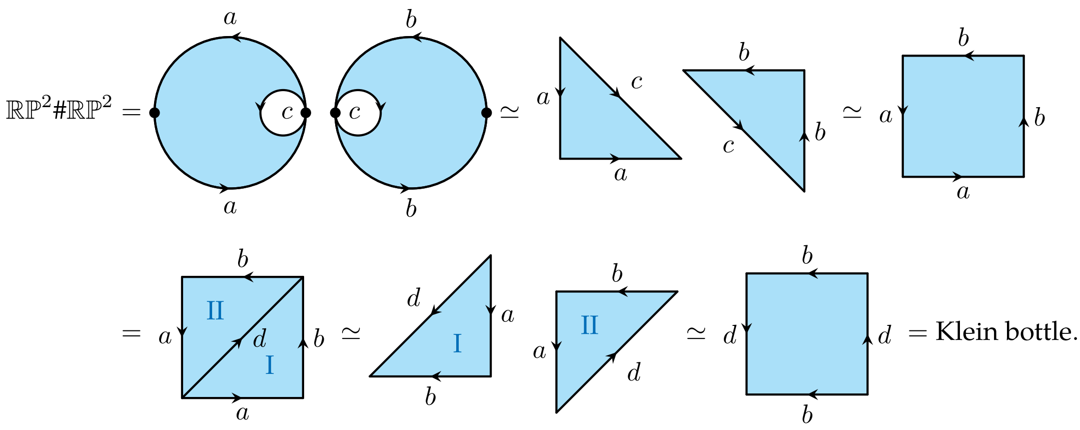
For latest updates of this note, visit https://xiaoshuo-lin.github.io/001707E.

Problem 1 Show that $\mathbb{RP}^2\#\mathbb{RP}^2$ is homeomorphic to the Klein bottle.

Proof Viewing $\mathbb{RP}^2$ as a unit disk with antipodal points on the boundary identified, one gets



$\square$

Problem 2 Show that the **comb space**

$$C \coloneqq (\{0\} \times [0,1]) \cup (\{1/n : n \in \mathbb{N}\} \times [0,1]) \cup ([0,1] \times \{0\}) \subset \mathbb{R}^2$$

is not triangulable.

Proof By investigating the neighborhood of the point $(0, 1/2)$, one sees that C is not locally path-connected. However, any simplicial complex is locally path-connected, so C cannot be homeomorphic to the underlying space of any simplicial complex. $\square$

Problem 3 For any triangulation of a *closed* surface with v vertices, e edges, and f faces, show that

(1) $3f = 2e$.

(2) $e = 3(v - \chi)$.

(3) $v \geqslant \frac{1}{2}\left(\sqrt{49 - 24\chi} + 7\right)$.

Here $\chi \coloneqq v - e + f$ is the **Euler characteristic** of the surface.

Proof (1) Each face has three edges, and each edge is shared by two faces, so $3f = 2e$.

(2) From the definition of χ and the result in (1), we have $\chi = v - e + \frac{2}{3}e = v - \frac{1}{3}e$.

(3) Using the result in (1), we have

$$\begin{aligned}(2v-7)^2 - (49 - 24\chi) &= 4\big(v^2 - 7v + 6\chi\big) \\ &= 4\big[v^2 - 7v + 6(v - e + \tfrac{2e}{3})\big] \\ &= 4\big[\tbinom{v}{2} - e\big].\end{aligned}$$

Since any two vertices can be connected by at most one edge, we have $e \leqslant \binom{v}{2}$, so $(2v-7)^2 \geqslant 49 - 24\chi$. Solving this inequality for v gives the desired result. $\square$

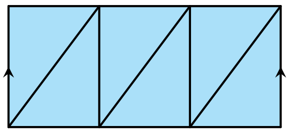
Remark Combining these results, one obtains

$$f = 2v - 2\chi \geqslant 2\left\lceil \frac{1}{2}\left(\sqrt{49 - 24\chi} + 7\right)\right\rceil - 2\chi. \tag{P3–1}$$

The results of Ringel and Jungerman[1] show that the equality in (P3–1) is attained for all closed surfaces except for $\mathbb{T}^2\#\mathbb{T}^2$, $\mathbb{RP}^2\#\mathbb{RP}^2$ (the Klein bottle), and $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. In these three exceptional cases, the minimal number of faces in a triangulation is 24, 16, and 20, respectively. //

Problem 4 Construct a triangulation for the cylinder.

Solution A triangulation for the cylinder is given by



□

Remark Some might think that somewhat simpler choices exist, Figure 1, for example. This is, however, not a triangulation since, for $\sigma_2 = \langle A, B, C\rangle$ and $\sigma_2' = \langle C, D, A\rangle$, we find that $\sigma_2 \cap \sigma_2' = \langle A\rangle \cup \langle C\rangle$ is not a face of either σ_2 or σ_2'.

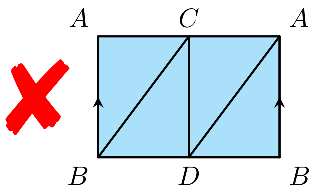
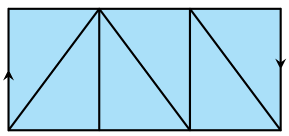


Figure 1: An incorrect triangulation for the cylinder

Problem 5 Construct a triangulation for the Möbius strip.

Solution A triangulation for the Möbius strip is given by

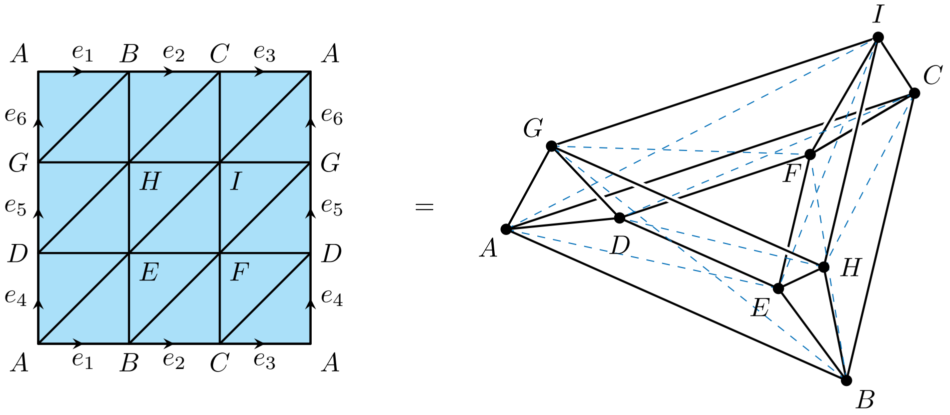


□

Problem 6 Construct a triangulation for the torus $\mathbb{T}^2$.

