Final Project for Advanced FEM (ME46050)

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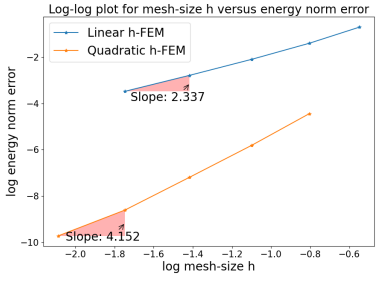
I. Introduction

This report addresses two advanced finite element problems. The first problem constructs a solver for a one-dimensional Poisson equation using both h-version and p-version finite elements. The second problem develops a solver for a two-dimensional stress distribution of elliptical inhomogeneity in plane elasticity, employing the h-version FEM with T3 element and Q4 elements.

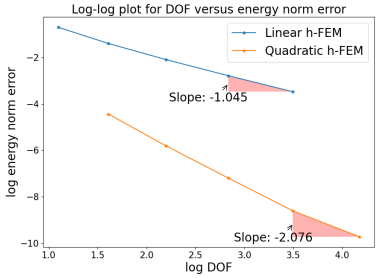
II. PROBLEM 1

i. Question 1

The code for finite element method, shape functions as well as the gaussian integration method in Appendix.A, B, and C.



(a) *log-log plot for the error versus meshsize*



(b) *log-log plot for the error versus DOF*

Figure 1: *The log-log figure for the energy norm error versus mesh size and DOF.*

The precise strain energy for this problem is given as U=0.03559183822564316. To determine the rates of convergence in the energy norm for both element types, we focus on terminal convergence by considering the last two points in the convergence plots.

The formula for the convergence rate can be found in Eq.1, which can also be defined as the slope of the log-log plot. For both elements, the error decreases with the increase of the DOFs and the decrease of the mesh size. It's noteworthy that the convergence rate for the quadratic elements is approximately greater than that for the linear elements. Given the smoothness of the solution, the theoretical rates of convergence are typically 2 for linear elements and 4 for quadratic elements. For the computed errors, the linear elements exhibit an error of approximately 0.031, while the quadratic elements have a significantly smaller error of about 6.0×10^{-5}. These computed rates align closely with the theoretical expectations.

$$\text{Rate} = \frac{\log(\text{error}_2) - \log(\text{error}_1)}{\log(\text{DOF}_2) - \log(\text{DOF}_1)} \tag{1}$$

ii. Question 2

In the log-log plot in Fig.2 of the relative error in the energy norm versus the number of DOFs, the slopes of the plotted lines represent these rates.

Given the computed convergence rates for the different finite element methods, we observe the following rates:

- For Linear h-FEM: The rate of convergence is approximately -1.045.
- For Quadratic h-FEM: The rate of convergence is approximately -2.076.
- For p-FEM: The rate of convergence is approximately -8.230.

The negative values for the convergence rates indicate that the error decreases as the number of DOFs increases, which is expected in a convergence study. Notably, the rate of convergence of the linear element is close to 1, and the quadratic element is close to 2, respectively, which indicates that the convergence rate of h-FEM is equal to the polynomial order. From the rates, it's evident that the p-FEM has the steepest convergence, indicating a faster reduction in error with increasing DOFs compared to the other methods.

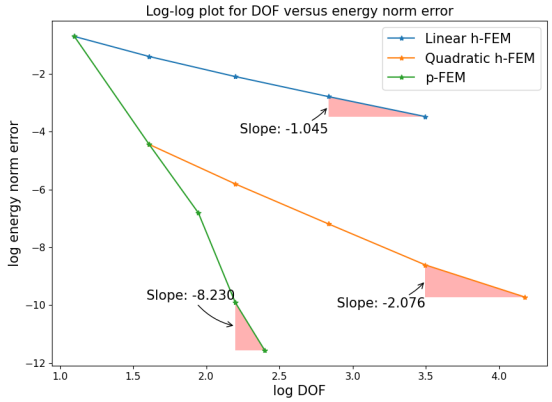


Figure 2: *log-log plot of the error versus DOF in h-version and p-version FEM.*

iii. Question 3

In order to estimate the error in our finite element method solutions, we use a posteriori error analysis based on the energy norms described in the following processes.

Considering the algebraic convergence of the energy norm error for exact solution u and the finite elements solution u_h in energy space $\epsilon(\Omega)$:

$$\left\| u - u^h \right\|_{\epsilon(\Omega)} \leqslant C_1 h^{\beta_h} \|u\|_{\epsilon(\Omega)} \tag{2}$$

We went:

$$\|u\|_{\epsilon(\Omega)} = \sqrt{U} \tag{3}$$

where U is the exact energy.

Considering the relation between the energy and binary term in the finite element methods.

$$\begin{aligned} u(u) &= \frac{1}{2} B(u,u), \\ \|u\|_e &= \sqrt{\frac{1}{2} B(u,u)}, \\ \|u - u^h\|_e &= \frac{1}{2} B\left(u - u^h, u - u^h\right) \\ &= \frac{1}{2} B(u,u) - \frac{1}{2} B\left(u^h, u^h\right), \end{aligned} \tag{4}$$

Now we obtain the error of the strain energy:

$$U_e = U - U^h. \tag{5}$$

By using the energy values obtained from three different mesh sizes, a system of equations can be constructed to determine the exact solution U:

$$\begin{aligned} U - U^{h_0} &= C_1^2 h_0^{2\beta_h} U \text{ (I)} \\ U - U^{h_1} &= C_1^2 h_1^{2\beta_h} U \text{ (II)} \\ U - U^{h_2} &= C_1^2 h_2^{2\beta_h} U \text{ (III)} \end{aligned} \tag{6}$$

In these equations:

- U^{h_0}, U^{h_1}, and U^{h_2} are the FEM approximated solutions for mesh sizes h_0, h_1, and h_2 respectively.
- C_1 is a coefficient.
- β_h is an exponent that determines the convergence rate of error reduction as mesh size decreases.

The logarithmic relationship between the errors for different mesh sizes can be obtained obtained by Eq.6:

$$\begin{aligned} \text{Take } \frac{\log(\text{I})}{\log(\text{II})} : \log\left(\frac{U - U^{h_0}}{U - U^{h_1}}\right) &= 2\beta_h \log\left(\frac{h_0}{h_1}\right) \\ \text{Take } \frac{\log(\text{II})}{\log(\text{III})} : \log\left(\frac{U - U^{h_1}}{U - U^{h_2}}\right) &= 2\beta_h \log\left(\frac{h_1}{h_2}\right) \end{aligned} \tag{7}$$

These equations provide insight into how the error changes logarithmically as the mesh size changes.

Using the above relationships, the a posteriori error estimate, which is a measure of the relative error, is expressed as:

$$\frac{\log\left(\frac{U-U^{h_0}}{U-U^{h_1}}\right)}{\log\left(\frac{U-U^{h_1}}{U-U^{h_2}}\right)} = \frac{\log\left(\frac{h_0}{h_1}\right)}{\log\left(\frac{h_1}{h_2}\right)} = \mathbf{Q} \tag{8}$$

Considering the relation between the mesh size h and the *DOF* (N):

$$h \cong \frac{1}{N^{1/dimensionality}} \tag{9}$$

The expression of $\mathbf{Q}$ is given by:

$$Q = \frac{\log\left(N_1/N_0\right)}{\log\left(N_2/N_1\right)} \tag{10}$$

The term $\mathbf{Q}$ gives a weighted comparison of the errors between different mesh sizes. This relationship becomes pivotal in understanding the error behavior across different mesh sizes.

