2 The (Galerkin) Finite Element Method

2.1 Approximate Solution and Nodal Values

In order to obtain a numerical solution to a differential equation using the Galerkin Finite Element Method (GFEM), the domain is subdivided into **finite elements**. The function is approximated by piecewise **trial functions** over each of these elements. This is illustrated below for the one-dimensional case, with *linear* functions used over each element, p being the dependent variable.

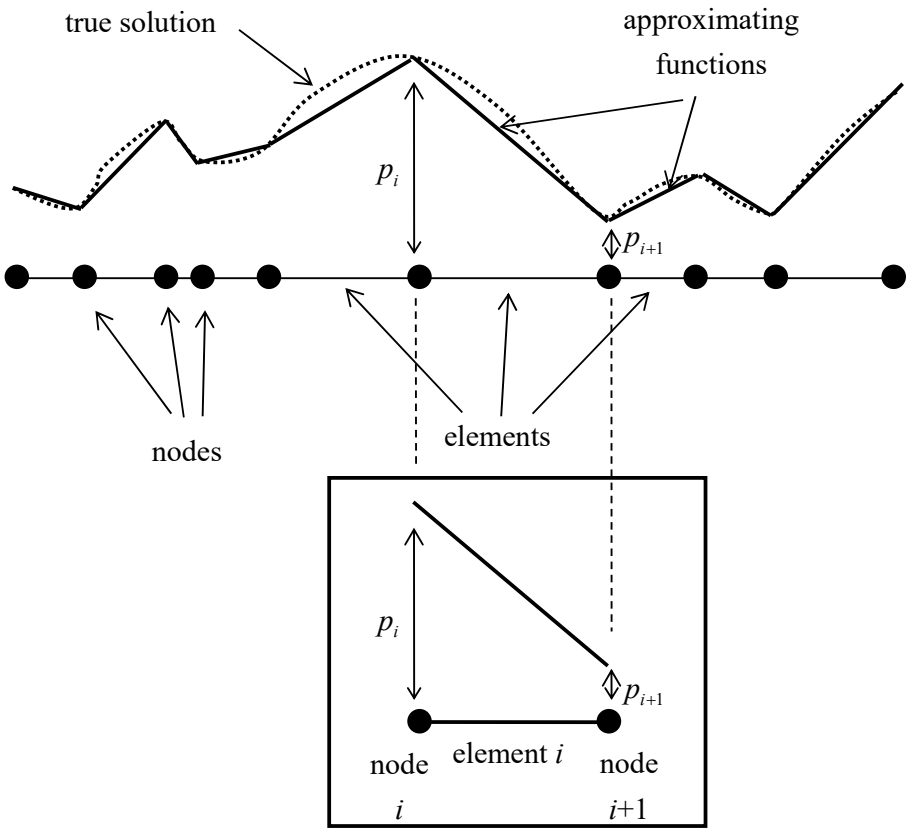

Figure 2.1: A mesh of *N* one dimensional Finite Elements

The unknowns of the problem are the **nodal values** of p, $p_i \;\; i = 1 \ldots N+1$, at the **element boundaries** (which in the 1D case are simply points). The (approximate) solution within each element can then be constructed once these nodal values are known.

2.2 Trial Functions

2.2.1 Lagrange and Hermite Elements

There are an endless number of different trial functions which one can use. In practice, these trial functions can be grouped into two broad types. The first consists of the **Lagrange** or C^0 trial functions (and corresponding Lagrange or C^0 element). These are trial functions which are continuous across element boundaries, but whose first derivatives are not continuous across boundaries. The elements in Fig. 2.1 are C^0 linear elements – there is a clear jump in the first derivative of the trial functions at the element boundaries (nodes) – the first derivative is piecewise continuous.

The second group consists of the **Hermite** or C^1 trial functions (elements). These are functions which are not only continuous across element boundaries, but whose first derivatives are also continuous across boundaries.

In general, a C^n element is one for which the trial functions are continuous up to the nth derivative, but elements with $n > 1$ are rarely used. In fact, three of the most commonly encountered types of element are those with a

(1) linear Lagrange trial function (C^0)
(2) quadratic Lagrange trial function (C^0)
(3) cubic Hermite trial function (C^1)

These three trial functions / elements will be discussed in what follows.

Obviously, the higher the order and the higher the continuity of the element, the better the accuracy one would expect, but the more computation which is required.

2.2.2 The C^0 Linear Element

The C^0 linear element is by far the most commonly used finite element. Consider one typical element of the domain, with end-points x_1, x_2, Fig. 2.2. Assuming a linear interpolation,

$$\tilde{p}(x) = a + bx\,. \tag{2.1}$$

Let the (unknown) end-point values be $\tilde{p}(x_1) = \tilde{p}_1$, $\tilde{p}(x_2) = \tilde{p}_2$. This gives two algebraic equations in two unknowns *a* and *b*,

$$\begin{aligned} \tilde{p}_1 &= a + bx_1 \\ \tilde{p}_2 &= a + bx_2 \end{aligned} \tag{2.2}$$

Solving the equations gives

$$a = \frac{\tilde{p}_1 x_2 - \tilde{p}_2 x_1}{x_2 - x_1}, \quad b = \frac{\tilde{p}_2 - \tilde{p}_1}{x_2 - x_1} \tag{2.3}$$

so that, after some algebra, one can write $\tilde{p}(x)$ in terms of the two unknowns $\tilde{p}_1$, $\tilde{p}_2$ (instead of in the form of Eqn. 2.1, which is in terms of the two values *a* and *b*)

Linear Trial Function:

$$\begin{aligned} \tilde{p}(x) &= N_1(x)\tilde{p}_1 + N_2(x)\tilde{p}_2 \\ N_1(x) &= \frac{x_2 - x}{L}, \quad N_2(x) = \frac{x - x_1}{L} \end{aligned} \tag{2.4}$$

where *L* is the length of the element, $L = x_2 - x_1$.

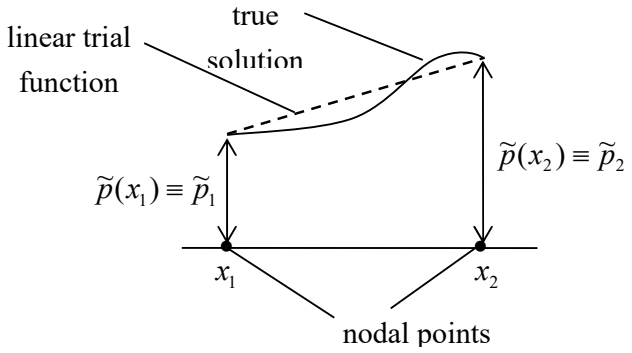

Figure 2.2: Linear trial function approximation over an element

The function $p(x)$ can now be approximated over each interval through

$$\tilde{p}(x) = \begin{array}{ll} N_1(x)\tilde{p}_1 + N_2(x)\tilde{p}_2, & x_1 < x < x_2 \\ N_1(x)\tilde{p}_2 + N_2(x)\tilde{p}_3, & x_2 < x < x_3 \\ \vdots & \\ N_1(x)\tilde{p}_N + N_2(x)\tilde{p}_{N+1}, & x_N < x < x_{N+1} \end{array} \tag{2.5}$$

Here, the **shape** (or **basis**) **functions** N_1, N_2 are the same over each interval (although they don't have to be – they could be interspersed with, for example, quadratic shape functions – see later).

Structure of the Linear Shape Functions

The shape functions, Eqns. 2.4, have a number of interesting properties. Most importantly, they have a value of either 0 or 1 at a node - the variation of the shape functions over an element is shown in Fig. 2.3. A second property of the shape functions is that they sum to 1, $\sum_{i=1}^{2} N_i = 1$.

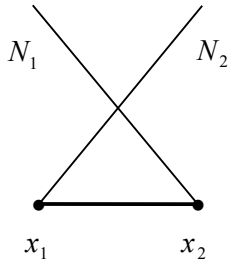

Figure 2.3: Shape functions for the linear trial function

2.3 The Standard Galerkin FEM

The Galerkin FEM for the solution of a differential equation consists of the following steps:

(1) multiply the differential equation by a **weight function** $\omega(x)$ and form the integral over the whole domain
(2) if necessary, integrate by parts to reduce the order of the highest order term

(3) choose the order of interpolation (e.g. linear, quadratic, etc.) and corresponding shape functions $N_i, i = 1 \ldots m$, with trial function $p = \tilde{p}(x) = \sum_{i=1}^{m} N_i(x) p_i$

(4) evaluate all integrals over each element, either exactly or numerically, to set up a system of equations in the unknown p_i's

(5) solve the system of equations for the p_i's.

The linear C^0 element will be used in what follows. Quadratic and cubic elements will be considered later.

2.3.1 A Single-Element Example

Consider the following problem: solve the following differential equation using *one* linear element:

$$\frac{d^2 p}{dx^2} = 0, \quad p'(0) = 1, \quad p(2) = 0 \tag{2.6}$$

[the exact solution is $p(x) = x - 2$]

First, multiply the equation across by $\omega(x)$ and integrating over $[0,2]$ to get the **weighted residual integral**

$$I = \int_0^2 \left(\frac{d^2 p}{dx^2} \right) \omega(x) dx = 0 \tag{2.7}$$

Integrating by parts (and multiplying across by -1) leads to the **weak form**

$$I = \int_0^2 \left(\frac{dp}{dx} \frac{d\omega}{dx} \right) dx - \left[\frac{dp}{dx} \omega \right]_0^2 = 0 . \tag{2.8}$$

This step is crucial to the FEM when linear trial functions are used, since if the trial function $\tilde{p}$ is linear, then $\tilde{p}'' = 0$, and one cannot work with (2.7). However, by first integrating by parts, there is no longer any second derivative and (2.8) is no longer the trivial $0 = 0$.

Choose the linear trial function[1] and, from Eqn. 2.4,

$$\tilde{p}(x) = N_1 p_1 + N_2 p_2 \quad N_1 = 1 - \frac{x}{2} \qquad N_2 = \frac{x}{2} \tag{2.9}$$

Now in the Galerkin FEM, one lets the weight functions simply be equal to the shape functions, i.e. $\omega_i = N_i$, so that[2] $\omega_i = \partial\tilde{p} / \partial p_i$. Thus one has two equations in two unknowns, *one equation for each weight function*. Note also that *the boundary term is not discretised* using (2.9), it is left as dp/dx, so that boundary conditions can be applied (see below),

$$\begin{aligned} \int_0^2 \left(\frac{d\left(N_1 p_1 + N_2 p_2\right)}{dx} \frac{dN_1}{dx} \right) dx - \left[\frac{dp}{dx} N_1 \right]_0^2 = 0 \\ \int_0^2 \left(\frac{d\left(N_1 p_1 + N_2 p_2\right)}{dx} \frac{dN_2}{dx} \right) dx - \left[\frac{dp}{dx} N_2 \right]_0^2 = 0 \end{aligned} \tag{2.10}$$

As mentioned, the shape functions have the following property:

$$\begin{aligned} N_1(0) = 1, \quad N_1(2) = 0 \\ N_2(0) = 0, \quad N_2(2) = 1 \end{aligned} \tag{2.11}$$

i.e. they are zero or one at one of the end-points, and so the boundary terms simplify to

$$\begin{aligned} p_1 \int_0^2 \frac{dN_1}{dx} \frac{dN_1}{dx} dx + p_2 \int_0^2 \frac{dN_2}{dx} \frac{dN_1}{dx} dx + \left[\frac{dp}{dx} \right]_{\text{node 1}} = 0 \\ p_1 \int_0^2 \frac{dN_1}{dx} \frac{dN_2}{dx} dx + p_2 \int_0^2 \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \left[\frac{dp}{dx} \right]_{\text{node 2}} = 0 \end{aligned} \tag{2.12}$$

Substituting in the shape functions and evaluating the integrals leads to the equations

$$\frac{1}{2} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -p'(0) \\ +p'(2) \end{bmatrix} \tag{2.13}$$

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[1] this example is similar to the Galerkin examples of Chapter 1, the only difference being that here the unknowns in the trial function are the end-point values, rather than the *a*, *b* of (2.1)

[2] this is by far the most commonly used version of the FEM. However, there are some problems which are not well solved using the Galerkin FEM; in these cases other variations of the FEM can be used in which the weight functions chosen are not simply the shape functions

Applying the essential boundary condition $p(2)=0$, the first equation with the natural BC $p'(0)=1$ gives $\frac{1}{2}p_1 = -p'(0) = -1 \rightarrow p_1 = -2$. From the second equation, one finds that $p'(2)=1$. The full solution is

$$p = (1-x/2)p_1 + (x/2)p_2 = x-2 \tag{2.14}$$

Note that the solution is exact – because the solution was assumed to be linear, and it is.

