Project

Advanced Finite Element Methods (ME46050)

PROBLEM 1.— BOUNDARY VALUE PROBLEM WITH $u \in C^{\infty}$

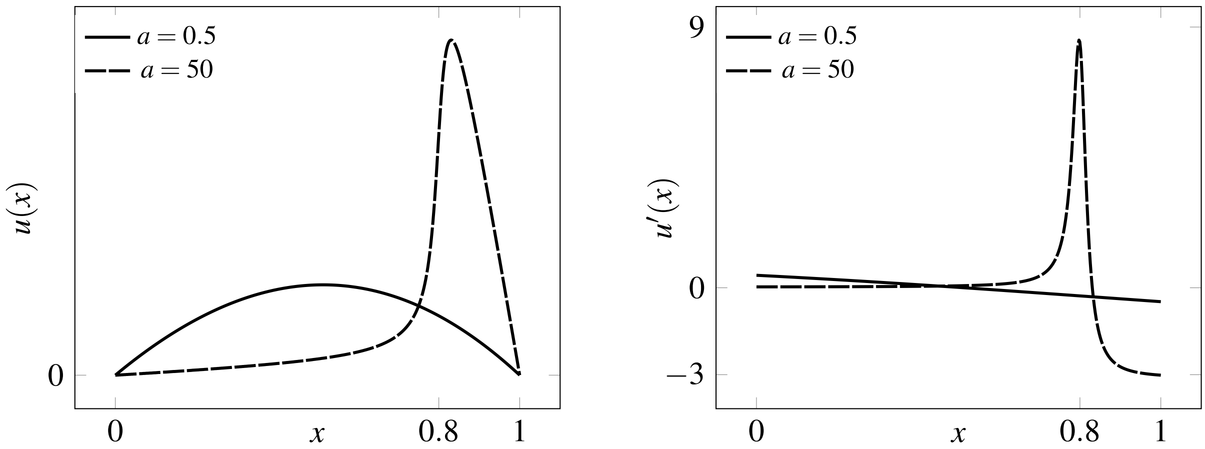
In this problem, we study the performance of h–FEM and p–FEM for solving problems with smooth or rough solutions. Consider the differential equation

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{2\left(a + a^3 b(b - x + 1)\right)}{\left(a^2 b^2 + 1\right)^2}, \tag{1}$$

with boundary conditions (BCs) $u(0) = u(1) = 0$, where a is constant and $b = x - x_b$. The solution to this boundary value problem (BVP) is given by

$$u(x) = (1 - x)\left(\arctan(ab) + \arctan(a x_b)\right). \tag{2}$$

This solution has a peak centered near $x = x_b = 0.8$ whose sharpness depends on the value of a. For large a the solution has a "near discontinuity" while for small a the solution is very smooth. Figures 1a and 1b show the exact solution and its derivative, respectively, for $a = \{0.5, 50\}$.



(a) (b)

Figure 1: Exact solution (a) and its derivative (b) for problem (1) considering $a = \{0.5, 50\}$.

1. h–version: Solve problem (1) for $x_b = 4/5$ and $a = 0.5$. Use 2, 4, 8, 16 and 32 evenly spaced linear and quadratic *Lagrangian* elements (10 runs total). Plot the numerical solutions against the exact solution in a single figure. Make a log-log plot of the relative error in the energy norm in ordinates, defined as