By repeatedly applying the aforementioned process for multiple mesh sizes and averaging the computed energies, a more accurate representation of the solution's energy is achieved, which provides a reliable posterior error estimate.

Certainly, based on the table provided, here's a suitable answer:

	Energy	Relative Error
Linear	0.034626674	2.7117(%)
Quadratic	0.035591726	0.000314(%)
Exact solution	0.035591838	/

Table 1: *Energy obtained by a posterior estimate and Relative Error values for different FEM methods*

The table.1 presents the energy values obtained using different Finite Element Methods (FEM) and their respective relative errors when compared to the exact solution.

For the linear FEM, the energy is computed to be 0.03463, which results in a relative error of 2.7117%. This indicates a slight deviation from the exact solution. On the other hand, the quadratic FEM provides an energy value of 0.03559, which is extremely close to the exact solution with a minuscule relative error of 0.000314%. This suggests that the quadratic FEM is significantly more accurate than the linear FEM for this problem.

In summary, while the linear FEM offers a reasonable approximation, the quadratic FEM provides an almost exact match to the true solution in terms of energy.

The code for a posterior estimate is provided in the Appendix.E.

iv. Question 4

In the h-version study using the quadratic finite element method, we analyzed the model with varying mesh sizes, namely 5, 10, 20, and 40 evenly spaced elements. Fig.3 represents the h-FEM solutions with four mesh sizes. A comparison of the numerical solutions against the exact solution provided insights into the accuracy of the employed method. From Fig.4 it was discernible that the graph wasn't strictly linear. However, by focusing on the terminal two data points, we derived an asymptotic rate of convergence of -2.122. This suggests a quadratic rate of reduction in error relative to the refinement in element size. For this specific problem, the exact strain energy is given by $U = 1.585854059271320$, and our computed results closely mirrored this value.

v. Question 5

From the log-log plot in Fig.6, the computed rate of convergence for the p-version was approximately -4.882, whereas for the h-version, it was -2.2122. The convergence rate of quadratic p-FEM is faster than h-FEM

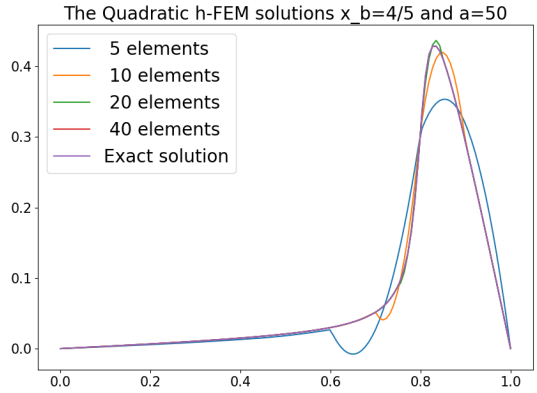


Figure 3: *The Quadratic h-FEM solutions x_b=4/5 and a=50 with different element numbers.*

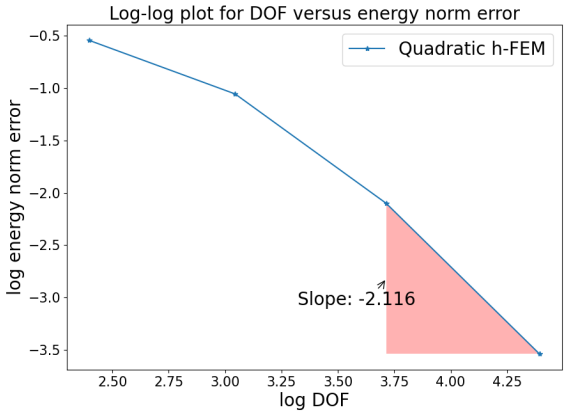


Figure 4: *Log-log plot for DOF versus energy norm error.*

in this problem. As a result, the p-FEM can achieve higher accuracy with fewer degrees of freedom.

vi. Question 6

The stability of numerical methods in finite element analysis can be assessed using the condition number of the stiffness matrix.

Observing the log-log plot in Fig.7:

- **p-FEM**: The condition number remains constant regardless of the DOFs increase, indicating its robustness.
- **Quadratic h-FEM**: Condition number growth is consistent with increasing DOFs and is unaffected by equation parameter changes (both for $a = 0.5$ and $a = 50$).
- **Linear vs Quadratic h-FEM**: Both show similar growth trends, but the linear version has a slightly

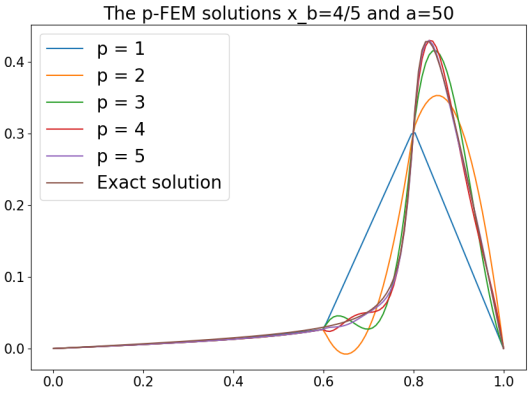


Figure 5: *The p-FEM solutions x_b=4/5 and a=50 with different element numbers.*

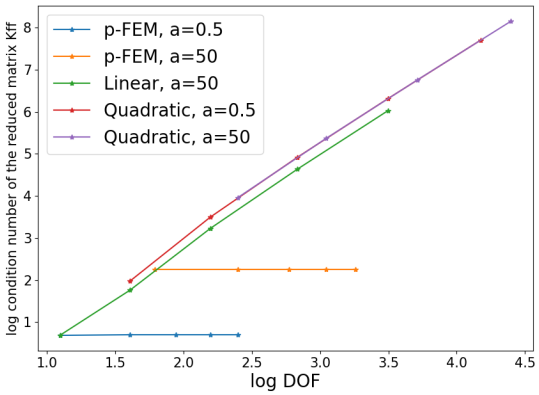


Figure 7: *Log-log plot for the condition number of the reduced matrix K_{ff} versus energy norm error.*

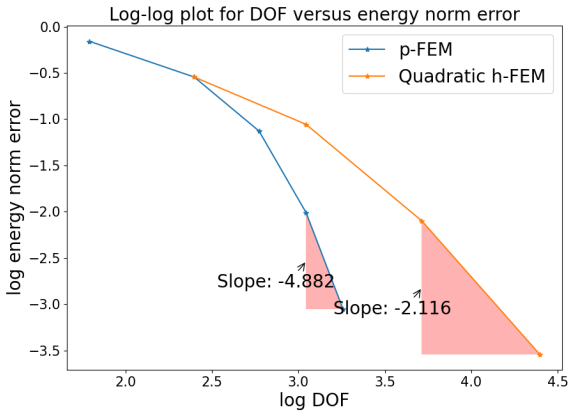


Figure 6: *Log-log plot for DOF versus energy norm error in p-FEM and h-FEM.*

lower condition number for similar DOFs.

In summary, p-FEM stands out in stability, while h-FEM versions show predictable growth trends, with the linear version in slightly better condition.

vii. Question 7

vii.1 Comparison of Results and Conclusions on Strong Gradients

From the results obtained, several conclusions can be drawn regarding the behavior of the finite element methods under study, especially in problems with strong gradients or sharp features.

