Notes on Representations of Groups

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1 Definitions

We use 1_G or 1 to denote the neutre element of group G.

Def 1.1. A (complex) representation of a group G on a complex linear space is a group homomorphism $\rho : G \to GL(V)$.Its dimension is the dimension of V as a linear space.

In the case GL(V) is the general linear group $GL_n(\mathbb{C})$ (the multiple group of all inversible matrix de taille n ayant elements in $\mathbb{C}$), ρ is called a matrix representation.

In the rest of this note we shall denote $\rho(g)(-)$ as $\rho_g(-)$ or $g(-)$ when there is no confuse.

Prop 1.1. Let G be a group and (ρ, V) and (π, W) be representations of G.Then $(\rho^\vee, V^\vee)$, $(\rho \oplus \pi, V \oplus W)$ and $(\rho \otimes \pi, V \otimes W)$ are also representations of G ($V^\vee$ means the dual space of V).

Sketch of Proof. Follow definition. □

Def 1.2. A representation $\rho : G \to GL(V)$ is said to be faithful if ρ is an injective.

In the finite-dimensional case, each basis of V induces an isomorphism between $GL(V)$ and $GL_n(\mathbb{C})$,and therefore each representation ρ induces a matrix representation:

$$\mathrm{P} : G \to GL_n(\mathbb{C}) \quad , \quad g \mapsto \mathrm{Mat}(\rho(g))$$

Obviously a change of basis in V changes a representation to a conjugate.

$v \in V$ is said to be G-invariant if for all $g \in G$ on obtient $g(v) = v$. A subspace W of V is said to be G-invariant if for any $g \in G$ we have $gW = W$.

In order to produce a G-invariant vector we have a useful technique called averaging :

Lemma 1.1. Let G be a finite group and (ρ, V) be a representation of G. For any $v \in V$, $v' := \frac{1}{|G|} \sum_{g \in G} g(v)$ is G-invariant.

Sketch of Proof. Trival. □

We have some first results on the decomposition of a representation of a finite group.

Def 1.3. Let G be a group and (ρ, V) and (π, W) be representations of G. A homomorphism from (ρ, V) to (π, W) is a linear application $f : V \to W$ so that the diagram suivant soit commutatif:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle (g)} & & \downarrow{\scriptstyle (g)} \\ V & \xrightarrow{f} & W \end{array}$$

Denote all homomorphism from (ρ, V) to (π, W) by $\mathrm{Hom}_G(V, W)$.

Prop 1.2. Let G be a finite group and (ρ, V) and (π, W) be representations of G. Suppose that f is a homomorphism from (ρ, V) to (π, W). Then both $\ker f$ and $\mathrm{im} f$ are G-invariant.

Sketch of Proof. Trival. $\square$

Def 1.4. Let G be a group and (ρ, V) a representation of G. If V is the direct sum of G-invatiant subspaces, say $V = W_1 \oplus W_2$, the representation is called the direct sum of its restrictions to W_1 and W_2. If V has no proper G-invariant direct sum factor, (ρ, V) is said to be irreducible.

Prop 1.3. (Schur's Lemma) Let G be a finite group and (ρ, V) (π, W) be irreducible representations of G. Then $\mathrm{Hom}_G(V, W)$ is a $\mathbb{C}$-linear space of dimension 0 or 1(iff V is isomorphic to W).

Sketch of Proof. For all $f \in \mathrm{Hom}_G(V, W)$, by the irreducibility we have $\ker f = 0$ or $\ker f = V$ i.e. f is an isomorphism or $f = 0$. If W and V is isomorphic, fix $f \in \mathrm{Hom}_G(V, W)$ non-zero. Clearly $f^{-1}\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, V)$. For $h \in \mathrm{Hom}_G(V, V)$ non-zero on considère l'un de ses valeurs propres $\lambda \neq 0$ et $v \in V$
0 tel que $h(v) = \lambda v$. We use again the irreducibility on $\ker(h - \lambda 1_V)$ and therefore $\ker(h - \lambda 1_V) = V$ d'où $h = \lambda 1_V$. $\square$

Now we turn to the case of unitary representation and we shall show that each representation produce a unitary one, then we can prove our main theorem of this section. Recall: A complex linear space V together with a positive definite hermitian form $(-, -)$ is called an Hermitian space. $f \in \mathrm{End}(V)$ is said to be unitary if $(f(u), f(v)) = (u, v)$ for all $u, v \in V$. A unitary representation of G on a hermitian space V is a representation (ρ, V) satisfying for all $g \in G, \rho(g)$ is unitary, or equivalently$(\rho, U(n))$.

Prop 1.4. (Weyl) Let G be a finite group (ρ, V) be a representation of G where V is an Hermitian space. Then there exist a positive definite Hermitian form that is G-invariant on V.

Sketch of Proof. Define $\langle -, - \rangle$ as: $\langle u, v \rangle = \sum_{g \in G}(g(u), g(v))$. Clearly this is a positive definite Hermitian form. Prop 1.2 s'applique, we can see it is G-invariant. $\square$

Thm 1.2. (Mascheke) Each finite-dimensional representation on a finite group G is semisimple, i.e. a direct sum of some irreducible representations.

Sketch of Proof. Suppose that (ρ, V) is a finite-dimensional representation of G.If ρ is reducible, let W be a G-invariant proper subspace of V. By Prop 1.4 there exists a positive definite Hermitian form on V (as $\dim V < \infty$).

Let $W^\perp$ be the orthogonal supplement of W with respect to this Hermitian form. Comme V est de dimension finie nous obtenons $V = W \oplus W^\perp$ and $W^\perp$ is G-invariant. Note que dim$W$ est inférieure strictement à laquelle de V, par réccurence nous finissons notre démonstration. $\square$

By considering the decomposition of representation we can get some information about the structure of $GL_n(\mathbb{C})$.

Cor 1.3. Every finite subgroup of $GL_n(\mathbb{C})$ is conjugate to some subgroup of $\mathrm{U}(n)$.