Consider now the general ordinary differential equation with constant coefficients:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = d \tag{2.15}$$

The weighted residual integral is

$$I = \int_{x_1}^{x_2}\left(a\frac{du}{dx}\frac{d\omega}{dx} - b\frac{du}{dx}\omega - cu\omega \right)dx + \int_{x_1}^{x_2} d\,\omega dx - a\left[\frac{du}{dx}\omega\right]_{x_1}^{x_2} = 0. \tag{2.16}$$

Using the linear trial function, and after some algebra, one arrives at the system of two equations

Equations for Linear Trial Function:

$$\left\{ a\frac{1}{L}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix} - b\frac{1}{2}\begin{bmatrix}-1 & +1\\ -1 & +1\end{bmatrix} - c\frac{L}{6}\begin{bmatrix}+2 & +1\\ +1 & +2\end{bmatrix}\right\}\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix} + d\frac{L}{2}\begin{bmatrix}1\\ 1\end{bmatrix} = a\begin{bmatrix}-u'(x_1)\\ +u'(x_2)\end{bmatrix} \tag{2.17}$$

As an example, let $a=1, b=-2, c=d=1$ and $x_1=0, x_2=1, L=1$. The exact solution for the boundary conditions $u(x_1)=\tilde{u}_1=1, u'(x_2)=2$ is $u(x)=1+xe^{x-1}$. Putting these values directly into the linear equations Eqn. 2.17 immediately yields $\tilde{u}_2 = 2.2$ and $u'(x_1)=0.2$ (compared with the exact solutions 2 and 0.368 respectively) so

$$\tilde{u}(x) = \tilde{u}_1(1-x) + \tilde{u}_2 x = 1 + 1.2x \tag{2.18}$$

This solution is compared with the exact solution in Fig. 2.4.

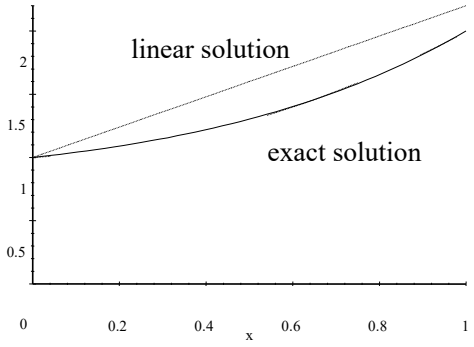


Figure 2.4: Single Linear C^0 Element Solution to Eqn. 2.15

2.3.2 Global and Local Formulations of the FEM

There are two ways in which the FEM can be formulated – the **global** and **local** formulations. In what follows, a simple example will be examined using both formulations; the emphasis here is on how the **global stiffness matrix** is formed, and how the boundary conditions are applied.

Consider the differential equation

$$\frac{d^2p}{dx^2} + 1 = 0, \qquad p(0) = 1, \quad p'(2) = 1 \tag{2.19}$$

[the exact solution is $p(x) = 1 + 3x - \frac{1}{2}x^2$]

The weighted residual, after an integration by parts, is

$$I = \int_0^2 \left(\frac{dp}{dx}\frac{d\omega}{dx} - \omega \right) dx - \left[\frac{dp}{dx}\omega \right]_0^2 = 0 \tag{2.20}$$

This equation will now be solved using two finite elements.

Solution: The Global Formulation

In the global formulation, one uses a single trial function which extends over the *complete domain*[3],

$$\tilde{p}(x) = \tilde{N}_1(x)p_1 + \tilde{N}_2(x)p_2 + \tilde{N}_3(x)p_3 \tag{2.21}$$

and these shape functions are shown below.

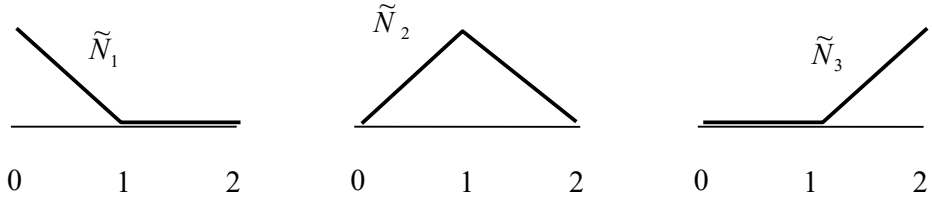


Figure 2.5: Linear shape functions

$$\tilde{N}_1 = \begin{cases} 1-x & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases} \qquad \tilde{N}_2 = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \end{cases} \qquad \tilde{N}_3 = \begin{cases} 0 & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \end{cases} \tag{2.22}$$

Three weighted residuals are now formed, one for each shape function, leading to

$$\begin{aligned}
&p_1\left[+\int_0^1 dx\right] + p_2\left[-\int_0^1 dx\right] + \left[-\int_0^1 (1-x)dx\right] && + p'(0) = 0 \quad \left(\omega = \tilde{N}_1\right) \\
&p_1\left[-\int_0^1 dx\right] + p_2\left[+\int_0^1 dx\right] + \left[-\int_0^1 x dx\right] && \\
&\qquad + p_2\left[+\int_1^2 dx\right] + p_3\left[-\int_1^2 dx\right] + \left[-\int_1^2 (2-x)dx\right] && = 0 \quad \left(\omega = \tilde{N}_2\right) \\
&\qquad + p_2\left[-\int_1^2 dx\right] + p_3\left[+\int_1^2 dx\right] + \left[-\int_1^2 (x-1)dx\right] && - p'(2) = 0 \quad \left(\omega = \tilde{N}_3\right)
\end{aligned} \tag{2.23}$$

and the system of equations

[3] as in the standard Galerkin Method of Chapter 1

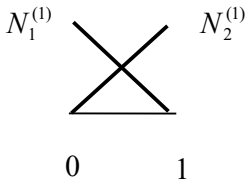
$$\begin{bmatrix} +1 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - p'(0) \\ 1 \\ \frac{1}{2} + p'(2) \end{bmatrix} \tag{2.24}$$

Global Stiffness Matrix

Solution: The Local Formulation

The local or *element* viewpoint is the traditional approach in engineering and is the more useful one when it comes to coding equations. Here, the calculations are done for each element separately:

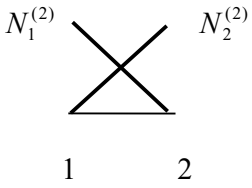
Element 1



$$N_1^{(1)} = 1 - x$$
$$N_2^{(1)} = x$$
$$\tilde{p}^{(1)} = (1-x)p_1 + xp_2$$

$$\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - p'(0) \\ \frac{1}{2} + p'(1) \end{bmatrix}$$

Element 2



$$N_1^{(2)} = 2 - x$$
$$N_2^{(2)} = x - 1$$
$$\tilde{p}^{(2)} = (2-x)p_2 + (x-1)p_3$$

$$\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - p'(1) \\ \frac{1}{2} + p'(2) \end{bmatrix}$$

(2.25)

$$\begin{bmatrix} +1 & -1 & 0 \\ -1 & +1 & 0 \\ \hline 0 & +1 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - p'(0) \\ \frac{1}{2} + p'(1) \\ \hline \frac{1}{2} - p'(1) \\ \frac{1}{2} + p'(2) \end{bmatrix}$$

boundary terms cancel at internal boundaries

the full global matrix is constructed from the individual element matrices – from the global formulation, it can be seen that one must sum the contributions for common nodes

$$\begin{bmatrix} +1 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - p'(0) \\ 1 \\ \frac{1}{2} + p'(2) \end{bmatrix}$$

Boundary Conditions

Apply now the BC's $p(0) = 1,\ p'(2) = 1$ to get

$$\begin{bmatrix} 2 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} \end{bmatrix}, \qquad p'(0) = p_2 - \tfrac{1}{2} \tag{2.26}$$

which can be solved for

$$p_2 = \tfrac{7}{2},\ p_3 = 5, \quad p'(0) = 3 \tag{2.27}$$

The solution here happens to be exact at the nodes – this will be the case for problems of the form $p'' = f(x)$.

Returning now to the trial functions (2.21) or (2.25), the full solution, graphed in Fig. 2.6, is

Element 1: $\widetilde{p}^{(1)} = 1 + \frac{5}{2}x$
Element 2: $\widetilde{p}^{(2)} = 2 + \frac{3}{2}x$

Note that, when programming the FE, it is best if rows and columns of the coefficient matrix are not eliminated as in going from (2.24) to (2.26), when applying boundary conditions. The essential BC corresponds to node 1, so we replace the first row with the essential BC. If one wants to preserve a symmetric coefficient matrix, one can also then replace the first column with zeros (and a 1). In this way, application of the boundary conditions to the above system of 3 equations would lead to (note how the "–1" in the first column, second row, of the original coefficient matrix, is brought over to the right hand side, changing the 1 to a 2)

$$\begin{bmatrix} +1 & 0 & 0 \\ 0 & +2 & -1 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{3}{2} \end{bmatrix} \tag{2.28}$$

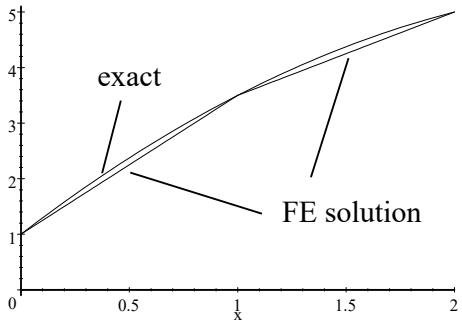


Figure 2.6: Two (Linear C^0) Element Solution to the ODE (2.19)

The First Derivative

The solution for the primary variable p is more accurate than the solution for the derivative p'; p was approximated by a linear C^0 function, so here p' can only be approximated as constant over each element, and will be discontinuous at element boundaries:

Element 1: $\widetilde{p}'^{(1)} = \frac{5}{2}$
Element 2: $\widetilde{p}'^{(2)} = \frac{3}{2}$

The amount of discontinuity can be used as a guide to the inaccuracy of the overall FE solution.

A **smoothed solution** for the first derivative, plotted in Fig. 2.7, can be obtained as follows: take the average of the two solutions at the boundary point, so set $p'(1) = \left(\frac{5}{2} + \frac{3}{2}\right)/2 = 2$, and also use the values at the end points, $p'(2) = 1$ (specified) and $p'(0) = 3$ (evaluated from the FE equations 2.24), then join them up linearly. In this simple example, the smoothed solution actually equals the exact solution. Note that although the smoothed solution looks good, it tends to hide the inaccuracy of the solution. FE software typically outputs smoothed results by default – one should remove this option if one is interested in examining the accuracy/reliability of FE software solutions.

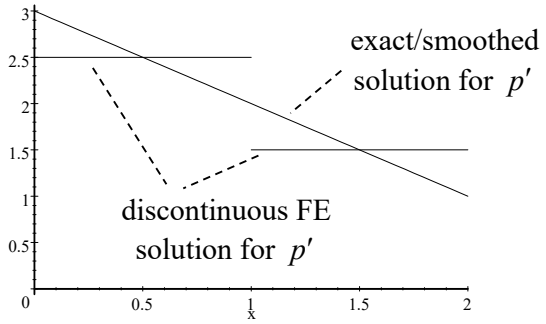


Figure 2.7: FEM solution for the Derivative using Linear Elements

Note the following:

(1) the solution can be made more accurate by either (or both)
   (i) dividing the domain into more elements (the ***h*-method**)
   (ii) selecting higher order elements (e.g. quadratic elements) (the ***p*-method**)

(2) This mesh has three **degrees of freedom**: there are three nodes, and each node has a single degree of freedom (there is only one variable, p, associated with each node). The final system of equations to be solved will be of size $n \times n$, where n is the number of degrees of freedom.

(3) It is not difficult to reformulate the problem with elements of unequal length

(4) The coefficient (Global Stiffness) matrix for this problem, in (2.24), is singular. Thus if one puts two natural boundary conditions into the system of equations one cannot obtain a solution.

2.4 Adaptive Meshing

The difference between the discontinuous FE solution for p' and the smoothed solution allows many FE softwares to automatically refine the mesh in regions where the accuracy is not good[4]. The FE and smoothed solutions are first obtained: $(p')_{\text{FE}}$, $(p')_{\text{smoothed}}$. The "error" is then $(p')_{\text{FE}} - (p')_{\text{smoothed}}$. Measures of these over each element can be determined by integrating over the element, say $\int_{L_i} \left[(p')_{\text{FE}} \right]^2 dx$, $\int_{L_i} \left[(p')_{\text{smoothed}} \right]^2 dx$ and the

[4] that is supposing that the first derivative *shouldn't* be discontinuous – it may well be discontinuous, in certain problems, for example where a stress field is discontinuous at the interface between different materials

element error (squared) $e_i = \int_{L_i} \left[(p')_{\text{FE}} - (p')_{\text{smoothed}} \right]^2 dx$. An example of a global relative error η would then be

$$\eta = \sqrt{\frac{\sum e_i}{\sum \left\{ \int_{L_i} \left[(p')_{\text{FE}} \right]^2 dx \right\} + \sum e_i}} \tag{2.29}$$

η is a global parameter – it doesn't measure error at a particular point/element (element mesh-adaptation parameters might be unreliable, since they would be large simply if $\int_{L_i} \left[(p')_{\text{FE}} \right]^2 dx$ were small). An algorithm might then be: evaluate η; if $\eta < 0.05$ terminate, otherwise refine the mesh in regions where e_i is large; stop after, say, 5 iterations. Note that neither η nor e can reveal the percentage error in the analysis, since the true solution is unknown.