- **Convergence Rate and Accuracy**: The convergence rate, represented as the slope of the log-log plot, provides insights into the efficacy of the different finite element methods. The error decreased with the increase of the DOFs and the decrease of the mesh size. Notably, the quadratic elements exhibited a more significant convergence rate than the linear ones, reflecting the theoretical expectations. Meanwhile, the rate of convergence for h-FEM is equal to the polynomial order.
- **Linear h-version**: For the linear FEM, the energy was computed to be somewhat deviated from the exact solution, especially in problems with sharp features. This indicates that while the linear h-FEM offers a reasonable approximation, there's a clear margin for improvement in accuracy for such problems.
- **Quadratic h-version**: In sharp gradient problems, the quadratic h-FEM showed its strength by providing an energy value that was extremely close to the exact solution, emphasizing its higher accuracy.
- **p-version vs. h-version in Sharp Problems**: In problems with sharp gradients, the p-version exhibited remarkable resilience and adaptability. Despite its slower convergence rate, it outperformed both the linear and quadratic h-versions in terms of accuracy for comparable DOFs. This suggests that the p-version, with its adaptability, can better capture local variations and sharp features without requiring extensive mesh refinements that h-version methods might demand.
- **Effect of Strong Gradients**: The p-FEM is particularly effective for problems with strong gradients or sharp features. Its higher-order polynomial approximations and local refinement capabilities

allow it to capture complex variations in the solution more accurately than methods like h-FEM. This adaptability often results in higher accuracy with fewer computational resources.

- **Stability and Robustness**: The stability of finite element methods, assessed by the condition number of the stiffness matrix, highlighted the robustness of the p-FEM. Its condition number remains invariant with increasing DOFs, ensuring consistent performance. On the other hand, the h-FEM versions, both linear and quadratic, exhibit predictable growth in condition numbers, with the linear version showing a slight edge in conditioning. The quadratic h-FEM's stability remains consistent even with changes in equation parameters.

vii.2 Quadrature Points and Computation Efficiency

In p-FEM, higher-order shape functions demand precise integral evaluations, achieved effectively with Gauss quadrature. Given the complexity of these functions, 9 Gauss points were selected to ensure accurate integration of non-linear shape functions. While more Gauss points increase precision, they also require more computational effort. Nonetheless, for our specific setup, the added computational time was minimal, making the choice justifiable for enhanced accuracy without sacrificing efficiency.

III. PROBLEM 2

i. Questions 1

For the mesh size $h/L = 0.05$ and Q4 mesh, the displacement fields for different a/b are represented in Fig.8

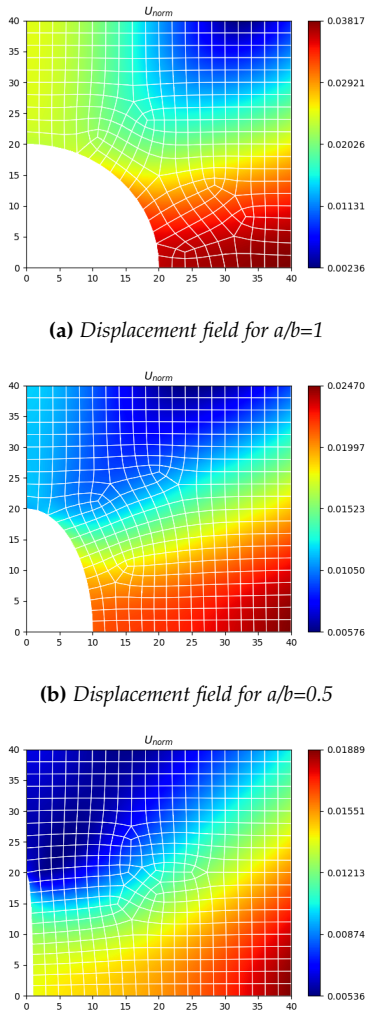
The code for finite element method, shape functions as well as the Gaussian integration method in Appendix.F, G, and H.

ii. Question 2

Considering a plate subjected to plane stress conditions, we assume the following material properties and dimensions:

- Young's Modulus, $E = 200$ GPa.
- Poisson's ratio, $\nu = 0.3$.
- Plate thickness, $h = 1$ mm.

Given the applied total force $P = 2000$ N and side length $L = b = 40$ mm, we aim to estimate the displacements.



(a) *Displacement field for a/b=1*

(b) *Displacement field for a/b=0.5*

(c) *Displacement field for a/b=0.05*

Figure 8: *displacement fields for different a/b : (a) a/b=1; (b) a/b=0.5; and (c) a/b=0.05.*

The stress in the x-direction due to the applied force is:

$$\sigma_x = \frac{P}{h \times b} \tag{11}$$

No force in the y-direction implies $\sigma_y = 0$.

The strains in the x and y directions, under plane stress conditions, are:

$$\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \end{aligned} \tag{12}$$

Using the strains, the displacements at the boundaries are estimated as:

$$u_x(L,0) = \epsilon_x \times L$$
$$u_y(0,L) = \epsilon_y \times L \tag{13}$$

The estimated displacements at the boundaries are:

$$u_x(L,0) \approx 0.01\,\text{mm}$$
$$u_y(0,L) \approx -0.003\,\text{mm} \tag{14}$$

The above displacements are approximations based on plane stress and linear elasticity assumptions.

iii. Question 3

iv. Question 4

The log-log plot for energy norm error versus mesh size and DOF are represented in Fig.9

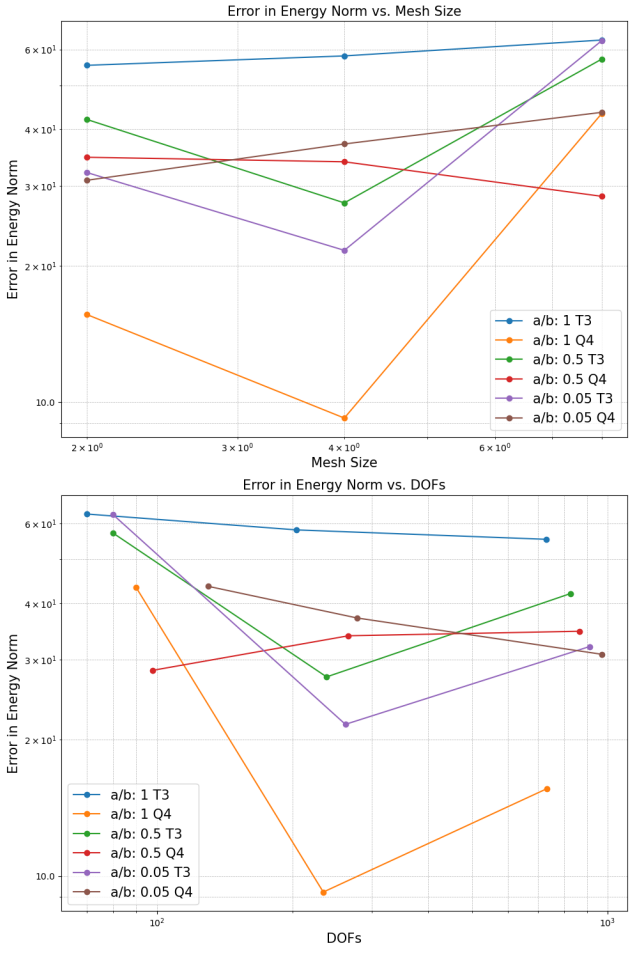


Figure 9: *Log-log plot of energy norm error versus DOFs.*

Based on the provided data, the convergence rates in the energy norm were computed and compared to the theoretical values. Considering the fluctuation of the error is sensitive to the mesh size (the error reaches the lowest when the mesh size is 4), the convergence rates are calculated by the first point and the last point of the energy list.

