Notes on Doubly Periodic Functions

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1 Doubly Periodic Functions

For any meromorphic function f let M_f denote the set of all its periods, i.e.

$$M_f = \{\omega \in \mathbb{C} \mid f(z+\omega) = f(z) \text{ for any } z \in \mathbb{C}\}.$$

Prop 1.1. M_f forms a free $\mathbb{Z}$-module.

Sketch of Proof. $M_f \in \mathbb{C} \Rightarrow$ no torsion $\square$

Prop 1.2. f non-constasnt $\Longrightarrow$ M_f is isolated in $\mathbb{C}$; $\mathrm{rank} M_f \leq 2$.

Sketch of Proof. $f|_{M_f}$ is const. By the uniqueness we know that M_f cannot have an accumulation since f is non-constant. By Prop1.1 we take a base $(\omega_i)_{i\in I}$ of M_f. If it contains 3 elements there exist $r_1, r_2 \in \mathbb{R}$ so that $r_1\omega_1 + r_2\omega_2 = \omega_3$. Note that $(\omega_i)_{i=1}^3$ are $\mathbb{Z}$-independent, both of r_1 and r_2 cannot be rational. WLOG $r_1 \notin \mathbb{Q}$. Then

$$\{nr_1\omega_1 + mr_2\omega_2 \mid m, n \in \mathbb{Z}\}$$

drops infinitely many times into the segment connecting 0 and ω_3, a contradiction! $\square$

remark. Therefore there are any 3 cases of $\mathrm{rank} M_f$. The case 0 is to say that f is not periodic; the case 1 can be studied by its Fourier expansion; hence we will be interested when $\mathrm{rank} M_f = 2$. Such f is said to be a doubly periodic function or eliptic function. In the following part we will write DPF for short.

Let f be a DPF. By the periodicity we can naturally consider f as a map over $\mathbb{C}/M_f \simeq T^2$ (the torus).

Thm 1.1. The sum of residues of a DPF on T^2 is zero.

Sketch of Proof. Choose a "fundamental parallelogram" P of f whose boundary contains no poles. Then the sum of residues of f on T^2 is equal to

$$\frac{1}{2\pi i}\int_{\partial P} f \mathrm{d}z = 0 \ .$$

$\square$

Cor 1.2. A DPF can not have only one pole on T^2.

Sketch of Proof. A single pole must have a non-zero residue. □

Cor 1.3. A DPF has equally many zeros and poles on T^2.

Sketch of Proof. Note that f'/f is a DPF having the same periodic module. □

A more exact result.

Thm 1.4. Let $(a_i)_{i=1}^n$ be all its zeros in T^2 and $(b_i)_{i=1}^n$ be all its poles in T^2, then

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \mod M_f.$$

Sketch of Proof. Do some calculations. By the residue theorem

$$\frac{1}{2\pi i}\int_{\partial P} \frac{zf'(z)}{f(z)}\mathrm{d}z = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i.$$

□

Cor 1.5. All complex values are assumed the same times by a DPF on T^2.

Sketch of Proof. $f(z) - c$ has the same poles as f. □

Def 1.1. The order of a DPF: $\mathrm{Ord}(f) :=$ number of its poles on T^2.

2 Weierstrass $\wp$

In this section we introduce the "simplest" nonconst DPF: the Weierstrass $\wp$.

Def 2.1. Fix a free $\mathbb{Z}$-module M contained in $\mathbb{C}$ of rank 2. Define

$$\wp(z; M) = \frac{1}{z^2} + \sum_{\omega \in M - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Lemma 2.1. The series

$$\sum_{\omega \in M - \{0\}} \omega^{-s}$$

converges iff $s > 2$.

Sketch of Proof. There exists $\Delta > \delta > 0$ such that for all pair of integers (m, n)

$$\Delta|m\omega_1 + n\omega_2|^2 \geq m^2 + n^2 \geq \delta|m\omega_1 + n\omega_2|^2.$$

Therefore $\sum_{\omega \in M-\{0\}} \omega^{-s}$ converges iff

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{0\}} (m^2+n^2)^{-s/2}$$

converges, which is equivalent to

$$\iint_{x^2+y^2\geq 1} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{-s/2}} = \int_0^{2\pi}\int_1^{+\infty} \frac{\mathrm{d}r\mathrm{d}\theta}{r^{s-1}}.$$

□

Lemma 2.2. $\wp(z;M)$ defines an even meromorphic function of order 2 whose poles situated in M.