Sketch of Proof. Let H be a finite subgroup of $GL_n(\mathbb{C})$ and consider the inclusion $\iota : H \hookrightarrow GL_n(\mathbb{C})$ as a faithful representation of H. Fix a H-invariant positive definite Hermitian form on $GL_n(\mathbb{C})$ and choose an orthogonal basis of $GL_n(\mathbb{C})$ under this Hermitian form. ι becomes a unitary representation on this Hermitian form, because of the change of basis it takes H to a conjugate of some subgroup of $\mathrm{U}(n)$. $\square$

As a corollary of this corollary, since all unitary matrix is diagonalizable, a matrix A of finite order in $GL_n(\mathbb{C})$ generate a finite cyclic subgroup of $GL_n(\mathbb{C})$,particularly A is also diagonalizable.

Exercice 1.1. Irreducible representations on a finite group are finite-dimensional.

Exercice 1.2. All irreducible representation of the dihedral group D_n is of dimension 1 or 2. Then determine all the irreducible representations of D_n.

Prop 1.5. Supposons que G soit an abelian group. Then any finite-dimensional irreducible representation of G is 1-dimensional.

Sketch of Proof. We take an irreducible representation (ρ, V). For all $g \in G$, ρ_g admits an eigenvalue $\lambda \in \mathbb{C}$ and the eigenspace V_λ. Note that G is abelian implies ρ_h and ρ_g are commutative, we have V_λ is G-invariant. Hence $V_\lambda = V$ i.e. ρ_g must be a scalar multiplication. By the irreducibility we get dim$V = 1$. $\square$

For a cyclic group it's easy to determine all its 1-dimensional representations, thus we have found all ireducible complex representations of an abelian group.

Prop 1.6. All 1-dimensional representation of a finite group G can be owned as a lifting of laquelle de $G/[G, G]$.

Sketch of Proof. A 1-dimensional representation of G can be viewed as a group homomprphism from G to $\mathbb{C}^\times$. By the universal property of the quotient $G/[G, G]$ given 1-dimensional representation of G provides a unique 1-dimensional representation of $G/[G, G]$ and we've done. $\square$

2 Characters

Inthe following part of this section G is always required to be a finite group. All linear space we will use is finite-dimensional , as we've seen before irreducible representations on a finite group are finite-dimensional. We use V^G to denote all of the G-invariant elements in V.

Def 2.1. Let (π, V) a representation of G. The character of π is a complex-valued fonction on G defined as $\chi_{\pi,V}(g) := \mathrm{tr}(\pi(g))$.

We will omit V and π in such notation when no confuse. Here we list some immediate properties of character.

Prop 2.1. Let (π, V) and (σ, W) be representations of G.

1) $\chi_{\pi\oplus\sigma} = \chi_\pi + \chi_\sigma$; $\chi_{\pi\otimes\sigma} = \chi_\pi\chi_\sigma$.

2) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)} = \chi_{\pi^\vee}(g)$.

3) $\chi_\pi(1) = \mathrm{dim}V$.

4) $\chi_\pi(h^{-1}gh) = \chi_\pi(g)$. That is to say, χ_π is constant on conjugacy classes.

5) Isomorphic representations have the same character.

6) If g is a k-order element of G, then the eigenvalues of $\pi(g)$ are some of the k-th roots of unity and therefore $\chi_\pi(g)$ is a sum of them.

Sketch of Proof. Trival. □

remark. As G is finite, by (6) we know that $\chi_\pi(g)$ is always an algebraic integer.

Prop 2.2.

$$\mathrm{dim}V^G = \frac{1}{|G|}\sum_{g\in G}\chi(g)$$

.

Sketch of Proof. Consider $T = \frac{1}{|G|}\sum_{g\in G}\pi(g) \in \mathrm{End}(V)$. Then $V^G = \ker(T - \mathrm{id}_V)$. By Lemma1.1 we have $T^2 = T$, hence $V = \ker(T - \mathrm{id}_V) \oplus \ker(T)$. Therefore $\frac{1}{|G|}\sum_{g\in G}\chi(g) = \mathrm{tr}(T)$ is the sum of $\mathrm{dimker}(T)$ times 0 and $\mathrm{dimker}(T - \mathrm{id}_V)$ times 1. □

Thm 2.1. (orthogonality relation) Let (π, V) and (σ, W) be irreducible representations of G.Then

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_\sigma(g)} = \begin{cases} 1, & \pi \simeq \sigma \\ 0, & \pi \nsim \sigma \end{cases}$$

.

Sketch of Proof. By using Prop2.1 (1) and (2) and Prop 2.2, we have

$$\sum_{g\in G}\chi_\pi(g)\overline{\chi_\sigma(g)} = \sum_{g\in G}\chi_{\pi\otimes\sigma^\vee}(g) = \mathrm{dim}(V\otimes W^\vee)^G = \mathrm{dimHom}_G(V, W)$$

By Schur's Lemma this is exactly what we expect. □

remark. We have the canonical isomorphism $\mathrm{Hom}(X\otimes Y, Z) = \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$. Let $X = V$, $Y = W^\vee$ and $Z = \mathbb{C}$, we get $V\otimes W^\vee = \mathrm{Hom}(V, W)$, hence $(V\otimes W^\vee)^G = \mathrm{Hom}_G(V, W)$. We can believe that $\mathrm{Hom}_G(V, W)$ is also a representation of G.

Cor 2.2. Let (π, V) and (σ, W) be representations of G and σ be irreducible. Then the times of W in V n (i.e. W appears for n times as a direct sum factor of V in the decomposition of V) is equal to $\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_\sigma(g)}$.

Sketch of Proof. Prop2.1(1) + Thm2.1 . □

Cor 2.3. (π, V) is irreducible iff $\sum_{g\in G} |\chi_\pi(g)|^2 = |G|$.

Sketch of Proof. $\Rightarrow$ is owned directly by Thm2.1. $\Leftarrow$ by contradiction. $\square$

Then we turn to the group algebra $\mathbb{C}[G]$. It is the space containing all finite sum of the form $\sum_{g\in G} c_g g$ where $c_g \in \mathbb{C}$, or equivalently the linear space generated by all complex-valued fonction on G. We define the convolution on $\mathbb{C}[G]$ by

$$f_1 * f_2(g) := \sum_{h\in G} f_1(h) f_2(h^{-1}g)$$

$\mathbb{C}[G]$ together with the usual addition and the convolution forms a finite-dimensional $\mathbb{C}$-algebra. One can check that a representation of G can be viewed as a $\mathbb{C}[G]$-module, which provides another way to study the representations. We have the two regular operations on G on $\mathbb{C}[G]$ defined naturally as follows.