The convergence rates for different elements with different a/b ratios concerning the mesh size and DOF are represented in Table.2 and Table.3

Table 2: *Convergence Rates based on Mesh Size*

a/b Ratio	Element Type	Convergence Rate (Mesh Size)
1	T3	0.092621
1	Q4	0.736803
0.5	T3	0.221261
0.5	Q4	-0.143155
0.05	T3	0.483199
0.05	Q4	0.249004

Table 3: *Convergence Rates based on DOFs*

a/b Ratio	Element Type	Convergence Rate (DOFs)
1	T3	-0.054701
1	Q4	-0.486695
0.5	T3	-0.131251
0.5	Q4	0.091080
0.05	T3	-0.275252
0.05	Q4	-0.171758

Convergence Rates based on Mesh Size

- For T3 elements with a/b ratios of 1, 0.5, and 0.05, the observed convergence rates are 0.092621, 0.221261, and 0.483199, respectively.
- For Q4 elements with a/b ratios of 1, 0.5, and 0.05, the observed convergence rates are 0.736803, -0.143155, and 0.249004, respectively.

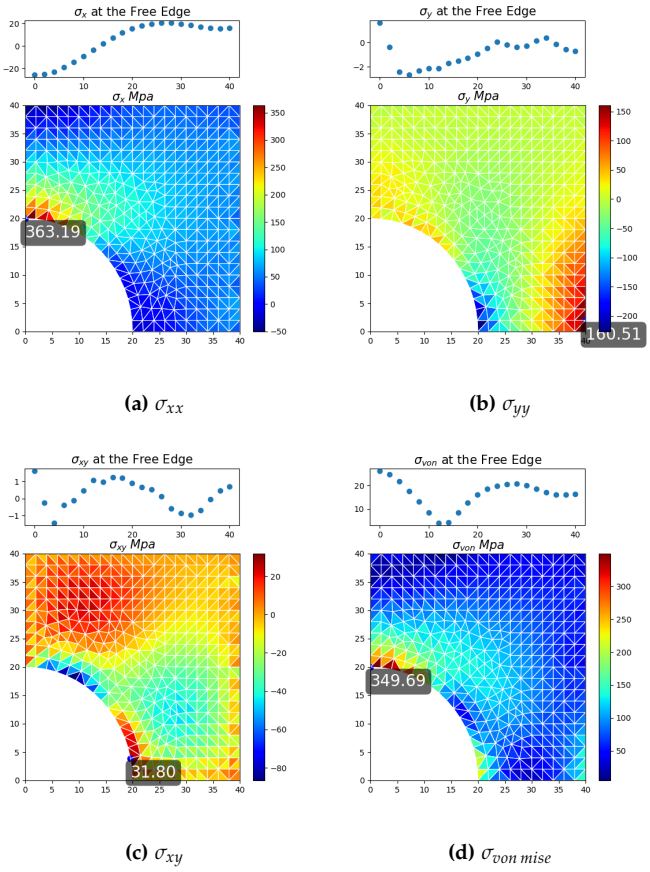
Convergence Rates based on DOFs

- For T3 elements with a/b ratios of 1, 0.5, and 0.05, the observed convergence rates are -0.054701, -0.131251, and -0.275252, respectively.
- For Q4 elements with a/b ratios of 1, 0.5, and 0.05, the observed convergence rates are -0.486695, 0.091080, and -0.171758, respectively.

It is observed that only for $a/b = 1$ in T3 mesh and $a/b = 0.5$ in Q4 mesh, the log-log plots appear to be linear. However, the convergence rates for all curves are lower than the expected theoretical values. The convergence rates of each mesh type in h-FEM in 2D are equal to the order of the interpolation polynomials over

2. Therefore, the experimentally observed convergence rates do not align with the theoretical predictions.

v. Question 5



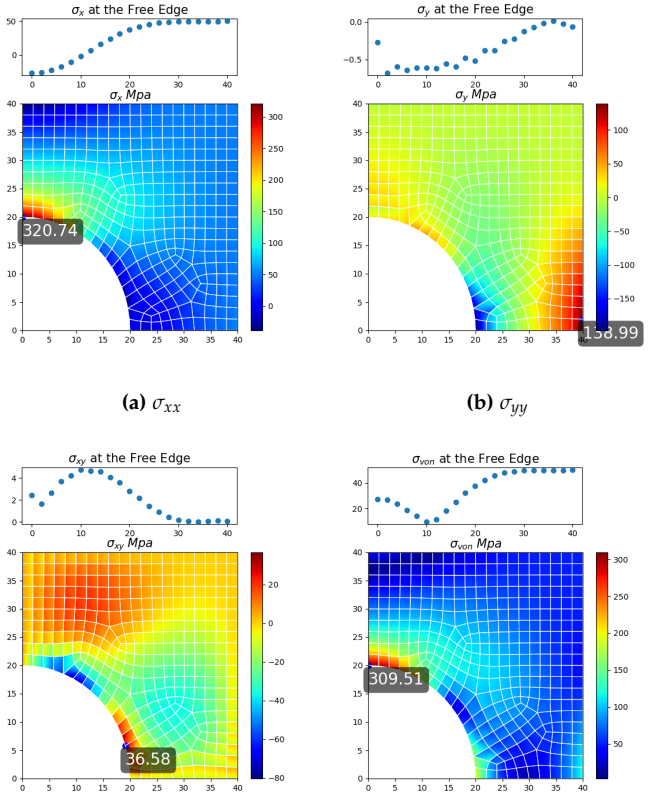
(a) σ_{xx} **(b)** σ_{yy}

(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 10: *Stress fields for a/b=1 with T3 mesh.*

The Table.4 presents the stress concentration factors (SCF) for T3 and Q4 elements at different a/b ratios. It is evident that the experimentally observed SCFs are significantly different from the theoretical predictions.

- For an a/b ratio of 1, both T3 and Q4 elements show SCFs (7.3 and 6.4, respectively) that are much higher than the theoretical value of 3. The average SCF is 6.85, which is more than twice the theoretical prediction.
- At an a/b ratio of 0.5, the SCFs for T3 and Q4 are 8.4 and 7.8, respectively, with an average of 8.1. This is also significantly higher than the theoretical value of 5.
- Interestingly, for an a/b ratio of 0.05, the SCFs are lower than the theoretical value. The SCFs for T3 and Q4 are 5.8 and 7, respectively, with an average of 6.4, which is far below the theoretical value of 41.



(a) σ_{xx} **(b)** σ_{yy}

(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 11: *Stress fields for a/b=1 with Q4 mesh.*

vi. Question 6

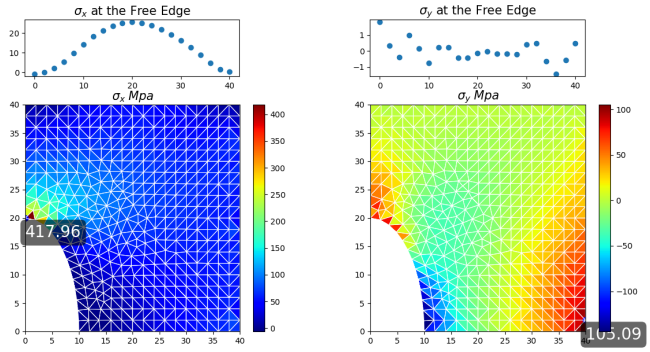
Table 4: *Stress Concentration Factors for T3 and Q4 Elements*

a/b Ratio	T3	Q4	Average	Theory
1	7.3	6.4	6.85	3
0.5	8.4	7.8	8.1	5
0.05	5.8	7	6.4	41