2.5 Local Coordinate Systems

It is convenient to rewrite the FE expressions in terms of a **local** (or **natural**) **coordinate system** ξ. Here, all elements will be normalised over the interval $\xi = [-1,+1]$[5]. To this end, let

Change of Variable to Local Coordinates:

$$x = \frac{1}{2}(1-\xi)x_i + \frac{1}{2}(1+\xi)x_{i+1}, \qquad \xi = \frac{2(x-x_i)}{L} - 1, \qquad J \equiv \frac{dx}{d\xi} = \frac{L}{2} \tag{2.30}$$

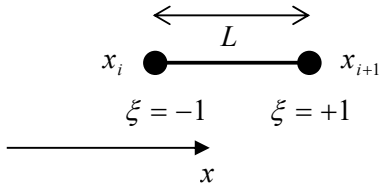


Figure 2.8: Local coordinates over an element

[5] here, the coordinate system used is $\xi = [-1,+1]$. Note that the interval $\xi = [0,+1]$ is sometimes used

From Eqn. 2.4,

Linear Shape Functions (Local Coordinates)

$$N_1(\xi)=\frac{1}{2}(1-\xi),\quad N_2(\xi)=\frac{1}{2}(1+\xi) \tag{2.31}$$

This change of coordinates results in integrals which appear again and again in different FE problems, and they only have to be evaluated on a *once-and-for-all basis*. A number of this type of important integral are evaluated in the Appendix to this chapter. In higher dimensional problems (2-D and 3-D), this normalisation allows one to obtain approximate solutions to the integrals using numerical integration rules.

As an example of using the local coordinate system, consider this problem: solve the following differential equation using two linear elements of equal length:

$$\frac{d^2p}{dx^2}+\frac{dp}{dx}=1,\quad p(1)=1,\quad \left.\frac{dp}{dx}\right|_{x=2}=2 \tag{2.32}$$

[the exact solution is $p(x)=e+x-e^{2-x}$]

One has

$$I=\int_1^2\left(\frac{dp}{dx}\frac{d\omega}{dx}-\frac{dp}{dx}\omega+w\right)dx-\left[\frac{dp}{dx}\omega\right]_1^2=0\,. \tag{2.33}$$

In the global coordinate system,

$$\begin{aligned}
&\underline{\text{Element 1:}}\quad \sum_{i=1}^{2}p_i\int_0^L\frac{dN_i^{(1)}}{dx}\frac{dN_j^{(1)}}{dx}dx-\sum_{i=1}^{2}p_i\int_0^L\frac{dN_i^{(1)}}{dx}N_j^{(1)}dx+\int_0^L N_j^{(1)}dx+\delta_{j1}p'(1)\\
&\underline{\text{Element 2:}}\quad \sum_{i=1}^{2}p_{i+1}\int_L^{2L}\frac{dN_1^{(2)}}{dx}\frac{dN_j^{(2)}}{dx}dx-\sum_{i=1}^{2}p_{i+1}\int_L^{2L}\frac{dN_1^{(2)}}{dx}N_j^{(2)}dx+\int_L^{2L}N_j^{(2)}dx-\delta_{j2}p'(2)
\end{aligned} \tag{2.34}$$

with $j=1,2$. Changing to local coordinates, note first the shape functions,

$$\underline{\text{Element 1}}:\ p^{(1)} = N_1^{(1)} p_1 + N_2^{(1)} p_2 = \left(\frac{1-\xi^{(1)}}{2}\right) p_1 + \left(\frac{1+\xi^{(1)}}{2}\right) p_2 \qquad \left(\xi^{(1)} = 2\,\tfrac{x-1}{L} - 1\right)$$

$$\underline{\text{Element 2}}:\ p^{(2)} = N_1^{(2)} p_2 + N_2^{(2)} p_3 = \left(\frac{1-\xi^{(2)}}{2}\right) p_2 + \left(\frac{1+\xi^{(2)}}{2}\right) p_3 \quad \left(\xi^{(2)} = 2\,\tfrac{x-3/2}{L} - 1\right) \tag{2.35}$$

with $L = \frac{1}{2}$. Using the chain rule, $dN/dx = \left(dN/d\xi\right)\left(d\xi/dx\right)$ and, from Eqn. 2.30, $d\xi/dx = 2/L$, this leads to

$$\begin{aligned} &\sum_{i=1}^{2} p_i \left\{ \left[\frac{2}{L} \int_{-1}^{+1} \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} d\xi \right] - \left[\int_{-1}^{+1} \frac{dN_i}{d\xi} N_j d\xi \right] \right\} + \left[\frac{L}{2} \int_{-1}^{+1} N_j d\xi \right] + \delta_{j1} p'(1) \\ &\sum_{i=1}^{2} p_i \left\{ \left[\frac{2}{L} \int_{-1}^{+1} \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} d\xi \right] - \left[\int_{-1}^{+1} \frac{dN_i}{d\xi} N_j d\xi \right] \right\} + \left[\frac{L}{2} \int_{-1}^{+1} N_j d\xi \right] - \delta_{j2} p'(2) \end{aligned} \tag{2.36}$$

The square bracketed terms/integrals are evaluated in the Appendix to this Chapter so that

Element 1:
$$\begin{aligned} \tfrac{5}{2} p_1 - \tfrac{5}{2} p_2 + \tfrac{1}{4} + p'(1) &= 0 \\ -\tfrac{3}{2} p_1 + \tfrac{3}{2} p_2 + \tfrac{1}{4} &= 0 \end{aligned}$$

Element 2:
$$\begin{aligned} \tfrac{5}{2} p_2 - \tfrac{5}{2} p_3 + \tfrac{1}{4} &= 0 \\ -\tfrac{3}{2} p_2 + \tfrac{3}{2} p_3 + \tfrac{1}{4} - p'(2) &= 0 \end{aligned}$$

and the following system of three equations is obtained:

$$\begin{bmatrix} +\frac{5}{2} & -\frac{5}{2} & 0 \\ -\frac{3}{2} & +4 & -\frac{5}{2} \\ 0 & -\frac{3}{2} & +\frac{3}{2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} - p'(1) \\ -\frac{1}{2} \\ -\frac{1}{4} + p'(2) \end{bmatrix} \tag{2.37}$$

The essential boundary condition is applied at 1 and the natural boundary condition at 2, which leads to

$$\begin{bmatrix} 4 & -\frac{5}{2} \\ -\frac{3}{2} & +\frac{3}{2} \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix} \ \rightarrow \ \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{47}{18} \\ \frac{34}{9} \end{bmatrix} = \begin{bmatrix} 2.6111 \\ 3.7777 \end{bmatrix} \tag{2.38}$$

This can be compared to the exact solution, which is $p_2 = 2.5696,\ p_3 = 3.7183$. The complete solution (plotted in Fig. 2.9) is then

Element 1: $p^{(1)} = (3-2x)p_1 + (2x-2)p_2 = -\frac{20}{9} + \frac{29}{9}x$

Element 2: $p^{(2)} = (4-2x)p_2 + (2x-3)p_3 = -\frac{8}{9} + \frac{21}{9}x$

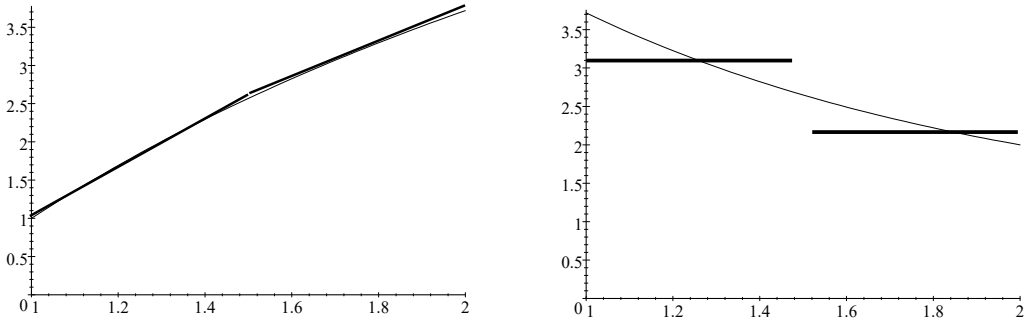


Figure 2.9: FEM solution for the ODE in (2.32) – left: *p*, right: d*p*/d*x*

Note that the error at $x = 2$ reduces as more elements are taken. The error is shown here:

L	*N*	*Error*
1	1	0.28172
0.5	2	0.05949
0.25	4	0.01433

When the element size is halved, the error is reduced by a factor of approximately 4. This general convergence behaviour occurs for linear differential equations and is explained in section 2.6.1 below.

Finally, consider the following general differential equation with constant coefficients using *n* linear elements of length L_i, $i = 1 \ldots n$, Fig. 2.10:

$$a\frac{d^2p}{dx^2} + b\frac{dp}{dx} + cp = d \tag{2.39}$$

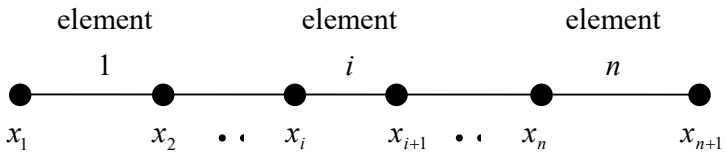


Figure 2.10: a 1-D FE mesh

The weighted residual integral for element *i* is

$$I = \int_{x_1}^{x_{n+1}} \left(a\frac{dp}{dx}\frac{d\omega}{dx} - b\frac{dp}{dx}\omega - cpw \right) dx + \int_{x_1}^{x_{n+1}} dw dx - a\left[\frac{dp}{dx}\omega \right]_{x_1}^{x_{n+1}} = 0 . \qquad (2.40)$$

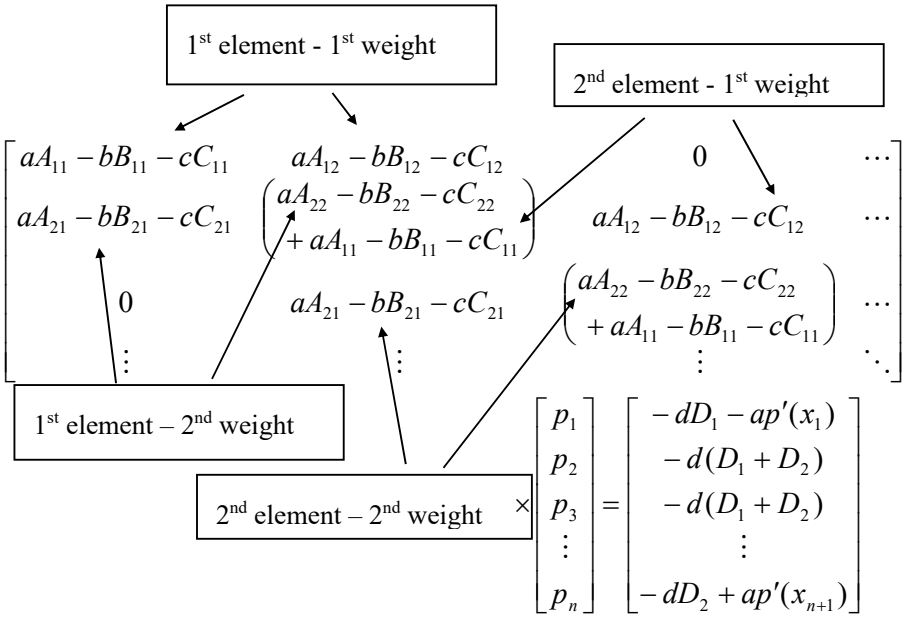
The integral is subdivided into two separate integrals: the first will give rise to matrices, the second to column vectors. Changing to local coordinates leads to

$$p_i \left\{ aA_{j1} - bB_{j1} - cC_{j1} \right\} + p_{i+1} \left\{ aA_{j2} - bB_{j2} - cC_{j2} \right\} + dD_j + \delta_{j1} p'(x_1) - \delta_{jn} p'(x_{n+1}) \qquad (2.41)$$

for element *i* and weight *j*, with (see the Appendix to this Chapter)

$$\begin{aligned} A_{jm} &= \frac{2}{L_i}\int_{-1}^{+1} \frac{dN_j}{d\xi}\frac{dN_m}{d\xi} d\xi = \frac{1}{L_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, & B_{jm} &= \int_{-1}^{+1} N_j \frac{dN_m}{d\xi} d\xi = \frac{1}{2}\begin{bmatrix} -1 & +1 \\ -1 & +1 \end{bmatrix} \\ C_{jm} &= \frac{L_i}{2}\int_{-1}^{+1} N_j N_m d\xi = \frac{L_i}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, & D_j &= \frac{L_i}{2}\int_{-1}^{+1} N_j d\xi = \frac{L_i}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \qquad (2.42)$$

The final system of equations are therefore as shown here.



$$\begin{bmatrix} aA_{11}-bB_{11}-cC_{11} & aA_{12}-bB_{12}-cC_{12} & 0 & \cdots \\ aA_{21}-bB_{21}-cC_{21} & \begin{pmatrix} aA_{22}-bB_{22}-cC_{22} \\ +aA_{11}-bB_{11}-cC_{11} \end{pmatrix} & aA_{12}-bB_{12}-cC_{12} & \cdots \\ 0 & aA_{21}-bB_{21}-cC_{21} & \begin{pmatrix} aA_{22}-bB_{22}-cC_{22} \\ +aA_{11}-bB_{11}-cC_{11} \end{pmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} -dD_1 - ap'(x_1) \\ -d(D_1+D_2) \\ -d(D_1+D_2) \\ \vdots \\ -dD_2 + ap'(x_{n+1}) \end{bmatrix} \tag{2.43}$$

Note that the system matrix is **tridiagonal** meaning only the diagonal and adjacent elements are non-zero. Special efficient sparse-matrix algorithms can be used to solve such systems of equations.