Def 2.2. For $f \in \mathbb{C}[G]$ and $g, h \in G$, the left regular operation is defined as

$$\mathcal{L}_g(f)(x) := f(g^{-1}x)$$

the right regular operation is defined as

$$\mathcal{R}_g(f)(x) := f(xg)$$

Prop 2.3.

$$\chi_{\mathcal{R}}(g) = \begin{cases} |G|, & g = 1_G \\ 0, & g \neq 1_G \end{cases}$$

Sketch of Proof. Choose the base $\{f_g \in \mathbb{C}[G] \mid f_g(g) = 1; f_g(h) = 0 \text{ for } h \neq g,\ g \in G\}$ of $\mathbb{C}[G]$. Then

$$\chi_{\mathcal{R}}(g) = \mathrm{tr}(\mathcal{R}_g) = \sum_{h\in G} \mathcal{R}_g(f_h)(h) = \sum_{h\in G} f_h(hg)$$

$\square$

Prop 2.4.

$$|G| = \sum_{\pi} \chi_\pi(1)^2$$

The sum of RHS traversal all different irreducible representation (isomorphism type of) π.

Sketch of Proof. By Cor2.2 we have

$$\bigoplus_{\pi} (\pi, V)^{\oplus \dim V} \simeq (\mathcal{R}, \mathbb{C}[G])$$

where π traversal all different (isomorphism type of) irreducible representation of G. Calcul the dimension of the two sides and again we use Prop2.1. Then we have

$$|G| = \sum_{\pi} (\dim V)^2 = \sum_{\pi} \chi_\pi(1)$$

where π traversal all different (isomorphism type of) irreducible representation of G. $\square$

Now it is possible for us to determine the number of different irreducible representations on G. Let $\mathrm{cf}(G)$ be the complex linear space of all class fonction on G i.e. a complex-valued fonction that be constant on each conjugacy class of G. Note that there is a (complex) scalar multiplication on this space

$$\langle \chi, \phi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} = \sum_{C} \frac{|C|}{|G|} \chi(C)\overline{\psi(C)}$$

where C traversal all conjugacy class of G.

Thm 2.4. $\{\chi_\pi | \pi \text{ is irreducible}\}$ forms a orthonormal base of $\mathrm{cf}(G)$.

Sketch of Proof.

We only need to check that all the χ_π generate the whole space. This is equivalent to : for $f \in \mathrm{cf}(G)$,

$$\langle f, \chi_\pi \rangle = 0 \text{ for all } \pi \implies f = 0$$

Suppose that f is a such fonction. For all representation (ρ, V) define $\mathrm{P} := \frac{1}{|G|} \sum_{g \in G} f(g)\rho(g^{-1})$. For f is a class fonction

$$\rho_h^{-1} \mathrm{P} \rho_h = \frac{1}{|G|} \sum_{g \in G} f(g)\rho(h^{-1}g^{-1}h) = \frac{1}{|G|} \sum_{g \in G} f(g)\rho(g^{-1}) = \mathrm{P}$$

i.e. P is G-invariant. Note that $\mathrm{tr(P)} = \langle f, \chi_\rho \rangle = 0$. WLOG ρ is irreducible. Schur's Lemma tells that $\mathrm{P} \in \mathrm{End}_G(\rho, V)$ should be a scalar multiplication, so $\mathrm{tr(P)} = 0$ implies $\mathrm{P} = 0$. Now we take $(\mathcal{R}, \mathbb{C}[G])$ as (ρ, V). Particularly $(\mathcal{R}_g)_{g \in G}$ are linear independent, so the coefficients $f(g)$ must be 0, hence $f = 0$.

$\square$

Cor 2.5. The number of different isomorphism types of irreducible representations on G is the number of conjugacy classes in G, both of which is equal to $\dim \mathrm{cf}(G)$.

Now we can construct the character table of a group. The character table of a group G is the matrix $(\chi_i(C_j))_{1 \le i,j \le n}$ where $n = \dim \mathrm{cf}(G)$, χ_i and C_j lists all different irreducible characters and conjugacy classes of G. After knowing the conjugacy classes in G, the general method to construct the character table of a group is to write as many "evident" representations as possible, the use the sum of rows and columns which can be determined by the following orthogonality relations to fill the blanks:

$$\sum_{j=1}^{n} |C_j| \chi_i(C_j)\overline{\chi_{i'}(C_j)} = |G|\delta_{ii'}$$

$$\sum_{i=1}^{n} |C_j| \chi_i(C_j)\overline{\chi_i(C_{j'})} = |G|\delta_{jj'}$$

Exercice 2.1. Construct the character tables of S_3, S_4, A_4, A_5, D_n. D_8 and the quaternion group are non-isomorphic having the same character table.

Exercice 2.2. A group G is abelian iff all its irreducible representation is 1-dimensional. If G contains a abelian subgroup of indice m then the dimension of an irreducible representation of G is at most m.

Exercice 2.3. For $g \in G$, g and g^{-1} is conjugate iff its each irreducible character is in $\mathbb{R}$.

Then we put the two regular representations together. Consider the operation of $G \times G$ on $\mathbb{C}[G]$ owned by $\Phi_{g_1,g_2}(f) = (x \mapsto f(g_1^{-1} x g_2))$.

Def 2.3. A matrix coefficient of a representation (π, V) of G is a complex-valued fonction on G given by $v \in V$ and $f \in V^\vee$ defined as

$$\psi_{v,f}(g) = f(\pi_{g^{-1}}(v))$$

Clearly $\mathbb{C}[G]$ is always a Hilbert space when G is finite, then by Risez representation theorem, there exists a scalar multiplication $\langle -, - \rangle$ on V so that we can write the matrix coefficients as $\psi_{v,w}(g) = \langle \pi_{g^{-1}}(v), w \rangle = \langle v, \pi_g(w) \rangle$.