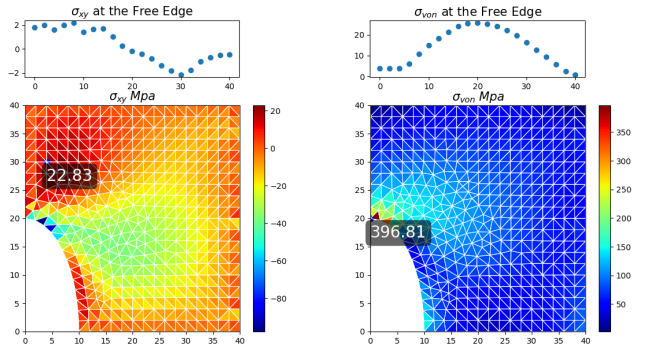
vii. Question 7

The allowable stress values for the material under non-failing conditions at different a/b ratios are presented in the following table:

The table shows that the material has relatively consistent allowable stress values across different a/b ratios. Specifically, the average allowable stress is approx-

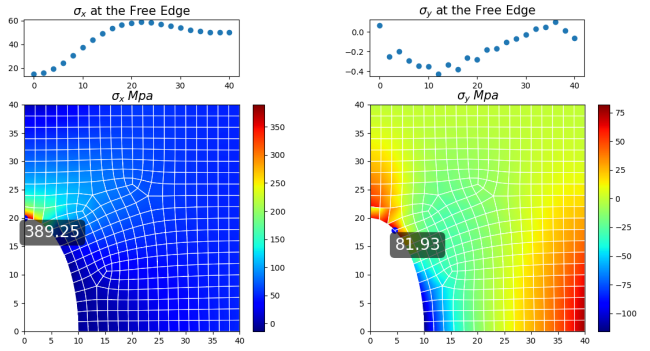


(a) σ_{xx} **(b)** σ_{yy}

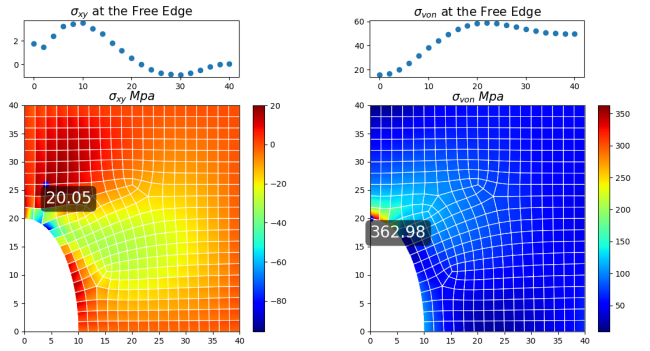


(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 12: *Stress fields for a/b=0.5 with T3 mesh.*



(a) σ_{xx} **(b)** σ_{yy}



(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 13: *Stress fields for a/b=0.05 with Q4 mesh.*

imately around 30, varying slightly from 27.65 to 31.95. This suggests that the material's allowable stress is not significantly influenced by the a/b ratio, indicating good material robustness under varying conditions.

IV. Discussion

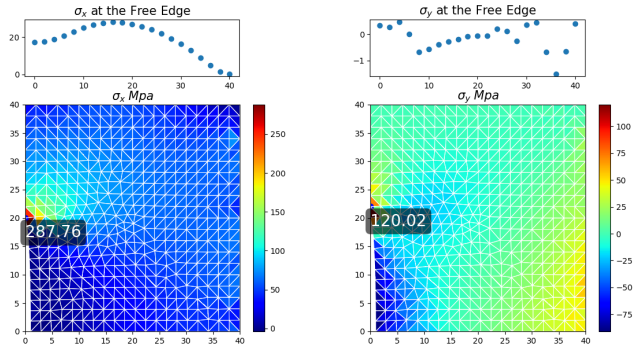
The discussion is a very important part of the report, so make sure you write it properly.[Figueredo and Wolf, 2009]

References

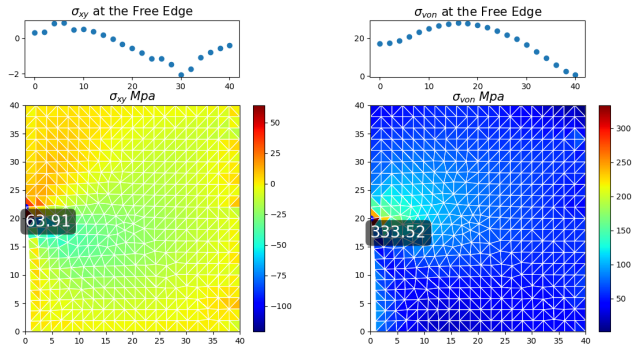
[Figueredo and Wolf, 2009] Figueredo, A. J. and Wolf, P. S. A. (2009). Assortative pairing and life history strategy - a cross-cultural study. *Human Nature*, 20:317–330.

Remember there is a 10-page limit!

References

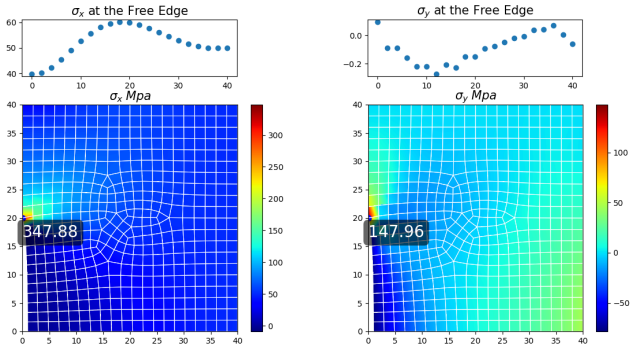


(a) σ_{xx} **(b)** σ_{yy}

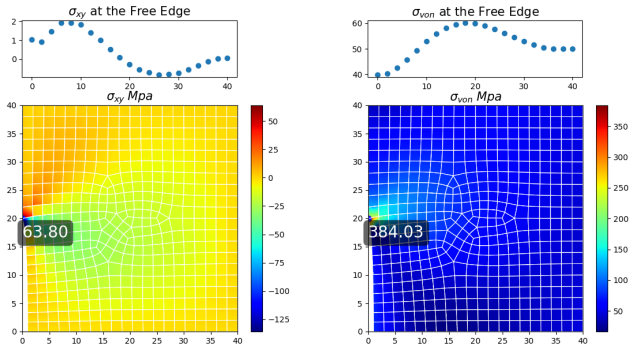


(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 14: *Stress fields for a/b=0.5 with T3 mesh.*



(a) σ_{xx} **(b)** σ_{yy}



(c) σ_{xy} **(d)** $\sigma_{von\ mise}$

Figure 15: *Stress fields for a/b=0.05 with Q4 mesh.*

A. Finite elements methods in 1-D main codes

```
def FEM_1D(shape_class = Hierarchical, p = 3, num_elems = 3, domain = (0, 1),rhs_func = rhs_fn(a=50, xb=0.8), exact_func=exact_fn(0.5,0.8), BCs = (0, 0), verbose = False):
    N=6
    mesh = np.linspace(domain[0], domain[1], num_elems+1)
    ori_phi_phip = {'phis': [], 'phips': []}
    for elem in range(num_elems):
        scale = [mesh[elem], mesh[elem+1]]
        phis, phips = shape_class(scale, p)
        ori_phi_phip['phis'].append(phis)
        ori_phi_phip['phips'].append(phips)


    linear_phi_phip = {'phis': [], 'phips': []}  # Linear
    for elem in range(num_elems):
        linear_phis = []
        linear_phips = []
        for idx in range(len(ori_phi_phip['phis'][elem])):
            if ori_phi_phip['phis'][elem][idx].p < 2:
                phi = ori_phi_phip['phis'][elem][idx]
                phip = ori_phi_phip['phips'][elem][idx]
                linear_phi_phip['phis'].append(phi)
                linear_phi_phip['phips'].append(phip)
```

