On Costa's Minimal Surface

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Outline

Weierstrass Representation

Theorem (The Weierstrass Representation Formula)

*Let f and g be functions on a simply connected domain $D \subset \mathbb{C}$, where g is meromorphic and f is holomorphic, such that wherever g has a pole of order m, f has a zero of order at least $2m$ (or equivalently, such that the product fg^2 is holomorphic). Fix $z_0 \in D$, and let c_1, c_2, c_3 be constants. Then the surface with coordinates (x_1, x_2, x_3) is **minimal**, where the x_k are defined as follows:*

$$x_k(z) = \mathrm{Re}\left\{\int_{z_0}^{z} \varphi_k(w)\,\mathrm{d}w\right\} + c_k, \quad k = 1, 2, 3.$$

$$\varphi_1 = \frac{f\left(1 - g^2\right)}{2}, \quad \varphi_2 = \frac{\mathrm{i}f\left(1 + g^2\right)}{2}, \quad \varphi_3 = fg.$$

Basic Example: The Catenoid

From the functions

$$f(z) = -e^{-z} \text{ and } g(z) = -e^{z}$$

we obtain (up to constants)

$$\begin{cases} x_1(u,v) = \cosh u \cos v, \\ x_2(u,v) = \cosh u \sin v, \\ x_3(u,v) = u, \end{cases}$$

which describes the catenoid.

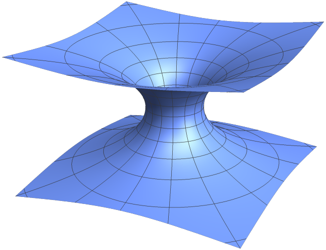


Figure: The catenoid

Weierstrass $\wp$ and ζ

We choose the lattice $\mathbb{Z}[\mathrm{i}] = \{m + \mathrm{i}n : m, n \in \mathbb{Z}\}$ so that the Weierstrass $\wp$ and ζ functions are defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Clearly, $\zeta'(z) = -\wp(z)$. Let us denote

$$e_1 = \wp(\tfrac{1}{2}), \quad e_2 = \wp\left(\tfrac{\mathrm{i}}{2}\right), \quad e_3 = \wp\left(\tfrac{1+\mathrm{i}}{2}\right).$$

Note that ζ here is not the Riemann zeta function.

Identities Involving $\wp$ and ζ

1. $\wp(z+m+\mathrm{i}n)=\wp(z)$ for all $m,n\in\mathbb{Z}$.
2. $\wp(z_1+z_2)=\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2-\wp(z_1)-\wp(z_2)$.
3. $\wp'(z)^2=\left[4\wp(z)^2-g_2\right]\wp(z)$.
4. $\wp'(z)^2=4\wp(z)\left[\wp(z)^2-e_1^2\right]$.
5. $\wp\left(z+\frac{1}{2}\right)=e_1+\frac{2e_1^2}{\wp(z)-e_1}$.
6. $\wp\left(z+\frac{\mathrm{i}}{2}\right)=e_2+\frac{2e_2^2}{\wp(z)-e_2}=-e_1+\frac{2e_1^2}{\wp(z)+e_1}$.
7. $\wp\left(z-\frac{1}{2}\right)-\wp\left(z-\frac{\mathrm{i}}{2}\right)-2e_1=\frac{16e_1^3\wp(z)}{\wp'(z)^2}$.

Identities Involving $\wp$ and ζ

8. $\zeta(z+m+\mathrm{i}n) = \zeta(z) + 2m\zeta\left(\frac{1}{2}\right) + 2n\zeta\left(\frac{\mathrm{i}}{2}\right)$ for all $m, n \in \mathbb{Z}$.
9. $\mathrm{i}\zeta(\mathrm{i}z) = \zeta(z)$.
10. $\zeta\left(\frac{1}{2}\right) = \mathrm{i}\zeta\left(\frac{\mathrm{i}}{2}\right) = \frac{\pi}{2}$.
11. $\zeta\left(\frac{1+\mathrm{i}}{2}\right) = \frac{(1-\mathrm{i})\pi}{2}$.
12. $\zeta(z+u) - \zeta(z) - \zeta(u) = \frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$.