Lemma 2.6. Suppose that (π, V) and (σ, W) are irreducible representations of G. Then

$$\frac{1}{|G|} \sum_{g \in G} f(\pi_g(v)) h(\pi_{g^{-1}}(w)) = \begin{cases} 0, & \pi \nsim \sigma \\ \dfrac{1}{\dim(\pi)} f(w)h(v), & \pi \simeq \sigma \end{cases}$$

Sketch of Proof. ? $\square$

Thm 2.7. We have the (canonical) isomorphism of $G \times G$ representations

$$\bigoplus_\pi (\pi, V) \otimes (\pi^\vee, V^\vee) \simeq (\mathbb{C}[G], \Phi)$$

Sketch of Proof. First we have a homomorphism from $V \otimes V^\vee$ to $\mathbb{C}[G]$ by $v \otimes f \mapsto \psi_{v,f}$. One can check it is well-defined. The direct sum of these homomorphisms gives a homomorphism from $\bigoplus_\pi (\pi, V) \otimes (\pi^\vee, V^\vee)$ to $(\mathbb{C}[G], \Phi)$. Note that the two space own the same dimension and by Lemma2.6 ? this homomorphism is injective, it must be an isomorphism. $\square$

Lemma 2.8. Suppose that (π, V) is irreducible, then for all $g \in G$, $\frac{|C_g|\chi_\pi(g)}{\dim V}$ is an algebraic integer (C_g is the conjugacy class of g).

Sketch of Proof. Consider the linear map

$$T : V \to V, \quad v \mapsto \sum_{h \in C_g} \pi_h(v)$$

We have $\mathrm{tr}(T) = |C_g|\chi_\pi(g)$. By Schur's Lemma T is the scalar multiplication $\frac{|C_g|\chi_\pi(g)}{\dim V}\mathrm{id}_V$. Note that π is a subrepresentation of $\mathcal{R}$, T is the restriction of $S : v \mapsto \sum_{h \in C_g} \mathcal{R}_h(v)$ on V. Choose the base we used in Prop2.3 on which all elements of the matrix of S is 0 and 1, hence the eigenvalues of S are algebraic integers, and so is T. $\square$

Thm 2.9. Suppose that (π, V) is irreducible, then $\dim V \mid |G|$.

Sketch of Proof. Let $(C_i)_{1 \le i \le n}$ be all conjugacy classes of G. By the orthogonality relation we have

$$\frac{|G|}{\dim V} = \sum_{i=1}^{n} |C_i| \chi_\pi(C_i) \overline{\chi_\pi(C_i)}$$

By Lemma 2.8 the RHS of this equation is algebraic integer, but $\frac{|G|}{\dim V}$ is a rational, so it must be an integer. □

The theorem about sovablity of Burnside. We shall quote the following lemma from algebraic number theory.

Lemma 2.10. If $\alpha \in \mathbb{C}$ is a sum of n roots of unity and $\frac{\alpha}{n}$ is an algebraic integer, then either $\alpha = 0$ or all these roots of unity are equal.

Sketch of Proof. Wolaobaixingbudong □

Lemma 2.11. Let C be a conjugacy class of G, $|C| = p^t$ where p is a prime and $t \in \mathbb{N}^*$, then G has a normal subgroup.

Sketch of Proof. Obviously G is not abelian. Fix $g \in C$. Let $(\pi_i)_{1 \le i \le n}$ denote all irreducible representations of G where π_1 is trival. By orthogonality we have

$$\sum_{i=1}^{n} \chi_i(1)\overline{\chi_i(g)} = 0$$

Therefore

$$\sum_{i=2}^{n} \chi_i(1)\overline{\chi_i(g)} = -1$$

modulo p we get that there exists i WLOG 2 such that $\chi_2(g) \neq 0$ and $p \nmid \chi_2(1)$. Then $\gcd(|C|, \chi_2(1)) = 1$. Choose $a, b \in \mathbb{Z}$ such that $a|C| + b\chi_2(1) = 1$. Thus

$$\frac{\chi_2(g)}{\chi_2(1)} = a\frac{|C|\chi_2(g)}{\chi_2(1)} + b\chi_2(g)$$

By Lemma2.10 all the eigenvalues of $\pi_2(g)$ are the same. Hence $\pi_2(g)$ (and of the same reason, any elements of C) acts as a scalar multiplication. Therefore for all $h_1, h_2 \in C$, $h_1^{-1}h_2$ acts trivally on π_2. Let H the subgroup of G generated by $h_1^{-1}h_2$, $h_1, h_2 \in C$. Clearly it is n ormal and non-trival (clearly $H \neq \{1_G\}$ and $H \neq G$ for H acts trivally on π_2 which is a non-trival representation of G). □

Thm 2.12. (Burnside $p^a q^b$) A group of order $p^a q^b$, where p, q are primes, is solvable.

Sketch of Proof. We only focus on the non-abelian case. If else let G be the smallest non-solvable group of order $p^a q^b$. By Lemma2.11, any conjugacy class C of G is singleton or $pq \mid |C|$. Modulo pq on the class equation of G, we have G contains a non-trival centre. As G is non-abelian, $Z(G)$ is a such group strictly smaller then G, a contradiction! □

The Fourier transformation of an Finite Abelian group. When G is abelian we know that all its elements are self-conjugated. Hence $\mathrm{cf}(G) = \mathbb{C}[G]$. Then we have the following prop.

Prop 2.5. Suppose that G is abelian of order n. The n 1-dimensional characters of G $(\chi_i)_{1 \le ilen}$ forms a orthonormal base of $\mathbb{C}[G]$, thus any $f \in \mathbb{C}[G]$ can be written in the form

$$f = \sum_{i=1}^{n} \langle f, \chi_i \rangle \chi_i$$

.

Def 2.4. Define $\hat{f}(g) = \sqrt{|G|}\langle f, \chi_i \rangle$. $f \mapsto \hat{f}$ provides a transformation Φ on $\mathbb{C}[G]$. Φ is called the Fourier transformation on G.

Prop 2.6.

$$\hat{f}(g) = \frac{1}{\sqrt{|G|}} \sum_{x \in G} f(x)\overline{\chi_g(x)}.$$

$$f(x) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \hat{f}(g)\chi_g(x).$$

Φ is unitary over $\mathbb{C}[G]$.