[1] Ringel, G. Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann. *Math. Ann.* 130, 317–326 (1955). `https://doi.org/10.1007/BF01343898`; Jungerman, M., Ringel, G. Minimal triangulations on orientable surfaces. *Acta Math.* 145, 121–154 (1980). `https://doi.org/10.1007/BF02414187`.

Solution A triangulation for the $\mathbb{T}^2$ is given by



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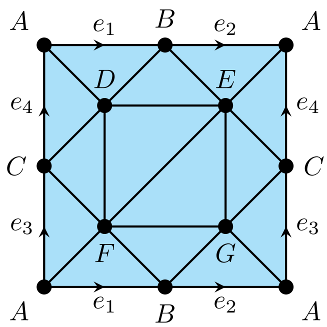
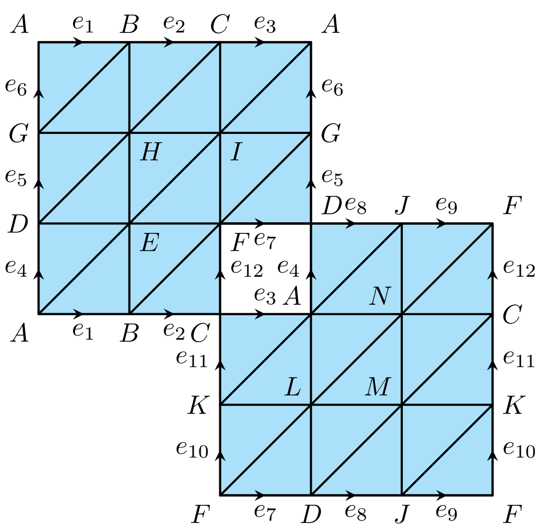
Remark Since $\chi(\mathbb{T}^2) = 0$, by Problem 3 any triangulation of the torus must have at least 7 vertices, 21 edges, and 14 faces. Such a minimal triangulation is given below:



//

Problem 7 Construct a triangulation for the double torus $\mathbb{T}^2\#\mathbb{T}^2$.

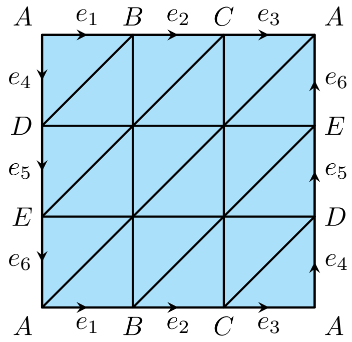
Solution Based on the triangulation of the torus in Problem 6, one can construct a triangulation for the double torus as follows:



□

Problem 8 Construct a triangulation for the Klein bottle.

Solution A triangulation for the Klein bottle is given by



□

Problem 9 Show that the Klein bottle depicted in Problem 8 is homeomorphic to the quotient of the torus of revolution in Problem 6 by the action of the antipodal map.

Proof The torus $\mathbb{T}^2$ in Problem 6 is the quotient space

$$\mathbb{T}^2 = \frac{[0,1]\times[0,1]}{(s,0)\sim(s,1),(0,t)\sim(1,t)}.$$

The action of the antipodal map (see Figure 2) gives on $\mathbb{T}^2$ the equivalence relation

$$(s,t)\sim\begin{cases}(1-s,t+1/2), & \text{if } 0\leqslant t\leqslant 1/2,\\ (1-s,t-1/2), & \text{if } 1/2\leqslant t\leqslant 1.\end{cases}$$

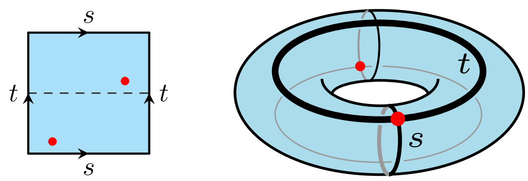
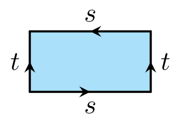


Figure 2: Antipodal map on the torus

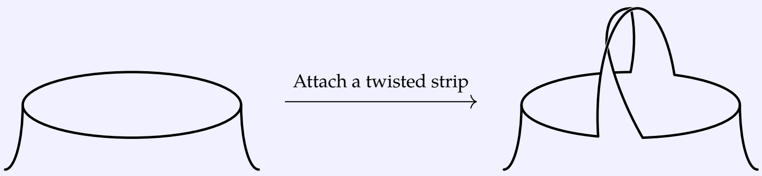
Since every point in the upper half $[0,1]\times[1/2,1]$ is identified with a point in the lower half $[0,1]\times[0,1/2]$, and since those two halves are compact[2], we can rewrite the quotient by discarding the upper half and adjusting the identifications on the edges:

$$\mathbb{T}^2/\sim\; = \frac{[0,1]\times[0,1/2]}{(s,0)\sim(1-s,1/2),(0,t)\sim(1,t)} =$$  $=$ Klein bottle.

□

[2] The restriction of the quotient map to $[0,1]\times[1/2,1]$ is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism.

Problem 10 Let N be a surface without boundary. Show that if one removes an open disk *(whose closure is homeomorphic to a closed disk)* from N and then attaches the ends of a twisted strip to the resulting boundary circle, the resulting surface is homeomorphic to $N\#\mathbb{RP}^2$ with an open disk removed.



N − open disk $\qquad\qquad$ $(N\#\mathbb{RP}^2)$ − open disk

**This result is used in the classification of compact surfaces with boundary via Morse theory.*

Proof We can assume $N = \mathbb{S}^2$ since removing an open disk and attaching a twisted strip are local operations that do not depend on the global topology of N. Then it suffices to show that attaching a twisted strip to a closed disk results in $\mathbb{RP}^2$ with an open disk removed, or equivalently, that if we further attach a disk to the boundary circle of the resulting surface, we obtain $\mathbb{RP}^2$ (see Figure 3).

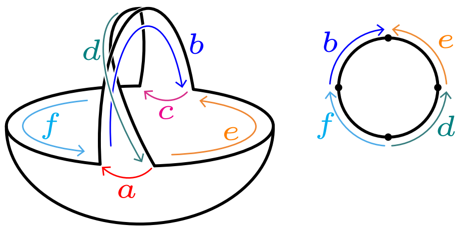
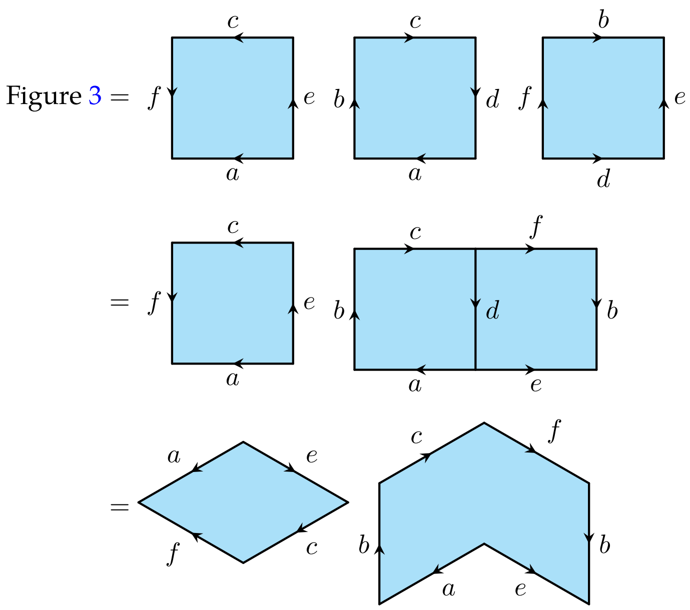
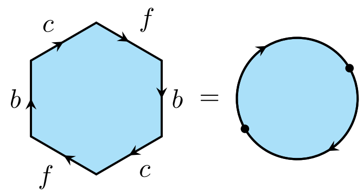