1. Clear from the definition of $\wp$.
2. A well-known addition formula that can be found in most textbooks on elliptic functions. So is 12.
3. Corollary 2.3 of Chapter 9 in *Complex Analysis* by Stein and Shakarchi gives the identity

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where

$$g_2 = 60 \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \frac{1}{\omega^6}.$$

In our case, $g_3 = 0$, since

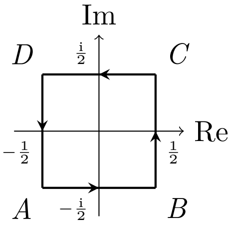
$$(m - \mathrm{i}n)^6 + (m + \mathrm{i}n)^6 + (n - \mathrm{i}m)^6 + (n + \mathrm{i}m)^6 = 0.$$

4. It is known that $1/2$, $\mathrm{i}/2$ and $(1+\mathrm{i})/2$ are the roots of the cubic polynomial $\left[4\wp(z)^2 - g_2\right]\wp(z)$, and $e_3 = \wp\left(\frac{1+\mathrm{i}}{2}\right) = 0$. Hence $4e_1^2 = g_2$, and the identity follows from 3.
5. Apply 2 and then 4.
6. Apply 2 and then 4. Note that $e_2 = -e_1$.
7. By 1, $\wp\left(z - \frac{1}{2}\right) - \wp\left(z - \frac{\mathrm{i}}{2}\right) = \wp\left(z + \frac{1}{2}\right) - \wp\left(z + \frac{\mathrm{i}}{2}\right)$. Then combine 5 and 6 to get the identity.
8. Since $\zeta'(z) = -\wp(z)$ and $\wp(z+1) = \wp(z)$, the two functions $\zeta(z+1)$ and $\zeta(z)$ differ by a constant, say $\zeta(z+1) = \zeta(z) + c$. Take $z = -\frac{1}{2}$ and use the fact that ζ is odd to get $c = 2\zeta\left(\frac{1}{2}\right)$. The same argument gives $\zeta(z+\mathrm{i}) = \zeta(z) + 2\zeta\left(\frac{\mathrm{i}}{2}\right)$.
9. Clear from the definition of ζ and the fact that $\mathrm{i}\mathbb{Z}[\mathrm{i}] = \mathbb{Z}[\mathrm{i}]$.

10. The residue theorem gives

$$\int_{ABCDA} \zeta(z)\,\mathrm{d}z = 2\pi\mathrm{i}.$$

On the other hand, by 8 we have



$$\int_{CD} \zeta(z)\,\mathrm{d}z = \int_{BA} \zeta(z)\,\mathrm{d}z - 2\zeta\left(\tfrac{\mathrm{i}}{2}\right), \int_{BC} \zeta(z)\,\mathrm{d}z = \int_{AD} \zeta(z)\,\mathrm{d}z + 2\mathrm{i}\zeta\left(\tfrac{1}{2}\right).$$

Combining these equations gives $\zeta\left(\frac{1}{2}\right) + \mathrm{i}\zeta\left(\frac{\mathrm{i}}{2}\right) = \pi$. Then use 9.

11. Take $z = -\frac{1+\mathrm{i}}{2}$ and $m = n = 1$ in 8 and use the fact that ζ is odd to get $\zeta\left(\frac{1+\mathrm{i}}{2}\right) = \zeta\left(\frac{1}{2}\right) + \zeta\left(\frac{\mathrm{i}}{2}\right)$. Then 10 applies.

The Weierstrass Data

Costa's minimal surface is defined as a Weierstrass patch using the functions

$$f(z) = \wp(z) \quad \text{and} \quad g(z) = \frac{A}{\wp'(z)}.$$

In order that Costa's minimal surface has no self-intersections, we need to take[1]

$$A = 2\sqrt{2\pi}\, e_1 \approx 34.46707.$$

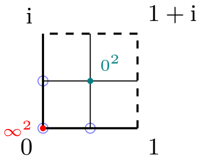


Figure: Zeros and poles of f

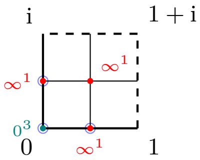


Figure: Zeros and poles of g

To avoid integrals, we shall use ζ to express the coordinates.

[1]The choice of the constant A is forced by the requirement that the components $\varphi_1(z)\,\mathrm{d}z, \varphi_2(z)\,\mathrm{d}z, \varphi_3(z)\,\mathrm{d}z$ have no real periods.

Using 7 we obtain

$$
\begin{aligned}
f(w)\left[1-g(w)^2\right] &= \wp(w) - \frac{A^2\wp(w)}{\wp'(w)^2} \\
&= \wp(w) - \frac{A^2}{16e_1^3}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right) - 2e_1\right] \\
&= \wp(w) - \tfrac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right) - 2e_1\right] \\
&= \wp(w) + \pi - \tfrac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right)\right].
\end{aligned}
$$