Sketch of Proof. Tr □

Prop 2.7. (Parseval Equation)

$$\sum_{g \in G} \hat{f}_1(g)\overline{\hat{f}_2(g)} = \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Sketch of Proof. Calculer-le. □

3 Induced Representations

Induced representations give us a method to construct a representation of a large group given a representations of a smaller group.

Def 3.1. Suppose that G is a group and H its subgroup, (σ, V) is a representation of H. The induced representation of $(\mathrm{Ind}_H^G \sigma, \mathrm{Ind}_H^G V)$ is defined as

$$\mathrm{Ind}_H^G V = \{f : G \to V \mid f(hg) = \sigma_h(f(g))\}$$

and

$$\mathrm{Ind}_H^G \sigma : g \mapsto (f \mapsto (x \mapsto f(xg)))$$

We write Ind instead of Ind_H^G when no confuse.

remark. $\mathrm{Ind}_H^G V$ is naturally isomorphic to $\mathrm{Hom}_H(\mathbb{C}[G], V)$. The canonical isomorphism given by

$$f \mapsto (\sum_i c_i g_i \mapsto \sum_i c_i f(g_i))$$

The definition of induced representation is to say that write $G = \bigcup_{i=1}^{[G:H]} g_i H$(the union of H-cosets), then give $[G : H]$ copies of V and let each $g_i H$ acts on one copy of V of the same H.

Prop 3.1.

$$\mathrm{dim}\mathrm{Ind}\sigma = [G : H]\mathrm{dim}\sigma$$

Sketch of Proof. Trival... □

Thm 3.1. (Mackey Formula) Write $G = \bigcup_{i=1}^{[G:H]} Hg_i$. Let σ be a representation of H and let $\mathrm{Ind}\sigma = \pi$. Then

$$\chi_\pi(g) = \sum_i \chi_\sigma(g_i g g_i^{-1}) = \frac{1}{|H|} \sum_{x \in G, x^{-1}gx \in H} \chi_\sigma(x^{-1}gx)$$

Sketch of Proof. Define $V_i = \{f \in \mathrm{Ind}V \mid \mathrm{supp}(f) \subset Hg_i\}$. Clearly we have $\mathrm{Ind}V = \bigoplus_i V_i$, hence $\chi_\pi(g) = \sum_i \chi_{V_i}(g)$, where χ_{V_i} is the trace of $\pi_g|_{V_i}$. For $f \in V_i$ since $\pi_g(f)(xg_i) = f(xg_i g)$ we have $\pi_g(f)(xg_i) \neq 0$ iff $xg_i g \in Hg_i$. In other words if $Hg_i g \neq Hg_i$ then $\pi_g(f)(Hg_i) = 0$. Note that $Hg_i g = Hg_i$ iff $g_i g g_i^{-1} \in H$ therefore

$$\chi_\pi(g) = \sum_i \chi_{V_i}(g) = \sum_{g_i g g_i^{-1} \in H} \chi_{V_i}(g) = \sum_i \chi_\sigma(g_i g g_i^{-1})$$

$\square$

Thm 3.2. (Frobenius Reciprocity) H is a subgroup of G. Let π be a representation of G and σ be a representation of H. We have the natural isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}\sigma, \pi) = \mathrm{Hom}_H(\sigma, \pi|_H)$$

In other words Ind_H^G and $|_H$ is a pair of adjoint functors.

Sketch of Proof. We define

$$S : \mathrm{Hom}_G(\mathrm{Ind}\sigma, \pi) \to \mathrm{Hom}_H(\sigma, \pi|_H),\ \alpha \longmapsto (v \mapsto \alpha(v))$$

v is considered as a constant map in $\mathrm{Ind}(\sigma)$, and

$$T : \mathrm{Hom}_H(\sigma, \pi|_H) \to \mathrm{Hom}_G(\mathrm{Ind}\sigma, \pi),\ \mathcal{A} \longmapsto (f \mapsto \mathcal{A}f(1_G))$$

One can check that S and T are well-defined homomorphisms, $S \circ T = \mathrm{id}_{\mathrm{Hom}_H(\sigma, \pi|_H)}$ and that $T \circ S = \mathrm{id}_{\mathrm{Hom}_G(\mathrm{Ind}\sigma, \pi)}$. I will not type it here using tex because I am lazy, but it is strongly recommended to do this exercice. $\square$

Thm 3.3. (Frobenius Reciprocity) Let μ be a character of H, χ be a character of G. Then

$$\langle \mathrm{Ind}\mu, \chi \rangle_G = \langle \mu, \chi|_H \rangle_H.$$

Sketch of Proof. By Mackey Formula we have

$$\langle \mathrm{Ind}\mu, \chi \rangle_G = \frac{1}{|G|} \sum_g \mathrm{Ind}\mu(g)\overline{\chi(g)} = \frac{1}{|G||H|} \sum_{g \in G} \sum_{x^{-1}gx \in H} \mu(x^{-1}gx)\overline{\chi(g)} = \frac{1}{|G||H|} \sum_{x^{-1}gx \in H} \sum_{g \in G} \mu(x^{-1}gx)\overline{\chi(g)}$$

$$= \frac{1}{|G||H|} \sum_{h \in H} \sum_{xhx^{-1} \in G} \mu(h)\overline{\chi(xhx^{-1})} = \frac{1}{|G||H|} \sum_{h \in H} \sum_{h \in G} \mu(h)\overline{\chi(h)} = \frac{1}{|H|} \sum_{h \in H} \mu(h)\overline{\chi(h)} = \langle \mu, \chi|_H \rangle_H.$$

$\square$

Let H be a subgroup of G and suppose that we have an irreducible character χ of H, alors nous pouvons nous demander: $\mathrm{Ind}\chi$ est-il irreducible? By Cor2.3 il faut qu'on calcule la somme $\sum_{g \in G} |\mathrm{Ind}\chi(g)|^2$. The Frobenius Reciprocity tells us that $\langle \mathrm{Ind}\chi, \mathrm{Ind}\chi \rangle_G = \langle \chi, (\mathrm{Ind}\chi)|_H \rangle_H$.