Sketch of Proof. For $|\omega| \geq 2|z|$ we have

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \frac{|z||(z-2\omega)|}{|z-\omega|^2|\omega|^2} \leq \frac{4|z|}{|\omega|^3}.$$

Hence it converges uniformly in any compact. □

Prop 2.1.

$$\wp'(z;M) = -2\sum_{\omega\in M}\frac{1}{(z-\omega)^3}$$

defines an odd DPF of order 3 whose poles situated in M.

Sketch of Proof. odd trival; order 3 trival; □

Thm 2.3. $\wp(z;M)$ defines an even DPF of order 2 whose poles situated in M.

Sketch of Proof. We only need to check its periodicity. Let ω be 1/2 generator of M we have $\wp'(z+\omega) = \wp'(z)$, hence $\wp(z+\omega) - \wp(z)$ is constant. Let $z = -\frac{1}{2}\omega \notin M$ we get $\wp(z+\omega) - \wp(z) = \wp(\frac{1}{2}\omega) - \wp(-\frac{1}{2}\omega) = 0$. □

Note that there is an odd periodic function we can give some immediate properties.

Prop 2.2. $\wp'(z;M)$ has 3 simply zeros on T^2.

Sketch of Proof. Write $M = \mathrm{Span}(\omega_1, \omega_2)$. Note that $\mathrm{Ord}(\wp') = 3$ and that $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$ are 3 different zeros. □

remark. As a corollary $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$ is independent of the choice of the base of M (?).

Thm 2.4. Define $\wp(\omega_1/2) = e_1$, $\wp(\omega_2/2) = e_2$, $\wp((\omega_1+\omega_2)/2) = e_3$. Then

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Sketch of Proof. Prop2.2 tells us that

$$\frac{(\wp')^2}{(\wp - e_1)(\wp - e_2)(\wp - e_3)}$$

is holomorphic a la fois doubly periodic. Hence it must be a constant. We determine this const by viewing the coefficients of their Laurent expansion around 0... □

By the similar method one can construct all DPF using $\wp$. Clearly all DPF having the periodic module M forms a field. We denote it by $E(M)$.

Thm 2.5.

$$E(M) = \mathbb{C}(\wp) + \wp'\mathbb{C}(\wp)$$

Sketch of Proof. Note that for any $F \in E(M)$ we can write F as a sum of an odd function F_o and an even function F_e, and that $F_o/\wp'$ is even, we only need to prove that even DPF having period M is given by $\mathbb{C}(\wp)$. We choose a fundamental parallelogram of F. Clearly $\mathrm{Ord}(F)$ is even. Let $(a_i, -a_i)_{i=1}^n$ be all its zeros and $(b_i, -b_i)_{i=1}^n$ be all its poles. Let

$$G(z) = \frac{\prod_{i=1}^n (\wp(z) - \wp(a_i))}{\prod_{i=1}^n (\wp(z) - \wp(b_i))}.$$

Recall that $\wp$ is of order 2, hence G has exactly the same zeros and poles as F. Therefore F/G is constant. □

Determine the Laurent expansion of $\wp$ around 0. Note that $\wp(z) - \frac{1}{z^2}$ is holomorphic when $0 < |z| < \min_{\omega \in M} |\omega|$. Write

$$\wp(z) - \frac{1}{z^2} = f(z) = \sum_{i=1}^{\infty} a_{2n} z^{2n}.$$

Knowing that

$$f^{(n)}(z) = (-1)^n (n+1)! \sum_{\omega} \frac{1}{(z-\omega)^{n+2}},$$

hence $a_2 n = \frac{f^{(2n)}(0)}{(2n)!} = (2n+1) \sum_{\omega} \frac{1}{\omega^{2n+2}}$.

Prop 2.3. For z near 0 we have

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2} z^{2k},$$

where

$$E_k = \sum_{\omega \in M - \{0\}} \omega^{-k}.$$

E_k is called the Eisenstein series of order k.