Take $z_0 = \frac{1+\mathrm{i}}{2}$. Integrating both sides and using 10 and 11, we get

$$\begin{aligned}
&\int_{z_0}^{z} f(w)\left[1-g(w)^2\right]\mathrm{d}w \\
&=\left\{-\zeta(w)+\pi w+\tfrac{\pi}{2e_1}\left[\zeta\left(w-\tfrac{1}{2}\right)-\zeta\left(w-\tfrac{\mathrm{i}}{2}\right)\right]\right\}\Big|_{z_0}^{z} \\
&=-\zeta(z)+\pi z+\tfrac{\pi}{2e_1}\left[\zeta\left(z-\tfrac{1}{2}\right)-\zeta\left(z-\tfrac{\mathrm{i}}{2}\right)\right] \\
&\quad+\zeta\left(\tfrac{1+\mathrm{i}}{2}\right)-\tfrac{\pi(1+\mathrm{i})}{2}-\tfrac{\pi}{2e_1}\left[\zeta\left(\tfrac{\mathrm{i}}{2}\right)-\zeta\left(\tfrac{1}{2}\right)\right] \\
&=-\zeta(z)+\pi z+\tfrac{\pi}{2e_1}\left[\zeta\left(z-\tfrac{1}{2}\right)-\zeta\left(z-\tfrac{\mathrm{i}}{2}\right)\right]-\mathrm{i}\pi+\tfrac{\pi^2(1+\mathrm{i})}{4e_1}.
\end{aligned}$$

Dividing by 2 and taking the real part, we get x_1.

Similarly,

$$f(w)\left[1+g(w)^2\right]=\wp(w)-\pi+\tfrac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right)-\wp\left(w-\tfrac{\mathrm{i}}{2}\right)\right]$$

and then

$$\begin{aligned}&\int_{z_0}^{z} f(w)\left[1+g(w)^2\right]\mathrm{d}w\\&=\left\{-\zeta(w)-\pi w-\tfrac{\pi}{2e_1}\left[\zeta\left(w-\tfrac{1}{2}\right)-\zeta\left(w-\tfrac{\mathrm{i}}{2}\right)\right]\right\}\Big|_{z_0}^{z}\\&=-\zeta(z)-\pi z-\tfrac{\pi}{2e_1}\left[\zeta\left(z-\tfrac{1}{2}\right)-\zeta\left(z-\tfrac{\mathrm{i}}{2}\right)\right]+\pi-\frac{\pi^2(1+\mathrm{i})}{4e_1}.\end{aligned}$$

From this we can find x_2.

Using 4 we obtain

$$
\begin{aligned}
\int_{z_0}^{z} f(w) g(w) \, \mathrm{d}w &= A \int_{z_0}^{z} \frac{\wp(w)}{\wp'(w)} \, \mathrm{d}w = \frac{A}{4} \int_{z_0}^{z} \frac{\wp'(w) \, \mathrm{d}w}{\wp(w)^2 - e_1^2} \\
&= \frac{A}{8e_1} \int_{z_0}^{z} \left(\frac{\wp'(w)}{\wp(w) - e_1} - \frac{\wp'(w)}{\wp(w) + e_1} \right) \mathrm{d}w \\
&= \frac{\sqrt{2\pi}}{4} \log \left(\frac{\wp(w) - e_1}{\wp(w) + e_1} \right) \Bigg|_{z_0}^{z} \\
&= \frac{\sqrt{2\pi}}{4} \left\{ \log \left(\frac{\wp(z) - e_1}{\wp(z) + e_1} \right) - \log \left(\frac{e_3 - e_1}{e_3 + e_1} \right) \right\} \\
&= \frac{\sqrt{2\pi}}{4} \left\{ \log \left(\frac{\wp(z) - e_1}{\wp(z) + e_1} \right) - \pi \mathrm{i} \right\}.
\end{aligned}
$$

Taking the real part gives x_3.

Coordinates of the Surface

Costa's minimal surface is given by (x_1, x_2, x_3) where

$$\begin{cases} x_1(u,v) = \frac{1}{2}\,\mathrm{Re}\left\{-\zeta(u+\mathrm{i}v) + \pi u + \frac{\pi^2}{4e_1} \right. \\ \qquad\qquad \left. + \frac{\pi}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\frac{1}{2}\right) - \zeta\left(u+\mathrm{i}v-\frac{\mathrm{i}}{2}\right)\right]\right\}, \\ x_2(u,v) = \frac{1}{2}\,\mathrm{Re}\left\{-\mathrm{i}\zeta(u+\mathrm{i}v) + \pi v + \frac{\pi^2}{4e_1} \right. \\ \qquad\qquad \left. - \frac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\frac{1}{2}\right) - \zeta\left(u+\mathrm{i}v-\frac{\mathrm{i}}{2}\right)\right]\right\}, \\ x_3(u,v) = \frac{\sqrt{2\pi}}{4}\log\left|\frac{\wp(u+\mathrm{i}v) - e_1}{\wp(u+\mathrm{i}v) + e_1}\right|. \end{cases}$$

