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Problem 1 (Maps to $\mathbb{S}^n$)

(1) Prove that any non-surjective continuous map $f\colon X\to\mathbb{S}^n$ is null-homotopic.

(2) Let $f,g\colon X\to\mathbb{S}^n$ be continuous maps. Suppose they are never antipodal, i.e., $g(x)\neq -f(x)$ holds for all x. Prove that f is homotopic to g.

(3) Prove that $f\in C(X,Y)$ is null-homotopic if and only if f has a continuous extension $F\in C(C(X),Y)$, where $C(X)$ denotes the cone over X.

(4) Let $\mathbb{D}^{n+1}$ be the closed unit ball in $\mathbb{R}^{n+1}$. Prove that there exists a retraction $f\in C(\mathbb{D}^{n+1},\mathbb{S}^n)$ if and only if $\mathrm{Id}_{\mathbb{S}^n}$ is null-homotopic.

Proof (1) Without loss of generality, assume $N=(0,\ldots,0,1)\notin f(X)$ and consider the stereographic projection

$$\sigma\colon \mathbb{S}^n\setminus\{N\}\to\mathbb{R}^n,\quad (x_1,\ldots,x_{n+1})\mapsto\frac{1}{1-x_{n+1}}(x_1,\ldots,x_n).$$

It is a homeomorphism with inverse

$$\sigma^{-1}\colon\mathbb{R}^n\to\mathbb{S}^n\setminus\{N\},\quad (u_1,\ldots,u_n)\mapsto\frac{1}{|u|^2+1}\big(2u_1,\ldots,2u_n,|u|^2-1\big).$$

Then $F(t,x)=\sigma^{-1}((1-t)\sigma(f(x)))$ is a homotopy from f to the constant map $x\mapsto(0,\ldots,0,-1)$.

(2) Since $f(x)$ and $g(x)$ are never antipodal, their convex combination $(1-t)f(x)+tg(x)$ is never zero. Thus the map $H(t,x)=\dfrac{(1-t)f(x)+tg(x)}{\|(1-t)f(x)+tg(x)\|}$ is well-defined and is a homotopy from f to g.

(3) Let us specify $C(X)=([0,1]\times X)/(\{1\}\times X)$.

(⇒) Let $H\colon[0,1]\times X\to Y$ be a homotopy from f to a constant map c_{y_0}, for some $y_0\in Y$. Then $H(0,x)=f(x)$ and $H(1,x)=y_0$ for all $x\in X$. Since H is constant on the subspace $\{1\}\times X$, it induces a continuous map $\widetilde{H}\colon C(X)\to Y$ which agrees with f on $\{0\}\times X$. Thus $\widetilde{H}$ is an extension of f to $C(X)$.

(⇐) Suppose $F\in C(C(X),Y)$ is an extension of f. Then the map

$$H\colon[0,1]\times X\to Y\quad (t,x)\mapsto F([(t,x)])$$

is a homotopy from f to the constant map $x\mapsto F([(1,x)])$.

(4) (⇒) Suppose $f\in C(\mathbb{D}^{n+1},\mathbb{S}^n)$ is a retraction. Then the map

$$F\colon[0,1]\times\mathbb{S}^n\to\mathbb{S}^n,\quad (t,x)\mapsto f((1-t)x)$$

is a homotopy from $\mathrm{Id}_{\mathbb{S}^n}$ to the constant map $x\mapsto f(0)\in\mathbb{S}^n$.

(⇐) Suppose $F\colon[0,1]\times\mathbb{S}^n\to\mathbb{S}^n$ is a homotopy from $\mathrm{Id}_{\mathbb{S}^n}$ to a constant map. Fix any point $p_0\in\mathbb{S}^n$. Then the map $f\in C(\mathbb{D}^{n+1},\mathbb{S}^n)$, defined by

$$f(x)=\begin{cases}F(1,p_0), & x=0,\\ F\Big(1-\|x\|,\frac{x}{\|x\|}\Big), & x\neq 0,\end{cases}$$

is a retraction. It is continuous since $F(1,x)$ is constant for all $x \in \mathbb{S}^n$. $\square$

Problem 2 Suppose $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a map which is not homotopic to the identity. Show that $f(x) = -x$ for some $x \in \mathbb{S}^1$.

