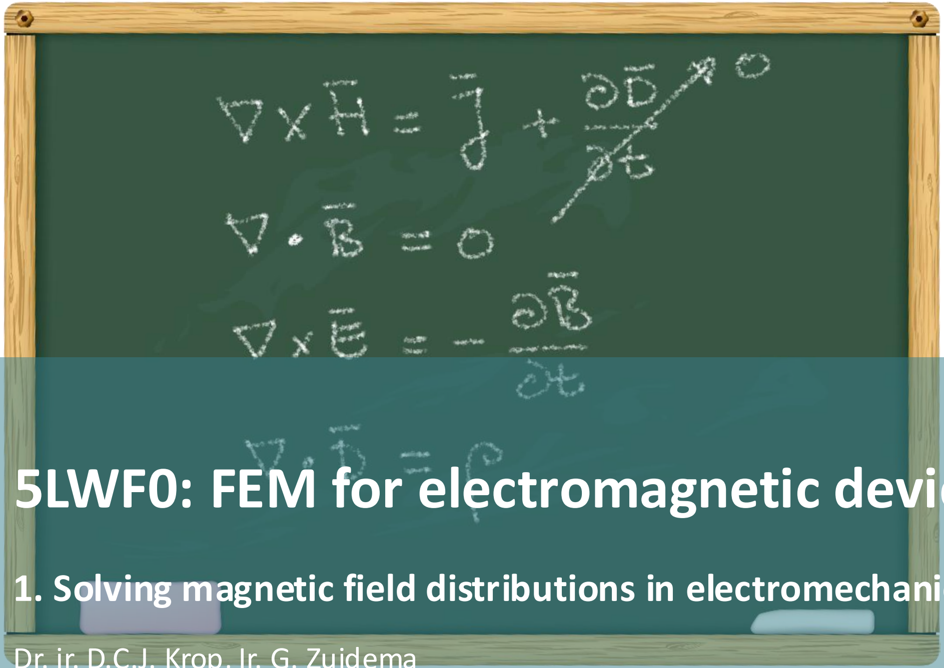

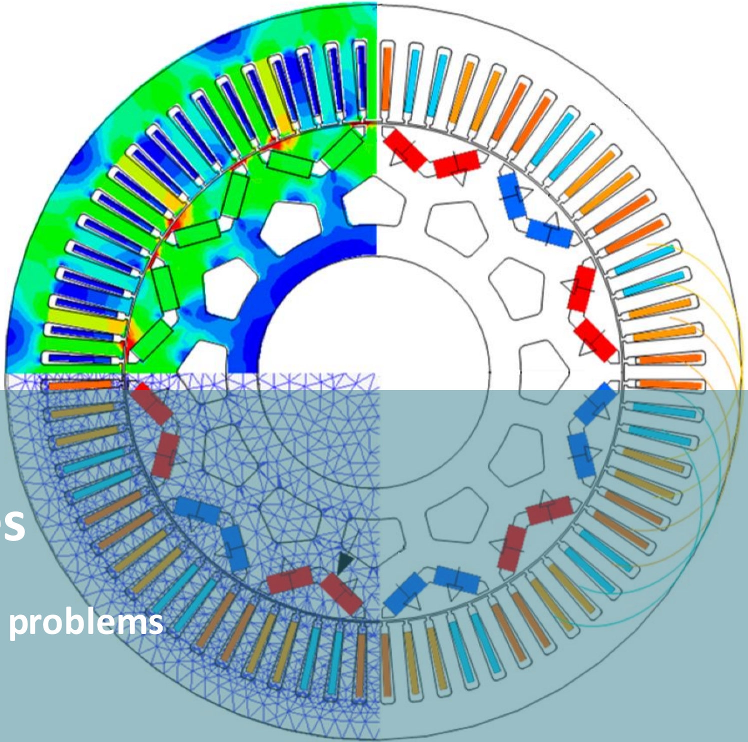

5LWF0: FEM for electromagnetic devices

1. Solving magnetic field distributions in electromechanical problems

Dr. ir. D.C.J. Krop, Ir. G. Zuidema
10 February 2023

Outline for the lecture of week 1

1. General information
   - Course material
   - Structure of the course
2. Recapitulation of Maxwell's equations
   - Differential and integral form
   - Physical interpretation
3. Solving <u>B</u>oundary <u>V</u>alue <u>P</u>roblems (BVPs) analytically (1D)
   - Recapitulation of solving differential equations for a toroidal inductor
4. Numerical methods for solving BVPs (1D)
   - Method of weighted residuals
   - Solving a BVP numerically (1D)

General information

Contact information

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Required teaching material for 5LWF0

- Notebook

- Finite Element software package:
  - Altair Flux2D 2022.0.1
  - Download from Canvas
    - *Software* folder
    - Unzip hwFlux2020.1_win64.7z
  - The software package FLUX is used under an **academic** license: **commercial use is strictly prohibited!**
  - The license server keeps logs of all FLUX activities!

Connection to other courses

Prior courses 5LWF0 builds on

- **5EWA0: Electromechanics**
  - Magnetic Equivalent Circuits (MECs)
  - DC-machines
  - Induction machines
  - Synchronous machines
- **5SWA0: Rotary PM machines**
  - Brushless PM machines (AC and DC)
  - Winding configurations and saturation
  - EMF, MMF, flux linkage, inductance, and torque production
  - dq0-axes decomposition

Follow-up courses

- **5LWC0: Advanced actuator design**
  - Analytical magnetic field calculation methods for electromagnetic actuators
- **5LWE0: Control of rotating-field machines**
  - Controlling BLPM machines
- **Graduation project within the EPE**
  - for electromechanics (5LWF0 required!)

Objectives of 5LWF0

- Setting up a 2D electromagnetic problem in the Finite Element Method (FEM) software environment FLUX2D
- Discretization aspects, i.e. how to properly discretize (or mesh) the problem in order to obtain a reliable electromagnetic field distribution
- Using the ‘overlay’ feature in the FLUX2D FEM package for the analysis/design of electromagnetic devices with focus on BLPM machines
- Interpretation of the magnetic field distribution on the performance of a BLPM machine

Exam

- Assignment :
  - Analysis of an electrical machine topology: 5 ECTS
  - Distributed to the students around the halfway-mark: week 4

- Written paper on your design approach and final design (60% , min. grade to pass: 5)
  - Max. 8 pages and stick to the predefined template
  - To be handed in 1 week prior to oral exam

- Oral exam (40%, min. grade to pass: 5)
  - Defend your design based on simulation results in combination with physical insight and design constraints
  - Time-slot based on student-ballot, but **not** later than 2 weeks after Q4 ends

Recapitulation of Maxwell’s equations

Recapitulation of Maxwell's equations

	Integral form	Differential form
Ampère's law	$\oint_{\mathcal{C}} \vec{H}(\vec{r},t)\cdot \mathrm{d}\vec{\ell} = \iint_{\mathcal{S}} \left[\vec{J}(\vec{r},t) + \frac{\mathrm{d}\vec{D}(\vec{r},t)}{\mathrm{d}t}\right]\cdot \mathrm{d}\vec{A}$	$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$
Gauss's law (magnetism)	$\oiint_{\mathcal{S}} \vec{B}(\vec{r},t)\cdot \mathrm{d}\vec{A} = 0$	$\nabla \cdot \vec{B}(\vec{r},t) = 0$
Faraday's law	$\oint_{\mathcal{C}} \vec{E}(\vec{r},t)\cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathcal{S}} \vec{B}(\vec{r},t)\cdot \mathrm{d}\vec{A}$	$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$
Gauss's law	$\oiint_{\mathcal{S}} \vec{D}(\vec{r},t)\cdot \mathrm{d}\vec{A} = \iiint_{\mathcal{V}} \rho(\vec{r},t)\,\mathrm{d}V$	$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t)$

Classical electromagnetic problems are governed by four coupled **partial differential equations** (**PDE**): Maxwell's equations

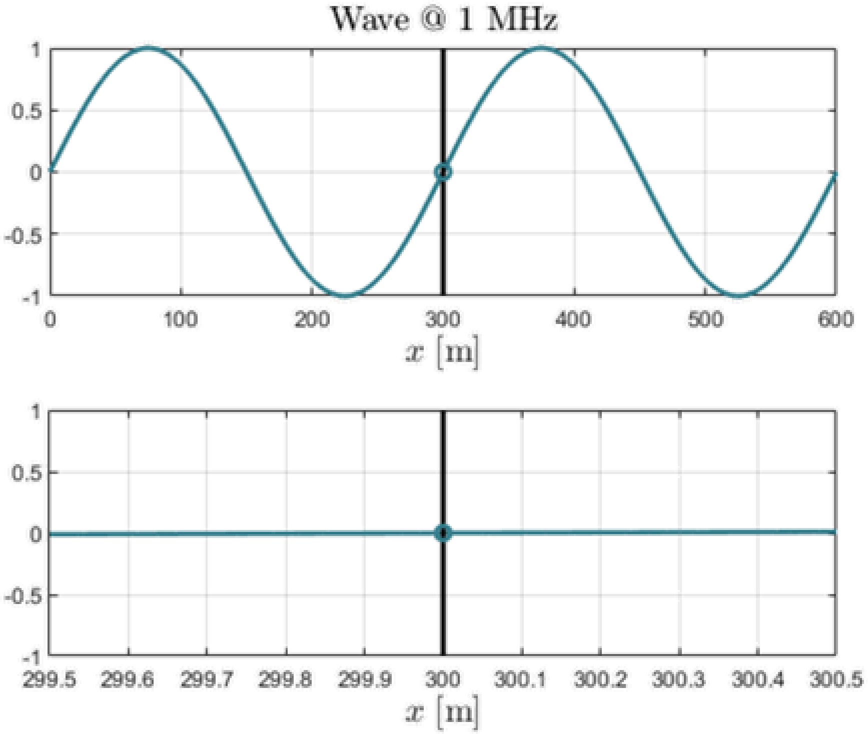
Recapitulation of the quasi-static formulation