Figure 3: Attaching a disk to the boundary circle of the resulting surface

This can be seen by using polygonal presentations of surfaces:



$=$ 

$= \mathbb{RP}^2.$ $\square$

For a smooth function $f\colon M \to \mathbb{R}$ on a smooth n-manifold M, a point $p \in M$ is called a **critical point** of f if the differential $\mathrm{d}f_p\colon T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is the zero map. In local coordinates $(x^1, \ldots, x^n)$ around p, this is equivalent to the vanishing of all partial derivatives at p:

$$\frac{\partial f}{\partial x^1}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0.$$

The real number $f(p)$ is called a **critical value** of f. A critical point p is called **non-degenerate** if the Hessian

$$\operatorname{Hess} f(p) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)_{1 \leqslant i,j \leqslant n}$$

is invertible. The **index** of a non-degenerate critical point p is the number of negative eigenvalues of $\operatorname{Hess} f(p)$. The degeneracy and index of a critical point are independent of the choice of the local coordinate system used, as shown by Sylvester's law of inertia.

A smooth function $f\colon M \to \mathbb{R}$ is called a **Morse function** if all its critical points are non-degenerate.

Problem 11 (Morse lemma) Let p be a non-degenerate critical point of a smooth function $f\colon M \to \mathbb{R}$ on a smooth n-manifold M, with index λ. Show that there exist local coordinates $(x^1, \ldots, x^n)$ in an open neighborhood U of p such that $x^i(p) = 0$ for all i and

$$f(x) = f(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2.$$

Remark (1) The proof of the Morse lemma can be found in John Milnor's *Morse Theory*, Lemma 2.2.

(2) As a consequence of the Morse lemma, non-degenerate critical points are isolated. //

Problem 12 Find a Morse function on the torus $\mathbb{T}^2$ with exactly 4 critical points.

Solution Embed $\mathbb{T}^2$ in $\mathbb{R}^3$ as a standing torus (see Figure 4) by

$$X(u,v) = ((R + r\cos v)\cos u, r\sin v, (R + r\cos v)\sin u), \quad u, v \in [0, 2\pi),$$

with $R > r > 0$.

The height function is the projection onto the z-axis, which restricted to $\mathbb{T}^2$ is given by

$$h(u,v) = (R + r\cos v)\sin u.$$

The partial derivatives are

$$\frac{\partial h}{\partial u} = (R + r\cos v)\cos u, \quad \frac{\partial h}{\partial v} = -r\sin v \sin u.$$

Thus

$$\nabla h = 0 \iff \cos u = 0 \text{ and } \sin v = 0 \iff u = \frac{\pi}{2} \text{ or } \frac{3\pi}{2},\ v = 0 \text{ or } \pi.$$

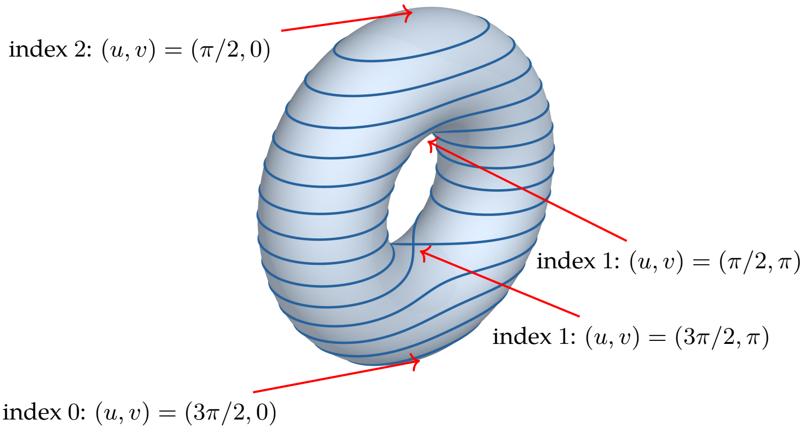


Figure 4: Height function on the standing torus with 4 critical points

The Hessian is

$$\operatorname{Hess} h(u,v)=\begin{pmatrix}-(R+r\cos v)\sin u & -r\sin v\cos u\\ -r\sin v\cos u & -r\cos v\sin u\end{pmatrix}.$$

Evaluating at the critical points, we find that they are all non-degenerate:

$$\operatorname{Hess} h\left(\frac{\pi}{2},0\right)=\begin{pmatrix}-(R+r) & 0\\ 0 & -r\end{pmatrix},\quad \operatorname{Hess} h\left(\frac{\pi}{2},\pi\right)=\begin{pmatrix}-(R-r) & 0\\ 0 & r\end{pmatrix},$$
$$\operatorname{Hess} h\left(\frac{3\pi}{2},0\right)=\begin{pmatrix}R+r & 0\\ 0 & r\end{pmatrix},\quad \operatorname{Hess} h\left(\frac{3\pi}{2},\pi\right)=\begin{pmatrix}R-r & 0\\ 0 & -r\end{pmatrix}.$$

Therefore, h is a Morse function with exactly 4 critical points. □

Remark The Morse property of the height function can be entirely determined by the function's behavior on the fundamental polygon $[0,2\pi]\times[0,2\pi]$ used to represent the torus. Since the function $h(u,v)$ respects the equivalence relations of the torus, one only needs to ensure that the critical points—treated as points in the (u,v)-plane—have a non-vanishing Hessian determinant, effectively ignoring the specific embedding geometry. //

Problem 13 Find a Morse function on the Klein bottle.