Proof This is an immediate corollary of Problem 1 (2), by taking $g = \mathrm{Id}_{\mathbb{S}^1}$. $\square$

Problem 3 Show that any two continuous maps $f, g\colon X \to C(Y)$ are homotopic, where $C(Y)$ denotes the cone over Y.

Proof It suffices to show that any continuous map $f\colon X \to C(Y)$ is null-homotopic, and by Problem 1 (3), this is equivalent to showing that f has a continuous extension $F \in C(C(X), C(Y))$.

Since $C(Y) = ([0,1] \times Y)/(\{1\} \times Y)$, we may write $f(x) = [(t(x), y(x))]$ for some continuous maps $t\colon X \to [0,1]$ and $y\colon X \to Y$. Now define the continuous map

$$F\colon C(X) = ([0,1] \times X)/(\{1\} \times X) \to C(Y), \quad [(s,x)] \mapsto [((1-s)t(x) + s, y(x))].$$

$\boxed{F \text{ is well-defined}}$ We have $F([(1,x)]) = [1, y(x)]$ which is the cone point of $C(Y)$, for all $x \in X$.

$\boxed{F|_{\{0\}\times X} = f}$ We have $F([(0,x)]) = [(t(x), y(x))] = f(x)$ for all $x \in X$.

Thus F is a continuous extension of f to $C(X)$, and the proof is complete. $\square$

Problem 4 Let X be a path-connected space. Recall that any path $\lambda\colon [0,1] \to X$ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$ induces a group isomorphism

$$\Gamma_\lambda\colon \pi_1(X, x_0) \to \pi_1(X, x_1), \quad \langle\gamma\rangle \mapsto \langle\lambda^{-1} * \gamma * \lambda\rangle.$$

Prove that $\pi_1(X, x_0)$ is abelian if and only if for any two paths λ_1, λ_2 from x_0 to x_1, we have $\Gamma_{\lambda_1} = \Gamma_{\lambda_2}$.

Proof ($\Rightarrow$) If there exist two paths λ_1, λ_2 from x_0 to x_1 such that $\Gamma_{\lambda_1}(\langle\gamma\rangle) \neq \Gamma_{\lambda_2}(\langle\gamma\rangle)$ for some $\langle\gamma\rangle \in \pi_1(X, x_0)$, i.e., $\langle\lambda_1^{-1} * \gamma * \lambda_1\rangle \neq \langle\lambda_2^{-1} * \gamma * \lambda_2\rangle$, then

$$\langle\lambda_2 * \lambda_1^{-1} * \gamma\rangle = \langle\lambda_2\rangle\langle\lambda_1^{-1} * \gamma * \lambda_1\rangle\langle\lambda_1^{-1}\rangle \neq \langle\lambda_2\rangle\langle\lambda_2^{-1} * \gamma * \lambda_2\rangle\langle\lambda_1^{-1}\rangle = \langle\gamma * \lambda_2 * \lambda_1^{-1}\rangle.$$

This shows that the elements $\langle\gamma\rangle$ and $\langle\lambda_2 * \lambda_1^{-1}\rangle$ do not commute, a contradiction.

($\Leftarrow$) For any two loops γ_1, γ_2 based at x_0, by assumption we have $\Gamma_{\gamma_1}(\langle\gamma_2\rangle) = \Gamma_{\gamma_2}(\langle\gamma_2\rangle)$, i.e.,

$$\langle\gamma_1^{-1} * \gamma_2 * \gamma_1\rangle = \langle\gamma_2^{-1} * \gamma_2 * \gamma_2\rangle = \langle\gamma_2\rangle.$$

Thus $\langle\gamma_1\rangle\langle\gamma_2\rangle = \langle\gamma_1\rangle\langle\gamma_1^{-1} * \gamma_2 * \gamma_1\rangle = \langle\gamma_2\rangle\langle\gamma_1\rangle$. $\square$

Problem 5 Suppose X is connected, and $p\colon \widetilde{X} \to X$ be a covering map. Show that the cardinality of the fiber $p^{-1}(x)$ is the same for all $x \in X$.

Proof Any $x \in X$ has an open neighborhood U_x homeomorphic to an open neighborhood V_α in $\widetilde{X}$ with $p(V_\alpha) = U_x$. For any $y \in U_x$, since $p^{-1}(y) \cap V_\alpha$ contains exactly one point, we have $|p^{-1}(y)| = |p^{-1}(x)|$.