Quasi-static formulation for the majority of ***electromechanical*** problems

- For problems much smaller in size than the wavelength, the displacement term vanishes

- Time changes of field quantities within small domains can be considered instantaneous

- In electromechanical problems the free-charge, ρ, is often not of interest



Recapitulation of the quasi-static formulation

	Integral form	Differential form
Ampère's law	$\oint_{\mathcal{C}} \vec{H}(\vec{r},t) \cdot \mathrm{d}\vec{\ell} = \iint_{\mathcal{S}} \vec{J}(\vec{r},t) \cdot \mathrm{d}\vec{A}$	$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t)$
Gauss's law (magnetism)	$\oiint_{\mathcal{S}} \vec{B}(\vec{r},t) \cdot \mathrm{d}\vec{A} = 0$	$\nabla \cdot \vec{B}(\vec{r},t) = 0$
Faraday's law	$\oint_{\mathcal{C}} \vec{E}(\vec{r},t) \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathcal{S}} \vec{B}(\vec{r},t) \cdot \mathrm{d}\vec{A}$	$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$

Constitutive relations (material properties)

The four Partial Differential Equations (PDEs) are coupled via the electromagnetic material properties

For **isotropic**, **linear** materials the constitutive relations due to materials are

Magnetic

$$\vec{B}(\vec{r},t) = \mu_0\mu_r\vec{H}(\vec{r},t) + \mu_0\vec{M}_0(\vec{r},t)$$

- $\boldsymbol{H}$: magnetic field strength [A/m]
- $\boldsymbol{B}$: magnetic flux density [T]
- $\boldsymbol{M_0}$: remanent magnetization [A/m]
- $\boldsymbol{\mu_0}$: (magnetic) permeability of vacuum [H/m]
  - value: $\mu_0 = \mathbf{4\pi \cdot 10^{-7}}$ H/m
- $\boldsymbol{\mu_r}$: relative (magnetic) permeability [-]

Electric

$$\vec{D}(\vec{r},t) = \varepsilon_0\varepsilon_r\vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$$

Of minor interest in 5LWF0!

- $\boldsymbol{E}$: electric field strength [V/m]
- $\boldsymbol{D}$: electric flux density [C/m^2]
- $\boldsymbol{P}$: polarization [C/m^2]
- $\boldsymbol{\varepsilon_0}$: (electric) permittivity of vacuum [F/m]
  - value†: $\varepsilon_0 = \mu_0^{-1}c^{-2} = \mathbf{8.854 \cdot 10^{-12}}$ F/m
- $\boldsymbol{\varepsilon_r}$: relative (electric) permittivity [-]

† c denotes the speed of light in vacuum, i.e. $c = 2.998 \cdot 10^8$ m/s

Constitutive relations (material properties)

- ***Hard*-magnetic materials: $M_0 \neq 0$**
  - Assumed to be **linear** when utilized properly! (In later lectures more)
- **Non-linear *soft*-magnetic materials: $M_0 = 0$ and $\mu_r\,(H)$**

Isotropic

***Scalar** material properties that depend only on the local **magnitude** of the fields*

$$\mu_r(\vec{r}) = f_\mu\left(\left|\vec{H}(\vec{r})\right|\right)$$

$$\Downarrow$$

$$\vec{B}(\vec{r}) = \mu_0 f_\mu\left(\left|\vec{H}(\vec{r})\right|\right)\vec{H}(\vec{r})$$

Anisotropic

***Tensor** material properties that depend on the local **spatial** components of the fields*

Of minor interest in 5LWF0!

$$\mu_r(\vec{r}) = f_\mu\left(\vec{H}(\vec{r})\right)$$

$$\Downarrow$$

$$\vec{B}(\vec{r}) = \mu_0 f_\mu\left(\vec{H}(\vec{r})\right) \cdot \vec{H}(\vec{r})$$

The magnetic vector potential (MVP) for 2D problems

By introducing the MVP the number of PDEs can be reduced

- Gauss's law for magnetism: $\nabla \cdot \vec{B} = 0$
- Applying the vector identity $\nabla \cdot (\nabla \times \vec{A}) = 0$. Hence,

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

$$\nabla \times (\nabla \varphi) = 0$$

- Reduction of a **system** of coupled PDEs to a single PDE (where reluctivity $\nu = \frac{1}{\mu}$).

2D elliptic PDE →

$$\nabla \cdot [\nu(x,y)\nabla A_z(x,y,t)] = -J_z(x,y,t)$$

Poisson equation for linear materials: constant reluctivity →

$$\nu \nabla^2 A_z(x,y,t) = -J_z(x,y,t)$$

Vector potential in 2D problems

In 2D problems the **flux density field** is confined to a single plane ➔ The **MVP** has only one nonzero spatial component as result of the curl operation

2D Cartesian	***B*** confined to the ***xy***-plane	$A_x(x,y,t) = A_y(x,y,t) = 0$	$A_z(x,y,t) \neq 0$
2D polar	***B*** confined to the ***rθ***-plane	$A_r(r,\theta,t) = A_\theta(r,\theta,t) = 0$	$A_z(r,\theta,t) \neq 0$
2D cylindrical	***B*** confined to the ***rz***-plane	$A_r(r,z,t) = A_z(r,z,t) = 0$	$A_\theta(r,z,t) \neq 0$

Additional source terms

Decomposition of the current density source term, $J_z = J_{\text{free}} + J_\sigma + J_M$:

- **Free-current term only:** an independently imposed current source

$$J_z(x, y, t) = J_{\text{free}}(x, y, t)$$

- **Eddy-current term only:** current flow in (homogeneous) conductive media (σ>0) when exposed to EMF inducing time-varying magnetic fields (Faraday's law)

$$J_z(x, y, t) = J_\sigma(x, y, t) = \sigma \vec{E}_z(x, y, t) = -\sigma \frac{\partial A_z(x,y,t)}{\partial t}$$

- **Permanent magnet term only:** hard-magnetic material ($J_{\text{free}} = 0$, $M \neq 0$)

$$0 = \nabla \times \vec{H}(x, y) = \nabla \times \left(\nu_0 \nu_{\text{r}}(x, y) \left(\vec{B}(x, y) - \mu_0 \vec{M}(x, y) \right) \right) \Rightarrow$$

$$J_z(x, y) = \nabla \times \left(\nu_{\text{r}}(x, y) \vec{M}(x, y) \right)$$

General PDE for additional source terms (parabolic PDE)

PDE for 2D electromechanical, **non-linear** problems:

$$\left(\nabla \cdot (\nu(x,y)\nabla) - \sigma \frac{\mathrm{d}}{\mathrm{d}t}\right) A_z(x,y,t) = -J_z(x,y,t) - \nabla \times \left(\nu_r(x,y)\vec{M}_0(x,y)\right)$$

- Depending on the physical nature of the problem (or assumptions) terms might vanish
- In general, problems consist of a **combination** of **multiple** domains of **different** physical nature
- The magnetization term is the only source term that is not defined in the *z*-direction.
  - *Is this a mathematical violation?*
- For most practical cases, the PDE cannot be solved analytically, not even for linear materials
- *What is still missing to uniquely solve this parabolic PDE?*

Electromagnetic Boundary Conditions (BC)

- **Normal BC**
  - The **normal flux density** components are equal on either side of the boundary, Γ

$$\vec{n} \cdot \left(\vec{B}_2 - \vec{B}_1 \right) = 0$$

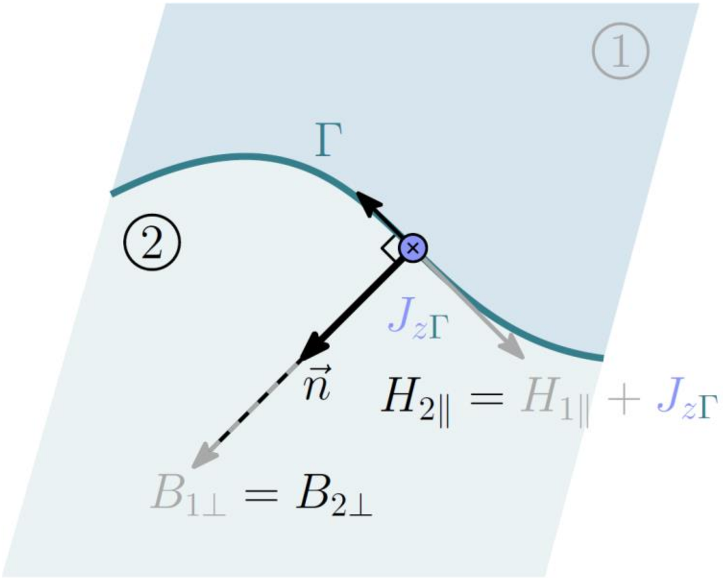
- **Tangential BC**
  - The difference in the **magnetic field strength** between the **tangential** components is equal to the **surface current density** on the boundary, Γ