Solution Represent the Klein bottle as the square $[0,2\pi]\times[0,2\pi]$ with the identifications $(0,v)\sim(2\pi,v)$ and $(u,0)\sim(2\pi-u,2\pi)$. Consider the function $f(u,v)=\cos u+\cos v$ defined on the square. It respects the equivalence relations of the Klein bottle since

$$f(0,v)=1+\cos v=f(2\pi,v),\qquad f(u,0)=\cos u+1=f(2\pi-u,2\pi).$$

The partial derivatives of f are

$$\frac{\partial f}{\partial u}=-\sin u,\quad \frac{\partial f}{\partial v}=-\sin v.$$

Thus

$$\nabla f=0\iff \sin u=0 \text{ and } \sin v=0\iff u=0 \text{ or } \pi,\ v=0 \text{ or } \pi.$$

The Hessian is

$$\operatorname{Hess} f(u,v) = \begin{pmatrix} -\cos u & 0 \\ 0 & -\cos v \end{pmatrix}.$$

Evaluating at the critical points, we find that they are all non-degenerate. Therefore, f is a Morse function on the Klein bottle. $\square$

Remark For a closed connected n-manifold M, the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable, and $\mathbb{Z}_2$ if M is nonorientable. Hence by Alexander duality, a closed[3] nonorientable n-manifold cannot be embedded as a subspace of $\mathbb{R}^{n+1}$. In particular, the Klein bottle cannot be embedded in $\mathbb{R}^3$. Therefore, unlike the torus, we cannot use a height function derived from an embedding in $\mathbb{R}^3$ to obtain a Morse function on the Klein bottle. However, one can easily verify that the function

$$g\colon \mathbb{R}^2 \to \mathbb{R}^4, \quad (u,v) \mapsto \Big(\sin v \sin \frac{u}{2}, \sin v \cos \frac{u}{2}, (2+\cos v)\sin u, (2+\cos v)\cos u\Big)$$

descends to an embedding[4] of the Klein bottle

$$K = \frac{[0,2\pi]\times[0,2\pi]}{(u,0)\times(u,2\pi), (0,v)\sim(2\pi, 2\pi - v)}$$

into $\mathbb{R}^4$. Composing this embedding with the projection onto the fourth coordinate gives the function $h(u,v) = (2+\cos v)\cos u$, with

$$\nabla h = 0 \iff -(2+\cos v)\sin u = 0 \text{ and } -\cos u \sin v = 0 \iff u = 0 \text{ or } \pi,\ v = 0 \text{ or } \pi.$$

The Hessian is

$$\operatorname{Hess} h(u,v) = \begin{pmatrix} -(2+\cos v)\cos u & \sin u \sin v \\ \sin u \sin v & -\cos u \cos v \end{pmatrix}.$$

Evaluating at the critical points, we find that they are all non-degenerate:

$$\operatorname{Hess} h(0,0) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \quad \operatorname{Hess} h(0,\pi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\operatorname{Hess} h(\pi,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \operatorname{Hess} h(\pi,\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, h is another Morse function on the Klein bottle, with exactly 4 critical points. In particular,

- The index-0 critical point corresponds to height $h(\pi,0) = -3$.
- The two index-1 critical points correspond to heights $h(0,\pi) = 1$ and $h(\pi,\pi) = -1$ (see Figure 5).
- The index-2 critical point corresponds to height $h(0,0) = 3$. //

[3] Here *closed* means compact and without boundary. For example, the Möbius band is not closed, and can be embedded in $\mathbb{R}^3$.

[4] Indeed, to show the injectivity of the induced map, notice that the $(2+\cos v)$ factors in the last two coordinates are always positive, so the ratio of the last two coordinates determines u within $[0,2\pi)$. Then, knowing u, the last two coordinates determine $\cos v$ and the first two coordinates determine $\sin v$, which together determine v within $[0,2\pi)$.

Figure 5: Hyperplane sections[5] of the embedded Klein bottle near index-1 critical points

Problem 14 Prove that the fundamental group of any compact orientable 3-manifold is the free product of fundamental groups of prime 3-manifolds.

Proof According to the Kneser–Milnor prime decomposition theorem for 3-manifolds, any compact orientable 3-manifold is the connected sum of a unique (up to homeomorphism) finite collection of prime 3-manifolds. By induction on the number of summands, it suffices to show that if

$$\pi_1(M\#N) = \pi_1(M) * \pi_1(N)$$

for any two 3-manifolds M and N.

Let $M\#N$ be obtained by removing an open 3-ball from each of M and N and gluing along the resulting boundary 2-spheres. Choose a collar neighborhood $\mathbb{S}^2 \times [0,1]$ of the gluing sphere, and set

$$U = \big(M \setminus \mathbb{B}^3\big) \cup \big(\mathbb{S}^2 \times [0, 3/4)\big), \quad V = \big(N \setminus \mathbb{B}^3\big) \cup \big(\mathbb{S}^2 \times (1/4, 1]\big).$$

Then U and V are open and path-connected, $U \cup V = M\#N$, and $U \cap V$ is homotopy equivalent to $\mathbb{S}^2$, which is simply connected. Moreover, removing a 3-ball does not change the fundamental group[6], so

$$\pi_1(U) \simeq \pi_1(M), \quad \pi_1(V) \simeq \pi_1(N).$$

Now, the desired result follows directly from van Kampen's theorem. $\square$

Problem 15 The **compact-open topology** on the set $C(X,Y)$ of continuous maps from a topological space X to a topological space Y is defined by taking as a subbasis the sets of the form

$$M(K,U) \coloneqq \{f \in C(X,Y) : f(K) \subset U\}$$

where $K \subset X$ is compact and $U \subset Y$ is open. In this problem, we always endow $C(X,Y)$ with the compact-open topology.

(1) The evaluation map

$$\mathrm{ev}\colon X \times C(X,Y) \to Y, \quad (x,f) \mapsto f(x)$$

is continuous if X is **locally compact** *(i.e., for each point $x \in X$ and each neighborhood U of x there is a compact*

[5] By the regular level set theorem, the preimage of a regular value of h is a 1-manifold. A visualization of hyperplane sections of the embedded Klein bottle is available at `https://github.com/Xiaoshuo-Lin/4D-Klein-Sliced`.

[6] Try to prove this fact using van Kampen's theorem.

neighborhood V of x contained in U).

(2) If $f\colon X \times Z \to Y$ is continuous then so is the map $\hat{f}\colon Z \to C(X, Y)$, $\hat{f}(z)(x) = f(x, z)$.

(3) The converse to (2) holds when X is locally compact.

Proof (1) For $(x, f) \in X \times C(X, Y)$ let $U \subset Y$ be an open neighborhood of $f(x)$. Since X is locally compact, continuity of f implies there is a compact neighborhood $K \subset X$ of x such that $f(K) \subset U$. Then $K \times M(K, U)$ is a neighborhood of (x, f) in $X \times C(X, Y)$ taken to U by ev, so ev is continuous at (x, f).

(2) Suppose $f\colon X \times Z \to Y$ is continuous. To show continuity of $\hat{f}$ it suffices to show that for a subbasic set $M(K, U) \subset C(X, Y)$, the set $\hat{f}^{-1}(M(K, U)) = \{z \in Z : f(K, z) \subset U\}$ is open in Z. Let $z \in \hat{f}^{-1}(M(K, U))$. Since $f^{-1}(U)$ is an open neighborhood of the compact set $K \times \{z\}$, by the tube lemma there exist open sets $V \subset X$ and $W \subset Z$ such that

$$K \times \{z\} \subset V \times W \subset f^{-1}(U).$$

So W is a neighborhood of z contained in $\hat{f}^{-1}(M(K, U))$, proving that $\hat{f}^{-1}(M(K, U))$ is open in Z.