Parameter Space

By 8 and 10 we have $\zeta(z+1)=\zeta(z)+\pi$, hence

$$\begin{aligned} x_1(u+1,v) &= \tfrac{1}{2}\operatorname{Re}\Big\{-\zeta(u+\mathrm{i}v) - \pi + \pi u + \pi + \tfrac{\pi^2}{4e_1} \\ &\qquad + \tfrac{\pi}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\tfrac{1}{2}\right) + \pi - \zeta\left(u+\mathrm{i}v-\tfrac{\mathrm{i}}{2}\right) - \pi\right]\Big\} \\ &= x_1(u,v). \end{aligned}$$

Similarly, one can show that $x_1(u,v+1)=x_1(u,v)$ and

$$\begin{aligned} x_2(u+1,v) &= x_2(u,v), \quad & x_2(u,v+1) &= x_2(u,v), \\ x_3(u+1,v) &= x_3(u,v), \quad & x_3(u,v+1) &= x_3(u,v). \end{aligned}$$

Therefore, we can restrict u and v to the unit square $[0,1)\times[0,1)$.

Meromorphic functions of $\mathbb{T}^2$ $\leftrightsquigarrow$ Elliptic functions of $\mathbb{Z}[\mathrm{i}]$.

Shape of the Surface

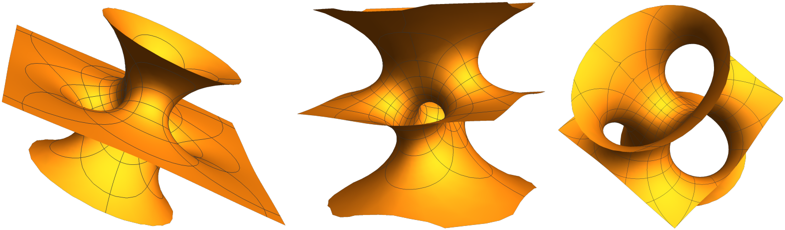


Figure: Close-up views of Costa's surface (left to right)

Ends of Complete Minimal Surfaces

Let M_g be a compact surface of genus g, and let $Q_1, \cdots, Q_r$ be distinct points on M_g. Consider a complete minimal immersion

$$x\colon M = M_g \setminus \{Q_1, \cdots, Q_r\} \to \mathbb{R}^n.$$

For each j, let $D_j \subset M_g$ be a topological disk centered at Q_j, with $Q_i \notin D_j$ for all $i \neq j$. The image

$$F_j = x(D_j \cap M)$$

is called an **end** of the immersion x. We say that x is a **complete minimal immersion in $\mathbb{R}^n$ of genus g with r ends**.

Osserman's Classification

A surface M is said to have **finite topology** if it is homeomorphic to a compact surface (i.e., has finite genus) from which a finite number of points have been removed (i.e., has finitely many ends).

In 1986, Osserman described all complete, properly embedded minimal surfaces in $\mathbb{R}^3$ of finite topology.

The ends are all graphs over the same plane, asymptotic to

$$x_3 = a + b \log \sqrt{x_1^2 + x_2^2},$$

for suitable constants a and b.

Costa's Groundbreaking Discovery

It had been a longstanding conjecture that the only complete embedded minimal surfaces in $\mathbb{R}^3$ of finite topology are the plane, the catenoid, the helicoid.

Theorem (Costa, 1984)

Costa's surface is a complete minimal immersion in $\mathbb{R}^3$, of ***genus one****, with* ***three ends*** *and the following properties:*

- *The total curvature is -12π.*
- *The ends are embedded.*

Two Catenoidal Ends and One Planar End

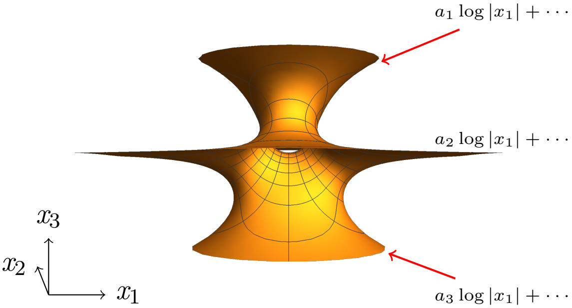


Figure: Front view of Costa's surface

We aim to determine the coefficients a_1, a_2, a_3**.**

Key Observation (i)

- $\wp(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$.

$$\overline{\wp(x)} = \frac{1}{x^2} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left(\frac{1}{(x - \overline{\omega})^2} - \frac{1}{\overline{\omega}^2} \right) = \wp(x).$$

- $\zeta(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$.

$$\overline{\zeta(x)} = \frac{1}{x} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left(\frac{1}{x - \overline{\omega}} + \frac{1}{\overline{\omega}} + \frac{x}{\overline{\omega}^2} \right) = \zeta(x).$$