Thm 3.4. (Mackey) Let H_l be subgroups of G, σ_l is an irreducible representation of H_l for $l = 1, 2$. Write $G = \bigcup_{i=1}^r H_2 g_i H_1$. We have the isomorphism of representations of H_1

$$(\mathrm{Ind}_{H_2}^G \sigma_2)|_{H_1} \simeq \bigoplus_{i=1}^r \mathrm{Ind}_{H_1 \cap g_i^{-1} H_2 g_i}^{H_1} \sigma_2^{g_i}$$

where σ_2^g is a representation of $g^{-1} H_2 g$ defined as $\sigma_2^g(h) = \sigma_2(g^{-1} h g)$.

Sketch of Proof. Define $V_i = \{f \in \mathrm{Ind}_{H_2}^G \sigma_2 \mid \mathrm{supp}(f) \subset H_2 g_i H_1\}$. Then

$$(\mathrm{Ind}_{H_2}^G \sigma_2)|_{H_1} \simeq \bigoplus_{i=1}^r V_i$$

Il nous reste encore de montrer que $V_i \simeq \mathrm{Ind}_{H_1 \cap g_i^{-1} H_2 g_i}^{H_1} \sigma_2^{g_i}$. We define

$$T_i : V_i \to \mathrm{Ind}_{H_1 \cap g_i^{-1} H_2 g_i}^{H_1} \sigma_2^{g_i},\ f \longmapsto (h \mapsto f(g_i h))$$

One can check that this gives an isomorphism.

□

Cor 3.5. Write $G = \bigcup_{i=1}^r H g_i H$, $g_1 = 1_G$ and $H_i = H \cap g_i H g_i^{-1}$. Then for an irreducible character μ of H, Indμ is irreducible iff $\langle \mathrm{Ind}_{H_i}^H \mu^{g_i}, \mu \rangle_{H_i} = 0$ for $2 \leq i \leq r$.

When N is a normal subgroup of G, we have $NgN = gN$ and $g^{-1} N g = N$, thus $\mathrm{Ind}_{N_i}^N \mu^{g_i} = \mu^{g_i}$, d'où we can simplify this corollary.

Cor 3.6. Write $G = \bigcup_{i=1}^r g_i N$, $g_1 = 1_G$. Then for an irreducible character of N μ, Indμ is irreducible iff μ and μ^{g_i} is not same for $2 \leq i \leq r$.

We end this section by citer two theorems profondes without proof car wolaobaixingbudong.

Thm 3.7. (Artin) Suppose π is a representation of G. Then for all cyclic subgroup C of G there exists $a_C \in \mathbb{Q}$ such that $\chi_\pi = \sum_C a_C \mu_C$ where μ_C is the trival representation of C.

A subgroup of G is said to be elementary if it is isomorphic au produit of a p-group and a cyclic group of order ne divise pas p.

Thm 3.8. (Brouwer) Suppose π is a representation of G. Then for all elementary subgroup E of G there exists $a_E \in \mathbb{Z}$ such that $\chi_\pi = \sum_E a_E \mu_E$ where μ_E is the trival representation of E.

4 Complex Representations of $GL_2(F_q)$

F_q is the finite field containing q elements with $2 \nmid q$.

4.1 The conjugacy classes of $GL_2(F_q)$, some definitions

Prop 4.1. $|GL_2(F_q)| = (q^2-1)(q^2-q)$.

Prop 4.2. Fix $\tau \in F_q^\times \backslash (F_q^\times)^2$. There are 4 types of conjugacy classes in $GL_2(F_q)$:

1)Scarlar(Center), $\begin{pmatrix} x & \\ & x \end{pmatrix}$, $x \in F_q^\times$ provides $q-1$ conjugacy classes containing 1 element.

2)Parabolic, $\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$, $x \in F_q^\times$ provides $q-1$ conjugacy classes containing q^2-1 element.

3)Hyperbolic, $\begin{pmatrix} x & \\ & y \end{pmatrix}$, $x, y \in F_q^\times,\ x \neq y$ provides $\frac{1}{2}(q-1)(q-2)$ conjugacy classes containing q^2+q element.

4)Eliptic, $\begin{pmatrix} x & y \\ \tau y & x \end{pmatrix}$, $x \in F_q,\ y \in F_q^\times.\ x^2 - \tau y^2 \neq 0$ provides $\frac{1}{2}q(q-1)$ conjugacy classes containing q^2-q element.

remark. Clearly here conjugacy classes is equivalent to determine its Jordan form, but to verify this prop il suffit d'ajouter the number of conjugacy types of the 4 types listed above. We only expliquons how we get (4). Knowing that $F_{q^2} = F_q(\sqrt{\tau})$. For $A \in GL_2(F_q)$, if A admits two eigenvalues in F_q then it is of type (1)(2)or(3). For those who admit not an eigenvalue in F_q they will have two eigenvalues over $F_q(\sqrt{\tau})$ of form $x \pm y\sqrt{\tau}$ associées aux vecteurs v et $\overline{v}$. We choose the base $v + \overline{v},\ \sqrt{\tau}(v - \overline{v})$ and we got the matrix. Recall: To fix $\tau \in F_q^\times \backslash (F_q^\times)^2$ forces the character cannot be 2.

Def 4.1. We nominate 4 subgroups of $GL_2(F_q)$:

$$\text{Borel } B = \text{upper triangular matrices}$$

$$N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \simeq F_q$$

$$\text{Split Torus } T_s = \text{diagonal matrices, non zero} \simeq (F_q^\times)^2$$

$$\text{Asplit Torus } T_a = \left\{ \begin{pmatrix} x & y \\ \tau y & x \end{pmatrix} : x \in F_q,\ y \in F_q^\times.\ x^2 - \tau y^2 \neq 0 \right\} \simeq (F_{q^2})^\times$$

Prop 4.3. (Presentation of $SL_2(F_q)$)

$$SL_2(F_q) = \langle t_x, n_y, w \ \mid \ t_x t_y = t_{xy},\ n_x n_y = n_{x+y},\ t_x n_y t_x^{-1} = n_{x^2 y},\ w t_x w^{-1} = t_{x^{-1}},\ w n_x w^{-1} = t_{x^{-1}} n_{-x} w n_{-x^{-1}}. \rangle$$

Where

$$t_x = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix},\ n_y = \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix},\ w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Sketch of Proof. WHO cares $\square$

4.2 1-dimensional representations

By Prop1.6 we shall conmmencer par consider $[GL_2(F_q), GL_2(F_q)]$.