a/b ratio	T3	Q4	Average
1	30	33.9	31.95
0.5	26.4	28.9	27.65
0.05	31.4	27.3	29.35

Table 5: *Allowable stress values for different $\frac{a}{b}$ ratios.*

```
                linear_phis.append(phi)
                linear_phips.append(php)
        linear_K_sub = np.zeros((len(linear_phips), len(linear_phips)))
        for indx, x in np.ndenumerate(linear_K_sub):
            linear_K_sub[indx] = G_integrate(mul(linear_phips[indx[0]], linear_phips[indx[-1]]), N=6, scale=linear_phips[indx[0]].scale)
            if abs(linear_K_sub[indx]) < 1e-10:
                linear_K_sub[indx] = 0
        linear_F_sub = np.zeros(len(linear_K_sub))
        for indx in range(len(linear_F_sub)):
            linear_F_sub[indx] = G_integrate(mul(rhs_func, linear_phis[indx]), N=N, scale=linear_phis[indx].scale)
        if elem == 0:
            K = linear_K_sub
            F = linear_F_sub
        else:
            K = assemble(K, linear_K_sub)
            F = assemble(F, linear_F_sub)

    linear_num = len(F)

    nonlinear_phi_php = {'phis': [], 'phips': []}
    for order in range(2, p+1):  # Non Linear
        for elem in range(num_elems):
            for idx in range(len(ori_phi_php['phis'][elem])):
                if (ori_phi_php['phis'][elem][idx].p == order) or (ori_phi_php['phips'][elem][idx].p == order):
                    nonlinear_phi = ori_phi_php['phis'][elem][idx]
                    nonlinear_php = ori_phi_php['phips'][elem][idx]
                    nonlinear_phi_php['phis'].append(nonlinear_phi)
                    nonlinear_phi_php['phips'].append(nonlinear_php)
                    nonlinear_K_sub = np.zeros((2, 2))

                    nonlinear_K_sub[-1, -1] = G_integrate(mul(nonlinear_php, nonlinear_php),N=N, scale=nonlinear_php.scale)
                    nonlinear_F_sub = np.zeros(2)
                    nonlinear_F_sub[-1] = G_integrate(mul(rhs_func, nonlinear_phi), N=N, scale=nonlinear_phi.scale)
```

```
                    K = assemble(K,
                        nonlinear_K_sub
                        )
                    F = assemble(F,
                        nonlinear_F_sub
                        )
                else:
                    pass

        # Applying boundary condition

        K[0, 1:] = 0.0
        K[linear_num-1, :linear_num-1] =
            0.0
        F[0] = BCs[0]* K[0, 0]
        F[linear_num-1] = BCs[-1] * K[
            linear_num-1, linear_num-1]

        U = -la.solve(K, F)
        phi_php = {'phis': [], 'phips':
            []}
        phi_php['phis'] = joint_funcs(
            linear_phi_php['phis']) +
            nonlinear_phi_php['phis']
        phi_php['phips'] = joint_funcs(
            linear_phi_php['phips']) +
            nonlinear_phi_php['phips']
        u_list = []
        for i in range(len(phi_php['phis
            ''])):
            u_list.append(mul(U[i],
                phi_php['phis'][i]))
        uh = plus(u_list)
        if verbose == True:
            print(f"Shape class: {
                shape_class.__name__}, 
                Number of elements: {
                num_elems}, Polynomial 
                order:{p}, Domain: {
                domain}, Boundary 
                conditions: {BCs}")
            x_data = np.linspace(domain
                [0], domain[1], 101)
            plt.plot(x_data, exact_func(
                x_data), label='
                Analytical solution')
            plt.plot(x_data, uh(x_data),
                label='FEM solution {} 
                elements'.format(
                num_elems))
            for i in range(len(phi_php['
                phis'])):
                func = phi_php['phis'][i
                    ]
                plt.plot(x_data, U[i]*
                    func(x_data))
            plt.legend()
            plt.show()
        eigenvalues = np.linalg.eigvals(K
            )
        cont_K = max(eigenvalues)/min(
            eigenvalues)
        return U, phi_php, uh, cont_K
```

Listing 1: *Finite elements methods in 1-D main code*

B. Defeinition of the shape functions in 1D

```
def Legendre(x=np.linspace(-1, 1, 100),
    p=5):

  if p == 0:
      return 1
  elif p == 1:
      return x

  else:
      return ((2*p-1)*x*Legendre(x, p
          -1)+(1-p)*Legendre(x, p-2))/p

class shape_function:
      def __init__(self, scale=[-1, 1])
          :
          self.scale = scale
          self.x_l = scale[0]
          self.x_r = scale[1]
          self.range = [-1, 1]

      def expression(self, x):
          return 1 - (x - self.x_l) / (
              self.x_r - self.x_l)

      def mapping(self, x):
          scale = self.scale
          range = self.range
          x_normalized = (x - scale[0])
              / (scale[1] - scale[0])
          return range[0] +
              x_normalized * (range[1]
              - range[0])

      def __call__(self, x):
          x = np.asarray(x)  # convert
              x to a numpy array if it'
              s not already
          expression_vectorized = np.
              vectorize(self.expression
              , otypes=['d'])
```

```
            return np.where((self.scale
                [0] <= x) & (x <= self.
                scale[-1]),
                expression_vectorized(x),
                 0)

class phi_func_1(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.p = p
      self.range = [0, 1]
  def expression(self, x):
      if self.p == 0:
          phi = 1-self.mapping(x)
      elif self.p == 1:
          phi = self.mapping(x)
      else:
          raise AssertionError("p␣
              should␣be␣0␣or␣1␣in␣
              linear␣shape␣function,␣
              not{}".format(self.p))
      return phi

class phip_func_1(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.range = [0, 1]
      self.p = p
  def expression(self, x):
      scale_up = 1/(self.scale[1]-self.
          scale[0])

      if self.p == 0:
          phip =  np.zeros_like(self.
              mapping(x))-1
      elif self.p == 1:
          phip = np.zeros_like(self.
              mapping(x))+1
      else:
          raise AssertionError("p␣
              should␣be␣0␣or␣1␣in␣
              linear␣shape␣function,␣
              not{}".format(self.p))
      return phip*scale_up

class phi_func_q(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.range = [0, 1]
      self.p = p
  def expression(self, x):
      xx = self.mapping(x)
      if self.p == -1:
          phi = (xx-1)*(xx-0.5)*2
      elif self.p == 0:
          phi = -xx*(xx-1)*4
      elif self.p ==1:
          phi = xx*(xx-0.5)*2
      else:
          raise AssertionError("p␣
              should␣be␣-1,␣0␣or␣1␣in␣
              quadratic␣shape␣function,
              ␣not{}".format(self.p))
      return phi

class phip_func_q(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.range = [0, 1]
      self.p = p
  def expression(self, x):
      scale_up = 1/(self.scale[1]-self.
          scale[0])
      xx = self.mapping(x)
      if self.p == -1:
          phip = 4*xx - 3.0
      elif self.p == 0:
          phip = 4-8*xx
      elif self.p ==1:
          phip = 4*xx - 1.0
      else:
          raise AssertionError("p␣
              should␣be␣-1,␣0␣or␣1␣in␣
              quadratic␣shape␣function,
              ␣not{}".format(self.p))
      return phip*scale_up

class phi_func_h(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.p = p
  def expression(self, x):
      scale = self.scale
      i =self.p
      if i == 0:
          phi = (1-self.mapping(x))/2
      elif i == 1:
          phi = (1+self.mapping(x))/2
      else:
          phi = 1/np.sqrt(4*i-2)*(
              Legendre(self.mapping(x),
               i)-Legendre(self.mapping
              (x), i-2))
      return phi

class phip_func_h(shape_function):
  def __init__(self, scale, p):
      super().__init__(scale)
      self.p = p
  def expression(self, x):
      scale_up = 2/(self.scale[1]-self.
          scale[0])
```

```
        i =self.p

        if i == 0:
            phip =  np.zeros_like(self.
                mapping(x))-0.5
        elif i == 1:
            phip = np.zeros_like(self.
                mapping(x))+0.5
        else:
            phip = np.sqrt(i-1/2)*(
                Legendre(self.mapping(x),
                 i-1))
        return phip*scale_up

def Hierarchical(scale, p):
  phis = []
  phips = []
  start=0

  for i in range(start, p+1):
      new_phi = phi_func_h(scale, i)
      new_phip = phip_func_h(scale,i)
      phis.append(new_phi)
      phips.append(new_phip)
  return phis, phips

def linear(scale, p):
  phis = []
  phips = []
  p = 1
  for i in range(p+1):
      new_phi = phi_func_l(scale, i)
      new_phip = phip_func_l(scale,i)
      phis.append(new_phi)
      phips.append(new_phip)
  return phis, phips

def quadratic(scale, p):
  phis = []
  phips = []
  p = 1
  for i in range(-1, p+1):
      new_phi = phi_func_q(scale, i)
      new_phip = phip_func_q(scale,i)
      phis.append(new_phi)
      phips.append(new_phip)
  return phis, phips
```