$M_0 = 0$

$$\vec{n} \times \left(\vec{H}_2 - \vec{H}_1 \right) = \vec{J}_\Gamma$$

$$\vec{n} \times \left(\mu_1 \vec{B}_2 - \mu_2 \vec{B}_1 \right) = \mu_1 \mu_2 \vec{J}_\Gamma$$

Convention:
normal vector points from 1 into 2



Reduced formulations of the PDE

Transient: full PDE solved by time stepping (slow)

Nonlinear materials, Faraday: time-varying sources + movement, PMs can be modeled

$$\left(\nabla \cdot (\nu(x,y)\nabla) - \sigma \frac{\mathrm{d}}{\mathrm{d}t}\right) A_z(x,y,t) = -J_z(x,y,t) - \nabla \times \left(\nu_r(x,y)\vec{M}_0(x,y)\right)$$

Magnetostatic: time-invariant field quantities and sources (fast)

Nonlinear materials, **no** Faraday: DC sources**, no** movement, PMs can be modeled

$$\nabla \cdot (\nu(x,y)\nabla)\, A_z(x,y) = -J_z(x,y) - \nabla \times \left(\nu_r(x,y)\vec{M}_0(x,y)\right)$$

Steady-state AC: frequency domain: complex-valued field quantities and sources (fast)

Linear materials, Faraday: single harmonic AC sources**, no** movement, **no** PMs (*why?*)

$$\left(\nabla \cdot [\nu(x,y)\nabla] - j\omega\sigma\right) \bar{A}_z(x,y) = -\bar{J}_z(x,y)$$

Summary on Maxwell’s equations and the PDE formulation

Basic electromagnetic principles of previous slides should already be (or become) ready knowledge

- Quasi-static Maxwell’s equation
  - Interpretation of field distribution images (**no derivation of equations!**)
  - Insight in basic principles of operation of electromechanical devices
- Boundary conditions
  - Placing, assigning and **exploiting** boundary conditions properly
- Formulation of the governing partial differential equation
  - Selection of the proper solver type in FLUX2D, i.e. advantages vs. limitations
  - Impact of material assumptions on performance of electromechanical problems
  - Optimize ‘cost’ of simulation effort

Solving a Boundary Value Problem (BVP) analytically

Generic expression of a (1D) differential equation

$$\mathcal{L}u(x) = S(x) \qquad \forall x \in \Omega$$

differential operator

source term

- u: the quantity/function of interest to be solved, e.g. MVP
- x: the spatial variable (a scalar or vector depending on the dimensions of the problem)
- S: the source term
  - Results in a nonzero particular/inhomogeneous solution of the PDE
- $\mathcal{L}$: the differential operator contains the expression that acts on u:

$$\mathcal{L} = \sum_{k=0}^{K} f_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k}$$

TU/e

Example of a (physically relevant) 1D 2nd order BVP

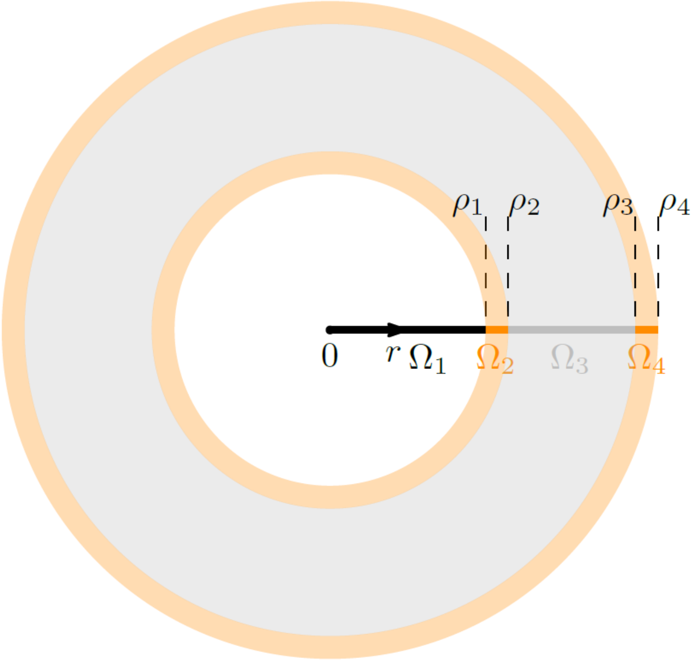
Model reduction for a toroidal inductor

- Axial length larger than outer radius:
  - 3D → 2D cross-section

- Assumption:
  - Uniform current distribution region instead of (ill-defined position of) individual wires

- Toroidal symmetry in the magnetic field:
  - 2D → 1D model with 4 regions in which

$$B_r(r) = 0,\ B_\theta(r) \neq 0$$



Generic solution of the linear magnetostatic BVP: toroid

Proper formulation of the PDE

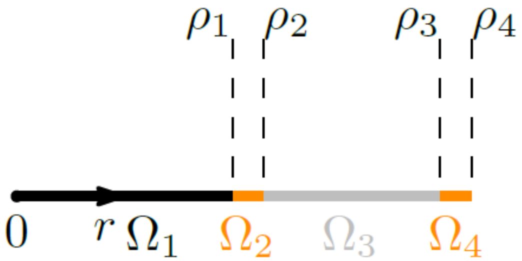
- Magneto**static**: $\frac{\mathrm{d}A_z}{\mathrm{d}t} = 0$
- No permanent magnets: $\vec{M}_0 = \vec{0}$
- Linear materials: $\mu(x,y,z) = \mu_0\mu_\mathrm{r} = \frac{1}{\nu}$

Governing PDE in **polar** coordinate system

$$\nu\nabla^2 A_z = -J_z$$

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \cancel{\frac{\nu}{r^2}\frac{\partial^2 A_z}{\partial \theta^2}} = -J_z$$

$$\Rightarrow$$



$$\mathcal{L} = \nu\frac{\partial}{\partial r} + \nu r\frac{\partial^2}{\partial r^2} = \nu\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$S = -J_z r$$

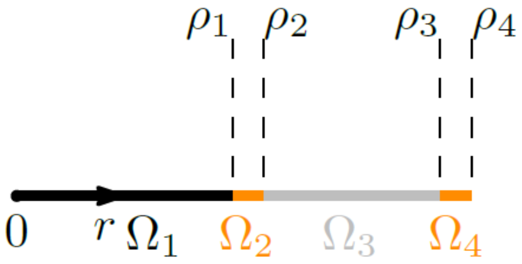
$$u = A_z$$

$$\nu\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -J_z r$$

Generic solution of the linear magnetostatic BVP: toroid

Integrate on both sides of the equal sign ($\nu^{-1} = \mu_0\mu_\mathrm{r}$)

$$\nu r \frac{\partial A_z}{\partial r} = -\int J_z r \,\mathrm{d}r = -\tfrac{1}{2} J_z r^2 + c$$



Integrate once more

$$A_z = \frac{1}{\nu}\int \left(-\tfrac{1}{2}J_z r + c r^{-1}\right)\,\mathrm{d}r$$
$$= -\frac{1}{4\nu}J_z r^2 + \frac{c}{\nu}\ln(r) + A_0$$

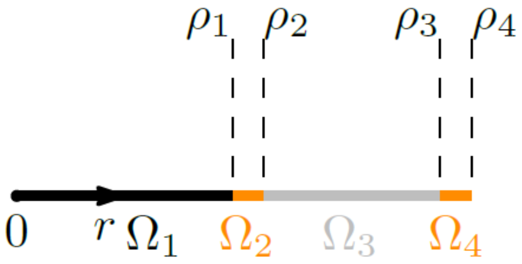
Solution holds for each region, Ω_k:

$$A_{z,k} = \frac{1}{\nu_k}\left(-\tfrac{1}{4}J_{z,k}r^2 + c_k \ln(r)\right) + A_{0,k}$$

k	J_z [A m^{-2}]	μ_r [-]	c	A_0 [Wb m^{-1}]
1	0	1	0 ($\lvert A_z(0)\rvert<\infty$)	arbitrary
2	$\dfrac{\mathcal{F}}{\pi(\rho_2^2-\rho_1^2)}$	1	unknown	arbitrary
3	0	>>1	unknown	arbitrary
4	$\dfrac{-\mathcal{F}}{\pi(\rho_4^2-\rho_3^2)}$	1	unknown	arbitrary