The complete system 2.43 of equations is of the form

$$\mathbf{K}\,\mathbf{u} = \mathbf{F} \tag{2.44}$$

As mentioned earlier, the coefficient matrix **K** is called the **global stiffness matrix**; the right-hand side **F** is usually called the **force vector**. This terminology arises from the formulation of structural mechanics problems in which the unknown variables are the nodal displacements **u**.

2.6 Approximation Error in the Finite Element Method

The first step in solving real world problems with FEM is to construct a mathematical model to represent a physical system. The question as to how well the ideal mathematical model represents reality is the domain of physics and validation. Errors associated with the FEM are those which arise when one constructs a computational model/representation of

the mathematical model and solve that computational model. Apart from bugs in software and "mistakes" made by the user, the main sources of error in the FEM are **discretization error** and **solution error**.

The discretization error arises when the mathematical model is converted into a discrete computational model. It is the error introduced when representing a function of a continuous variable by its values at a discrete set of nodes; or, equivalently, when representing a differential equation by its FEM matrix equations. This error will depend on the element order (linear, quadratic, etc.) and also on the number of elements used to represent the domain of interest. One would expect that the discretization error will tend to zero as the mesh of elements is made smaller by reducing element size.

The discretization error is associated with the complete problem and so is a global measure. The discretization error itself depends on the more local element level on the **interpolation error**. This is the error which arises when a function is approximated using shape functions over a particular element, and is discussed further below[6].

Other errors which can arise are those associated with numerical integration (all integrals are evaluated exactly in this chapter, but only approximate values can be obtained in higher dimensions), and numerical/rounding errors, for example in inverting the stiffness **K** matrix.

2.6.1 Interpolation Error

The **interpolation error** is usually the largest source of error in the FEM. It can be calculated for any typical element; for example, with the linear element, one can proceed as follows:

The FE approximation is given by

$$\tilde{p}(\xi) = N_1(\xi)p_1 + N_2(\xi)p_2 \tag{2.45}$$

[6] One is most often interested in the interpolation error for functions; one can also examine the interpolation errors associated with the gradients of functions

With the linear element, the shape functions are $\frac{1}{2}(1 \mp \xi)$. Expand the true solution in a Taylor series,

$$p(\xi + h) = p(\xi) + h\frac{\partial p}{\partial \xi}\bigg|_{\xi} + \frac{h^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\bar{\xi}\ \text{on}\ [\xi, \xi+h]} \tag{2.46}$$

Where the last term is the error associated with the Taylor series approximation, with $\bar{\xi}$ lying somewhere in the interval. Then, letting $h = \mp 1 - \xi$,

$$\begin{aligned} p(-1) = p_1 &= p(\xi) + (-1-\xi)\frac{\partial p}{\partial \xi}\bigg|_{\xi} + \frac{(-1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\bar{\xi}\ \text{on}\ [-1, \xi]} \\ p(+1) = p_2 &= p(\xi) + (+1-\xi)\frac{\partial p}{\partial \xi}\bigg|_{\xi} + \frac{(+1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\bar{\xi}\ \text{on}\ [+1, \xi]} \end{aligned} \tag{2.47}$$

When determining the interpolation error, we can assume that the nodal values p_1, p_2 are the exact values, since interpolation error is only associated with the approximation of the true function with a linear function, not with any error which might arise at the nodes. Thus, from (2.45), and assuming that p_1, p_2 are the exact nodal values,

$$\begin{aligned} \tilde{p}(\xi) = \{N_1(\xi) + N_2(\xi)\}p(\xi) + \{N_1(\xi)(-1-\xi) + N_2(\xi)(+1-\xi)\}\frac{\partial p}{\partial \xi}\bigg|_{\xi} \\ + \left\{ N_1(\xi)\frac{(-1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\bar{\xi}\ \text{on}\ [-1, \xi]} + N_2(\xi)\frac{(+1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\bar{\xi}\ \text{on}\ [+1, \xi]} \right\} \end{aligned} \tag{2.48}$$

The term inside the first curly bracket is 1, and that inside the second curly brackets is zero, so that the interpolation error is, using the change of coordinates (2.30), $\partial p / \partial \xi = (\partial p / \partial x)(\partial x / \partial \xi) = (L/2)(\partial p / \partial x)$,

$$\begin{aligned} e(\xi) &= \tilde{p}(\xi) - p(\xi) \\ &= N_1(\xi)\frac{(-1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{-1<\bar{\xi}<\xi} + N_2(\xi)\frac{(+1-\xi)^2}{2}\frac{\partial^2 p}{\partial \xi^2}\bigg|_{\xi<\bar{\xi}<1} \\ &= \frac{1}{4}\left(\frac{L}{2}\right)^2 (1-\xi^2)\left[(1+\xi)\frac{\partial^2 p}{\partial x^2}\bigg|_{-1<\bar{\xi}<\xi} + (1-\xi)\frac{\partial^2 p}{\partial x^2}\bigg|_{\xi<\bar{\xi}<1} \right] \end{aligned} \tag{2.49}$$

The maximum error in the element is then

$$\begin{aligned} |e|_0 &\le \frac{L^2}{16}\left|(1+\xi)\frac{\partial^2 p}{\partial x^2}\right|_{-1<\bar{\xi}<\xi} + (1-\xi)\frac{\partial^2 p}{\partial x^2}\Big|_{\xi<\bar{\xi}<1}\right| \\ &\le \frac{L^2}{8}\text{Max}\left|\frac{\partial^2 p}{\partial x^2}\right| \end{aligned} \tag{2.50}$$

Thus as the length of the element is decreased, the error decreases as the square of the element length. One says that the linear element is **second-order accurate**:

$$e \propto L^2 \tag{2.51}$$

In general, the error of an element is proportional to L^{k+1}, where L is the length, or a characteristic length, of the element, and k is the order of the interpolating polynomial. The error in the first derivative is proportional to L^k.

The same applies, to a certain extent, for higher dimensions. For example, the error for a linear 2-D triangular element is proportional to l^2, where l is a characteristic length of the element.

2.7 Quadratic C^0 Elements

Here the C^0 quadratic trial function is examined.

2.7.1 Quadratic Trial Function

A quadratic trial function is of the form $\tilde{u}(x) = a + bx + cx^2$, and one needs to choose three nodal points to evaluate the unknown coefficients. The obvious ones to take are the two end-points and the centre-point[7]. Assume then that the values at these nodal points are $\tilde{u}(x_1) = \tilde{u}_1$, $\tilde{u}(x_1 + L/2) = \tilde{u}_2$, $\tilde{u}(x_2) = \tilde{u}_3$, Fig. 2.11. There are then three equations to determine the three nodal values and one finds that {▲Problem 7}

[7] the third node does not have to be central; in some applications, e.g. fracture mechanics, it helps to place nodes one-quarter the way along elements

Quadratic Trial Function:

$$\tilde{u}(x) = N_1(x)\tilde{u}_1 + N_2(x)\tilde{u}_2 + N_3(x)\tilde{u}_3$$

$$N_1 = 1 - 3\frac{x - x_1}{L} + 2\left(\frac{x - x_1}{L}\right)^2 \qquad (2.52)$$

$$N_2 = 4\frac{x - x_1}{L} - 4\left(\frac{x - x_1}{L}\right)^2$$

$$N_3 = -\frac{x - x_1}{L} + 2\left(\frac{x - x_1}{L}\right)^2$$

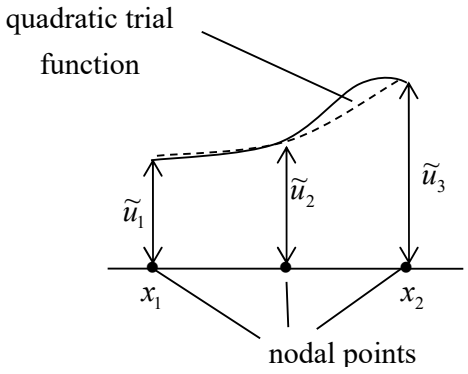


Figure 2.11: the quadratic trial function

Structure of the Weight Functions

As with the linear trial functions, the shape functions are either 0 or 1 at a node, Fig. 2.12, and they sum to 1.

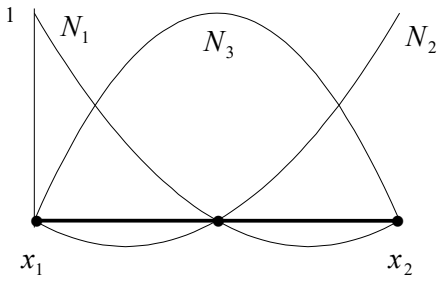


Figure 2.12: shape functions for the quadratic trial function

Looking again at the general ODE problem of Eqn. 2.15, with a single quadratic element one arrives at the system of three equations

Equations for Quadratic Trial Function:

$$\left\{ a\frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} - b\frac{1}{6}\begin{bmatrix} -3 & +4 & -1 \\ -4 & 0 & +4 \\ +1 & -4 & +3 \end{bmatrix} - c\frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \right\} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} + d\frac{L}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = a\begin{bmatrix} -u'(x_1) \\ 0 \\ +u'(x_2) \end{bmatrix} \tag{2.53}$$

Again looking at the example $a = 1, b = -2, c = d = 1$ and $x_1 = 0, x_2 = 1, L = 1$, application of the boundary conditions $u(x_1) = \tilde{u}_1 = 1, u'(x_2) = 2$ leads to two equations in two unknowns, which yield $u_2 = 1.290, u_3 = 1.993$ (compared with the exact solutions 1.303 and 2 respectively). The complete quadratic solution is, Fig. 2.13,

$$\begin{aligned} \tilde{u}(x) &= \tilde{u}_1\left(1 - 3x + 2x^2\right) + \tilde{u}_2\left(4x - 4x^2\right) + \tilde{u}_3\left(-x + 2x^2\right) \\ &= 1 + 0.167x + 0.826x^2 \end{aligned} \tag{2.54}$$

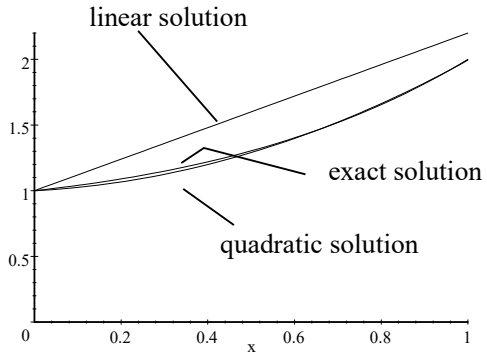


Figure 2.13: FEM solution for the ODE in (2.15) using Quadratic Elements

The quadratic shape functions can be expressed in terms of the local coordinates: from (2.30)

Quadratic Shape Functions (Local Coordinates)

$$N_1 = \frac{1}{2}\xi(\xi - 1), \quad N_2 = 1 - \xi^2, \quad N_3 = \frac{1}{2}\xi(\xi + 1) \tag{2.55}$$

Note that with a quadratic trial function, one does not need to integrate the higher order term by parts in order to obtain a solution. However, integration by parts will ensure that the resulting coefficient matrix is symmetric[8], and is done here.