Listing 2: *Defeinition of the shape functions in 1-D*

C. Defeinition of Gaussian integrate in 1D

```
def G_integrate(u, N=3, scale=(0, 1)):
  N = N
  a = scale[0]
  b = scale[1]
  x, w = roots_legendre(N)

  xp = x*(b-a)/2+(b+a)/2
  wp = w*(b-a)/2

  s = 0
  for i in range(N):
      s += wp[i]*u(xp[i])
  return s
```

Listing 3: *Defeinition of Gaussian integrate in 1D*

D. Energy calculator in 1D

```
def cal_energy(U_array, phi_phip_array)
    :
  U_energy = 0
  u_prime_list = []
  scales = []
  for i in range(len(phi_phip_array['
      phis'])):
      u_prime = mul(U_array[i],
          phi_phip_array['phips'][i])
      u_prime_list.append(u_prime)
      scales.append(u_prime.scale)
  flat_scales = [item for sublist in
      scales for item in sublist]
  rounded_scales = [round(num, 5) for
      num in flat_scales]
  nodes = list(set(rounded_scales))
  mesh = np.linspace(min(nodes), max(
      nodes), len(nodes))
  for i in range(len(mesh)-1):
      scale = [mesh[i], mesh[i+1]]
      U_energy+=G_integrate(mul(plus(
          u_prime_list), plus(
          u_prime_list)),N=9, scale=
          scale)
  return U_energy/2
```

Listing 4: *Energy calculator in 1D*

E. A posteriori error estimate

```
def posterior_energy(energy_list_array,
    DOFs_array):
  if len(energy_list_array)<3:
      raise AssertionError("The value
          of energy should be greater
          than three!")
  elif len(energy_list_array)!= len(
      DOFs_array):
```

```
        raise AssertionError("The number of energy values should be equal to the number of DOFs!")
    def equation(U, U0, U1, U2, Q):
        return ((U-U0)/(U-U1) / ((U-U1)/(U-U2))**Q - 1)**2

    i = 0
    U_list = []
    while i+3 <= len(energy_list_array):
        U0, U1, U2 = energy_list_array[i:i+3]
        h0, h1, h2 = 1/np.sqrt(DOFs_array[i:i+3])
        # print(h0, h1, h2)
        N0, N1, N2 = DOFs_array[i:i+3]
        # Q = np.log((h0/h1))/np.log((h1/h2))
        Q = np.log((N1/N0))/np.log((N2/N1))
        initial_guess = np.mean(energy_list_array)
        # Use minimize
        U_solution = minimize(equation, initial_guess, args=(U0, U1, U2, Q)).x
        U_list.append(U_solution )
        i+=1
return np.mean(U_list)
```

Listing 5: *A posteriori error estimate*

F. Finite elements methods in 2D

```
    def FEM(a_b, mesh_size, mesh_shape, GPN=2, show=False):
    Load_x = 50  # N/mm
    Load_y = 0  # N/mm
    A = 40  # mm^2
    nodes_coord, element_nodes = create_mesh(a_b, mesh_shape, mesh_size)
    nodes_list = Boundary(nodes_coord, a_b)
    element_list = []
    if mesh_shape == 0:
        element_nodes = element_nodes.reshape(-1, 3)
    elif mesh_shape == 1:
        element_nodes = element_nodes.reshape(-1, 4)

    for ele_lst in element_nodes:
        this_nodes = [node for id in ele_lst for node in nodes_list if node.id == id]
        try:
            elem = Q4(this_nodes, GPN=GPN)
        except:
            elem = T3(this_nodes, GPN=GPN)
        elem.a_b = a_b
        element_list.append(elem)
    DOFs = 2*len(nodes_list)
    glo_K = np.zeros((DOFs, DOFs))
    glo_F = np.zeros(DOFs)

    for elem in element_list:  # Assemble Force vector
        loc_F = elem.F
        for i, node_i in enumerate(elem.nodes):
            global_dof = 2 * node_i.id
            # print(loc_F[2*i])
            if abs(node_i.xy[0]-40) < 1e-3:
                glo_F[global_dof] += Load_x * loc_F[2*i]
                # glo_F[global_dof] += Load_x * 1
                glo_F[global_dof + 1] += Load_y * loc_F[2*i+1]

    for elem in element_list:  # Assemble Stiffness matrix
        loc_K = elem.K
        for i, node_i in enumerate(elem.nodes):
            for j, node_j in enumerate(elem.nodes):
                for dof_i in range(2):
                    for dof_j in range(2):
                        global_dof_i = 2 * node_i.id + dof_i
                        global_dof_j = 2 * node_j.id + dof_j

                        glo_K[global_dof_i][global_dof_j] += loc_K[2 * i + dof_i][2*j + dof_j]
```

```
    for elem in element_list:  # Boundary condition

        for i, node_i in enumerate(elem.nodes):
            for dof_i in range(2):
                global_dof_i = 2 * node_i.id + dof_i

                if node_i.BC[dof_i] == 1:

                    glo_K[global_dof_i, :] = 0
                    # glo_K[:, global_dof_i] = 0
                    glo_K[global_dof_i, global_dof_i] = 1e15
                    glo_F[global_dof_i] = 0

    U = np.linalg.solve(glo_K, glo_F)
    for id in range(len(nodes_list)):
        displacement = np.array([U[id*2], U[id*2+1]])
        nodes_list[id].value = displacement

    if show == True:
        x_coords = [node.xy[0] for node in nodes_list]
        y_coords = [node.xy[1] for node in nodes_list]

        temperatures = [np.linalg.norm(node.value) for node in nodes_list]

        tri = []
        for c in element_nodes:
            tri.append([c[0], c[1], c[2]])
            try:
                tri.append([c[0], c[2], c[3]])
            except:
                pass

        plt.tricontourf(x_coords, y_coords, temperatures, triangles=tri,  levels=15, cmap=plt.cm.jet)
        plt.colorbar(label='Displacement in magnitude')
        plt.title('Displacements Distribution')

        plt.show()
    return U, nodes_coord, copy.deepcopy(element_list)


def draw(elements_list, dir='xy', type='disp', show = True):
    global_min = min([np.min([output(test_element(xy[0], xy[1], type), dir, type)
                              for xy in test_element.sample_points(refine)])
                      for test_element in elements_list])

    global_max = max([np.max([output(test_element(xy[0], xy[1], type), dir, type)
                              for xy in test_element.sample_points(refine)])
                      for test_element in elements_list])

    for test_element in elements_list:
        test_inputs = test_element.sample_points(refine)
        test_mapping = test_element.mapping(test_inputs)
        test_output = [output(test_element(xy[0], xy[1], type), dir, type)
                       for xy in test_inputs]
        test_x, test_y, test_z = grid_to_mat(test_mapping, test_output)
        # plt.scatter(test_mapping[:, 0], test_mapping[:, 1], s=1, c=test_output)
        plt.imshow(test_z, extent=(test_mapping[:, 0].min(), test_mapping[:, 0].max(), test_mapping[:, 1].min(), test_mapping[:, 1].max()), origin='lower', aspect='auto'
```

```
            , interpolation='none',cmap='
            jet', vmin=global_min, vmax=
            global_max)
        vertices = test_element.vertices
        vertices = np.vstack([vertices,
            vertices[0]])
        vertices_x, vertices_y = zip(*
            vertices)
        plt.plot(vertices_x, vertices_y,
             color='white',
                linewidth=0.7)

    plt.xlim(0, 40)
    plt.ylim(0, 40)
    # Display the color bar
    cbar = plt.colorbar()
    ticks = np.linspace(global_min,
        global_max, num=5)
    cbar.set_ticks(ticks)
    if type == 'disp':
        type_str = 'U'
    elif type == 'strain':
        type_str = '\\epsilon'
    elif type == 'stress':
        type_str = '\\sigma'
    dir_str = "{ %s }" % dir
    plt.title(rf"${type_str}_{dir_str}$")
    if show:
    plt.show()
```