$$e_r = \sqrt{\frac{U - U_h}{U}}, \tag{3}$$

   versus the mesh size h and another one versus the number of degrees of freedom (DOFs) (2 plots and 4 curves total). The exact strain energy for this problem is $U = 0.03559183822564316$. Calculate the rates of convergence in the energy norm for both element types and indicate them in the convergence plots (use the last two points as we are interested in "terminal convergence." How do the computed values compare with the theoretical values (knowing that the solution is very smooth)?
2. p–version: Implement $p-$hierarchical elements for $1 \leq p \leq 5$ (or better for arbitrary p!). You will need a function that can provide Gauss quadrature of arbitrary order, for which you can use the Python function `grule` given below. Solve the same problem using 2 elements of degree $p = 1, \ldots, 5$. Plot the numerical solutions against the exact solution in a single figure. Make a log-log plot of the relative error in the energy norm versus the number of DOFs. Include the plots from h–version in the same figure for comparison purposes.
3. Using the data points (e_r and N) from the h–version and the *a posteriori* error estimate, evaluate the value of the exact strain energy. Compare this with the actual value.
4. h–version: Solve (1) for $x_b = 4/5$ and $a = 50$ and repeat the above h–version convergence study for *only* the quadratic element case. This time use 5, 10, 20 and 40 elements evenly spaced. Plot the numerical solutions against the exact solution in a single figure. Make a log-log plot and compute the rate of convergence in the energy norm. If the plot is not linear, use the last two data points to compute the (asymptotic) rate of convergence. The exact strain energy, U, for this problem is $U = 1.585854059271320$.

5. p–version: Solve problem above with 5 elements of degree $p = 1, \ldots, 5$. Plot the numerical solutions against the exact solution in a single figure. Make a log-log plot of the relative error in the energy norm versus the number of DOFs. Include the plots from the h–version in the same figure for comparison purposes.
6. Study the stability of all methods for this class of problems. Make a log-log plot of the condition number of the stiffness matrix as a function of the number of DOFs for all results obtained. Determine the rate of growth of the condition number and mark it in the plot. The condition number of the reduced matrix $\boldsymbol{K}_{\text{ff}}$ is defined as

$$\operatorname{cond}(\boldsymbol{K}_{\text{ff}}) \equiv \frac{\lambda_{\max}}{\lambda_{\min}}, \tag{4}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the matrix maximum and minimum eigenvalues.
7. Discuss (compare and draw conclusions) about your results. What conclusions can you draw about problems with strong gradients? h–version versus p–version? How many quadrature points you used in each problem and why? One page limit.

PROBLEM 2.— AN ELLIPTICAL CAVITY IN AN INFINITE PLATE UNDER REMOTE LOADING

Consider in Figure 2 a traction-free hole in an infinite plate that is subjected to a distant tensile stress σ_0 in the x direction. The whole can be described by an ellipse with axes $2a$ and $2b$ in the x and y directions, respectively. Take $b = 2\,\text{cm}$.

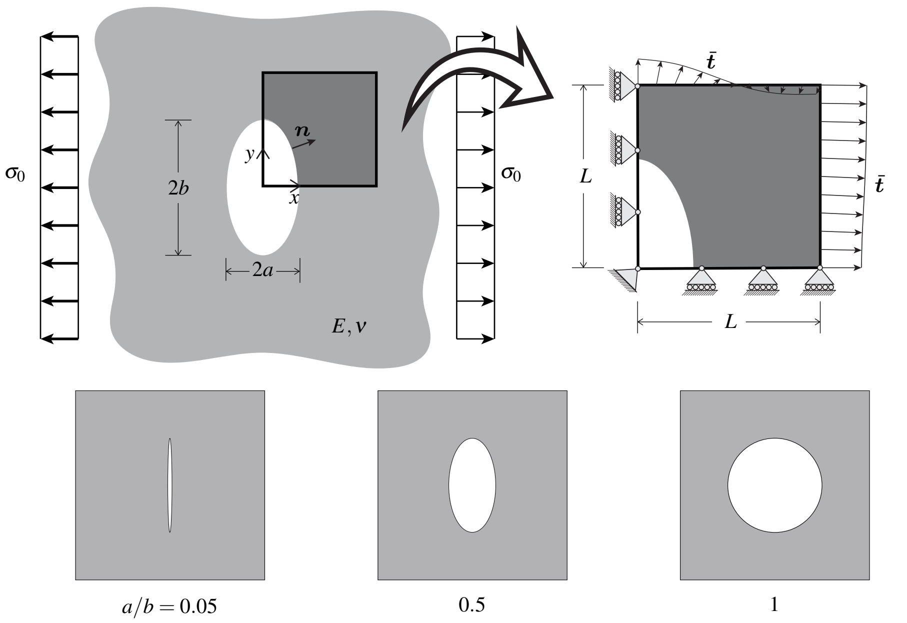


Figure 2: Schematic of an elliptical hole in an infinite plate subjected to a remote loading.