Consider next the following differential equation, to be solved using two quadratic elements of equal length:

$$\frac{d^2p}{dx^2} + p = 1, \quad \left.\frac{dp}{dx}\right|_{x=0} = 1, \quad p(\pi) = 0 \tag{2.56}$$

[the exact solution is $p(x) = \cos x + \sin x + 1$]

The weighted residual integral is $I = \int_0^\pi (p'' + p - 1)\omega dx = 0$, leading to

$$I = \int_0^\pi \left(\frac{dp}{dx}\frac{d\omega}{dx} - p\omega + \omega \right) dx - \left[\frac{dp}{dx}\omega \right]_0^\pi = 0 \tag{2.57}$$

The finite element mesh consists of two elements and five nodes:

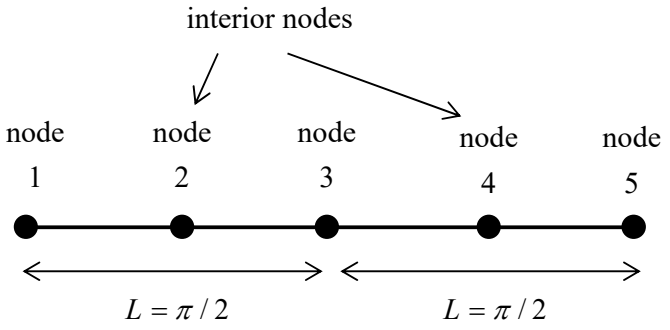


Figure 2.14: FEM solution for the ODE in (2.15) using Quadratic Elements

[8] if the DE to be solved is of the self-adjoint type, as discussed in Chapter 1

Element 1: $p^{(1)} = N_1^{(1)} p_1 + N_2^{(1)} p_2 + N_3^{(1)} p_3$

Element 2: $p^{(2)} = N_1^{(2)} p_3 + N_2^{(2)} p_4 + N_3^{(2)} p_5$

One now has the expressions:

Element 1:

$$\begin{aligned}
&\int_0^L \left(\frac{dp^{(1)}}{dx} \frac{dN_1^{(1)}}{dx} - p^{(1)} N_1^{(1)} + N_1^{(1)} \right) dx + p'(0) \\
&\int_0^L \left(\frac{dp^{(1)}}{dx} \frac{dN_2^{(1)}}{dx} - p^{(1)} N_2^{(1)} + N_2^{(1)} \right) dx = 0 \\
&\int_0^L \left(\frac{dp^{(1)}}{dx} \frac{dN_3^{(1)}}{dx} - p^{(1)} N_3^{(1)} + N_3^{(1)} \right) dx = 0
\end{aligned} \tag{2.58}$$

Element 2:

$$\begin{aligned}
&\int_L^{2L} \left(\frac{dp^{(2)}}{dx} \frac{dN_1^{(2)}}{dx} - p^{(2)} N_1^{(2)} + N_1^{(2)} \right) dx = 0 \\
&\int_L^{2L} \left(\frac{dp^{(2)}}{dx} \frac{dN_2^{(2)}}{dx} - p^{(2)} N_2^{(2)} + N_2^{(2)} \right) dx = 0 \\
&\int_L^{2L} \left(\frac{dp^{(2)}}{dx} \frac{dN_3^{(2)}}{dx} - p^{(2)} N_3^{(2)} + N_3^{(2)} \right) dx - p'(\pi) = 0
\end{aligned} \tag{2.59}$$

Substituting the shape functions into these six integrals, changing to the local variables, and making use of the integrals in the Appendix to this Chapter, leads to

Element 1:

$$\begin{aligned}
p_1\left[+\tfrac{7}{3L} - \tfrac{4L}{30}\right] + p_2\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_3\left[\tfrac{1}{3L} + \tfrac{L}{30}\right] + \tfrac{L}{6} + p'(0) &= 0 \\
p_1\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_2\left[+\tfrac{16}{3L} - \tfrac{16L}{30}\right] + p_3\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + \tfrac{4L}{6} &= 0 \\
p_1\left[+\tfrac{1}{3L} + \tfrac{L}{30}\right] + p_2\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_3\left[+\tfrac{7}{3L} - \tfrac{4L}{30}\right] + \tfrac{L}{6} &= 0
\end{aligned}$$

Element 2:

$$\begin{aligned}
p_3\left[+\tfrac{7}{3L} - \tfrac{4L}{30}\right] + p_4\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_5\left[+\tfrac{1}{3L} + \tfrac{L}{30}\right] + \tfrac{L}{6} &= 0 \\
p_3\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_4\left[+\tfrac{16}{3L} - \tfrac{16L}{30}\right] + p_5\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + \tfrac{4L}{6} &= 0 \\
p_3\left[+\tfrac{1}{3L} + \tfrac{L}{30}\right] + p_4\left[-\tfrac{8}{3L} - \tfrac{2L}{30}\right] + p_5\left[+\tfrac{7}{3L} - \tfrac{4L}{30}\right] + \tfrac{L}{6} - p'(\pi) &= 0
\end{aligned}$$

Summing the last equation for element 1 and the first equation for element 2, one arrives at the five equations (one for each degree of freedom)

$$\frac{1}{30L}\begin{bmatrix} +70-4L^2 & -80-2L^2 & +10+L^2 & 0 & 0 \\ -80-2L^2 & +160-16L^2 & -80-2L^2 & 0 & 0 \\ +10+L^2 & -80-2L^2 & +140-8L^2 & -80-2L^2 & +10+L^2 \\ 0 & 0 & -80-2L^2 & +160-16L^2 & -80-2L^2 \\ 0 & 0 & +10+L^2 & -80-2L^2 & +70-4L^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -L-6p'(0) \\ -4L \\ -2L \\ -4L \\ -L+6p'(\pi) \end{bmatrix} \tag{2.60}$$

Applying the natural boundary condition at $x=0$, $p'(0)=1$, and eliminating the last row by applying the essential boundary condition $p(\pi)=p_5=0$ leaves

$$\frac{1}{30L}\begin{bmatrix} +70-4L^2 & -80-2L^2 & +10+L^2 & 0 \\ -80-2L^2 & +160-16L^2 & -80-2L^2 & 0 \\ +10+L^2 & -80-2L^2 & +140-8L^2 & -80-2L^2 \\ 0 & 0 & -80-2L^2 & +160-16L^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -L-6 \\ -4L \\ -2L \\ -4L \end{bmatrix} \tag{2.61}$$

Solving the equations, with $L=\pi/2$, gives

$$\begin{aligned} p_1 &= 2.011591 \\ p_2 &= 2.417818 \\ p_3 &= 2.000278 \\ p_4 &= 1.000196 \end{aligned} \quad . \qquad \text{The exact solution is} \qquad \begin{aligned} p_1 &= 2 \\ p_2 &= 2.41421 \\ p_3 &= 2 \\ p_4 &= 1 \end{aligned}$$

The full solution is

Element 1:

$$\begin{aligned} p^{(1)} &= \tfrac{1}{2}\xi^{(1)}\left(\xi^{(1)}-1\right)p_1 + \left(1-\xi^{(1)2}\right)p_2 + \tfrac{1}{2}\xi^{(1)}\left(\xi^{(1)}+1\right)p_3, \qquad \xi^{(1)} = \tfrac{2x}{L}-1 \\ &= \left[1-3\frac{x}{L}+2\left(\frac{x}{L}\right)^2\right]p_1 + \left[4\frac{x}{L}-4\left(\frac{x}{L}\right)^2\right]p_2 + \left[-\frac{x}{L}+2\left(\frac{x}{L}\right)^2\right]p_3 \\ &= 2.012 + 1.042x - 0.668x^2 \end{aligned}$$

Element 2:

$$\begin{aligned} p^{(2)} &= \tfrac{1}{2}\xi^{(2)}\left(\xi^{(2)}-1\right)p_3 + \left(1-\xi^{(2)2}\right)p_4 + \tfrac{1}{2}\xi^{(2)}\left(\xi^{(2)}+1\right)p_5, \qquad \xi^{(2)} = \tfrac{2x}{L}-3 \\ &= \left[6-7\frac{x}{L}+2\left(\frac{x}{L}\right)^2\right]p_3 + \left[-8+12\frac{x}{L}-4\left(\frac{x}{L}\right)^2\right]p_4 + \left[3-5\frac{x}{L}+2\left(\frac{x}{L}\right)^2\right]p_5 \\ &= 4.000 - 1.273x - 0.00009x^2 \end{aligned}$$

which is very accurate.

2.8 Cubic Hermite Finite Elements

Here the C^1 cubic Hermite trial function is examined.

2.8.1 Cubic Hermite Trial Function

The purpose of introducing the cubic Hermite element is to ensure that the first derivatives of the trial function are continuous at the nodes. To this end, consider four unknowns, the two values of p at the element ends and the two values of p' at the element ends. With four unknowns, one can use the cubic trial function

$$p(x) = a + bx + cx^2 + dx^3 \tag{2.62}$$

Using the four equations

$$p(x_1) = p_1, \quad p(x_2) = p_2, \quad p'(x_1) = p_1', \quad p'(x_2) = p_2' \tag{2.63}$$

to re-write the trial function in terms of the unknown nodal values, one arrives at (after some lengthy algebra)

Cubic Hermite Trial Function:

$$p(x) = N_1(x)p_1 + N_2(x)p_2 + N_3(x)p_1' + N_4(x)p_2'$$

where

$$N_i = \frac{1}{(x_1 - x_2)^3}\left[a_i + b_i x + c_i x^2 + d_i x^3\right]$$

with

$$a_i = \begin{Bmatrix} x_2^2\left((3x_1 - x_2\right) \\ x_1^2\left((x_1 - 3x_2\right) \\ -x_1x_2^2(x_1 - x_2) \\ -x_1^2x_2\left((x_1 - x_2\right) \end{Bmatrix}, \quad b_i = \begin{Bmatrix} -6x_1x_2 \\ +6x_1x_2 \\ x_2(2x_1 + x_2)(x_1 - x_2) \\ x_1(x_1 + 2x_2)(x_1 - x_2) \end{Bmatrix}$$

$$c_i = \begin{Bmatrix} +3(x_1 + x_2) \\ -3(x_1 + x_2) \\ -(x_1 + 2x_2)(x_1 - x_2) \\ -(2x_1 + x_2)(x_1 - x_2) \end{Bmatrix}, \quad d_i = \begin{Bmatrix} -2 \\ +2 \\ x_1 - x_2 \\ x_1 - x_2 \end{Bmatrix}$$

(2.64)

The cubic Hermite shape functions can be expressed in terms of the local coordinates: from (2.30),

Cubic Hermite Trial Function (Local Coordinates):

$$\begin{aligned} N_1 &= \frac{1}{4}\left(2 - 3\xi + \xi^3\right) = \frac{1}{4}(1 - \xi)^2(2 + \xi) \\ N_2 &= \frac{1}{4}\left(2 + 3\xi - \xi^3\right) = \frac{1}{4}\left(1 + \xi^2\right)(2 - \xi) \\ N_3 &= \frac{L}{8}\left(+1 - \xi - \xi^2 + \xi^3\right) = +\frac{L}{8}\left(1 - \xi^2\right)(1 - \xi) \\ N_4 &= \frac{L}{8}\left(-1 - \xi + \xi^2 + \xi^3\right) = -\frac{L}{8}\left(1 - \xi^2\right)(1 + \xi) \end{aligned} \tag{2.65}$$

Note that the third and fourth shape functions depend on the element length. Also, although these are zero at the nodes, their derivatives are one or zero there, as required.

Consider the following differential equation, to be solved using a single cubic Hermite element:

$$\frac{d^2p}{dx^2} = 1, \quad p(0) = 1, \quad \left.\frac{dp}{dx}\right|_{x=1} = 2 \tag{2.66}$$

[the exact solution is $p(x) = \frac{1}{2}x^2 + x + 1$]

The weighted residual integral is $I = \int_0^1 (p'' - 1)\omega dx = 0$, leading to

$$\int_0^1 \frac{dp}{dx}\frac{d\omega}{dx}dx = -\int_0^1 \omega dx + \left[\frac{dp}{dx}\omega\right]_0^\pi \tag{2.67}$$

Using the interpolation (2.64), re-labelling $p_1' = p_3$, $p_2' = p_4$, and using the integrals in the Appendix (Eqns. 2A.13, 2A.14), leads to the system of equations

$$\frac{1}{30}\begin{bmatrix} +36\frac{1}{L} & -36\frac{1}{L} & +3 & +3 \\ -36\frac{1}{L} & +36\frac{1}{L} & -3 & -3 \\ +3 & -3 & +4L & -L \\ +3 & -3 & -L & +4L \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \frac{L}{12}\begin{bmatrix} -6 \\ -6 \\ -L \\ +L \end{bmatrix} + \begin{bmatrix} -p_1' \\ +p_2' \\ 0 \\ 0 \end{bmatrix} \tag{2.68}$$

The boundary conditions then lead to the exact solution, $p_2 = 5/2$, $p_1' = 1$.