Key Observation (ii)

$$\begin{aligned} x_2(u,0) &= \tfrac{1}{2}\,\mathrm{Re}\left\{-\mathrm{i}\zeta(u) + \tfrac{\pi^2}{4e_1} - \tfrac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\ &= \tfrac{1}{2}\left\{\tfrac{\pi^2}{4e_1} - \tfrac{\pi}{2e_1}\,\mathrm{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right\}. \end{aligned}$$

By 12 we have

$$\zeta\left(u-\tfrac{\mathrm{i}}{2}\right) = \zeta(u) + \zeta\left(-\tfrac{\mathrm{i}}{2}\right) + \frac{1}{2}\frac{\wp'(u) - \wp'\left(\frac{\mathrm{i}}{2}\right)}{\wp(u) - \wp\left(\frac{\mathrm{i}}{2}\right)}.$$

Since $\wp\left(\frac{\mathrm{i}}{2}\right) = e_2 = -e_1 \in \mathbb{R}$ and $\wp'\left(\frac{\mathrm{i}}{2}\right) = 0$, we obtain

$$\mathrm{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\} = \mathrm{Im}\left\{\zeta\left(-\tfrac{\mathrm{i}}{2}\right)\right\} = \mathrm{Im}\left\{\tfrac{\pi\mathrm{i}}{2}\right\} = \tfrac{\pi}{2}.$$

Hence $x_2(u,0) = 0$.

Key Observation (iii)

$$
\begin{aligned}
& x_2\left(u, \tfrac{1}{2}\right) \\
&= \tfrac{1}{2}\operatorname{Re}\left\{-\mathrm{i}\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\tfrac{\pi}{2}+\tfrac{\pi^2}{4e_1}-\tfrac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\} \\
&= \tfrac{1}{2}\left\{\operatorname{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)\right\}+\tfrac{\pi}{2}+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\operatorname{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)\right\}\right\}.
\end{aligned}
$$

As in the case with (ii), we have

$$
\operatorname{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)\right\}=\operatorname{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)-\pi\mathrm{i}\right\}=\tfrac{\pi}{2}-\pi=-\tfrac{\pi}{2},
$$
$$
\operatorname{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)\right\}=\operatorname{Im}\left\{\zeta\left(\left(u-\tfrac{1}{2}\right)+\tfrac{\mathrm{i}}{2}\right)\right\}=-\tfrac{\pi}{2}.
$$

Therefore, $x_2\left(u, \frac{1}{2}\right)=0$.

Key Observation (iv)

- $x_1(u,0) \to -\infty$ as $u \searrow 0$.
- $x_1(u,0) \to -\infty$ as $u \nearrow \frac{1}{2}$.
- $x_1(u,0) \to +\infty$ as $u \searrow \frac{1}{2}$.
- $x_1(u,0) \to +\infty$ as $u \nearrow 1$.
- $x_3(u,0) \to 0$ as $u \to 0$.
- $x_3(u,0) \to -\infty$ as $u \to \frac{1}{2}$.

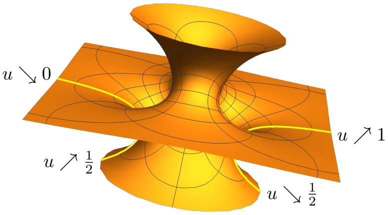


Figure: The curve $v = 0$

Proof of Observation (iv)

When $u \searrow 0$,

$$
\begin{aligned}
& x_1(u,0) \\
&= \tfrac{1}{2}\,\mathrm{Re}\left\{-\zeta(u) + \pi u + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
&= \tfrac{1}{2}\left\{-\zeta(u) + \pi u + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathrm{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \tfrac{1}{2}\left\{-\tfrac{1}{u} + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(-\tfrac{1}{2}\right) - \mathrm{Re}\left\{\zeta\left(-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim -\frac{1}{2u} \to -\infty.
\end{aligned}
$$

Proof of Observation (iv)

When $u \nearrow \frac{1}{2}$,

$$
\begin{aligned}
& x_1(u,0) \\
&= \frac{1}{2}\operatorname{Re}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\frac{1}{2}\right) - \zeta\left(u-\frac{\mathrm{i}}{2}\right)\right]\right\} \\
&= \frac{1}{2}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\frac{1}{2}\right) - \operatorname{Re}\left\{\zeta\left(u-\frac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \frac{1}{2}\left\{-\zeta\left(\frac{1}{2}\right) + \frac{\pi}{2} + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\frac{1}{2}\right) - \operatorname{Re}\left\{\zeta\left(\frac{1-\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \frac{\pi}{4e_1}\zeta\left(u-\frac{1}{2}\right) \\
&\sim \frac{\pi}{4e_1}\frac{1}{u-1/2} \to -\infty.
\end{aligned}
$$

Proof of Observation (iv)