(3) Note that $f\colon X \times Z \to Y$ is the composition of Id $\times \hat{f}$ and the evaluation map:

$$\begin{tikzcd} X \times Z \arrow[rr, "f"] \arrow[dr, "\mathrm{Id} \times \hat{f}"'] & & Y \\ & X \times C(X, Y) \arrow[ur, "\mathrm{ev}"'] & \end{tikzcd}$$

So part (1) gives the result. □

Problem 16 Show that if $q\colon X \to Y$ is a quotient map then so is $q \times \mathrm{Id}\colon X \times Z \to Y \times Z$ whenever Z is locally compact.

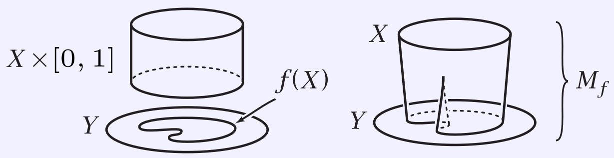
Proof Consider the diagram below, where W is $Y \times Z$ with the quotient topology from $X \times Z$, with g the quotient map and h the identity.

$$\begin{tikzcd} X \times Z \arrow[rr, "q \times \mathrm{Id}"] \arrow[dr, "g"'] & & Y \times Z \arrow[dl, "h"] \\ & W & \end{tikzcd}$$

Every open set in $Y \times Z$ is open in W since $q \times \mathrm{Id}$ is continuous, so it will suffice to show that h is continuous.

Since g is continuous, so is the associated map $\hat{g}\colon X \to C(Z, W)$, by Problem 15 (2). This implies that $\hat{h}\colon Y \to C(Z, W)$ is continuous since q is a quotient map. Applying Problem 15 (3), we conclude that h is continuous. □

Problem 17 For a continuous map $f\colon X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times [0, 1]) \sqcup Y$ obtained by identifying each $(x, 1) \in X \times [0, 1]$ with $f(x) \in Y$.



Show that Y is a deformation retract of M_f, where we identify Y with its image in M_f.

Proof Let $W = (X \times [0,1]) \sqcup Y$ be the disjoint union before taking the quotient. Define the continuous map $\widetilde{H}\colon W \times [0,1] \to W$ by

$$\begin{aligned} \widetilde{H}((x,s),t) &= (x, s(1-t)+t), \quad \text{for } (x,s) \in X \times [0,1], \\ \widetilde{H}(y,t) &= y, \quad \text{for } y \in Y. \end{aligned}$$

Let $q\colon W \to M_f$ be the quotient map defining the mapping cylinder. By Problem 16, the map $q \times \mathrm{Id}\colon W \times [0,1] \to M_f \times [0,1]$ is a quotient map since $[0,1]$ is locally compact.

$$\begin{tikzcd} W \times [0,1] \arrow[r, "\widetilde{H}"] \arrow[d, "q \times \mathrm{Id}"'] & W \arrow[d, "q"] \\ M_f \times [0,1] \arrow[r, dashed, "H"'] & M_f \end{tikzcd}$$

Note that $q \circ \widetilde{H}$ is constant on the fibers of $q \times \mathrm{Id}$:

$$q\big(\widetilde{H}((x,1),t)\big) = q(x,1) = [f(x)] = q(f(x)) = q\big(\widetilde{H}(f(x),t)\big), \quad \text{for } x \in X \text{ and } t \in [0,1].$$

Thus, by the universal property of quotient maps, there is a unique continuous map $H\colon M_f \times [0,1] \to M_f$ such that $H \circ (q \times \mathrm{Id}) = q \circ \widetilde{H}$. This map H is the desired deformation retraction of M_f onto Y since

$$\begin{aligned} H([x,s],0) &= q\big(\widetilde{H}((x,s),0)\big) = [x,s] \quad \text{for } x \in X \text{ and } s \in [0,1], \\ H([x,s],1) &= q\big(\widetilde{H}((x,s),1)\big) = [x,1] = [f(x)] \in Y \quad \text{for } x \in X \text{ and } s \in [0,1], \\ H([y],t) &= q\big(\widetilde{H}(y,t)\big) = [y] \in Y \quad \text{for } y \in Y \text{ and } t \in [0,1]. \end{aligned}$$

□

Problem 18 For a continuous map $f\colon X \to X$, the **mapping torus** T_f is the quotient space of $X \times [0,1]$ obtained by identifying each $(x,1)$ with $(f(x),0)$ for $x \in X$.

Suppose X is path-connected with basepoint x_0 and $f\colon X \to X$ is a homeomorphism with $f(x_0) = x_0$. Moreover, assume that x_0 has a contractible open neighborhood N in X, and let $\tau_0 = [x_0, 1/2] \in T_f$ be the basepoint of T_f. Show that the fundamental group of the mapping torus T_f is a semidirect product:

$$\pi_1(T_f, \tau_0) \simeq \pi_1(X, x_0) \rtimes_{f_*} \mathbb{Z}.$$

Proof Denote by $q\colon X \times [0,1] \to T_f$ the quotient map defining the mapping torus. Consider the open cover of T_f given by the sets U and V defined below:

$$U = q(X \times (0,1)), \quad V = q\big((X \times [0,1/3)) \cup (X \times (2/3,1]) \cup (N \times [0,1])\big).$$

Then U and V are path-connected, $U \cup V = T_f$, and $U \cap V$ is homotopy equivalent to the wedge sum of two copies of X. Thus,

$$\pi_1(U\cap V,\tau_0)\simeq \pi_1(X,x_0)*\pi_1(X,x_0)=:G_1*G_2,$$

where G_1 represents loops in the lower part (near 0) and G_2 represents loops in the upper part (near 1).