Sketch of Proof. ↑ □

Some properties of the Eisenstein series.

Prop 2.4.

$$(\wp')^2 = 4\wp^3 - 60 E_4 \wp - 140 E_6.$$

Sketch of Proof. $(\wp')^2 - 4\wp^3 + 60E_4\wp + 140E_6$ is holomorphic near 0 and s'annule at 0. □

Thm 2.6. Given M, $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ in T^2. There exists a DPF of period M having zeros $(a_i)_{i=1}^n$ and poles $(b_i)_{i=1}^n$ in T^2 iff

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \mod M.$$

Sketch of Proof. ⇒ Thm1.4. ⇐ Imagine that we have an entire function σ having one simply zero at $0 + M$, then

$$F(z) = \frac{\prod_{i=1}^n \sigma(z - a_i)}{\prod_{i=1}^n \sigma(z - b_i)}$$

has the zeros and poles required. To get the periodicity note that

$$F(z+\omega)/F(z) = \frac{\prod_{i=1}^n \sigma(z + \omega - a_i)/\sigma(z - a_i)}{\prod_{i=1}^n \sigma(z + \omega - b_i)/\sigma(z - b_i)}$$

we may ask $\sigma(z+\omega)/\sigma(z) = e^{u(\omega)z+t(\omega)}$ and choose some parallelogram so that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, then

$$F(z+\omega)/F(z) = \frac{\prod_{i=1}^n e^{u(\omega)(z-a_i)+t(\omega)}}{\prod_{i=1}^n e^{u(\omega)(z-b_i)+t(\omega)}} = e^{u(\omega)(\sum_{i=1}^n a_i - \sum_{i=1}^n b_i)} = 1.$$

Then we give the definition of σ. Define

$$\sigma(z) = z \prod_{\omega \in M-\{0\}} (1 - \frac{z}{\omega}) e^{P(z/\omega)}$$

where $P(z) = z + \frac{z^2}{2}$. We will explique the reason why σ satisfies our require.

□

Because $\wp$ has zero residues it is the derivative of a single-valued function. We denote it by $-\zeta$.

Prop 2.5.

$$\sum_{\omega \in M-\{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

converges near 0.

Sketch of Proof. derivate □

Def 2.2.

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in M-\{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Prop 2.6. $\zeta'(z) = \wp(z)$; $\zeta(z+\omega) = \zeta(z) + \eta(\omega)$ for $\omega \in M$.

Sketch of Proof. T □

Prop 2.7. Let ω_1, ω_2 be a base of M and $\eta_i = \zeta(z+\omega_i) - \zeta(z)$ for $i = 1, 2$. Then

$$\det \begin{pmatrix} \eta_1 & \omega_1 \\ \eta_2 & \omega_2 \end{pmatrix} = 2\pi i.$$

Sketch of Proof. Choose some parallelogram P. By the residue theorem

$$\frac{1}{2\pi i} \int_{\partial P} \zeta \mathrm{d}z = 1.$$

□

Prop 2.8. $\sigma'/\sigma = \zeta$.

Sketch of Proof. Do some calculations! □

This yields

$$\frac{\sigma'(z+\omega)}{\sigma(z+\omega)} = \frac{\sigma'(z)}{\sigma(z)} + \eta$$

therefore $\log \sigma(z+\omega) = \log \sigma(z) + \eta z + C_1$ for some C_1 (we now complete Thm2.6). On setting $z = \omega/2$ we get:

Prop 2.9.

$$\sigma(z+\omega) = -\sigma(z) e^{\eta(z+\omega/2)}.$$

Sketch of Proof. $\uparrow$ $\square$

Thm 2.7. Given M and $z, u \in \mathbb{C} - M$. We have

$$\wp(z+u) = 0.25\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2 - \wp(z) - \wp(u).$$

Sketch of Proof. In fact we can check this formula by directly check their Laurent developpemenets, but I choose to copy the book of Ahlfors. By the usual way one can proof that

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)\sigma(u)\sigma(z)\sigma(u)}.$$

Taking logarithmic derivative to turn σ into ζ. Then derivate and symmetrize it. $\square$

Equivalently (?)