When $u \searrow \frac{1}{2}$,

$$
\begin{aligned}
& x_1(u,0) \\
&= \frac{1}{2}\,\mathrm{Re}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
&= \frac{1}{2}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathrm{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \frac{1}{2}\left\{-\zeta\left(\tfrac{1}{2}\right) + \frac{\pi}{2} + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathrm{Re}\left\{\zeta\left(\tfrac{1-\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \frac{\pi}{4e_1}\zeta\left(u-\tfrac{1}{2}\right) \\
&\sim \frac{\pi}{4e_1}\frac{1}{u-1/2} \to +\infty.
\end{aligned}
$$

Proof of Observation (iv)

When $u \nearrow 1$,

$$
\begin{aligned}
& x_1(u,0) \\
&= \tfrac{1}{2}\operatorname{Re}\left\{-\zeta(u) + \pi u + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
&= \tfrac{1}{2}\left\{-\zeta(u) + \pi u + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \operatorname{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim \tfrac{1}{2}\left\{-\tfrac{1}{u-1} - \pi + \pi + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(\tfrac{1}{2}\right) - \operatorname{Re}\left\{\zeta\left(1-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
&\sim -\tfrac{1}{2(u-1)} \to +\infty.
\end{aligned}
$$

Proof of Observation (iv)

$$x_3(u,0) = \frac{\sqrt{2\pi}}{4} \log \left| \frac{\wp(u) - e_1}{\wp(u) + e_1} \right| .$$

- Since $\wp(u) \sim \frac{1}{u^2}$ as $u \to 0$, we have $x_3(u,0) \to 0$ as $u \to 0$.
- Since $\wp\left(\frac{1}{2}\right) = e_1$, we have $x_3(u,0) \to -\infty$ as $u \to \frac{1}{2}$.

Coefficients a_2 and a_3

It is obvious from observation (iv) that $a_2 = 0$.
To find a_3, first note that

$$a_3 = \lim_{u \searrow \frac{1}{2}} \frac{x_3(u,0)}{\log x_1(u,0)} = \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{\log \left| \frac{\wp(u+1/2)-e_1}{\wp(u+1/2)+e_1} \right|}{\log \frac{\pi}{4 e_1 u}}.$$

Using 5 we have

$$\frac{\wp\left(u+\frac{1}{2}\right)-e_1}{\wp\left(u+\frac{1}{2}\right)+e_1} = 1 - \frac{2e_1}{\wp\left(u+\frac{1}{2}\right)+e_1} = 1 - \frac{2e_1}{2e_1 + \frac{2e_1^2}{\wp(u)-e_1}}$$
$$= \frac{e_1}{\wp(u)} \sim e_1 u^2 \quad \text{as } u \to 0.$$

Coefficients a_2 and a_3

Now

$$\begin{aligned} a_3 &= \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{\log\left(e_1 u^2\right)}{\log \frac{\pi}{4 e_1 u}} \\ &= \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{2 \log u + \log e_1}{-\log u + \log \frac{\pi}{4 e_1}} \\ &= -\sqrt{\frac{\pi}{2}} \approx -1.25331. \end{aligned}$$

Key Observation (v)

- $x_1\left(u, \frac{1}{2}\right) \to -\infty$ as $u \searrow 0$.
- $x_1\left(u, \frac{1}{2}\right) \to +\infty$ as $u \nearrow 1$.

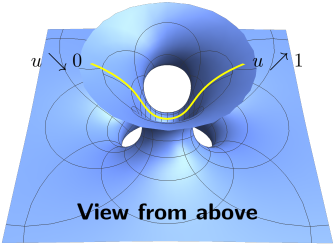


Figure: The curve $v = \frac{1}{2}$

Proof of Observation (v)

$$x_1\left(u,\tfrac{1}{2}\right)$$
$$=\tfrac{1}{2}\,\mathrm{Re}\left\{-\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\pi u+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\}.$$

When $u \searrow 0$,

$$\begin{aligned}x_1\left(u,\tfrac{1}{2}\right) &\sim \tfrac{1}{2}\,\mathrm{Re}\left\{-\zeta\left(\tfrac{\mathrm{i}}{2}\right)+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left[\zeta\left(\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\}\\ &\sim \tfrac{1}{2}\left\{\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left(-\tfrac{\pi}{2}-\tfrac{1}{u}\right)\right\}\\ &\sim -\tfrac{\pi}{4e_1 u}\to-\infty.\end{aligned}$$

Proof of Observation (v)

$$x_1\left(u,\tfrac{1}{2}\right) = \tfrac{1}{2}\operatorname{Re}\left\{-\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\pi u+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\}.$$