It is clear that $\pi_1(U,\tau_0)\simeq\pi_1(X,x_0)$ since U deformation retracts onto $X\times\{1/2\}$. For V, we can decompose it as the union of $q(N\times[0,1])$ (which is homeomorphic to $N\times\mathbb{S}^1$) and $q\big((X\times[0,1/3))\cup(X\times(2/3,1])\big)$. Define

$$\Phi\colon q\big((X\times[0,1/3))\cup(X\times(2/3,1])\big)\to X\times(2/3,4/3),\quad [x,t]\mapsto\begin{cases}(x,t+1), & t\in[0,1/3),\\ (f(x),t), & t\in(2/3,1].\end{cases}$$

Note that $\Phi([x,1])=(f(x),1)=\Phi([f(x),0])$, so Φ is well-defined and continuous. Since f is a homeomorphism, Φ has an inverse given by

$$\Psi\colon X\times(2/3,4/3)\to q\big((X\times[0,1/3))\cup(X\times(2/3,1])\big),\quad (y,s)\mapsto\begin{cases}[f^{-1}(y),s], & s\in(2/3,1],\\ [y,s-1], & s\in[1,4/3).\end{cases}$$

The continuity of Ψ follows from the gluing lemma. Thus, Φ is a homeomorphism. Moreover, the intersection of $q(N\times[0,1])$ and $q\big((X\times[0,1/3))\cup(X\times(2/3,1])\big)$ is contractible since N is contractible. Therefore, by van Kampen's theorem,

$$\pi_1(V,\tau_0)\simeq\pi_1(X,x_0)*\mathbb{Z}.$$

Now, the inclusion map $i_U\colon U\cap V\to U$ induces the homomorphism[7]

$$(i_U)_*\colon G_1*G_2\to\pi_1(X,x_0),\quad g\mapsto\begin{cases}g, & g\in G_1,\\ f_*(g), & g\in G_2.\end{cases}$$

Here $f_*\colon\pi_1(X,x_0)\to\pi_1(X,x_0)$ is the isomorphism induced by the basepoint-preserving homeomorphism f. Similarly, the inclusion map $i_V\colon U\cap V\to V$ induces the homomorphism[8]

$$(i_V)_*\colon G_1*G_2\to\pi_1(X,x_0)*\mathbb{Z},\quad g\mapsto\begin{cases}g, & g\in G_1,\\ tgt^{-1}, & g\in G_2,\end{cases}$$

where t is a generator of the $\mathbb{Z}$ factor in $\pi_1(V,\tau_0)$.

By van Kampen's theorem, the fundamental group $\pi_1(T_f,\tau_0)$ is the free product $\pi_1(U,\tau_0)*\pi_1(V,\tau_0)$ modulo the relations $(i_U)_*(\omega)=(i_V)_*(\omega)$ for all $\omega\in G_1*G_2$.

- From G_1, we get $g=g$, which identifies the $\pi_1(X,x_0)$ factors in $\pi_1(U,\tau_0)$ and $\pi_1(V,\tau_0)$.

- From G_2, we get $f_*(g)=tgt^{-1}$.

[7] To see why $(i_U)_*(g)=f_*(g)$ for $g\in G_2$, observe that an element $g\in G_2$ is represented by a loop γ in $X\times\{1-\varepsilon\}$. As $\varepsilon\to 0$, the quotient map q identifies the points $(x,1)$ with $(f(x),0)$. Thus, the loop at the top of the cylinder U is identified with the loop $f\circ\gamma$ at the bottom of the cylinder.

[8] With the decomposition of V into two parts as above, to reach the upper copy of X from our basepoint at $t=0$, we must travel through the bridge $q(N\times[0,1])$. This path is precisely the generator t.

Therefore,[9]

$$\pi_1(T_f,\tau_0)\simeq\left\langle \pi_1(X,x_0),t \,\middle|\, tgt^{-1}=f_*(g),\forall g\in\pi_1(X,x_0)\right\rangle\simeq\pi_1(X,x_0)\rtimes_{f_*}\mathbb{Z}. \qquad \square$$

Remark (1) Since $f\colon X\to X$ is a homeomorphism, there is a natural projection

$$p\colon T_f\to\mathbb{S}^1,\quad [x,t]\mapsto e^{2\pi i t},$$

which makes T_f into a fiber bundle over $\mathbb{S}^1$ with fiber X. If we identify the fiber $p^{-1}(1)$ with X, then the long exact sequence of homotopy groups for this fibration yields

$$\cdots\longrightarrow\pi_2\left(\mathbb{S}^1,1\right)\longrightarrow\pi_1(X,x_0)\xrightarrow{i_*}\pi_1(T_f,[x_0,0])\xrightarrow{p_*}\pi_1\left(\mathbb{S}^1,1\right)\longrightarrow\pi_0(X)$$

where $i\colon X\to T_f$ is the inclusion map. Since $\pi_2\left(\mathbb{S}^1,1\right)=0$ and $\pi_0(X)=0$ (as X is path-connected), we have the short exact sequence

$$0\longrightarrow\pi_1(X,x_0)\xrightarrow{i_*}\pi_1(T_f,[x_0,0])\xrightarrow{p_*}\mathbb{Z}\longrightarrow 0$$

Choose $t\in\pi_1(T_f,[x_0,0])$ mapping to the generator $1\in\mathbb{Z}$. The action of $\mathbb{Z}=\pi_1\left(\mathbb{S}^1,1\right)$ on $\pi_1(X,x_0)$ is given by monodromy: lifting the generator of $\pi_1\left(\mathbb{S}^1,1\right)$ to a path in T_f transports loops in the fiber by the map f. Thus, this short exact sequence splits, and we recover the semidirect product structure as before.

(2) Let us illustrate the result with some examples:

- When $f=\mathrm{Id}_X$, the mapping torus T_f is homeomorphic to the product space $X\times\mathbb{S}^1$, and f_* is the identity automorphism of $\pi_1(X)$. In this case,

$$\pi_1(T_f)\simeq\langle\pi_1(X),t\,|\,tg=gt,\forall g\in\pi_1(X)\rangle\simeq\pi_1(X)\times\mathbb{Z}\simeq\pi_1(X)\times\pi_1\left(\mathbb{S}^1\right).$$

- When $X=\mathbb{S}^1\subset\mathbb{C}$ and $f\colon\mathbb{S}^1\to\mathbb{S}^1$ is the conjugation map $f(z)=\bar{z}$, the mapping torus T_f is homeomorphic to the Klein bottle K, and f_* acts on $\pi_1\left(\mathbb{S}^1\right)$ by inversion. In this case,

$$\pi_1(T_f)\simeq\left\langle s,t\,\middle|\,tst^{-1}=s^{-1}\right\rangle\simeq\pi_1(K). \qquad /\!\!/$$

[9] Suppose that we are given a group G with a normal subgroup N and a subgroup H, such that every element $g\in G$ may be written uniquely in the form $g=nh$ where $n\in N$ and $h\in H$. Let $\varphi\colon H\to\mathrm{Aut}(N)$ be the homomorphism (written $\varphi(h)=\varphi_h$) given by $\varphi_h(n)=hnh^{-1}$ for all $n\in N$ and $h\in H$. Then G is isomorphic to the semidirect product $N\rtimes_\varphi H$. The isomorphism $\lambda\colon G\to N\rtimes_\varphi H$ is well-defined by $\lambda(g)=(n,h)$ where $g=nh$ due to the uniqueness of the decomposition. In G, we have

$$(n_1h_1)(n_2h_2)=n_1h_1n_2\left(h_1^{-1}h_1\right)h_2=(n_1\varphi_{h_1}(n_2))(h_1h_2).$$

Thus, for $g_1=n_1h_1$ and $g_2=n_2h_2$ in G, we obtain

$$\lambda(g_1g_2)=\lambda(n_1\varphi_{h_1}(n_2)h_1h_2)=(n_1\varphi_{h_1}(n_2),h_1h_2)=(n_1,h_1)\bullet(n_2,h_2)=\lambda(g_1)\bullet\lambda(g_2),$$

which shows that λ is a homomorphism. Since λ is clearly bijective, it is an isomorphism.