Unique solution of the linear magnetostatic BVP: toroid

3 Unknown constants, c_k, to be found via BCs, $k \in \{1,2,3\}$



$$H_{\theta,k}(r=\rho_k) = H_{\theta,k+1}(r=\rho_k)$$

$$\underbrace{-\frac{\partial A_{z,k}(r=\rho_k)}{\partial r}}_{B_{\theta,k}(\rho_k)} \nu_k = \underbrace{-\frac{\partial A_{z,k+1}(r=\rho_k)}{\partial r}}_{B_{\theta,k+1}(\rho_k)} \nu_{k+1}$$

Substitution results in a system of linear equations:

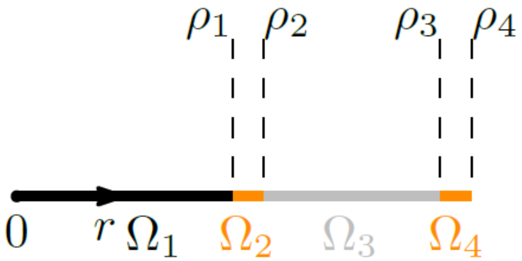
$$\frac{1}{\nu_k}\left(\tfrac{1}{2}J_{z,k}\rho_k - \frac{c_k}{\rho_k}\right)\nu_k = \frac{1}{\nu_{k+1}}\left(\tfrac{1}{2}J_{z,k+1}\rho_k - \frac{c_{k+1}}{\rho_k}\right)\nu_{k+1}$$

$$\boxed{c_k - c_{k+1} = \frac{\rho_k^2}{2}\left(J_{z,k} - J_{z,k+1}\right)}$$

Unique solution of the linear magnetostatic BVP: toroid

Remaining constants, $A_{0,k}$, can **all** be chosen freely, as B_k is not affected by this choice!

- Common to only choose one and require continuity, e.g.:

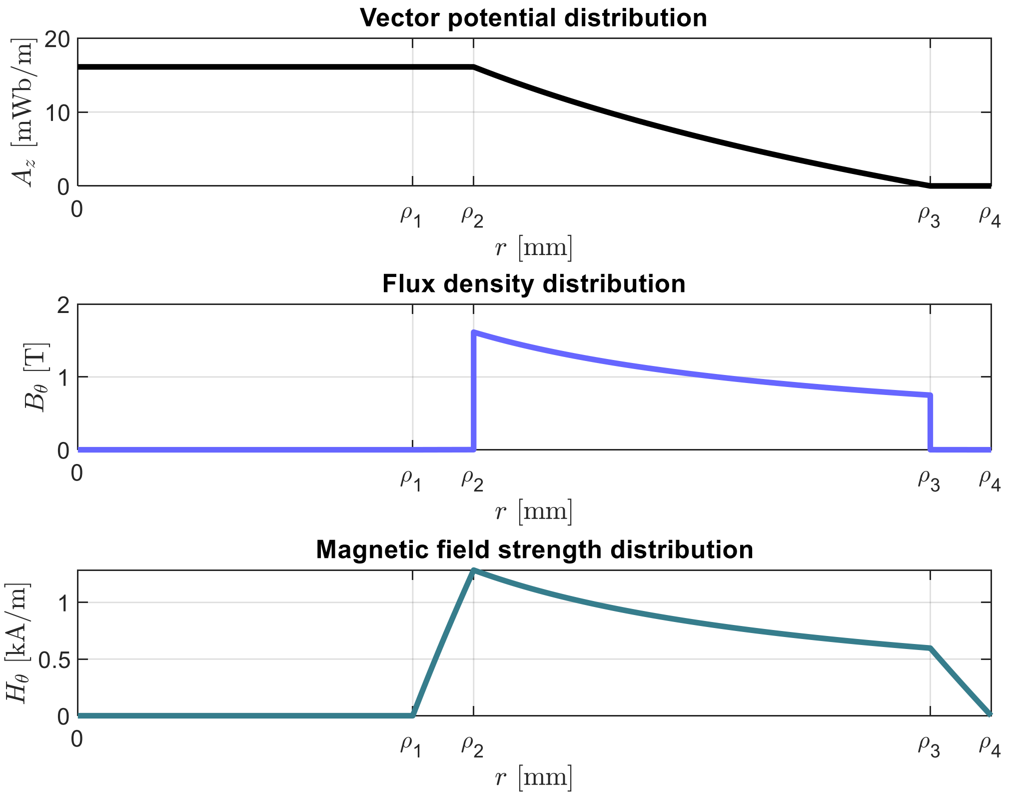


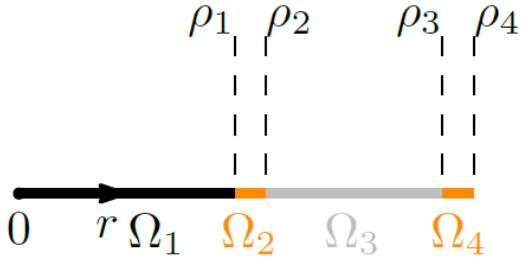
$$A_{z,4}(r = \rho_4) = A_{\text{free}}$$
$$A_{z,k}(r = \rho_k) = A_{z,k+1}(r = \rho_k)$$

$$\Downarrow$$

$$A_{0,4} = A_{\text{free}} + \frac{1}{\nu_4}\left(\frac{J_{z,4}}{4}\rho_4^2 - c_4 \ln(\rho_4)\right)$$
$$A_{0,k} - A_{0,k+1} = \frac{\rho_k^2}{4}\left(\frac{J_{z,k}}{\nu_k} - \frac{J_{z,k+1}}{\nu_{k+1}}\right) - \left(\frac{c_k}{\nu_k} - \frac{c_{k+1}}{\nu_{k+1}}\right)\ln \rho_k$$

Unique solution of the linear magnetostatic BVP: toroid



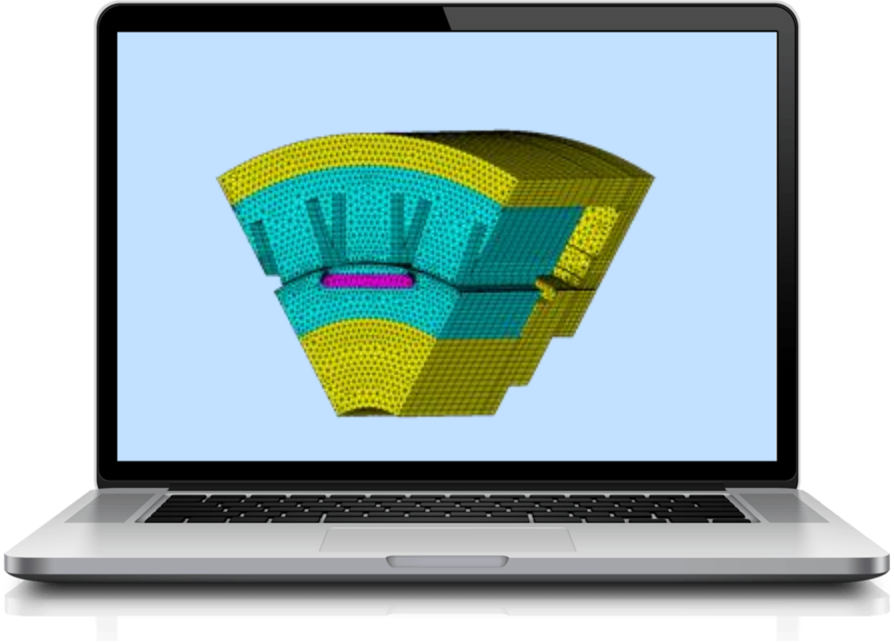


$$A_{z,k} = A_{0,k} - \frac{1}{\nu_k}\left(\tfrac{1}{4}J_{z,k}r^2 - c_k \ln(r)\right)$$

$$B_{\theta,k} = -\frac{\partial A_{z,k}}{\partial r} = \frac{1}{\nu_k}\left(\tfrac{1}{2}J_{z,k}r - \frac{c_k}{r}\right)$$

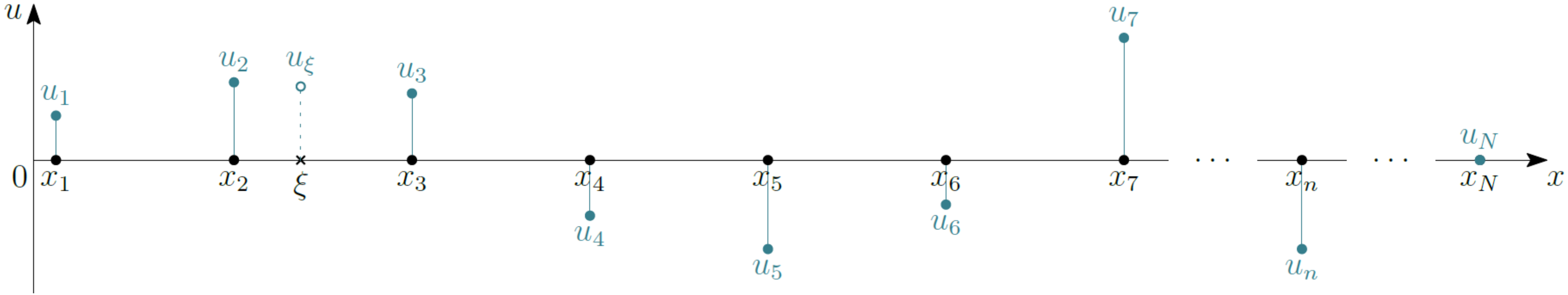
$$H_{\theta,k} = \nu_k B_{\theta,k} = \tfrac{1}{2}J_{z,k}r - \frac{c_k}{r}$$