The second derivatives of the shape functions are non-zero for the cubic Hermite trial function, and so the solution can be obtained using the weighted residual *without* the integration by parts, using the integrals in (2A.15), but the coefficient matrix will not be symmetric. In this case, when one is changing to local coordinates, one must use

$$\begin{aligned} \frac{dN_i}{dx} &= \frac{dN_i}{d\xi}\frac{d\xi}{dx} \\ \frac{d}{dx}\left(\frac{dN_i}{dx}\right) &= \frac{d}{dx}\left(\frac{dN_i}{d\xi}\frac{d\xi}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{dN_i}{d\xi}\right)\frac{d\xi}{dx} + \frac{dN_i}{d\xi}\frac{d^2\xi}{dx^2} \\ &= \frac{d^2N_i}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 + \frac{dN_i}{d\xi}\frac{d^2\xi}{dx^2} \end{aligned} \tag{2.69}$$

2.9 Finite Differences

Finite Differences (FD) is an alternative method of obtaining a numerical solution to differential equations. Here, FD and its relation to FE is described. Focusing on a simple second order problem,

$$\frac{\partial^2 p}{\partial x^2} = 1, \qquad p(0) = 1, \quad p'(1) = 2 \tag{2.70}$$

[exact solution: $\frac{1}{2}x^2 + x + 1$]

the FD approach is to approximate derivatives of functions using truncated Taylor series. For example, consider the expansions

$$\begin{aligned} p(x_0 + \Delta x) &\approx p(x_0) + \Delta x \left.\frac{dp}{dx}\right|_{x_0} + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 p}{dx^2}\right|_{x_0} \\ p(x_0 - \Delta x) &\approx p(x_0) - \Delta x \left.\frac{dp}{dx}\right|_{x_0} + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 p}{dx^2}\right|_{x_0} \end{aligned} \tag{2.71}$$

Adding these leads to the approximation

$$\left.\frac{d^2 p}{dx^2}\right|_{x_0} = \frac{1}{(\Delta x)^2}\left[p(x_0 - \Delta x) - 2p(x_0) + p(x_0 + \Delta x)\right] + O(\Delta x)^2 \tag{2.72}$$

Using a grid [9] with $N+1$ nodes, so that $\Delta x = 1/N$, and for the natural boundary condition using the approximation

[9] the *mesh* of Finite Elements is usually called a *grid* in Finite Differences

$$
\begin{aligned}
p(x_0-2\Delta x) &\approx p(x_0)-2\Delta x\left.\frac{dp}{dx}\right|_{x_0}+2(\Delta x)^2\left.\frac{d^2p}{dx^2}\right|_{x_0}\\
-4p(x_0-\Delta x) &\approx -4p(x_0)+4\Delta x\left.\frac{dp}{dx}\right|_{x_0}-2(\Delta x)^2\left.\frac{d^2p}{dx^2}\right|_{x_0}\\
&\rightarrow\\
\left.\frac{dp}{dx}\right|_{x_0} &= \frac{1}{2(\Delta x)}\left[p(x_0-2\Delta x)-4p(x_0-\Delta x)+3p(x_0)\right]+O(\Delta x)^2
\end{aligned}
\tag{2.73}
$$

the ODE (2.70) is transformed into the system of equations

$$
N^2\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 1 & & & \\
 & 1 & -2 & 1 & & \\
 & & 1 & -2 & & \\
 & & & & \ddots & \\
 & & & \frac{1}{2N} & -\frac{2}{N} & \frac{3}{2N}
\end{bmatrix}
\begin{bmatrix} p_1\\ p_2\\ p_3\\ \vdots\\ p_{n+1}\end{bmatrix}=
\begin{bmatrix} N^2\\ 1\\ 1\\ 1\\ 2\end{bmatrix}
\tag{2.74}
$$

This leads to the exact solution with 3 (or more) nodes.

Now consider a quadratic finite element. The interpolation can be written as

$$
p(x)=N_1p(x_0-\Delta x)+N_2p(x_0)+N_3p(x_0+\Delta x) \tag{2.75}
$$

where x_0 is the central node, the shape functions are given by (2.52), and Δx is the semi-element length. Forming the weighted residual of (2.70) and taking the weight function to be unity, $\omega(x)\equiv 1$ (or a constant), and integrating the shape functions over the element, but not integrating by parts, leads to

$$
\frac{1}{(\Delta x)^2}\left[p(x_0-\Delta x)-2p(x_0)+p(x_0+\Delta x)\right]=1 \tag{2.76}
$$

This scheme provides a single equation for this central node, exactly the same as the Finite Difference equation. Thus this FD scheme can be considered to be a FE scheme with a *constant* weight function.

2.10 Application: Static Elasticity (Structural Mechanics)

The FE methodology described in this Chapter is here used to analyse the problem of an elastic material subject to arbitrary loading conditions. The geometry of the problem is as shown below, a (one dimensional) rod of length *l*, possibly varying cross section $A(x)$ and Young's modulus $E(x)$, subjected to a given displacement or stress/force at its ends.

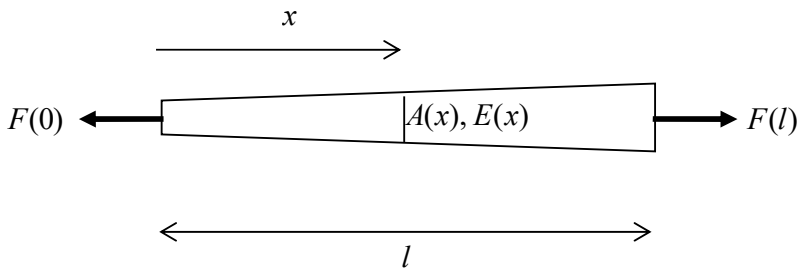


Figure 2.15: an elastic rod

This problem is discussed in detail in Solid Mechanics, Part II, section 2.1.

2.10.1 Governing Differential Equation

The equations governing the response of the rod are:

Governing Equations for Elastostatics:

Equation of Equilibrium:

$$\frac{d}{dx}(A\sigma) + Ab = 0 \tag{2.77}$$

Strain-Displacement Relation:

$$\varepsilon = \frac{du}{dx} \tag{2.78}$$

Constitutive Relation:

$$\sigma = E\varepsilon \tag{2.79}$$

The first two of these are derived in the Appendix to this Chapter, §2.12.2. The third is Hooke's experimental law, which is valid for elastic materials undergoing small strains.

In these equations, σ is the stress, ε is the small strain (change in length per original length) and u is the displacement. The constant of proportionality in the linear elastic constitutive law is E, the Young's modulus of the material; b is a body force (per unit volume), for example the force of gravity, and A is the cross-sectional area.

The strain-displacement relation and the constitutive equation can be substituted into the equation of equilibrium to obtain

1D Governing Equation for Static Elasticity:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + f = 0 \tag{2.80}$$

where $f(x)$ is the product of the body force (per unit volume) and the cross-sectional area, and so is a *force per unit length*.

Note the significance of the term inside the brackets in Eqn. 2.80: from Eqn. 2.78, the stresses acting on any cross-section of the rod are

$$\sigma(x) = E\frac{du}{dx} \tag{2.81}$$

and so the forces acting on any cross section are

$$F\left(x\right) = AE\frac{du}{dx} \tag{2.82}$$

An Exact Solution

When A, E and f are constant, the exact solution is obtained by integrating twice the governing differential equation to obtain

$$u = -\frac{f}{2AE}x^2 + \overline{A}x + \overline{B}, \qquad \varepsilon = -\frac{f}{AE}x + \overline{A}, \qquad \sigma = -\frac{f}{A}x + E\overline{A} \tag{2.83}$$

where $\overline{A}$ and $\overline{B}$ are constants to be determined from the boundary conditions.

2.10.2 FEM Formulation

Formally applying the Galerkin method to the one dimensional static elasticity problem and integrating by parts leads to

$$\int_0^l AE\frac{du}{dx}\frac{d\omega}{dx}dx = \int_0^l f\omega dx + \left[\left(AE\frac{du}{dx}\right)\omega\right]_0^l \tag{2.84}$$

Trial Function & Boundary Conditions

The trial function for the GFEM is of the form

$$\tilde{u}(x) = \sum_{i=1}^{n} \omega_i(x) u_i \tag{2.85}$$

where n is the number of nodes in the element, the u_i are the unknown nodal values and the ω_i are the weighting functions. With the shape functions N_i as the weights, substituting into Eqn. 2.84 gives

$$\sum_{i=1}^{n}\left[\int_0^l AE\frac{dN_i}{dx}\frac{dN_j}{dx}dx\right]u_i = \int_0^l f(x)N_j dx + \left[\left(AE\frac{du}{dx}\right)N_j\right]_0^l \tag{2.86}$$

The boundary conditions can involve the displacement u and its derivative du/dx. Boundary conditions on u are of the essential type and boundary conditions on du/dx are of the natural type. It can be seen that the natural boundary condition in effect involve a condition on the forces F acting at the ends of the rod:

Boundary conditions for Static Elasticity:

$$u(0) = u_0, \quad u(l) = u_l, \quad \left(AE \frac{du}{dx} \right)_{x=0} = A\sigma_0 = F_0, \quad \left(AE \frac{du}{dx} \right)_{x=l} = A\sigma_l = F_l \qquad (2.87)$$

essential boundary conditions

natural boundary conditions

The Stiffness Matrix and Force Vector

The FE equations corresponding to (2.84) take the form

$$\mathbf{Ku} = \mathbf{f} + \mathbf{F} \qquad (2.88)$$

where **f** is the distributed **body force vector** and **F** is the concentrated **force vector**. These two vectors represent the external loads acting on the rod, and together are called the **loads vector**. For example, taking A and E to be constant over an element, and neglecting the body force term, the element equations for a linear element of length L are

$$\frac{AE}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = \begin{bmatrix} -F(0) \\ +F(L) \end{bmatrix}, \qquad \mathbf{Ku} = \mathbf{F} \qquad (2.89)$$

The dimensions of each term here is one of force. The coefficient matrix **K**, including the AE/L term, is called the **stiffness matrix**. Note that the force $F(0)$ is positive when directed in the negative x direction (tension positive; see Fig. 2.15).

The formulation used here allows one to apply concentrated forces at internal nodes, i.e. within the rod. This can be achieved by having non-zero entries other than at the first and last nodes in the global force vector.

More generally, the form of the FE equations for an arbitrary linear element is, from the above,

$$\begin{bmatrix} \int_{x_i}^{x_{i+1}} AE\frac{dN_1}{dx}\frac{dN_1}{dx}dx & \int_{x_i}^{x_{i+1}} AE\frac{dN_1}{dx}\frac{dN_2}{dx}dx \\ \int_{x_i}^{x_{i+1}} AE\frac{dN_2}{dx}\frac{dN_1}{dx}dx & \int_{x_i}^{x_{i+1}} AE\frac{dN_2}{dx}\frac{dN_2}{dx}dx \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = \begin{bmatrix} -F_i \\ +F_{i+1} \end{bmatrix} + \begin{bmatrix} \int_{x_i}^{x_{i+1}} f(x) N_1 dx \\ \int_{x_i}^{x_{i+1}} f(x) N_2 dx \end{bmatrix} \tag{2.90}$$

The stiffness matrix **K** is singular. This means that if no essential BC is applied (so that a row can be eliminated), then no solution can be obtained. Physically, an absence of an essential BC means that there is only an application of forces at the rod-ends (natural BCs), but no applied displacement. A singular matrix occurs when a structure is not adequately supported, in this case the rod can undergo an arbitrary rigid body translation – this must be prevented by at least one essential BC.

The diagonal terms of **K** must be positive. To see this, suppose that $u_{i+1} = 0$. Then, neglecting the body force term, $K_{11}u_i = -F_i$. F_i is defined positive in the negative x direction, and one must have the displacement u_i directed the same way as the force (so $u_i < 0$ if $u_{i+1} = 0$), and hence $K_{11} > 0$, and similarly for other diagonal terms.