Thm 2.8.

$$\det\begin{pmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(z+u) & -\wp'(z+u) & 1 \end{pmatrix} = 0.$$

Sketch of Proof. This can be viewed as a corollary of Thm2.7, However we have the following shougeki no shinjitsu. Define

$$F(w) = \det\begin{pmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix}$$

Clearly F has form $A + B\wp(w) + C\wp'(w)$ with $C = \wp(z) - \wp(u) \neq 0$. Hence F is a DPF of order 3. Note that the poles of F are in M and that $F(z) = F(u) = 0$, therefore by Thm2.6 its third zero is $-(u+z)$. I really want to stay at your house. $\square$

remark. If we consider this theorem geometrically we may get a more shougeki no shinjitsu. This fact is so shougeki that I will start a new section.

3 Shougeki no Shinjitsu

An eliptic curve appears naturally in Prop2.4/Thm2.4. We shall study un peu its structure. Recall: $\wp$ and E_k is determined uniquely by M.

Prop 3.1. Fix M. Let $g_2 = 60E_4$ and $g_3 = 140E_6$ and

$$X = X(g_2, g_3) = \{(z.w) \in \mathbb{C}^2 \mid w^2 = 4z^3 - g_2 z - g_3\}.$$

Then we have the bijection

$$T^2 - \{0\} \to X(g_2, g_3),\ z + M \mapsto (\wp(z), \wp'(z)).$$

Sketch of Proof. Trival. $\square$

We consider X in $\mathbb{CP}^n$ (all 1-dimensional linear subspace of $\mathbb{C}^{n+1}$) and $\mathbb{A}^n(\mathbb{C})$ (of form $z_0 : (z_1 : z_2 : ... : z_n)$). Define

$$\hat{X} = \hat{X}(g_2, g_3) = \{(z : w : u) \in \mathbb{A}^2(\mathbb{C}) \mid w^2 u = 4z^3 - g_2 z u^2 - g_3 u^3\}.$$

Then we got the bijection

$$\Phi : \mathbb{C}/M \to \hat{X},\ z + M \mapsto (\wp(z) : \wp'(z) : 1).$$

Horonable mention: $\Phi(M) = (1 : 0 : 0)$. Note that $\mathbb{C}/M$ is an abelian group. Thus we can define a group structure on $\hat{X}$ such that Φ is an isomorphism. Its neutre element $O = \Phi(M) = (1 : 0 : 0)$. This definition explains clearly the group structure of an eliptic curve.

Thm 3.1. For $P, Q, R \in \hat{X}$,

$$P + Q + R = O \text{ iff } P, Q, R \text{ on a straight line}$$

Sketch of Proof. Let $\Phi^{-1}(P) = (1 : \wp(p), \wp'(p))$, cyc. P, Q, R on a straight line $\Leftrightarrow$

$$\det \begin{pmatrix} \wp(p) & \wp'(p) & 1 \\ \wp(q) & \wp'(q) & 1 \\ \wp(r) & \wp'(r) & 1 \end{pmatrix} = 0$$

by Thm 2.8 $\Leftrightarrow p + q + r \in M \Leftrightarrow P + Q + R = \Phi(p + q + r + M) = \Phi(M) = 0$. $\square$

Moreover Φ is an homeomorphism(?). To make sure that this explanation is general we need to check that: for all(?) pairs $(u, v) \in \mathbb{C} \times \mathbb{C}$ there exists a periodic module M so that $(g_2, g_3) = (u, v)$. We start a new section.

4 Unimodular Transformations

In this section we study the transformation between different periodic module. We consider an equivalent relation: periodic module $M_1 \sim M_2$ iff there exists $\alpha \in \mathbb{C}$ so that $M_1 = \alpha M_2$. A DPF $F(z)$ of period M_1 induce a DPF of M_2 by $F(z/\alpha)$.

Lemma 4.1. Each periodic module is equivalent to some

$$M(\tau) = \mathbb{Z} \oplus \mathbb{Z}\tau,\ \tau \in \mathbb{H}.$$

Sketch of Proof. T $\square$

Now suppose that $M(\tau) = \alpha M(\rho)$.

