(x) &= \frac{1}{2}x^\top Ax + b^\top x \\
&= \frac{1}{2}\sum_{i,j} A_{i,j}x_i x_j + \sum_i b_i x_i \\
\frac{\partial}{\partial x_k} f(x) &= \frac{1}{2}(\sum_{i\neq k} A_{ik}x_i + \sum_{j\neq k} A_{kj}x_j + 2A_{kk}x_k) + b_k \\
&= \frac{1}{2}(\sum_{i\neq k} A_{ik}x_i + \sum_{j\neq k} A_{jk}x_j + 2A_{kk}x_k) + b_k \\
&= \sum_{i\neq k} A_{ik}x_i + A_{kk}x_k + b_k \\
&= \sum_i A_{ik}x_i + b_k \\
\nabla f(x) &= \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix} \\
&= \begin{bmatrix} \sum_i A_{i1}x_i + b_1 \\ \vdots \\ \sum_i A_{in}x_i + b_n \end{bmatrix} \\
&= Ax + b
\end{aligned}$$

(b) **Answer:**

$$\begin{aligned}
\frac{\partial}{\partial x_k} f(x) &= \frac{\partial}{\partial x_k} g(h(x)) \\
&= \frac{\partial g(h(x))}{\partial h(x)} \frac{\partial h(x)}{\partial x_k} \\
&= g'(h(x)) \frac{\partial h(x)}{\partial x_k} \\
\nabla f(x) &= g'(h(x)) \nabla h(x)
\end{aligned}$$

(c) **Answer:**

$$\begin{aligned}
\nabla^2 f(x) &= \nabla(\nabla f(x)) \\
&= \nabla(Ax + b) \\
&= A
\end{aligned}$$

(d) **Answer:** From part (b), let $h(x) = a^\top x$. We have:

$$\begin{aligned}\nabla f(x) &= g'(a^\top x)\nabla h(x)\\ &= g'(a^\top x)a\\ \nabla^2 f(x) &= \nabla(\nabla f(x))\\ &= \nabla(g'(a^\top x)a)\\ &= g''(a^\top x)aa^\top\end{aligned}$$

2. (a) **Answer:** $A^\top = (zz^\top)^\top = zz^\top = A$
$\forall x \in \mathbb{R}^n$,

$$\begin{aligned}x^\top Ax &= x^\top zz^\top x\\ &= (z^\top x)^\top(z^\top x)\\ &= (z^\top x)^2 \geq 0\end{aligned}$$

Therefore, A is PSD.

(b) **Answer:** To calculate the null-space of A, set $Ax = zz^\top x = 0$ and solve for x, where $x \in \mathbb{R}^n$. For $i = 1, \cdots, n$, we have $z_i \sum_{j=1}^n z_j x_j = 0$. Since z is a non-zero n-vector, we require $\sum_{j=1}^n z_j x_j = 0$ only. Therefore, null-space of $A = \{x : x \in \mathbb{R}^n, z^\top x = 0\}$.

$$\text{Rank}(A) = \text{Rank}(zz^\top) = 1$$

(c) **Answer:** $(BAB^\top)^\top = B(BA)^\top = BA^\top B^\top = BAB^\top$
$\forall x \in \mathbb{R}^n$,

$$\begin{aligned}x^\top BAB^\top x &= (B^\top x)^\top A(B^\top x)\\ &\overset{y=B^\top x}{=} y^\top Ay \geq 0\end{aligned}$$

Therefore, $BAB^\top$ is PSD.

3. (a) **Answer:**

$$\begin{aligned}A &= T\Lambda T^{-1}\\ AT &= T\Lambda\\ A\begin{bmatrix}t^{(1)} & \cdots & t^{(n)}\end{bmatrix} &= \begin{bmatrix}t^{(1)} & \cdots & t^{(n)}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\\ \begin{bmatrix}At^{(1)} & \cdots & At^{(n)}\end{bmatrix} &= \begin{bmatrix}\lambda_1 t^{(1)} & \cdots & \lambda_n t^{(n)}\end{bmatrix}\end{aligned}$$

Therefore, we have: $At^{(i)} = \lambda_i t^{(i)}$.

(b) **Answer:**

$$
\begin{aligned}
A &= U \Lambda U^\top \\
AU &= U \Lambda \\
A \begin{bmatrix} u^{(1)} & \cdots & u^{(n)} \end{bmatrix} &= \begin{bmatrix} u^{(1)} & \cdots & u^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\
\begin{bmatrix} Au^{(1)} & \cdots & Au^{(n)} \end{bmatrix} &= \begin{bmatrix} \lambda_1 u^{(1)} & \cdots & \lambda_n u^{(n)} \end{bmatrix}
\end{aligned}
$$

Therefore, we have: $Au^{(i)} = \lambda_i u^{(i)}$.

(c) **Answer:** From part (b), $Au^{(i)} = \lambda_i u^{(i)}$.

$$
(u^{(i)})^\top A u^{(i)} = (u^{(i)})^\top \lambda_i u^{(i)} = \lambda_i (u^{(i)})^\top u^{(i)} = \lambda_i \geq 0
$$