Numerical methods for solving differential equations



Interpolating pairs of positions and corresponding values

A ***known*** finite, discrete set of length N of pairs of position and field-value in 1D



$u(\xi)$: $\xi \notin [x_1, x_N]$ via **polynomial** interpolation based on nodal information

$$u(\xi) = \sum_{n=1}^{N} u_n \phi_n(\xi, x_1, x_2, \ldots, x_N)$$

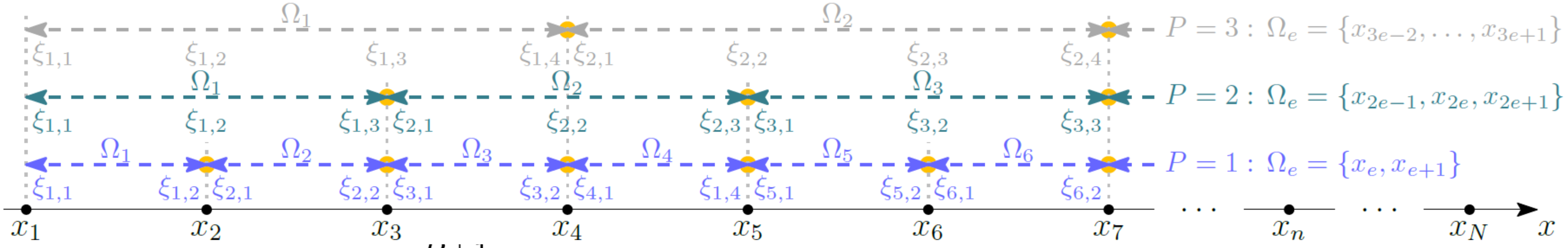
$\boxed{\phi\text{: basis function}}$

ϕ_n is a polynomial of order P expressed in terms of **all** nodal positions

Local interpolation within elements (subdomains)

Simple ‘local’ interpolation in an **element:** $\Omega_e = \{x_{Pe-P+1}, \ldots, x_{Pe+1}\} : 1 \leq e \leq E$

- Piecewise : value depends only on the **element** spanned by $\boldsymbol{P+1}$ nodes, in which it lies
- C^0-contuinuity : only require continuity of u between adjacent **elements**, not of its derivative(s)



$$\xi \in \Omega_e \;\Rightarrow\; u(\xi) = \sum_{p=1}^{P+1} \Psi_{e,p}(\xi, \xi_{e,1}, \ldots, \xi_{e,P+1}) u_{e,p} = \vec{\Psi}_e^T \vec{u}_e$$

Ψ: shape function

- $\left\{x_n : \frac{n-1}{P} \in \mathbb{Z}^+, n < N\right\}$ is the set of **coinciding end-nodes** of elements: $\xi_{e,P+1} = \xi_{e+1,1} = x_n$

Choice for the shape functions: Lagrange polynomials

The choice of the order P and number of elements E affect:

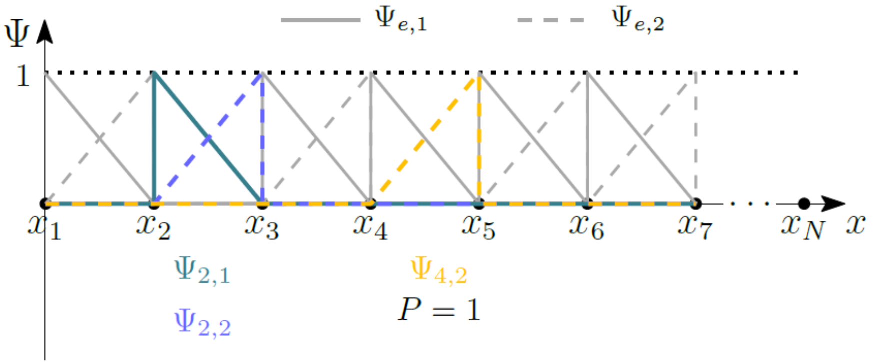
- $N = EP + 1$: Total number of nodes, where $E \in \mathbb{Z}^+$ (nonzero, positive integer)
- $E(P+1)$: The total number of shape functions
- $e \in \mathbb{Z}^+ : e \leq E$: **element** index

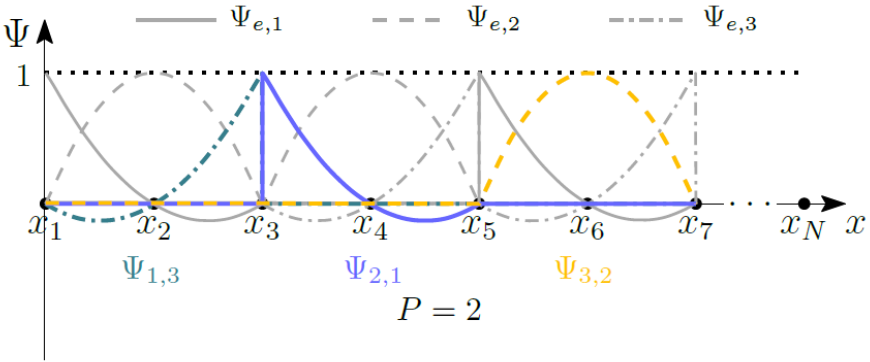
$$\Psi_{e,p} = \begin{cases} \prod_{\substack{k=1 \\ k \neq p}}^{P+1} \frac{x - x_{Pe-P+k}}{x_{Pe-P+p} - x_{Pe-P+k}}, & \text{if } x \in [x_{Pe-P+1}, x_{Pe+1}], \\ 0, & \text{elsewhere.} \end{cases}$$

No division by zero as $k \neq p$ excludes a zero-valued denominator from the **product**, where

$$\prod_{k=1}^{K} x_k = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_K$$

Properties of the Lagrange polynomials





- Only nonzero in its respective subdomain and
- Zero on all nodes x_n but one: at the nonzero node equal to one
- $u(x)$ is the sum of **scaled** shape functions; scaling factor is the **nodal value** $u_n : \Psi(x_n) \neq 0$
- More shape functions than nodes: $E(P+1) > EP + 1 : \{E, P\} \in \mathbb{Z}^+$
  - Shape functions nonzero at the same end-node are both scaled by the corresponding nodal value

Relevance of interpolation

The set of nodal field-values, u_n, are not known in advance, but actually are the ones that need to be calculated, such that:

- They satisfy the governing partial differential equation (PDE)
- Respect the physical boundary conditions

Can a ‘reverse’ method be formulated that allows the discrete nodal field distribution to be calculated once a proper interpolation is chosen?

YES!

By reformulating the governing PDE as a <u>weak formulation</u>

The weak formulation of a PDE

The weak formulation of a PDE relaxes requirements:

1. From local to global requirements by transforming to a **weighted** integral equation
2. Reducing continuity requirements by reducing the order via **integration by parts**.

The weak formulation can be obtained via the **method of weighted residuals**

- **Strong formulation (analytical)**

$$\mathcal{L}u - S = 0 \qquad \forall x \in \Omega$$

- **Weak formulation (numerical)**

$$\int_\Omega \underbrace{(\mathcal{L}u - S)}_{\textbf{residual } (\varepsilon)} w \mathrm{d}\Omega = 0$$

weighting function ($\neq$ 0)

$$\mathcal{L} = \nu \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)$$

$$S = -J_z r$$

$$u = A_z$$

Relaxing the order to obtain the weak form

Applying integration by parts to

$$\int_\Omega (\mathcal{L}u - S)\, w \,\mathrm{d}x = 0 \;\Rightarrow\; \int_\Omega \underbrace{(\mathcal{L}u)}_{v'}\, w \,\mathrm{d}x = \int_\Omega S w \,\mathrm{d}x$$

Integration by parts: $(vw)' = v'w + vw' \;\;\Rightarrow\;\; vw\Big|_a^b = \int_a^b v' w \,\mathrm{d}x + \int_a^b v w' \,\mathrm{d}x$

$$\int_\Omega v w' \,\mathrm{d}x = v(x_N) w(x_N) - v(x_1) w(x_1) - \int_\Omega S w \,\mathrm{d}x$$

Observations:

- The order of the PDE is reduced by 'shifting' it from $\mathcal{L}u$ to w via $v'(u) \rightarrow v(u)$
- *Information* on the field at the **boundaries** $\{x_1, xN\}$ is required