It can also be proved that the elasticity stiffness matrix **K** is positive definite, i.e. $\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} > 0$, for any arbitrary vector **x**. For example, in the example given above, Eqn. 2.89,

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \frac{AE}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{AE}{L}(x_1 - x_2)^2 > 0 \tag{2.91}$$

Example

Consider a rod with varying cross section $A(x) = A_0(1+x)$, constant E and f, and boundary conditions $u(0) = 0,\ F(l) = \overline{F}$. The exact solution to this problem is seen to be

$$u(x) = \frac{1}{EA_0}\left\{ f\left[(l+1)\ln(1+x) - x\right] + \overline{F}\ln(1+x)\right\} \tag{2.92}$$

For a quadratic element, the element equations are, assuming A to be constant,

$$\frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = f\frac{L}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -F(0) \\ 0 \\ +F(L) \end{bmatrix} \tag{2.93}$$

Here, A can be taken to be the average cross-sectional area in the element[10]. Taking the data $E = 10^9\,\text{Pa}$, $\overline{F} = 1\text{kN}$, $f = 500\,\text{N/m}$, $l = 1\text{m}$, $A_0 = 1\text{cm}$, the one and two-element solutions are as shown below.

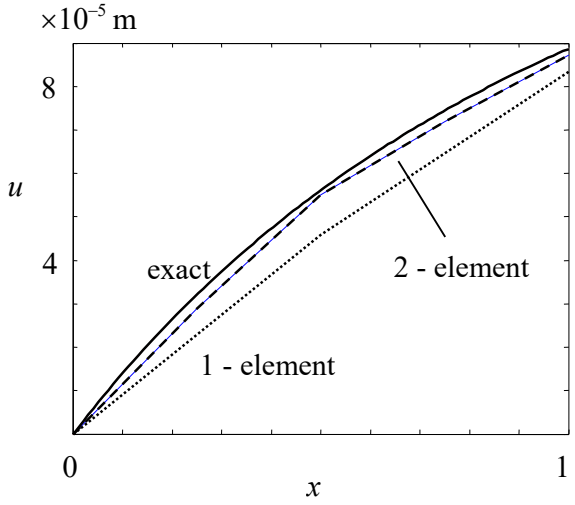


Figure 2.16: FEM solution for the Elastic Rod Example Problem

Work Equivalence of Distributed Loads

Consider now the term involving the body force load $f(x)$. In the real mathematical model (2.80), this force is distributed along the rod. In the FE model, forces cannot be represented at every material particle, only at the nodes; the Galerkin procedure distributes the load according to (see the integral on the right-hand side of Eqn. 2.84)

$$\int_0^L f(x) N_j dx \qquad \ldots \text{node } j \tag{2.94}$$

and this integral results in the distributed body force vector. For example, consider a constant force (per unit length) f distributed over an element of length L, so that the total force acting on the element is fL. For linear and quadratic elements, this force is distributed amongst the nodes as shown in Fig. 2.17.

[10] for greater accuracy, one could have used the explicit expression for $A(x)$ in (2.82) and carry out the integration to obtain a different stiffness matrix

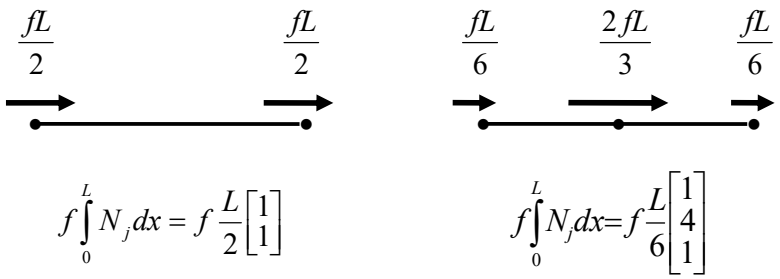


Figure 2.17: FEM distribution of loads at the nodes

In fact, the total force acting over an element in the real model is $F_{ext} = \int_0^L f(x)\,dx$. The total force acting over an element in the FE model is (see Eqn. 2.90)

$$\sum_j \int_0^L f(x) N_j(x)\,dx = \int_0^L f(x) \sum_j N_j(x) dx \tag{2.95}$$

which is seen to be the same as F_{ext}, since $\sum_j N_j(x) = 1$. Thus the forces in the FE model are the same as in the real model; they are **statically equivalent**. Not only that, the loads in both models are **work equivalent**. This is explained in what follows:

The small increment in work done by the body forces over an element in the real model as it undergoes a small displacement increment (over the length of the element) is $\delta W_{ext} = \int_0^L f(x)\,\delta u(x)\,dx$. Approximating the actual displacement by the FEM interpolation (which introduces some discretisation error),

$$\delta W_{ext} \approx \sum_j \delta u_j \int_0^L f(x) N_j(x)\,dx \tag{2.96}$$

The equivalent work in the FE model is

$$\sum_j \int_0^L f(x) N_j(x) dx \times \delta u_j \tag{2.97}$$

which is the same as that in the real model. Thus *the work done by the external loads is the same in the FE model as it is in the real model*. From a mechanics point of view, if the work done in both is the same, the FE model produces the "correct" results. Clearly, if the

forces were distributed in some other way, the work done in the FE model would not be the true work done, and the FE model would give spurious results.

Internal Forces and Internal Work

Internal forces arise within the elastic material to equilibrate the externally applied forces. Whereas the external forces are due to some external agency, for example gravity, or an applied load, the internal forces are a result of the stresses which arise within the deformed material. As the external forces perform work, so do the internal forces: the **internal work** is a result of the internal forces moving through some displacement.

As with the external forces, the internal forces are distributed amongst the nodes in the FE model. Since the FE equations are $\mathbf{Ku} = \mathbf{f} + \mathbf{F}$, and the right-hand side represents the externally applied forces, $\mathbf{Ku}$ must represent the equilibrating internal forces. For example, consider four linear elements, with $u(0) = 0$, a constant cross-section and concentrated forces F_2, F_3 applied at nodes 2 and 3. The FE equations are

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \\ 0 \end{bmatrix} \tag{2.98}$$

The internal forces at the nodes in this FE model are then as shown in Fig. 2.15.

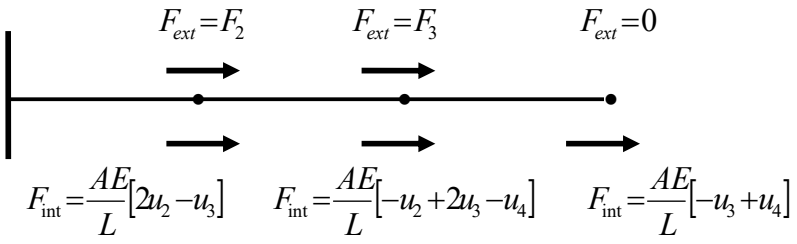


Figure 2.15: Internal forces in the FE model

The internal nodal forces in the FE model, also called the **element nodal point forces**, balance the external nodal forces at each node.

Since the internal forces equilibrate the external forces at each node, and the external forces are work equivalent to those in the real model, the internal forces are also work

equivalent to those in the real model. This point is better illustrated in the context of the potential energy of the deforming system, discussed in the following section.

2.10.3 Variational Formulation

The variational approach was introduced in Chapter 1, section 1.3. (See also the detailed discussion of the variational approach in sections 8.5 and 8.6 of Solid Mechanics, Part I.) Following that procedure, beginning again, the governing equation of static elasticity, Eqn. 2.80, is

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + f = 0 \tag{2.99}$$

Multiplying the equation across by a small displacement $\delta u(x)$,

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)\delta u + f\,\delta u = 0 \tag{2.100}$$

Recall that each term in Eqn. 2.99 has units of force (per unit length), and so the terms in Eqn 2.100 have units of work (per unit length). Integrating over the element gives the total work due to the change in displacement:

$$\int_o^l \frac{d}{dx}\left(AE\frac{du}{dx}\right)\delta u\,dx + \int_0^l f\,\delta u\,dx = 0 \tag{2.101}$$

Integrating by parts and using the results from the Calculus of Variations derived and given in section 1.3, this can now be re-expressed as

$$\delta\left\{\int_o^l \frac{1}{2}AE\left(\frac{du}{dx}\right)^2 dx - \left[\left(AE\frac{du}{dx}\right)u\right]_0^l - \int_0^l f\,u\,dx\right\} = 0 \tag{2.102}$$

It will be recognised that the term $\int_0^l \frac{1}{2}AE\left(du/dx\right)^2 dx$ is the strain energy in the bar (see, for example, Eqn. 8.2.3 in Solid Mechanics, Part I, with the force given by Eqn. 2.82). Thus Eqn. 2.102 can be expressed as

$$\delta\Pi = 0 \tag{2.103}$$

where Π is the potential (strain) energy. Taking the potential energy to be a function of the displacement, Eqn. 2.103 can be expressed as

$$\delta\Pi = \frac{d\Pi(u)}{du}\delta u = 0 \tag{2.104}$$

which is an expression of the principle of minimum potential energy: the solution to the structural mechanics problem is that which causes the potential energy to be stationary, $d\Pi / du = 0$.

Using now the displacement interpolation $u = \sum N_i u_i$,

$$\Pi \approx \int_o^l \frac{1}{2} AE\left(\sum \frac{dN_i}{dx} u_i\right)^2 dx - \left[\left(AE\frac{du}{dx}\right)\sum N_i u_i\right]_0^l - \int_0^l f \sum N_i u_i \, dx \tag{2.105}$$

Introducing row vectors for the shape functions:

$$\left[\mathbf{N}\right] = \left[N_1 \quad N_2\right], \qquad \left[\mathbf{B}\right] = \left[dN_1/dx \quad dN_2/dx\right] \tag{2.106}$$

and the column vector of unknown nodal displacements

$$\left\{\mathbf{u}\right\} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.107}$$

Eqn. 2.105 can be expressed as (for linear elements, but this can be generalised in the same way for other elements):

$$\int_o^l \frac{1}{2} AE \left\{\mathbf{u}\right\}^T \left[\mathbf{B}\right]^T \left[\mathbf{B}\right]\left\{\mathbf{u}\right\} dx - \left[\left(AE\frac{du}{dx}\right)\left[\mathbf{N}\right]\left\{\mathbf{u}\right\}\right]_0^l - \int_0^l f\left[\mathbf{N}\right]\left\{\mathbf{u}\right\} dx \tag{2.108}$$

The stationary value of the energy can now be obtained by differentiating this expression with respect to the vector **u**. For the purposes of differentiation with respect to matrices/vectors, note the following rules:

Consider the scalar function of a vector $\mathbf{u}$: $\phi(\mathbf{u}) = [\mathbf{N}]\{\mathbf{u}\} = N_1 u_1 + N_2 u_2$. Differentiation of a scalar with respect to a vector produces a vector:

$$\frac{\partial \phi(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \partial\phi / \partial u_1 \\ \partial\phi / \partial u_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = [\mathbf{N}]^T \tag{2.109}$$

Consider next the scalar function $\psi(\mathbf{u}) = \{\mathbf{u}\}^T [\mathbf{B}]^T [\mathbf{B}]\{\mathbf{u}\} = (B_1 u_1 + B_2 u_2)^2$. Differentiation gives:

$$\frac{\partial \psi(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \partial\psi / \partial u_1 \\ \partial\psi / \partial u_2 \end{bmatrix} = \begin{bmatrix} 2B_1 (B_1 u_1 + B_2 u_2) \\ 2B_2 (B_1 u_1 + B_2 u_2) \end{bmatrix} = 2[\mathbf{B}]^T [\mathbf{B}]\{\mathbf{u}\} \tag{2.110}$$

Thus the stationary point of Eqn. 2.108 is given by

$$\frac{d\Pi(\mathbf{u})}{d\mathbf{u}} = \int_o^l AE[\mathbf{B}]^T [\mathbf{B}]\{\mathbf{u}\}\, dx - \left[\left(AE \frac{du}{dx} \right) [\mathbf{N}]^{\mathbf{T}} \right]_0^l - \int_0^l f[\mathbf{N}]^{\mathbf{T}}\, dx \tag{2.111}$$

Setting this to zero then gives

$$\int_o^l AE[\mathbf{B}]^T [\mathbf{B}]\{\mathbf{u}\}\, dx = \left[\left(AE \frac{du}{dx} \right) [\mathbf{N}]^{\mathbf{T}} \right]_0^l + \int_0^l f[\mathbf{N}]^{\mathbf{T}}\, dx \tag{2.112}$$

which is exactly the same as Eqn. 2.90 derived earlier directly from the governing differential equation.

Thus the FE equations can be derived either directly from the governing differential equation or through the variational approach. When using the variational approach, the potential (strain) energy of the system enters directly, and in that sense one says that the finite element model is energy or work equivalent to the real model.

2.11 Problems

1. Solve the following problem using two linear elements of equal length:

$$\frac{d^2 u}{dx^2} = 6x, \quad u(0) = 1, \quad u(2) = 3 \qquad [\text{exact sln. } u(x) = x^3 - 3x + 1]$$

To deal with the non-homogeneous term, you might proceed in one of two ways:
(i) use the linear interpolation $6x \approx 6\left(x_i N_1 + x_{i+1} N_2\right)$ - which is in this case of course exact since the function $6x$ is linear
(ii) after converting to local coordinates, evaluate the resulting integrals $6\int x N_j d\xi$ exactly using the relation (2.A6)

$$\int_{-1}^{+1} N_i(\xi + A) d\xi = \frac{1}{3}\begin{bmatrix} 3A-1 \\ 3A+1 \end{bmatrix}$$

Evaluate also the FE solution for p'. Sketch the exact, FE and smoothed solutions for p'.