The analytical plane stress field for this problem is given by [1]:

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \sigma_0 \left(\left\{ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\} - \rho_a \rho_b \left(\frac{1}{2} \left\{ \begin{array}{c} H_1 \\ H_2 \\ H_3 \end{array} \right\} - \left(\frac{b}{a} + \frac{1}{2} \right) \left\{ \begin{array}{c} H_4 \\ H_1 \\ H_5 \end{array} \right\} \right) \right), \tag{5}$$

where

$$
\begin{aligned}
H_1 &= \frac{a^2\rho_a^2\rho_b^2 + b^2\rho_a^2 + ab\rho_a\rho_b}{(a\rho_b + b\rho_a)^2} - \rho_b^2 n_x^2 - \rho_a^2 n_y^2 + \left(5\rho_a^2 + 5\rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4\right) n_x^2 n_y^2,\\
H_2 &= \frac{\rho_b a\left(a\rho_b + b\rho_a + 2b\rho_a\rho_b^2 + a\rho_b^3\right)}{(a\rho_b + b\rho_a)^2} + n_y^2\left[2 - 6\rho_b^2 + \left(\rho_a^2 + 9\rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4\right) n_y^2\right],\\
H_3 &= n_x n_y\left[1 - 3\rho_b^2 + \left(3\rho_a^2 + 7\rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4\right) n_y^2\right],\\
H_4 &= \frac{\rho_a b\left(a\rho_b + b\rho_a + 2a\rho_a^2\rho_b + b\rho_a^3\right)}{(a\rho_b + b\rho_a)^2} + n_x^2\left[2 - 6\rho_a^2 + \left(9\rho_a^2 + \rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4\right) n_x^2\right],\\
H_5 &= n_x n_y\left[1 - 3\rho_a^2 + \left(7\rho_a^2 + 3\rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4\right) n_x^2\right],\\
\rho_a &= \frac{a}{\sqrt{a^2+\lambda}},\\
\rho_b &= \frac{b}{\sqrt{b^2+\lambda}}.
\end{aligned}
\tag{6}
$$

The outward unit normal vector $\boldsymbol{n}$ is expressed as

$$
\boldsymbol{n} = (n_x, n_y) = \frac{1}{\sqrt{x^2(b^2+\lambda)^2 + y^2(a^2+\lambda)^2}}\left(x(b^2+\lambda), y(a^2+\lambda)\right), \tag{7}
$$

where $\lambda = \frac{1}{2}\left(x^2 + y^2 - a^2 - b^2 + \sqrt{\left(x^2+y^2-a^2+b^2\right)^2 + 4\left(a^2-b^2\right)y^2}\right)$.

Due to the symmetry, consider only the $L \times L$ region (darker shade) for analysis, with $L = 2b$. Therefore, use symmetry boundary conditions along left and bottom edges, and prescribe tractions on top and right sides obtained from the analytical equation with $\sigma_0 = 50\,\text{MPa}$. Consider steel as the base material of the plate, so take Young's modulus $E = 200\,\text{GPa}$ and Poisson ratio $\nu = 0.3$.

For your analyses, choose a proper *consistent unit system* so that the results you obtain make sense (you need to report your results with units). By using any pre-processor (Gmsh for instance), create discretizations for $a/b = \{1, 0.5, 0.05\}$ using both linear-strain triangles (T3) and bilinear quadrangles (Q4). In addition, use mesh sizes $h/L = \{0.2, 0.1, 0.05\}$.