Problem 19 Let $f \in C(\mathbb{S}^1, \mathbb{S}^1)$ and $p\colon \mathbb{R} \to \mathbb{S}^1$ be the covering map given by $p(t) = e^{2\pi \mathrm{i} t}$.

$$\begin{tikzcd}
& & \mathbb{R} \arrow[d, "p"] \\
\mathbb{R} \arrow[r, "p"] \arrow[urr, dashed, "\tilde{f}"] & \mathbb{S}^1 \arrow[r, "f"] & \mathbb{S}^1
\end{tikzcd}$$

Since $\mathbb{R}$ is simply connected, the map $f \circ p\colon \mathbb{R} \to \mathbb{S}^1$ lifts to a continuous map $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$. Any two such lifts differ by an integer. The **degree** of f is defined to be the integer $\deg(f) = \tilde{f}(1) - \tilde{f}(0)$.

(1) If $f \in C(\mathbb{S}^1, \mathbb{S}^1)$ is not surjective, then $\deg(f) = 0$.

(2) If $f, g \in C(\mathbb{S}^1, \mathbb{S}^1)$, then $\deg(f \circ g) = \deg(f)\deg(g)$.

(3) Two maps $f, g \in C(\mathbb{S}^1, \mathbb{S}^1)$ are homotopic if and only if $\deg(f) = \deg(g)$.

Proof (1) If $f \in C(\mathbb{S}^1, \mathbb{S}^1)$ is not surjective, then $\left|\tilde{f}(1) - \tilde{f}(0)\right| < 1$ by the intermediate value theorem, so $\deg(f) = 0$.

(2) Using the commutative diagram above, we have

$$p\Big(\tilde{f}\big(\tilde{g}(t)\big)\Big) = f(p(\tilde{g}(t))) = f(g(p(t))) = (f \circ g)(p(t)), \quad \forall t \in \mathbb{R}.$$

Thus, $\tilde{f} \circ \tilde{g}$ is a lift of $f \circ g$. Note that the map $\tilde{f}(t+1) - \tilde{f}(t)$ is a continuous map from $\mathbb{R}$ to $\mathbb{Z}$, so it must be the constant map with value $\deg(f)$. Therefore, $\tilde{f}(t+k) - \tilde{f}(t) = k\deg(f)$ for $k \in \mathbb{Z}$, and

$$\begin{aligned}
\deg(f \circ g) &= \tilde{f} \circ \tilde{g}(1) - \tilde{f} \circ \tilde{g}(0) = \tilde{f}(\tilde{g}(0) + \deg(g)) - \tilde{f}(\tilde{g}(0)) \\
&= \tilde{f}(\tilde{g}(0)) + \deg(g)\deg(f) - \tilde{f}(\tilde{g}(0)) = \deg(f)\deg(g).
\end{aligned}$$

(3) Fix $x_0 = 1 \in \mathbb{S}^1$. Let $T_\theta(t) = t + \theta$ be the translation map on $\mathbb{R}$ by $\theta \in \mathbb{R}$, and let $R_\theta\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation map given by $R_\theta\big(e^{2\pi \mathrm{i} t}\big) = e^{2\pi \mathrm{i}(t+\theta)}$. It is clear that T_θ is a lift of R_θ. Given $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, let θ be such that $R_\theta \circ f(x_0) = x_0$. Since R_θ is homotopic to the identity map, it follows that f is homotopic to $R_\theta \circ f$. Now, $T_\theta \circ \tilde{f}$ is a lift of $R_\theta \circ f$ satisfying

$$T_\theta \circ \tilde{f}(1) - T_\theta \circ \tilde{f}(0) = \tilde{f}(1) - \tilde{f}(0),$$

so $\deg(R_\theta \circ f) = \deg(f)$. Similarly, any map $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is homotopic to a map $R_\phi \circ g$ with $R_\phi \circ g(x_0) = x_0$ and $\deg(R_\phi \circ g) = \deg(g)$. Thus, it suffices to show that two maps $f, g \in C(\mathbb{S}^1, \mathbb{S}^1)$ with $f(x_0) = g(x_0) = x_0$ are homotopic if and only if $\deg(f) = \deg(g)$.

($\Rightarrow$) Let $H\colon [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ be a homotopy from f to g.

$$\begin{tikzcd}
& & \mathbb{R} \arrow[d, "p"] \\
{[0,1] \times \mathbb{R}} \arrow[r, "\mathrm{Id} \times p"'] \arrow[urr, dashed, "\widetilde{H}"] & {[0,1] \times \mathbb{S}^1} \arrow[r, "H"'] & \mathbb{S}^1
\end{tikzcd}$$

Since $[0,1] \times \mathbb{R}$ is simply connected, the map $H \circ (\mathrm{Id} \times p)$ lifts to a continuous map $\widetilde{H}\colon [0,1] \times$

$\mathbb{R} \to \mathbb{R}$. Note that the map $\widetilde{H}(s, t+1) - \widetilde{H}(s, t)$ is a continuous map from $[0,1] \times \mathbb{R}$ to $\mathbb{Z}$, so it must be the constant map with value $\deg(f)$ when $s = 0$ and $\deg(g)$ when $s = 1$. Since $[0,1] \times \mathbb{R}$ is connected, we have $\deg(f) = \deg(g)$.

($\Leftarrow$) If $\deg(f) = \deg(g)$, then we can choose lifts $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f}(0) = \tilde{g}(0)$ and $\tilde{f}(1) = \tilde{g}(1)$. Define the homotopy

$$\widetilde{H} \colon [0,1] \times \mathbb{R} \to \mathbb{R}, \quad (s, t) \mapsto (1-s)\tilde{f}(t) + s\tilde{g}(t).$$

Since $\widetilde{H}$ satisfies $\widetilde{H}(s, t+1) - \widetilde{H}(s, t) = \deg(f) = \deg(g) \in \mathbb{Z}$ for all $(s, t) \in [0,1] \times \mathbb{R}$, it descends to a continuous homotopy $H \colon [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ between f and g. $\square$

Returning to the context of Problem 15, part (2) of that problem implies that there is a well-defined map $C(X \times Z, Y) \to C(Z, C(X, Y))$ sending f to $\hat{f}$. This is injective, and part (3) implies that it is surjective if X is locally compact.