Fix $x_0 \in X$ and let $A = \{x \in X : |p^{-1}(x)| = |p^{-1}(x_0)|\}$. Then both A and A^c are open. Since X is connected and $A \neq \varnothing$, we must have $A = X$, i.e., the cardinality of $p^{-1}(x)$ is the same for all $x \in X$. $\square$

Remark One can compare the above proof with the proof of Theorem 10.14 in the textbook. //

Problem 6 If $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are two covering maps, show that $p \times q\colon \widetilde{X} \times \widetilde{Y} \to X \times Y$ is also a covering map.

Proof Given $(x, y) \in X \times Y$, let U be an open neighborhood of x in X such that $p^{-1}(U)$ is a disjoint union of open sets V_α in $\widetilde{X}$ and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism for each α. Similarly, let U' be an open neighborhood of y in Y such that $q^{-1}(U')$ is a disjoint union of open sets V'_β in $\widetilde{Y}$ and $q|_{V'_\beta}\colon V'_\beta \to U'$ is a homeomorphism for each β. Then

- ◇ $U \times U'$ is an open neighborhood of (x, y) in $X \times Y$.
- ◇ $(p \times q)^{-1}(U \times U') = p^{-1}(U) \times q^{-1}(U')$ is a disjoint union of open sets $V_\alpha \times V'_\beta$ in $\widetilde{X} \times \widetilde{Y}$.
- ◇ $(p \times q)|_{V_\alpha \times V'_\beta} = (p|_{V_\alpha}) \times (q|_{V'_\beta})$ is a homeomorphism for all α, β. □

Remark It is then natural to ask whether the product of infinitely many covering maps is still a covering map. The answer is **negative** in general. To see this, one first needs to prove the following formula for the fundamental group of an arbitrary product. For an arbitrary collection of path-connected spaces X_α there is a group isomorphism (try it yourself!)

$$\pi_1\left(\prod_\alpha X_\alpha\right) \simeq \prod_\alpha \pi_1(X_\alpha). \tag{P6–1}$$

Now, let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the standard covering map. We claim that the infinite product

$$P = \prod_{n=1}^\infty p\colon \prod_{n=1}^\infty \mathbb{R} \to \prod_{n=1}^\infty \mathbb{S}^1$$

is **not** a covering map. Indeed, were it a covering map, then some open neighborhood U of $(0, 0, \ldots)$ in $\prod_{n=1}^\infty \mathbb{R}$ would be mapped by P homeomorphically to some neighborhood V of $(1, 1, \ldots)$ in $\prod_{n=1}^\infty \mathbb{S}^1$. By shrinking we can assume $U = (-a_1, a_1) \times \cdots \times (-a_k, a_k) \times \mathbb{R} \times \mathbb{R} \times \cdots$, so that U is contractible, and $P(U)$ has infinitely many $\mathbb{S}^1$ factors. By (P6–1),

$$\{e\} = \pi_1(U) \simeq \pi_1(P(U)) \simeq \prod_{n=1}^k \pi_1(p(-a_n, a_n)) \times \prod_{n=k+1}^\infty \pi_1(\mathbb{S}^1) \neq \{e\},$$

which is a contradiction. //

We say two paths $\gamma_0, \gamma_1\colon [0,1] \to X$ are path-homotopic, denoted as $\gamma_1 \underset{p}{\sim} \gamma_2$, if

- ◇ $\gamma_0(0) = \gamma_1(0) = x_0$, $\gamma_0(1) = \gamma_1(1) = x_1$, and
- ◇ there exists a continuous map $F\colon [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} F(0,t) &= \gamma_0(t) \quad \text{and} \quad F(1,t) = \gamma_1(t), \quad \forall t \in [0,1], \\ F(s,0) &= x_0 \quad \text{and} \quad F(s,1) = x_1, \quad \forall s \in [0,1]. \end{aligned}$$

Problem 7 (Fundamental group of a topological group) Let $(G, \bullet)$ be a path-connected topological group. We want to prove that $\pi_1(G, e)$ is an abelian group. Let γ_1, γ_2 be two loops in G based at e.

(1) (First proof) Denote by γ_e the constant loop at e. Verify that

$$F(s,t) = (\gamma_1 * \gamma_e)\big(\max\{0, t - \tfrac{s}{2}\}\big) \bullet (\gamma_e * \gamma_2)\big(\min\{1, t + \tfrac{s}{2}\}\big)$$

is a path-homotopy between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$, where $\bullet$ is the group multiplication.