When $u \nearrow 1$,

$$\begin{aligned} x_1\left(u,\tfrac{1}{2}\right) &\sim \tfrac{1}{2}\operatorname{Re}\left\{-\zeta\left(1+\tfrac{\mathrm{i}}{2}\right)+\pi+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left[\zeta\left(\tfrac{1+\mathrm{i}}{2}\right)-\zeta(u)\right]\right\} \\ &\sim \tfrac{1}{2}\left\{-\pi+\pi+\tfrac{\pi^2}{4e_1}+\tfrac{\pi}{2e_1}\left[\tfrac{\pi}{2}-\zeta(u-1)-\pi\right]\right\} \\ &\sim \tfrac{\pi}{4e_1(1-u)} \to +\infty. \end{aligned}$$

Coefficient a_1

As before, we write a_1 as

$$a_1 = \lim_{u \nearrow 1} \frac{x_3\left(u, \frac{1}{2}\right)}{\log x_1\left(u, \frac{1}{2}\right)} = \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{\log \left| \frac{\wp(u+\mathrm{i}/2)-e_1}{\wp(u+\mathrm{i}/2)+e_1} \right|}{\log \frac{\pi}{4e_1(1-u)}}.$$

Using 6 we have

$$\frac{\wp\left(u+\frac{\mathrm{i}}{2}\right)-e_1}{\wp\left(u+\frac{\mathrm{i}}{2}\right)+e_1} = 1 - \frac{2e_1}{\wp\left(u+\frac{\mathrm{i}}{2}\right)+e_1} = 1 - \frac{2e_1}{\frac{2e_1^2}{\wp(u)+e_1}}$$

$$= -\frac{\wp(u)}{e_1} \sim -\frac{1}{e_1(1-u)^2} \quad \text{as } u \to 1.$$

Coefficient a_1

Now

$$\begin{aligned} a_1 &= \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{\log \frac{1}{e_1 (1-u)^2}}{\log \frac{\pi}{4 e_1 (1-u)}} \\ &= \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{-2 \log(1-u) - \log e_1}{-\log(1-u) + \log \frac{\pi}{4e_1}} \\ &= \sqrt{\frac{\pi}{2}} \approx 1.25331. \end{aligned}$$

Two Straight Lines Meeting at Right Angles

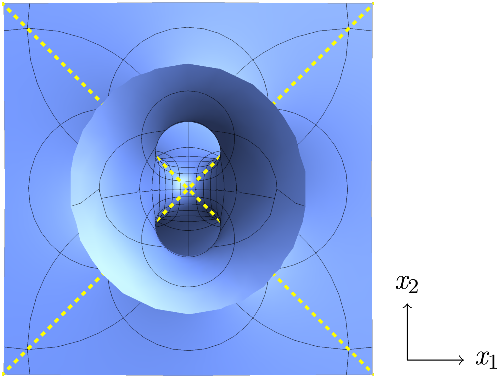


Figure: Vertical view of Costa's surface

Weierstrass $\wp$ on the Unit Square

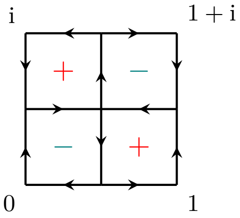


Figure: Sign of $\mathrm{Im}(\wp)$. Arrows in direction of increasing $\mathrm{Re}(\wp)$.

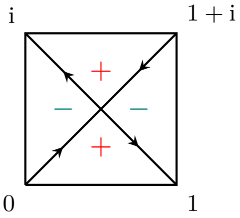


Figure: Sign of $\mathrm{Re}(\wp)$. Arrows in direction of increasing $\mathrm{Im}(\wp)$.

Since $\wp$ is purely imaginary on the diagonals of the unit square,

$$x_3(u,u) = \frac{\sqrt{2\pi}}{4} \log \frac{|\wp(u+\mathrm{i}u) - e_1|}{|\wp(u+\mathrm{i}u) + e_1|} = 0.$$

Moreover, with 9 and $\zeta(\bar{z}) = \overline{\zeta(z)}$ we see that

$$\begin{aligned} x_2(u,u) &= \tfrac{1}{2}\operatorname{Re}\Big\{-\mathrm{i}\zeta(u+\mathrm{i}u) + \pi u + \tfrac{\pi^2}{4e_1} \\ &\qquad -\tfrac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u+\mathrm{i}u-\tfrac{1}{2}\right) - \zeta\left(u+\mathrm{i}u-\tfrac{\mathrm{i}}{2}\right)\right]\Big\} \\ &= \tfrac{1}{2}\operatorname{Re}\Big\{-\zeta(u-\mathrm{i}u) + \pi u + \tfrac{\pi^2}{4e_1} \\ &\qquad -\tfrac{\pi}{2e_1}\left[\zeta\left(u-\mathrm{i}u+\tfrac{\mathrm{i}}{2}\right) - \zeta\left(u-\mathrm{i}u-\tfrac{1}{2}\right)\right]\Big\} \\ &= x_1(u,u). \end{aligned}$$