Problem 20 The map $C(X \times Z, Y) \to C(Z, C(X, Y))$, $f \mapsto \hat{f}$, is a homeomorphism if X is locally compact Hausdorff and Z is Hausdorff.

Proof First we show that a subbasis for $C(X \times Z, Y)$ is formed by the sets $M(A \times B, U)$ as A and B range over compact sets in X and Z respectively, and U ranges over open sets in Y. Given a compact $K \subset X \times Z$ and a continuous map $f \in M(K, U)$, let K_X and K_Z be the projections of K onto X and Z. Then $K_X \times K_Z$ is compact Hausdorff and hence normal. A normal space has the property that for each closed set C and each open set O containing C there is another open set O' containing C whose closure is contained in O. To see this, apply the normality property to the two closed sets C and the complement C' of O, taking O' to be the resulting open set containing C and disjoint from an open set containing C', so the closure of O' is contained in O. Applying this observation to the normal space $K_X \times K_Z$ with C a point $k \in K$ and $O = (K_X \times K_Z) \cap f^{-1}(U)$, the result is an open neighborhood of k in $K_X \times K_Z$ whose closure is contained in $f^{-1}(U)$. We can take this open neighborhood to be a product $V_k \times W_k \subset K_X \times K_Z$, so its closure is a compact neighborhood $A_k \times B_k \subset f^{-1}(U)$ of k in $K_X \times K_Z$. The sets $V_k \times W_k$ for varying $k \in K$ form an open cover of the compact set K, so a finite number of the products $A_k \times B_k$ cover K. After discarding the others we then have

$$f \in \bigcap_k M(A_k \times B_k, U) \subset M(K, U),$$

which shows that the sets $M(A \times B, U)$ form a subbasis for $C(X \times Z, Y)$ as claimed.

Under the bijection $C(X \times Z, Y) \to C(Z, C(X, Y))$, the sets $M(A \times B, U)$ correspond to the sets $M(B, M(A, U))$, so it will suffice to show that the latter sets form a subbasis for $C(Z, C(X, Y))$. We will show more generally that for any space Q a subbasis for $C(Z, Q)$ is formed by the sets $M(K, V)$ as V ranges over a subbasis for Q and K ranges over compact sets in Z, assuming that Z is Hausdorff. Then we let $Q = C(X, Y)$ with subbasis the sets $M(A, U)$.

Given $f \in M(K, U)$ with K compact in Z and U open in Q, write U as a union of basic sets U_α with each U_α an intersection of finitely many sets $V_{\alpha,j}$ of the given subbasis for Q. The cover of K by the open sets $f^{-1}(U_\alpha)$ has a finite subcover, say by the open sets $f^{-1}(U_i)$. Since K is compact Hausdorff, hence normal, we can write K as a union of compact subsets K_i with $K_i \subset f^{-1}(U_i)$, namely, each $k \in K$ has a compact neighborhood K_k contained in some $f^{-1}(U_i)$ with $k \in f^{-1}(U_i)$, so compactness of K implies that finitely many of these sets K_k cover K and we let K_i be the union of those contained in $f^{-1}(U_i)$.

Now f lies in $M(K_i, U_i) = M(K_i, \cap_j V_{ij}) = \bigcap_j M(K_i, V_{ij})$ for each i. Hence,

$$f \in \bigcap_{i,j} M(K_i, V_{ij}) = \bigcap_i M(K_i, U_i) \subset M(K, U).$$

Since $\bigcap_{i,j} M(K_i, V_{ij})$ is a finite intersection, this shows that the sets $M(K, V)$ form a subbasis for $C(Z, Q)$ as claimed. □

Let M be a topological manifold and $\mathrm{Homeo}(M)$ be the group of homeomorphisms of M endowed with the compact-open topology (as in Problem 15). The path-component of the identity map Id_M for this topology is denoted as $\mathrm{Homeo}_0(M)$. Since manifolds are always locally compact Hausdorff, and $[0, 1]$ is Hausdorff, Problem 20 gives a homeomorphism

$$C(M \times [0, 1], M) \simeq C([0, 1], C(M, M)).$$

Thus, $\mathrm{Homeo}_0(M)$ is a normal subgroup of $\mathrm{Homeo}(M)$, consisting of all homeomorphisms of M that are *isotopic* to the identity.

The **mapping class group** of M is defined as the quotient group

$$\mathrm{MCG}(M) := \mathrm{Homeo}(M) / \mathrm{Homeo}_0(M).$$

In other words, $\mathrm{MCG}(M)$ is the group of homeomorphism of M onto itself, modulo **isotopy**.

Problem 21 Show that $\mathrm{MCG}(\mathbb{S}^1) \simeq \mathbb{Z}_2$.

Proof By Problem 19 (2), for any $f \in \mathrm{Homeo}(\mathbb{S}^1)$, we have

$$1 = \deg(\mathrm{Id}_{\mathbb{S}^1}) = \deg(f \circ f^{-1}) = \deg(f) \deg(f^{-1}).$$

This implies $\deg(f) = \pm 1$. Using Problem 19 (2) again, we see that there is a group epimorphism

$$\deg\colon \mathrm{Homeo}(\mathbb{S}^1) \to \{\pm 1\}, \quad f \mapsto \deg(f).$$

By Problem 19 (3), $\mathrm{Homeo}_0(\mathbb{S}^1) \subset \ker(\deg)$. Conversely, if $f \in \ker(\deg)$, then it admits a lift $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for all $x \in \mathbb{R}$. Since $f \in \mathrm{Homeo}(\mathbb{S}^1)$, $\tilde{f}$ is a strictly increasing homeomorphism of $\mathbb{R}$. For each $t \in [0, 1]$, define $\tilde{f}_t(x) = (1-t)x + t\tilde{f}(x)$ where $x \in \mathbb{R}$. Then each $\tilde{f}_t$ is strictly increasing, and

$$\begin{aligned}
\tilde{f}_t(x+1) &= (1-t)(x+1) + t\tilde{f}(x+1) \\
&= (1-t)(x+1) + t\left[\tilde{f}(x) + 1\right] \\
&= \tilde{f}_t(x) + 1.
\end{aligned}$$

Thus, each $\tilde{f}_t$ is a homeomorphism of $\mathbb{R}$ and descends to a continuous bijection $f_t\colon \mathbb{S}^1 \to \mathbb{S}^1$. Since $\mathbb{S}^1$ is compact Hausdorff, each f_t is a homeomorphism. This shows that f is isotopic to the identity map, so $f \in \mathrm{Homeo}_0(\mathbb{S}^1)$. Therefore, by the first isomorphism theorem,

$$\mathrm{MCG}(\mathbb{S}^1) = \mathrm{Homeo}(\mathbb{S}^1) / \mathrm{Homeo}_0(\mathbb{S}^1) \simeq \{\pm 1\} \simeq \mathbb{Z}_2.$$ □