Clearly $\alpha M(\rho) \subset M(\tau)$ is equivalent to, for some $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{Z}$,

$$\alpha \begin{pmatrix} \rho \\ 1 \end{pmatrix} = X \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

$M(\tau) \subset \alpha M(\rho)$ is equivalent to, for some $Y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $a', b', c', d' \in \mathbb{Z}$,

$$\alpha Y \begin{pmatrix} \rho \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Therefore

$$YX \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \alpha Y \begin{pmatrix} \rho \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Note that elements of YX are real, thus YX has 2 independent eigenvectors of eigenvalue 1, which yields

$$YX = I$$

. Moreover we can do some calculations to prove that for $\mathrm{Im}\rho > 0$ we must have $\det X = 1$. Therefore we have proven the following

Thm 4.2. $M(\rho) \sim M(\tau)$ iff there exists

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$$

such that $\rho = X\tau$.

Sketch of Proof. $\uparrow$ □

remark. We shall write $\rho = X\tau$ to express $\rho = \frac{a\tau+b}{c\tau+d}$ when no confuse.

To solve the problem in section 3 we need to study some general properties of Eisenstein series.

Prop 4.1. Given M and $\alpha \in \mathbb{C}$. $E_k(\alpha M) = \alpha^{-k} E_k(M)$.

Sketch of Proof. T □

Prop 4.2. For $k \geq 3$ Define $E_k(\tau) = E_k(\mathbb{Z} \oplus \mathbb{Z}\tau)$. Then $E_k(\tau)$ is holomorphic in $\mathbb{H}$.

Sketch of Proof. T □

Prop 4.3. For $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$ we have $E_k(X\tau) = (c\tau + d)^k E_k(\tau)$.

Sketch of Proof. T □

Prop 4.4. For even $k \geq 4$

$$\lim_{\mathrm{Im}\tau \to \infty} E_k(\tau) = 2\zeta(\tau).$$

Sketch of Proof. ? □

Prop 4.5. For all $\tau \in \mathbb{H}$ there exists $X \in \Gamma$ s.t.

$$|X\tau| \geq 1;\ -1/2 \leq \mathrm{Re}X\tau \leq 1/2.$$

Sketch of Proof. T $\square$

Thm 4.3. There exists periodic module M so that $(g_2(M), g_3(M)) = (u, v)$ if $u^3 - 27v^2 \neq 0$.

Sketch of Proof. Define

$$j(M) = \frac{g_2^3(M)}{g_2^3(M) - 27g_3^2(M)};$$

by Prop4.1 $j(\alpha M) = j(M)$. Therefore we can well define

$$j(\tau) = j(\mathbb{Z} \oplus \mathbb{Z}\tau).$$

By Prop4.2 j is holomorphic in $\mathbb{H}$. By Prop4.3 $j(X\tau) = j(\tau)$ for $X \in \Gamma$. Knowing that $j(\mathbb{H})$ is open, to have $j : \mathbb{H} \to \mathbb{C}$ is surjective we only need to check that $j(\mathbb{H})$ is closed. Take a convergent seuquence $(j(\tau_n))_{n\in\mathbb{N}}$. Note the mysterious fact that

$$\mathrm{lim}_{\mathrm{Im}\tau\to\infty} j(\tau) = \frac{?}{(60\pi^4/90)^3 - 27(140\pi^6/645)} = \infty.$$

and by Prop4.5 one can check that $(\tau_n)_{n\in\mathbb{N}}$ must be bounded... Thus for $u^3 - 27v^2 \neq 0$ there exists τ s.t

$$j(\tau) = \frac{u^3}{u^3 - 27v^2}.$$

Hence there exists M s.t. $g_2^3(M)/g_3^2(M) = u^3/v^2$. By Prop4.1 replace M by some αM and we have done. $\square$

5 Reference

On considering the fact that I am lazy and these books are all very classical, wonderful and important, I will only list the name of the books and authors...

Complex Analysis, Lars Ahlfors ...

Complex Analysis, Elias M.Stein, Rami Shakarchi ...

SURON I: Fermat No Yume To RUITAIRON, Kazuya Kato, Nobushige Kurokawa, Takeshi Saito ...

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