Prop 4.4.

$$[GL_2(F_q), GL_2(F_q)] = SL_2(F_q).$$

Sketch of Proof. Checked by Prop4.3. $\square$

Note that $GL_2(F_q)/SL_2(F_q) \simeq F_q^\times$ via $g \mapsto \det(g)$ and that $F_q^\times$ is a cyclic group, all its irreducible representations are given by

$$g \longmapsto \sigma(\det(g)) \text{ for any group homomorphism } \sigma : F_q^\times \to \mathbb{C}^\times.$$

Since σ is determined uniquely up to a $q-1$ unity root, there are $q-1$ such representations.

4.3 Principle series representations

Def 4.2. Given (λ_1, λ_2) a pair of homomorphism from $F_q^\times$ to $\mathbb{C}^\times$ provides a representation of B by

$$\begin{pmatrix} x & * \\ & y \end{pmatrix} \longmapsto \lambda_1(x)\lambda_2(y).$$

Denote its induction on $GL(F_q)$ by $I(\lambda_1, \lambda_2)$.

Prop 4.5. $\dim I(\lambda_1, \lambda_2) = q+1$

Sketch of Proof. Prop3.1 $\square$

Then check its irreducibility. Malheureusement B is not normal thus we can only use Cor3.5. Heureusement we have the following decomposition.

Prop 4.6. $GL_2(F_q)$ has the following double coset decomposition

$$GL_2(F_q) = B \cup BwB.$$

for the w in Prop4.3.

Sketch of Proof. BwB $\square$

Prop 4.7. $I(\lambda_1, \lambda_2)$ is irreducible iff $\lambda_1 \neq \lambda_2$.

Sketch of Proof. Let μ be its character. Note that $T_s = B \cap wBw^{-1}$ and that μ^w is the character $\mu(wxw^{-1})$ of T_s. We can easily check that $\mu(wxw^{-1})$ is the character of (λ_2, λ_1) on T_s. Cor3.5 tells us that $I(\lambda_1, \lambda_2)$ is irreducible iff $\langle \mathrm{Ind}_{T_s}^B \mu^w, \mu \rangle_{T_s} = 0$. We have

$$\langle \mathrm{Ind}_{T_s}^B \mu^w, \mu \rangle_{T_s} = \frac{1}{(q-1)^2} \sum_{g \in T_s} \mathrm{Ind}_{T_s}^B \mu^w(g) \overline{\mu(g)} = \frac{1}{(q-1)^2} \sum_{x,y \in F_q^\times} \lambda_2(x)\lambda_1(y)\overline{\lambda_1(x)\lambda_2(y)}$$

$$= \frac{1}{(q-1)^2} \sum_{x \in F_q^\times} \lambda_2(x)\overline{\lambda_1(x)} \sum_{x \in F_q^\times} \lambda_1(x)\overline{\lambda_2(x)} = \frac{1}{(q-1)^2} \left| \sum_{x \in F_q^\times} \lambda_2(x)\overline{\lambda_1(x)} \right|^2 = \begin{cases} 1, & \lambda_1 \simeq \lambda_2 \\ 0, & \lambda_1 \not\simeq \lambda_2 \end{cases}$$

And we've done. $\square$

Prop 4.8. $I(\lambda_1, \lambda_2) \simeq I(\mu_1, \mu_2)$ iff $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$ or $(\lambda_2, \lambda_1) = (\mu_1, \mu_2)$.

Sketch of Proof. By Schur's Lemma, $I(\lambda_1, \lambda_2) \simeq I(\mu_1, \mu_2)$ iff

$$\dim\mathrm{Hom}_{GL_2(F_q^\times)}(I(\lambda_1, \lambda_2), I(\mu_1, \mu_2)) = 1.$$

By Frobenius Reciprocity,

$$\dim\mathrm{Hom}_{GL_2(F_q^\times)}(I(\lambda_1, \lambda_2), I(\mu_1, \mu_2)) = \dim\mathrm{Hom}_B(\lambda_1\lambda_2, I(\mu_1, \mu_2)|_B).$$

By Thm3.4 and the proof of Prop4.7,

$$I(\mu_1, \mu_2)|_B = \mu_1\mu_2 \oplus \mathrm{Ind}_{T_s}^B(\mu_2\mu_1).$$

By Frobenius Reciprocity,

$$\dim\mathrm{Hom}_B(\lambda_1\lambda_2, I(\mu_1, \mu_2)|_B) = \dim\mathrm{Hom}_B(\lambda_1\lambda_2, \mu_1\mu_2) + \dim\mathrm{Hom}_B(\lambda_1\lambda_2, \mathrm{Ind}_{T_s}^B(\mu_2\mu_1))$$

$$= \dim\mathrm{Hom}_B(\lambda_1\lambda_2, \mu_1\mu_2) + \dim\mathrm{Hom}_{T_s}(\lambda_1\lambda_2, \mu_2\mu_1).$$

Hence $I(\lambda_1, \lambda_2) \simeq I(\mu_1, \mu_2)$ iff

$$\dim\mathrm{Hom}_B(\lambda_1\lambda_2, \mu_1\mu_2) = 1 \text{ or } \dim\mathrm{Hom}_{T_s}(\lambda_1\lambda_2, \mu_2\mu_1) = 1.$$

And we've done. □

Now we get $\frac{1}{2}(q-1)(q-2)$ (the number of parabolics) different irreducible principle series representations. We study the rest of the form $I(\lambda, \lambda)$.

Prop 4.9. $I(\lambda, \lambda)$ is the direct sum of the 1-dimensional representation $\lambda \circ \det$ and a q-dimensional irreducible representation, denoted $\mathbf{St}_\lambda$.