The problem specific weak form of the PDE

Substitution of the problem specific expressions with $J_z = J$

$$x = r \in [0, \rho_4], \quad u = A_z, \quad S = -Jr, \quad v' = \mathcal{L}A_z = \nu \frac{\partial}{\partial r}\left(r \frac{\partial A_z}{\partial r}\right) \Rightarrow v = r\nu \frac{\partial A_z}{\partial r} = -rH_\theta$$

Substitution of the approximation/interpolation with **basis** functions

$$v = \nu r \frac{\partial}{\partial r}\left(\sum_{n=1}^{N} \phi_n A_n\right) = \nu r \sum_{n=1}^{N} \frac{\partial \phi_n}{\partial r} A_n = \nu r \sum_{n=1}^{N} \phi_n' A_n = -rH_\theta$$

Weak form:

$$\int_\Omega v w' \,\mathrm{d}r = \overbrace{v(r_N)}^{-r_N H_\theta(r=r_N)} w(r_N) - \overbrace{v(r_1)w(r_1)}^{r_1=0 \Rightarrow v=0} - \int_\Omega S w \,\mathrm{d}r$$

$$A_1\nu_1 \int_0^{\rho_4} \phi_1' w' \,r\mathrm{d}r + A_2\nu_2 \int_0^{\rho_4} \phi_1' w' \,r\mathrm{d}r + \ldots + A_N\nu_N \int_0^{\rho_4} \phi_N' w' \,r\mathrm{d}r = -\rho_4 H_\theta(\rho_4) w(\rho_4) + \int_0^{\rho_4} J w \,r\mathrm{d}r$$

Defining the weighting function

The problem specific weak formulation

- **weighting function**, w, not yet defined
- a **single** equation with N unknows → solvable via the choice of (a set of) **weighting function**, w

The choice for w determines the class of the Finite Element Method (FEM)

Method	εw	Principle
Least squares	$\nabla_{u_n} \varepsilon^2$	∇-operator optimizes the residual in the least squares sense by taking the **gradient** of the **squared** residual w.r.t. all N nodal values, u_n
Collocation	$\varepsilon \delta_n(x, x_n)$	Residual zero on each node by multiplication of a **set(!)** of N **δ-function** to obtain N equations (sifting property of the **δ-function**)
Galerkin	$\varepsilon \phi_n$	A **set(!)** of N weighting functions equal to the set of N **basis functions** to obtain N equations

Problem specific weak form: Galerkin method

Weak form: $$\sum_{n=1}^{N} A_n \nu_n \int_0^{\rho_4} \phi_1' \vec{\phi}'\, r\mathrm{d}r = \underbrace{-\rho_4 H_\theta(\rho_4)\vec{\phi}(\rho_4)}_{\text{Natural (Neumann) boundary condition!}} + \int_0^{\rho_4} J\vec{\phi}\, r\mathrm{d}r \Rightarrow$$

$$\underbrace{\begin{bmatrix} \nu_1 \int_0^{\rho_4} \phi_1'\phi_1'\, r\mathrm{d}r & \nu_2 \int_0^{\rho_4} \phi_2'\phi_1'\, r\mathrm{d}r & \dots & \nu_N \int_0^{\rho_4} \phi_N'\phi_1'\, r\mathrm{d}r \\ \nu_1 \int_0^{\rho_4} \phi_1'\phi_2'\, r\mathrm{d}r & \nu_2 \int_0^{\rho_4} \phi_2'\phi_2'\, r\mathrm{d}r & \dots & \nu_N \int_0^{\rho_4} \phi_N'\phi_2'\, r\mathrm{d}r \\ \vdots & \vdots & \ddots & \vdots \\ \nu_1 \int_0^{\rho_4} \phi_1'\phi_N'\, r\mathrm{d}r & \nu_2 \int_0^{\rho_4} \phi_2'\phi_N'\, r\mathrm{d}r & \dots & \nu_N \int_0^{\rho_4} \phi_N'\phi_N'\, r\mathrm{d}r \end{bmatrix}}_{\textbf{Stiffness matrix: K}} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} = \underbrace{\begin{bmatrix} \int_0^{\rho_4} J\phi_1\, r\mathrm{d}r - \rho_4 H_\theta(\rho_4)\phi_1(\rho_4) \\ \int_0^{\rho_4} J\phi_2\, r\mathrm{d}r - \rho_4 H_\theta(\rho_4)\phi_2(\rho_4) \\ \vdots \\ \int_0^{\rho_4} J\phi_N\, r\mathrm{d}r - \rho_4 H_\theta(\rho_4)\phi_N(\rho_4) \end{bmatrix}}_{\textbf{Load vector: } \vec{F}}$$

Problem specific weak form: integration per element

$$\xi_{e,1} = r_{(e-1)P+1}$$
$$L_e = r_{eP+1} - r_{(e-1)P+1}$$

Weak form: $\mathbf{K}\vec{A} = \vec{F} \Rightarrow \left[\sum_{e=1}^{E} \mathbf{K}_e\right] \vec{A}_e = \left[\sum_{e=1}^{E} \vec{F}_e\right]$

$$\mathbf{K}_e = \begin{bmatrix} \nu_1 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_1'\phi_1' \, r\mathrm{d}r & \nu_2 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_2'\phi_1' \, r\mathrm{d}r & \dots & \nu_N \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_N'\phi_1' \, r\mathrm{d}r \\ \nu_1 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_1'\phi_2' \, r\mathrm{d}r & \nu_2 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_2'\phi_2' \, r\mathrm{d}r & \dots & \nu_N \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_N'\phi_2' \, r\mathrm{d}r \\ \vdots & \vdots & \ddots & \vdots \\ \nu_1 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_1'\phi_N' \, r\mathrm{d}r & \nu_2 \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_2'\phi_N' \, r\mathrm{d}r & \dots & \nu_N \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \phi_N'\phi_N' \, r\mathrm{d}r \end{bmatrix}$$

For load vector $\vec{F}_e$, the limits of the integrals are also replaced by the respective **begin-** and **end-nodes** of the e^{th} element: $\xi_{e,1} = r_{(e-1)P+1}$ and $\xi_{e,1} + L_e = r_{eP+1}$

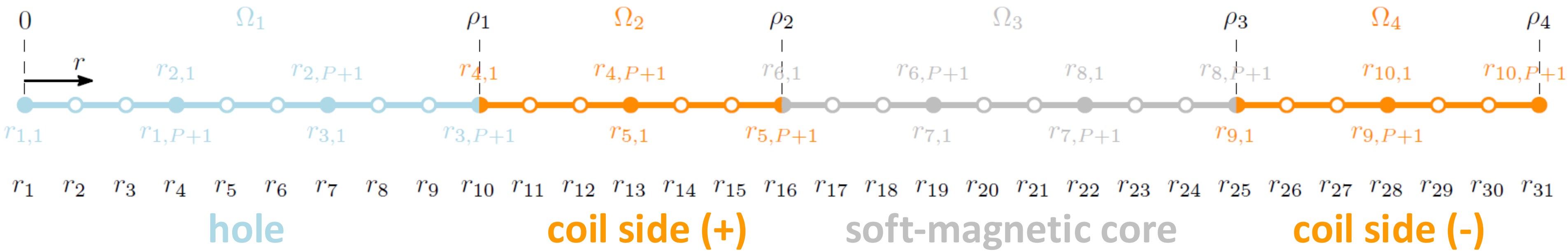
Problem specific weak form: meshing the domain

Dividing the spatial domain into elements

1. Identification of regions with different physical properties in terms of materials and sources
2. Meshing of each region into elements by creation of end-nodes (respect physical boundaries)
3. Addition of internal nodes within each element when $P > 1$

↓↓↓ Toroidal inductor ↓↓↓

$$r_{e,p} \to r_n : n = (e-1)P + p$$



Problem specific weak form: matrix construction

For integration on the element level substitute: $\phi \rightarrow \Psi$

- $\Psi_{m,p} = 0$: $m \neq e \rightarrow K_{e,ij} \neq 0 \;\forall\; \{i,j\} \in [(e-1)P+1, eP+1]$
- Only a 'small' **symmetric** $[P+1]^2$ submatrix is nonzero in $\mathbf{K}_e$

$$\mathbf{K}_e = \underbrace{\begin{bmatrix} 0 & \cdots & \mathbf{0} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{M}_e & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \cdots & 0 \end{bmatrix}}_{[EP+1]^2} \quad \mathbf{M}_e = \nu_e \underbrace{\begin{bmatrix} \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} (\Psi'_{e,1})^2 \, r\mathrm{d}r & \cdots & \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \Psi'_{e,1}\Psi'_{e,P+1} \, r\mathrm{d}r \\ \cdots & \ddots & \cdots \\ \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} \Psi'_{e,P+1}\Psi'_{e,1} \, r\mathrm{d}r & \cdots & \int\limits_{\xi_{e,1}}^{\xi_{e,1}+L_e} (\Psi'_{e,P+1})^2 \, r\mathrm{d}r \end{bmatrix}}_{[P+1]^2}$$