As before, one can show that as $u \searrow 0$

$$
\begin{aligned}
x_1(u,u) &= x_2(u,u) \\
&\sim \frac{1}{2}\,\mathrm{Re}\left\{-\frac{1}{u+\mathrm{i}u} + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(-\frac{1}{2}\right) - \zeta\left(-\frac{\mathrm{i}}{2}\right)\right]\right\} \\
&\sim -\frac{1}{4u} \to -\infty,
\end{aligned}
$$

and as $u \nearrow 1$

$$x_1(u,u) = x_2(u,u) \sim \frac{1}{4(1-u)} \to +\infty.$$

Therefore the straight line $(x,x,0)$ with $x \in \mathbb{R}$ lies on the surface. By reflection in the x_2-x_3 plane, we find the other straight line on the surface.

Dihedral Symmetry from Straight Lines

Theorem (Schwarz's Reflection Principle for Minimal Surfaces)

A minimal surface which contains a straight line on its boundary can be analytically extended by reflection across the line.

Corollary

If a minimal surface contains a straight line, then it is invariant under rotation by π about that line.

The symmetry group of Costa's surface is the dihedral group generated by

- Reflection in the x_1-x_3 plane; and
- Rotation about the x_3-axis by $\frac{\pi}{2}$ followed by reflection in the x_1-x_2 plane.

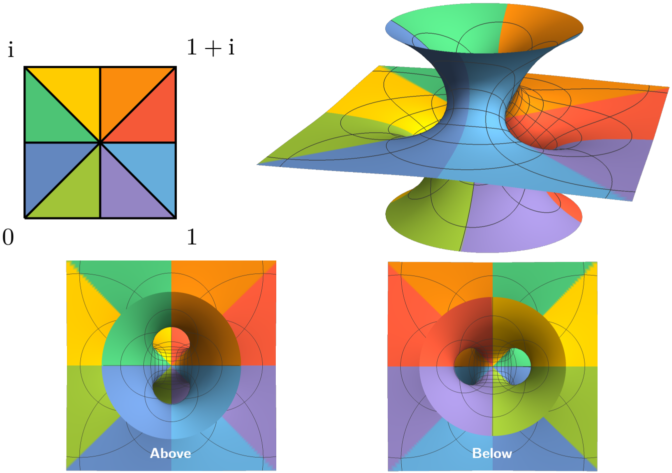


Figure: Fundamental triangles $\leftrightsquigarrow$ Congruent pieces of the surface

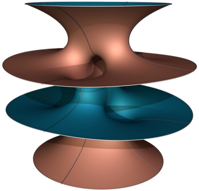
Genus vs. Ends

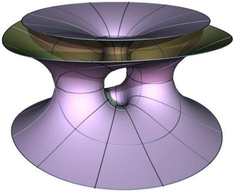
Examples with more ends and of higher genus followed rapidly, all nicely embedded. But there was a pattern: $\#\text{Ends} \leq \text{Genus} + 2$.

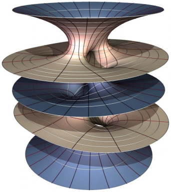
✔ **Left:** Genus 2 with 4 ends.

✘ **Middle:** All attempts to produce a torus with 4 ends have failed. Here the ends eventually intersect.

✔ **Right:** Genus 3 with 5 ends.







The Hoffman–Meeks Conjecture

Let $\mathcal{P}$ denote the space of all properly embedded connected minimal surfaces in $\mathbb{R}^3$ and let $\mathcal{M} \subset \mathcal{P}$ denote the subspace of examples with more than one end.

Finite Topology Conjecture (Hoffman and Meeks)

A noncompact orientable surface M of finite topology with genus g and r ends, $r \neq 2$, occurs in $\mathcal{P}$ if and only if $r \leq g + 2$.

This is possibly the most important open problem in the theory of minimal surfaces.

Known Results

- If $\Sigma \subset \mathcal{P}$ has finite topology, then:
  - If Σ has genus zero, then Σ is a plane, a helicoid or a catenoid;
  - If Σ has two ends, then Σ is a catenoid;
  - For every genus g, there exists an integer $e(g)$ such that if Σ has genus g, then the number of ends of Σ is at most $e(g)$.
- [Collin[1] and Schoen[2]] The only examples in $\mathcal{M}$ with finite topology and two ends are catenoids.
- [Collin[1] and Lopez–Ros[3]] If M has finite topology, genus zero and at least two ends, then M is a catenoid.

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[1] P. Collin. Topologie et courbure des surfaces minimales de $\mathbb{R}^3$. *Annals of Math. 2nd Series*, 145-1:1-31, 1997.

[2] R. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differential Geometry*, 18:791-809, 1983.

[3] F. J. Lopez and A. Ros. On embedded complete minimal surfaces of genus zero. *J. of Differential Geometry*, 33(1):293-300, 1991.

The End