Sketch of Proof. First we verify that $I(\lambda, \lambda)$ is a direct sum of two irreducible representations. By Cor2.3 il suffit de check that $\sum_{g \in GL_2(F_q^\times)} |\lambda(g)|^2 = 2|G|$. Then we verify that $I(\lambda, \lambda)$ contains a factor $\lambda \circ \det$. To get this we only need to check that $\langle \lambda \circ \det, \lambda\lambda \rangle = 1$. These two summands can be calculated by discussing each of the 4 types of matrices. □

In this way we get $q - 1$ (autant que les hyperbolics) q-dimensional irreducible representations.

4.4 Oscillators; Cuspidal Representations

To get the remaining type of representations we introduce the Weil representation. First we introduce some notations.

Prop 4.10. F_{q^2} and $F_q \oplus F_q$ together with the scalar product $\langle x, y \rangle = \mathrm{Tr}(x^\sigma y)$ are 2-dimensional prehilbertian space over F_q.

In the following part we use E to denote F_{q^2} and $F_q \oplus F_q$, in the former case let $\epsilon = -1$, in the latter case let $\epsilon = 1$, and $\mathcal{S}(E)$ the linear space of complex fontionals over E.

Prop 4.11. Fix a non-trival group homomorphism $\psi : F_q \to \mathbb{C}^\times$. We define the Fourier transformation over $\mathcal{S}(E) : \phi \mapsto \hat{\phi}$,

$$\hat{\phi}(x) = \frac{\epsilon}{q} \sum_{y \in E} \phi(y)\psi(\langle x, y\rangle).$$

We have $\hat{\hat{\phi}}(x) = \phi(-x)$.

Sketch of Proof. This can be implied by Prop2.6, but let we calculate one more time here. Let $\Phi : \mathcal{S}(E) \to \mathcal{S}(E)$ denote the Fourier Transformation. We define

$$\Psi : \mathcal{S}(E) \to \mathcal{S}(E), \Psi(\phi)(x) = \frac{\epsilon}{q} \sum_{y \in E} \frac{\phi(y)}{\psi(\langle x, y\rangle)}.$$

Clearly $\Psi(\phi)(-x) = \Phi(\phi)(x)$. We need only verify that $\Psi\Phi = \mathrm{id}_{\mathcal{S}(E)}$. We obtain

$$\Psi\Phi(\phi)(x) = \frac{1}{q^2} \sum_{y \in E} \sum_{z \in E} \phi(z) \frac{\psi(\langle y, z\rangle)}{\psi(\langle x, y\rangle)} = \frac{1}{q^2} \sum_{z \in E} \phi(z) \sum_{y \in E} \frac{\psi(\langle y, z\rangle)}{\psi(\langle x, y\rangle)}.$$

Note that fix x, $\psi(\langle -, y\rangle)$ is homomorphism from E to $\mathbb{C}^\times$ and that E is an abelian group, by the orthogonality relation we have

$$\sum_{y \in E} \frac{\psi(\langle y, z\rangle)}{\psi(\langle x, y\rangle)} = \sum_{y \in E} \psi(\langle y, z\rangle)\overline{\psi(\langle x, y\rangle)} = q^2 \delta_{xz}.$$

and we've done. □

Prop 4.12. Fix a non-trival group homomorphism $\psi : F_q \to \mathbb{C}^\times$. There exists a unique (up to ψ) representation ω of $SL_2(F_q)$ over $\mathcal{S}(E)$ such that

$$\omega(t_a)\phi(x) = \phi(ax);\ \omega(n_b)\phi(x) = \psi(b\mathrm{N}(x))\phi(x);\ \omega(w)\phi(x) = \hat{\phi}(x).$$

ω is called the Weil representation of $SL_2(F_q)$.

Sketch of Proof. Check by Prop4.3. □

Prop 4.13. Let E_1 denote all elements in E having the norm 1. Fix a character χ of $E^\times$ such that $\chi|_{E_1}$ is non-trival. Define

$$\mathcal{S}(E)_\chi = \{\phi \in \mathcal{S}(E) \mid \phi(tx) = \chi(t)^{-1}\phi(x),\ t \in E_1,\ x \in E\}$$

Then $\mathcal{S}(E)_\chi$ is $SL_2(F_q)$-invariant; $\dim\mathcal{S}(E)_\chi = q + \epsilon$.

Sketch of Proof. Tout d'abord $\phi(0) = \frac{1}{\chi(t)}(0)$, since $\chi|_{E_1}$ is non-trival we get $\phi(0) = 0$. When $E = F_q \oplus F_q$, $\mathrm{N}(a, b) = ab$. Thus $\phi(tx_1, t^{-1}x_2) = \frac{1}{\chi(t,t^{-1})}\phi(x_1, x_2)$. Note that $x \mapsto x^2$ is an automorphism of $F_q^\times$, ϕ is uniquely determined by its value on $(0, 1), (1, 0), (x, x)$, hence $\dim\mathcal{S}(E)_\chi = q+1$. When $E = F_q(\sqrt{\tau})$, $\mathrm{N}(a+b\sqrt{\tau}) = a^2 - \tau b^2$. Note that N is a surjective over $F_q^\times$ and that for each pair of $(a + b\sqrt{\tau}, c + d\sqrt{\tau})$ having the same norm we have $\phi(a + b\sqrt{\tau}) = \frac{1}{\chi(\frac{c+d\sqrt{\tau}}{a+b\sqrt{\tau}})}\phi(c + d\sqrt{\tau})$, hence $\phi(x)$ is uniquely determined by $\mathrm{N}(x)$,therefore $\dim\mathcal{S}(E)_\chi = q - 1$.

□

Prop 4.14. Fix ψ and χ as above. There exists a unique representation ω_χ of $GL_2(F_q)$ over $\mathcal{S}(E)_\chi$ satisfying

$$\omega_\chi \begin{pmatrix} \mathrm{N}(t) & \\ & 1 \end{pmatrix} \phi(x) = \chi(t)\phi(tx) \text{ for } \phi \in \mathcal{S}(E)_\chi,\ t \in F_q^\times$$

and ω_χ agrees with ω on $SL_2(F_q)$.

5 Representations of S_n

6 Representations on Any Fields

In this section we try to generalize our theory to any fields and infinite groups.

7 Representations of Infinite Groups

8 Tannakian

9 Referrences