1. By using a modified version of the Python code provided in class, solve the problems on both T3 and Q4 meshes (18 solutions in total), and plot the displacement fields (deformed configuration) for a given mesh size and element type (3 figures for different a/b).
2. Do a simple back-of-envelope calculation and estimate the displacements $u_x(L,0)$ and $u_y(0,L)$ (explain your rationale for the calculation). How does your estimate compare with numerically obtained values?
3. Compute the exact strain energy, and with the *a posteriori* error estimate, calculate the approximate value of the exact strain energy for each value of a/b. How do exact and approximate energy values compare with one another?
4. Create convergence plots, i.e., log-log plots of the error in energy norm as a function of both mesh size and total number of DOFs (2 plots total, with 6 curves each considering different values of a/b and element type). Compute the rate of convergence in the energy norm and compare it to theoretical values. Are convergence rates in agreement with theoretical predictions?
5. Plot the stress fields on the entire domain ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$, and von Mises stress) for the smallest mesh size and comment on stress concentrations and values at free boundaries. Notice that while for triangular elements the stress inside is constant, for Q4 elements stress values are most accurate at Gauss integration points (2×2 rule). Therefore, you need to extrapolate the stresses computed at Gauss points to nodes and then average (the so-called superconvergent patch recovery). Are stresses more or less accurate than displacements?
6. Compute the stress concentration factor, defined as the ratio of the maximum stress to the nominal applied tensile stress, and compare it to the analytical expression for an elliptical hole given by $K_c = 1 + 2\frac{b}{a}$.
7. Consider a yield strength of $\bar{\sigma} = 210\,\text{MPa}$. Does the material yield for the different values of a/b? Determine the exact value of the applied traction that is needed to produce yielding in each case. What yield criterion you used?
8. Discuss (compare and draw conclusions) about your results. One page limit. In view of the results, comment on the behavior of both T3 and Q4 elements regarding displacement and stresses, and on the validity of the simulations conducted.

Presentation of results

Each student must submit a final project report. This report should be in the format of a technical paper. You can consult an ASCE journal (*e.g.*, Journal of Engineering Mechanics) to get an idea of what is an appropriate format for a technical paper. Handwritten papers are not acceptable. All papers must be submitted in hardcopy and in pdf. You should make the report as concise as possible, and thus do not copy material from the course material unless it is really needed. For the report, *less is more*! Data generated in your investigation must be presented in an appropriate graphical or tabular format. Figures should be clear, informative, and well labeled. Source code supporting the project must be included and documented in an appendix to the report.

Note on collaboration

The final product (report, code, results and discussion, *etc.*) must be your own individual work. You are encouraged to discuss issues pertaining to the understanding of the projects with other students in the class within reasonable and customary bounds. You must *write your own code*, you must *select your own parameter values* in your studies, and *your discussion of results should be completely independent*. You should be able to divide your development into understanding and execution. To get to understanding of a given topic is generally OK for collaboration, execution steps generally are not. Certainly, if you are sharing electronic files, you have probably gone too far. If you are unsure of what constitutes acceptable collaboration, please ask.

Evaluation

Each enumerated item above will be graded and you may get total or partial credit depending on how you addressed the item. The final grade out of 100 points you obtain for the report (and for the course), will be the average you get between the two problems. Before submitting the report, consider the following:

- **Technical Execution** Have you conquered the technical aspects of your topic well? For example, have you written your computer program correctly? Are the equations used in your implementation correct? Do you have a strategy for deciding if your answers are right (verification of implementation and equations)?
- **Communication** Have you described what you did and what you found in a manner that I can appreciate by simply reading your paper? Do you use words well? Have you organized the paper well? Do you use graphical presentation of data effectively? Does the report have conclusions?
- **Technical Insight** (Discussion) Have you gone beyond simply calculating something to actually pondering the outcome of those calculations? Can you shed an interesting light on any computed results? Your conclusions are just a summary of what you did or actually bring new insight?

References

[1] X. Jin, Z. Wang, Q. Zhou, L. M. Keer, and Q. Wang. “On the solution of an elliptical inhomogeneity in plane elasticity by the equivalent inclusion method”. In: *Journal of Elasticity* 114.1 (2014), pages 1–18 (cited on page 2).