Listing 6: *Finite elements methods in 2D*

G. Defeinitions of shape functions for T3 and Q4 elements

```
class shape_fns:
    def __init__(self, scale_x = [0,
        1], scale_y = [0, 1], p=0):
        self.scale_x = scale_x
        self.scale_y = scale_y
        self.p = p

    def expression(self, xi, eta):
        return 1-xi-eta

    def __call__(self, x=0, y=0):

        return  self.expression(x, y)


class T3_phi(shape_fns):
    def expression(self, xi, eta):
        if self.p == 0:
            return xi
        elif self.p == 1:
            return eta
        elif self.p == 2:
            return 1-xi-eta
        else:
            raise ValueError("p should
                be 0, 1 or 2 in T3
                element shape functions
                , not {}".format(self.p
                ))


class T3_phipx(shape_fns):
    def expression(self, xi=0, eta=0):
        if self.p == 0:
            return 1
        elif self.p == 1:
            return 0
        elif self.p == 2:
            return -1
        else:
            raise ValueError("p should
                be 0, 1 or 2 in T3
                element shape functions
                , not {}".format(self.p
                ))


class T3_phipy(shape_fns):
    def expression(self, xi=0, eta=0):
        if self.p == 0:
            return 0
        elif self.p == 1:
            return 1
        elif self.p == 2:
            return -1
        else:
            raise ValueError("p should
                be 0, 1 or 2 in T3
                element shape functions
                , not {}".format(self.p
                ))


class Q4_phi(shape_fns):
    def expression(self, xi=0, eta=0):
        if self.p == 0:
            return (xi-1)*(eta-1)/4
        elif self.p == 1:
            return (1 + xi) * (1 - eta)
                /4
        elif self.p == 2:
            return (1 + xi) * (1 + eta)
                /4
        elif self.p == 3:
            return (1 - xi) * (1 + eta)
                /4
        else:
```

```
            raise ValueError("p should be 0, 1, 2 or 3 in Q4 element shape functions , not {}".format(self.p))

class Q4_phipx(shape_fns):
    def expression(self, xi=0, eta=0):
        if self.p == 0:
            return (eta - 1)/4
        elif self.p == 1:
            return (1 - eta)/4
        elif self.p == 2:
            return (1 + eta)/4
        elif self.p == 3:
            return -(1 + eta)/4
        else:
            raise ValueError("p should be 0, 1, 2 or 3 in Q4 element shape functions , not {}".format(self.p))


class Q4_phipy(shape_fns):
    def expression(self, xi=0, eta=0):
        if self.p == 0:
            return (xi - 1)/4
        elif self.p == 1:
            return -(xi + 1)/4
        elif self.p == 2:
            return (1 + xi)/4
        elif self.p == 3:
            return (1 - xi)/4
        else:
            raise ValueError("p should be 0, 1, 2 or 3 in Q4 element shape functions , not {}".format(self.p))
```

Listing 7: *Defeinitions of shape functions for T3 and Q4 elements*

H. Gaussian points in 2D

```
  def Gauss_points(element, order):
    if element.shape == 'quad':
        xi, wi = np.polynomial.legendre.leggauss(order)
        points = [(x, y) for x in xi for y in xi]
        weights = [wx * wy for wx in wi for wy in wi]

    elif element.shape == 'triangle':
        NGP_data = {
            1: {
                'points': np.array([(1/3, 1/3)]),
                'weights': np.array([1/2])
            },
            3: {
                'points': np.array([(1/6, 1/6), (2/3, 1/6), (1/6, 2/3)]),
                'weights': np.array([1/6, 1/6, 1/6])
            },
            4: {
                'points': np.array([(1/3, 1/3), (0.6, 0.2), (0.2, 0.6), (0.2, 0.2)]),
                'weights': np.array([-27/96, 25/96, 25/96, 25/96])
            }
        }
        if order == 2:
            order = 3
        points, weights = NGP_data[order]['points'], NGP_data[order]['weights']
    else:
        raise ValueError("Shape not supported")

    return points, weights
```

Listing 8: *Gaussian points in 2D*

I. Von Mise stress

```
  def Von_Mise(sigma_x, sigma_y, tau_xy):
        result = np.sqrt(sigma_x**2-sigma_x*sigma_y+sigma_y**2+3*tau_xy**2)
        return result
```

Listing 9: *Von Mise stress*

J. Strain Energy in 2D

```
def cal_energy(elements_list, GPN = 2):
  E = 200e3
```

```
    nu = 0.3
    D = E / (1 - nu**2)* np.array([
        [1, nu, 0],
        [nu, 1, 0],
        [0, 0, (1-nu)/2]
        ])
    energy = 0
    for elem in elements_list:
        elem_energy = 0
        points, Ws = Gauss_points(elem,
            GPN)
        loop = 0
        scale = 4 if elem.shape=="
            triangle" else 1
        for g in range(len(Ws)):
            xy = points[g]
            W = Ws[g]
            strain_list = elem(xy[0], xy
                [1], 'strain')
            dN = elem.gradshape(xy[0], xy
                [1])
            # J = jacobian(self.vertices,
                 dN)
            J = np.dot(dN , elem.vertices
                )
            J_det = np.linalg.det(J)
            B = elem.B_matrix(J, dN)
            this_energy = 0.5 * W *
                strain_list.T @ D @
                strain_list * J_det #*
                scale
            elem_energy += this_energy
            loop+=1
        energy+=elem_energy
    return energy[0][0]
```

Listing 10: *Strain energy in 2D*