- Similarly for the load vector: $m \neq e \rightarrow F_{e,i} \neq 0 \;\forall\; i \in [(e-1)P+1, eP+1]$

Problem specific weak form: matrix construction

$$\mathbf{K}\vec{A} = \vec{F} \Rightarrow \left[\sum_{e=1}^{E} \mathbf{K}_e\right] \vec{A}_e = \left[\sum_{e=1}^{E} \vec{F}_e\right]$$

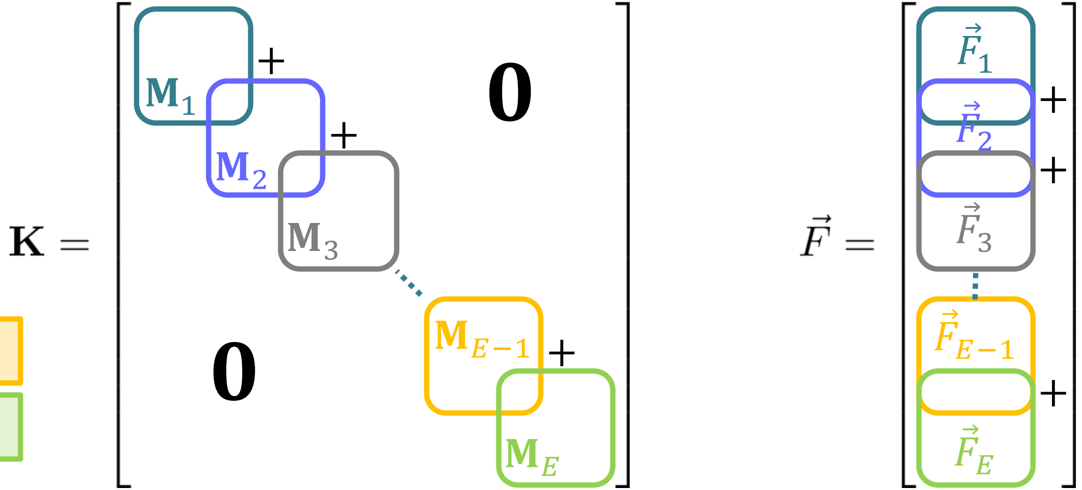
$\{i,j\}_1 \in [1, P+1]$

$\{i,j\}_2 \in [P+1, 2P+1]$

$\{i,j\}_3 \in [2P+1, 3P+1]$

$\{i,j\}_{E-1} \in [(E-2)P+1, (E-1)P+1]$

$\{i,j\}_E \in [(E-1)P+1, EP+1]$



Matrix entries (*uniform* nodal distribution in elements)

Shape functions for an element of length $L_e = r_{(e-1)P+1} - r_{(e-1)P+P+1}$

$$\Psi_{e,p} = \frac{P^P}{L_e^P} \prod_{\substack{k=1 \\ k \neq p}}^{P+1} \frac{r - r_{Pe-P+k}}{p-k}, \qquad \Psi'_{e,p} = \frac{P^P}{L_e^P} \sum_{\substack{m=1 \\ m \neq p}}^{P+1} \frac{1}{p-m} \prod_{\substack{k=1 \\ k \neq p \\ k \neq m}}^{P+1} \frac{r - r_{Pe-P+k}}{p-k}$$

Product rule:
$(fgh)' = ghf' + fhg' + fgh'$

Example for polynomial of order $P = 2$

$$\Psi_{e,1} = \frac{2}{L_e^2}\left(r - r_{2(e-1)+2}\right)\left(r - r_{2(e-1)+3}\right), \qquad \Psi'_{e,1} = \frac{2}{L_e^2}\left[\left(r - r_{2(e-1)+2}\right) + \left(r - r_{2(e-1)+3}\right)\right]$$

$$\Psi_{e,2} = \frac{-4}{L_e^2}\left(r - r_{2(e-1)+1}\right)\left(r - r_{2(e-1)+3}\right), \qquad \Psi'_{e,2} = \frac{-4}{L_e^2}\left[\left(r - r_{2(e-1)+1}\right) + \left(r - r_{2(e-1)+3}\right)\right]$$

$$\Psi_{e,3} = \frac{2}{L_e^2}\left(r - r_{2(e-1)+1}\right)\left(r - r_{2(e-1)+2}\right), \qquad \Psi'_{e,3} = \frac{2}{L_e^2}\left[\left(r - r_{2(e-1)+1}\right) + \left(r - r_{2(e-1)+2}\right)\right]$$

Stiffness submatrix entries (*uniform nodal distribution*)

Evaluating the integrals for the stiffness submatrix

$$M_{e,ij} = M_{e,ji} = \nu_e \int_{\xi_{e,1}}^{\xi_{e,1}+L_e} \Psi'_{e,i} \Psi'_{e,j} \, r \mathrm{d}r \;\Rightarrow\; \mathbf{M}_e = \nu_e \left(\frac{r_{(e-1)P+1}}{L_e} \mathbf{M}_a + \mathbf{M}_b \right)$$

$\mathbf{M}_a$ and $\mathbf{M}_b$ are both of size $[P+1]^2$ and only contain constants

$$P = 1 \Rightarrow \qquad \mathbf{M}_a = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{M}_b = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P = 2 \Rightarrow \qquad \mathbf{M}_a = \frac{1}{3} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, \qquad \mathbf{M}_b = \frac{1}{6} \begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix}$$

Load subvector entries (*uniform nodal distribution*)

Evaluating the integrals for the load subvector

$$F_{e,i} = J_e \int_{\xi_{e,1}}^{\xi_{e,1}+L_e} \Psi_{e,i}\, r \mathrm{d}r - \rho_4 H_\theta(\rho_4) \underbrace{\Psi_{e,i}(\rho_4)}_{\text{Only nonzero for } e=E \text{ and } i=P+1} \Rightarrow \vec{F}_e = J_e L_e \left(r_{(e-1)P+1} \vec{F}_a + L_e \vec{F}_b \right) - \vec{v}_{\mathrm{bc}}$$

Only nonzero for $e = E$ and $i = P + 1$

$\vec{F}_a$, $\vec{F}_b$ and $\vec{v}_{\mathrm{bc}}$ all have size $[(P+1)\times 1]$ and only contain constants

$$P = 1 \Rightarrow \qquad \vec{F}_a = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}, \qquad \vec{F}_b = \frac{1}{6}\begin{bmatrix}1\\2\end{bmatrix}, \qquad \vec{v}_{\mathrm{bc}} = \rho_4 H_\theta(\rho_4)\begin{bmatrix}0\\ \delta_{e,E}\end{bmatrix}$$

$$P = 2 \Rightarrow \qquad \vec{F}_a = \frac{1}{6}\begin{bmatrix}1\\4\\1\end{bmatrix}, \qquad \vec{F}_b = \frac{1}{6}\begin{bmatrix}0\\2\\1\end{bmatrix}, \qquad \vec{v}_{\mathrm{bc}} = \rho_4 H_\theta(\rho_4)\begin{bmatrix}0\\0\\ \delta_{e,E}\end{bmatrix}$$

Dealing with boundary conditions

Nodes at extremities of an *element* adjacent to other elements

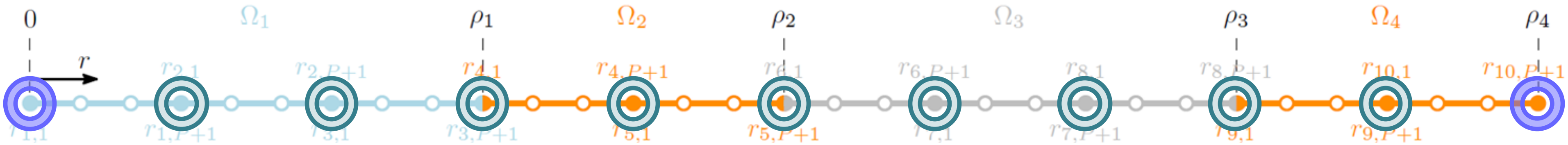
- Continuity of the MVP automatically ensured, no requirement on natural BCs

Any node at the extremities of the *domain* with *Neumann BC* (imposed field vector)

- Include and evaluate the natural boundary condition term accordingly for $e \in \{1, E\}$

Any node at the extremities of the *domain* with *Dirichlet BC* (imposed MVP)

- No requirement on the flux density for $e \in \{1, E\}$ → natural BC term ignored ($B_\theta = 0$)



Dealing with essential (Dirichlet) boundary conditions

- A **subset** of nodal values on the boundary of the domain are **known** in advance
- In 1D only nodal values A_1 and A_E are affected: $A_1 = a$ and $A_E = b$
- Only columns 1 and *E* of $\boldsymbol{K}$ operate on A_1 and A_E
- Move all **known values** to the load-vector side of the equal sign

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & K_{2,2} & \cdots & K_{2,E-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & K_{E-1,2} & \cdots & K_{E-1,E-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{E-1} \\ A_E \end{bmatrix} = \begin{bmatrix} a \\ F_2 - aK_{2,1} - bK_{2,E} \\ \vdots \\ F_{E-1} - aK_{E-1,1} - bK_{E-1,E} \\ b \end{bmatrix}$$