2. Re-solve the ODE of Problem 1, only now with a natural boundary condition at $x = 2$: $(dp/dx)_{x=2} = 6$ [the exact solution is $p(x) = x^3 - 6x + 1$]
3. Solve the problem $p'' + x = 0, \quad p(0) = 1, p'(3) = 0$ with two linear elements of lengths 1 and 2 respectively. Work through the problem in detail, using local coordinates. Derive an expression for p' in each element.
[answer: element 1 - $p' = \frac{13}{3}$, element 2 - $p' = \frac{7}{3}$]
4. What is a C^1 element?
5. What is adaptive meshing? How might an adaptive meshing procedure be implemented with C^0 elements (briefly explain)?
6. What are the sources of error in the FEM (explain any terminology you might use)?
7. Derive Eqns. 2.52, the shape functions for the quadratic element.
8. Consider the equation $p'' = f(x)$. Write down, as quick as you can, the global stiffness matrix for (consult the notes)
   a) a mesh of 5 linear 1-d elements, each of length 2.
   b) a mesh of 3 quadratic 1-d elements, each of length $\frac{1}{3}$.
9. For the (Galerkin) FEM, in a 2nd-order problem, why does one integrate by parts to reduce the order of the highest derivative to 1? Is this always necessary?
10. Note that, in the coefficient matrices encountered in this Chapter, e.g. Eqns. 2.24, 2.27, 2.53, 2.60, the entries of any row or column sum to zero. Using the properties of the shape functions, explain why this is so, and why it is not so for the matrix (2.68) of the cubic Hermite element.

2.12 Appendix to Chapter 2

2.12.1 Integrals involving the 1D Shape Functions

In the following, L is the length of the element, ξ is a local coordinate and A is a constant.

Linear Shape Functions

The shape functions for the standard linear element are

$$\begin{aligned} N_1 &= \frac{1}{2}(1-\xi), \qquad N_2 = \frac{1}{2}(1+\xi) \\ \frac{dN_1}{d\xi} &= -\frac{1}{2}, \qquad \frac{dN_2}{d\xi} = +\frac{1}{2} \end{aligned} \tag{2.A1}$$

The integrals are

1. $$\frac{2}{L}\int_{-1}^{+1}\frac{dN_1}{d\xi}\frac{dN_1}{d\xi}d\xi = +\frac{1}{L}, \qquad \frac{2}{L}\int_{-1}^{+1}\frac{dN_1}{d\xi}\frac{dN_2}{d\xi}d\xi = -\frac{1}{L}, \qquad \frac{2}{L}\int_{-1}^{+1}\frac{dN_2}{d\xi}\frac{dN_2}{d\xi}d\xi = +\frac{1}{L}$$

(2.A2)

2. $$\int_{-1}^{+1}\frac{dN_1}{d\xi}N_1 d\xi = -\frac{1}{2}, \quad \int_{-1}^{+1}\frac{dN_1}{d\xi}N_2 d\xi = -\frac{1}{2}, \quad \int_{-1}^{+1}\frac{dN_2}{d\xi}N_1 d\xi = +\frac{1}{2}, \quad \int_{-1}^{+1}\frac{dN_2}{d\xi}N_2 d\xi = +\frac{1}{2}$$

(2.A3)

3. $$\frac{L}{2}\int_{-1}^{+1}N_1 N_1 d\xi = \frac{L}{3}, \qquad \frac{L}{2}\int_{-1}^{+1}N_1 N_2 d\xi = \frac{L}{6}, \qquad \frac{L}{2}\int_{-1}^{+1}N_2 N_2 d\xi = \frac{L}{3}$$

(2.A4)

4. $$\frac{L}{2}\int_{-1}^{+1}N_1 d\xi = \frac{L}{2}, \qquad \frac{L}{2}\int_{-1}^{+1}N_2 d\xi = \frac{L}{2}$$

(2.A5)

5. $$\frac{L^2}{4}\int_{-1}^{+1}(\xi + A)N_1 d\xi = \frac{L^2}{12}(3A-1), \qquad \frac{L^2}{4}\int_{-1}^{+1}(\xi + A)N_2 d\xi = \frac{L^2}{12}(3A+1)$$

(2.A6)

Quadratic Shape Functions

The shape functions for the standard quadratic element are

$$N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = 1-\xi^2, \quad N_3 = \frac{1}{2}\xi(\xi+1)$$
$$\frac{dN_1}{d\xi} = \xi - \frac{1}{2}, \quad \frac{dN_2}{d\xi} = -2\xi, \quad \frac{dN_3}{d\xi} = \xi + \frac{1}{2} \tag{2.A7}$$

The integrals are

1.
$$\frac{2}{L}\int_{-1}^{+1}\frac{dN_1}{d\xi}\frac{dN_1}{d\xi}d\xi = +\frac{7}{3L}, \quad \frac{2}{L}\int_{-1}^{+1}\frac{dN_1}{d\xi}\frac{dN_2}{d\xi}d\xi = -\frac{8}{3L}, \quad \frac{2}{L}\int_{-1}^{+1}\frac{dN_1}{d\xi}\frac{dN_3}{d\xi}d\xi = +\frac{1}{3L}$$
$$\frac{2}{L}\int_{-1}^{+1}\frac{dN_2}{d\xi}\frac{dN_2}{d\xi}d\xi = +\frac{16}{3L}, \quad \frac{2}{L}\int_{-1}^{+1}\frac{dN_2}{d\xi}\frac{dN_3}{d\xi}d\xi = -\frac{8}{3L}, \quad \frac{2}{L}\int_{-1}^{+1}\frac{dN_3}{d\xi}\frac{dN_3}{d\xi}d\xi = +\frac{7}{3L} \tag{2.A8}$$

2.
$$\int_{-1}^{+1}\frac{dN_1}{d\xi}N_1 d\xi = -\frac{1}{2}, \quad \int_{-1}^{+1}\frac{dN_1}{d\xi}N_2 d\xi = -\frac{2}{3}, \quad \int_{-1}^{+1}\frac{dN_1}{d\xi}N_3 d\xi = +\frac{1}{6}$$
$$\int_{-1}^{+1}\frac{dN_2}{d\xi}N_1 d\xi = +\frac{2}{3}, \quad \int_{-1}^{+1}\frac{dN_2}{d\xi}N_2 d\xi = 0, \quad \int_{-1}^{+1}\frac{dN_2}{d\xi}N_3 d\xi = -\frac{2}{3}$$
$$\int_{-1}^{+1}\frac{dN_3}{d\xi}N_1 d\xi = -\frac{1}{6}, \quad \int_{-1}^{+1}\frac{dN_3}{d\xi}N_2 d\xi = +\frac{2}{3}, \quad \int_{-1}^{+1}\frac{dN_3}{d\xi}N_3 d\xi = +\frac{1}{2} \tag{2.A9}$$

3.
$$\frac{L}{2}\int_{-1}^{+1}N_1 N_1 d\xi = +\frac{2L}{15}, \quad \frac{L}{2}\int_{-1}^{+1}N_1 N_2 d\xi = +\frac{L}{15}, \quad \frac{L}{2}\int_{-1}^{+1}N_1 N_3 d\xi = -\frac{L}{30}$$
$$\frac{L}{2}\int_{-1}^{+1}N_2 N_2 d\xi = +\frac{8L}{15}, \quad \frac{L}{2}\int_{-1}^{+1}N_2 N_3 d\xi = +\frac{L}{15}, \quad \frac{L}{2}\int_{-1}^{+1}N_3 N_3 d\xi = +\frac{2L}{15} \tag{2.A10}$$

4.
$$\frac{L}{2}\int_{-1}^{+1}N_1 d\xi = +\frac{L}{6}, \quad \frac{L}{2}\int_{-1}^{+1}N_2 d\xi = +\frac{2L}{3}, \quad \frac{L}{2}\int_{-1}^{+1}N_3 d\xi = +\frac{L}{6} \tag{2.A11}$$

5. $$\frac{L^2}{4}\int_{-1}^{+1}(\xi+A)N_1 d\xi = \frac{L^2}{12}(A-1), \quad \frac{L^2}{4}\int_{-1}^{+1}(\xi+A)N_2 d\xi = \frac{L^2 A}{3}$$
$$\frac{L^2}{4}\int_{-1}^{+1}(\xi+A)N_3 d\xi = \frac{L^2}{12}(A+1)$$

(2.A12)

Cubic Hermite Shape Functions

The shape functions for the standard quadratic element are given by (2.64). The integrals are

1. $$\frac{2}{L}\int_{-1}^{+1}\frac{dN_i}{d\xi}\frac{dN_j}{d\xi}d\xi = \frac{1}{30}\begin{bmatrix} +36\frac{1}{L} & -36\frac{1}{L} & +3 & +3 \\ -36\frac{1}{L} & +36\frac{1}{L} & -3 & -3 \\ +3 & -3 & +4L & -L \\ +3 & -3 & -L & +4L \end{bmatrix}$$

(2.A13)

2. $$\frac{L}{2}\int_{-1}^{+1}N_j d\xi = \frac{L}{12}\begin{bmatrix} +6 \\ +6 \\ +L \\ -L \end{bmatrix}$$

(2.A14)

3. $$\frac{2}{L}\int_{-1}^{+1}N_i\frac{d^2N_j}{d\xi^2}d\xi = \frac{1}{30}\begin{bmatrix} -36\frac{1}{L} & +36\frac{1}{L} & -33 & -3 \\ +36\frac{1}{L} & -36\frac{1}{L} & +3 & +33 \\ -3 & +3 & -4L & +L \\ -3 & +3 & +L & -4L \end{bmatrix}$$

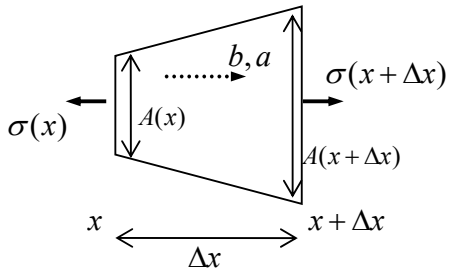
(2.A15)

2.12.2 Derivation of the Governing Equations for Elasticity

The following are discussed in more detail in Solid Mechanics, Part II, Sections 1.1, 1.2.

Force Balance

Consider a one-dimensional differential element of length Δx. The element has varying cross secton, with $A(x+\Delta x) = A(x) + O(\Delta x)$. Let a body force (per unit volume) b, e.g. the force of gravity, act on the element, again, with $b(x+\Delta x) = b(x) + O(\Delta x)$. Denote the acceleration and density of the element be a and ρ. Stresses σ act on the element.



The surface forces acting are $\sigma A\big|_{x+\Delta x} - \sigma A\big|_x$. The force due to b is $A(x)b(x)\Delta x + O(\Delta x)^2$. Applying Newton's second law,

$$\begin{aligned} \sigma A\big|_{x+\Delta x} - \sigma A\big|_x + A(x)b(x)\Delta x + O(\Delta x)^2 &= \rho a \Delta x \left(A\big|_{x+\Delta x} + A\big|_x \right)/2 \\ \rightarrow \frac{\sigma A\big|_{x+\Delta x} - \sigma A\big|_x}{\Delta x} + Ab &= A\rho a + O(\Delta x) \end{aligned} \tag{2A.16}$$

so that, by the definition of the derivative, in the limit as $\Delta x \rightarrow 0$, one has the *equation of motion*

$$\frac{d}{dx}(A\sigma) + Ab = A\rho a \tag{2A.17}$$

If the material is static, or if the accelerations are so low that they can be neglected, this equation reduces to the *equation of equilibrium*:

$$\frac{d}{dx}(A\sigma) + Ab = 0 \tag{2A.18}$$

Kinematics

When a material is subjected to a stress/force, it deforms. Define the strain $\varepsilon(x)$ to be the change in length per unit length that a small line element positioned at x undergoes.

To derive a relationship between the stress and the displacement of material particles, consider one such line element of length Δx and emanating from position x. During the deformation, the end at x undergoes a displacement $u(x)$ and the other end undergoes a displacement $u(x+\Delta x)$. From the definition then, the strain is

$$\varepsilon(x) = \frac{\left[u(x+\Delta x) + \Delta x - u(x)\right] - \Delta x}{\Delta x} \tag{2A.19}$$

In the limit as $\Delta x \to 0$ then, this reduces to the relation

$$\varepsilon(x) = \frac{\partial u}{\partial x} \tag{2A.20}$$