- ***<u>Only</u>*** $\boldsymbol{M}_1$ and $\boldsymbol{M}_E$ are affected: when constructing $\boldsymbol{K}$ (only nonzero contribution in 'corners')
- ***<u>All</u>*** entries in the load vector, $\vec{F}$, seem to be affected, but similarly ***<u>only</u>*** $\vec{F}_1$ and $\vec{F}_E$ will be

Dealing with boundary conditions

Stiffness-submatrices for 2nd order elements, $P = 2$

$$\mathbf{M}_e = \nu_e \left(\frac{r_{2(e-1)+1}}{L_e} \mathbf{M}_a + \mathbf{M}_b \right)$$

		Natural BC (Neumann)		Essential BC (Dirichlet)	
	$e \neq \{1, E\}$	$H_\theta(r_1) = \alpha$ $e = 1$	$H_\theta(r_E) = \beta$ $e = E$	$A(r_1) = a$ $e = 1$	$A(r_E) = b$ $e = E$
$\mathbf{M}_a$		$\frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$		$\frac{1}{3}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 16 & -8 \\ \mathbf{0} & -8 & 7 \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 7 & -8 & \mathbf{0} \\ -8 & 16 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$
$\mathbf{M}_b$		$\frac{1}{6}\begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix}$		$\frac{1}{6}\begin{bmatrix} \frac{\mathbf{6}}{\nu_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 16 & -12 \\ \mathbf{0} & -12 & 11 \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 3 & -4 & \mathbf{0} \\ -4 & 16 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{6}}{\nu_{\mathbf{E}}} \end{bmatrix}$

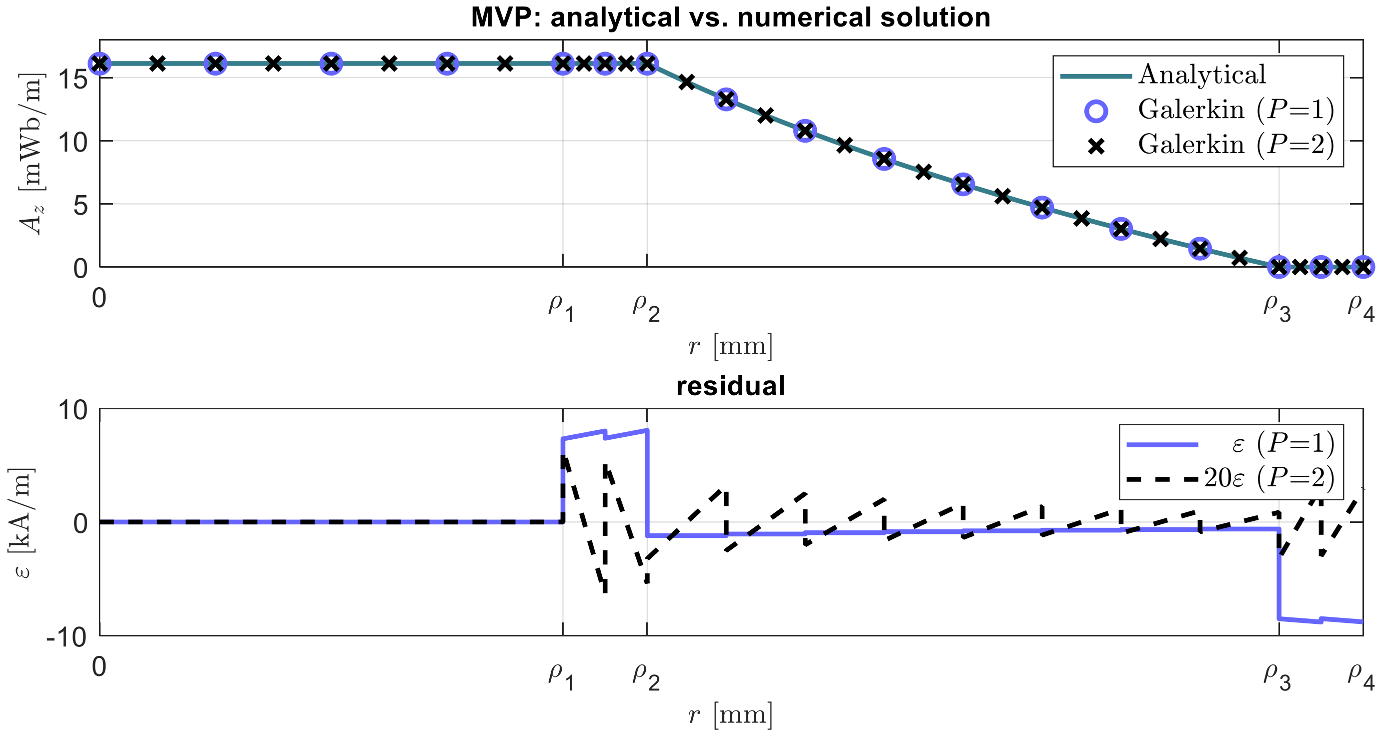
Dealing with boundary conditions

Load-subvectors for 2nd order elements, $P = 2$ **(Either BC-type applies for a node!!)**

$$\vec{F}_e = J_e L_e \left(r_{2(e-1)+1} \vec{F}_a + L_e \vec{F}_b \right) - \vec{v}_{\mathrm{bc}}$$

		Natural BC (Neumann)		Essential BC (Dirichlet)	
	$e \neq \{1, E\}$	$H_\theta(r_1) = \alpha$, $e = 1$	$H_\theta(r_E) = \beta$, $e = E$	$A(r_1) = a$, $e = 1$	$A(r_E) = b$, $e = E$
$\vec{F}_a^T$		$\frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \end{bmatrix}$		$\frac{1}{6}\begin{bmatrix} \mathbf{0} & 4 & 1 \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 1 & 4 & \mathbf{0} \end{bmatrix}$
$\vec{F}_b^T$		$\frac{1}{6}\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$		$\frac{1}{6}\begin{bmatrix} \mathbf{0} & 2 & 1 \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 0 & 2 & \mathbf{0} \end{bmatrix}$
$\vec{v}_{\mathrm{bc}}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -r_1\alpha \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ r_E\beta \end{bmatrix}$	$a \begin{bmatrix} -1 \\ M_{1_{2,1}} \\ M_{1_{3,1}} \end{bmatrix}$	$b \begin{bmatrix} M_{E_{1,3}} \\ M_{E_{2,3}} \\ -1 \end{bmatrix}$

Numeric vs. analytic solution: same mesh, different orders



Mesh

$$E_{\Omega_1} = 4$$
$$E_{\Omega_2} = 2$$
$$E_{\Omega_3} = 8$$
$$E_{\Omega_4} = 2$$

Implementation in Matlab

```matlab
function [A,xn,res] = GalerkinModelToroidLin(rho,mu,J,Afree,E,P,rescalc)

% A = GalerkinModelToroidLin(rho0,mu,J,Afree,E,P) ...


% number of physical domains
Ndom = numel(E);

% Matrices and vector with constants for constuction of the Stiffness ...
[Ma,Mb,Fa,Fb] = MatVecConstants(P);

% Total number of nodes in the entrire domain
Ntot = sum(E.*P)+1;

% creation of the stiffnes matrix, load vector and node vector
K  = zeros(Ntot);
F  = K(:,1);
xn = F;

% element counter
ElCnt = 0;

% indexing of the nodes in a domain
indxn         = ones(Ndom+1,1);
indxn(2:end) = P.*cumsum(E)+1;

for n = 1:Ndom
% column vector with position of the end-nodes of an element
x = linspace(rho(n),rho(n+1),E(n)+1).';

% column vector with the length of all elements in the domain
L = diff(x);

for e = 1:E(n)
    % stiffness sub-matrix and load sub-vector
        Me = (x(e).*Ma./L(e) + Mb)./mu(n);
        Fe = J(n).*L(e).*(x(e).*Fa + L(e).*Fb);

        if (n==Ndom)&&(e==E(n))
            % adjust the stiffness sub-matrix and load sub-vector to include an ...

            % boundary condition vector
            vbc = Afree.*[Me(1:P,end); -1];

            % stiffness sub-matrix
            Me(end,1:P) = 0;
            Me(1:P,end) = 0;
            Me(end)     = 1 ;

            % load sub-vector
            Fe0       = Fe;
            Fe0(end) = 0;
            Fe = Fe0 - vbc;

        end
        % Matrix/vector indexing to place the sub-matrix and -vector
        indx = ElCnt*P+1:(ElCnt+1)*P+1;
        K(indx,indx) = K(indx,indx) + Me;
        F(indx)      = F(indx) + Fe;

        ElCnt = ElCnt+1;

    end
    % position of nodes
    xn(indxn(n):indxn(n+1)) = linspace(rho(n),rho(n+1),E(n)*P+1);

    end
    % Solve the system of equations
    A = K\F;
```