(2) (Second proof) Construct explicit path-homotopies to verify that

① $\gamma_1(t) \bullet \gamma_2(t) \underset{p}{\sim} \gamma_2(t) \bullet \gamma_1(t)$.

② $(\gamma_1 * \gamma_2)(t) \underset{p}{\sim} \gamma_1(t) \bullet \gamma_2(t)$.

(3) (Third proof, the **Eckmann–Hilton argument**)

① Let S be a set on which there are two "semigroup with unitary" structures, $(S, \circ, 1_\circ)$ and $(S, \bullet, 1_\bullet)$. Moreover, suppose

$$(g \circ h) \bullet (g' \circ h') = (g \bullet g') \circ (h \bullet h'), \quad \forall g, g', h, h' \in S.$$

Prove that $1_\circ = 1_\bullet$, $g \bullet h = g \circ h$, and $g \circ h = h \circ g$.

② Define $\langle\gamma_1\rangle_p \bullet \langle\gamma_2\rangle_p = \langle\gamma_1 \bullet \gamma_2\rangle_p$. Show that $\bullet$ is well-defined on $\pi_1(G, e)$.

③ Use ① to prove that $\pi_1(G)$ is abelian.

Proof (1) Clearly $F(s,t) \in C([0,1] \times [0,1], G)$.

When $s = 0$, we have

$$\begin{aligned} F(0,t) &= (\gamma_1 * \gamma_e)(t) \bullet (\gamma_e * \gamma_2)(t) \\ &= \begin{cases} \gamma_1(2t) \bullet \gamma_e(2t) = \gamma_1(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_e(2t-1) \bullet \gamma_2(2t-1) = \gamma_2(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\ &= \gamma_1 * \gamma_2(t). \end{aligned} \tag{P7–1}$$

When $s = 1$, we have

$$\begin{aligned} F(1,t) &= (\gamma_1 * \gamma_e)\big(\max\{0, t - \tfrac{1}{2}\}\big) \bullet (\gamma_e * \gamma_2)\big(\min\{1, t + \tfrac{1}{2}\}\big) \\ &= \begin{cases} (\gamma_1 * \gamma_e)(0) \bullet (\gamma_e * \gamma_2)\big(t + \frac{1}{2}\big) = \gamma_2(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ (\gamma_1 * \gamma_e)\big(t - \frac{1}{2}\big) \bullet (\gamma_e * \gamma_2)(1) = \gamma_1(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\ &= \gamma_2 * \gamma_1(t). \end{aligned}$$

(2) ① Consider $F(s,t) = (\gamma_1(ts))^{-1} \bullet \gamma_1(t) \bullet \gamma_2(t) \bullet \gamma_1(ts)$. We have

$$F(0,t) = \gamma_1(t) \bullet \gamma_2(t), \quad F(1,t) = \gamma_2(t) \bullet \gamma_1(t).$$

② Consider $F(s,t)=(\gamma_1*\gamma_e)\big(t(1-\frac{s}{2})\big)\bullet(\gamma_e*\gamma_2)\big(t(1-\frac{s}{2})+\frac{s}{2}\big)$. We have

$$\begin{aligned} F(0,t)&=(\gamma_1*\gamma_e)(t)\bullet(\gamma_e*\gamma_2)(t)\xlongequal{\text{(P7–1)}}\gamma_1*\gamma_2(t),\\ F(1,t)&=(\gamma_1*\gamma_e)\big(\tfrac{t}{2}\big)\bullet(\gamma_e*\gamma_2)\big(\tfrac{t+1}{2}\big)=\gamma_1(t)\bullet\gamma_2(t). \end{aligned}$$

(3) ① The units of the two operations coincide:

$$1_\circ=1_\circ\circ 1_\circ=(1_\bullet\bullet 1_\circ)\circ(1_\circ\bullet 1_\bullet)=(1_\bullet\circ 1_\circ)\bullet(1_\circ\circ 1_\bullet)=1_\bullet\bullet 1_\bullet=1_\bullet.$$

For any $g,h\in S$, we have

$$\begin{aligned} \boxed{g\circ h}&=(1\bullet g)\circ(h\bullet 1)=(1\circ h)\bullet(g\circ 1)=\boxed{h\bullet g}\\ &=(h\circ 1)\bullet(1\circ g)=(h\bullet 1)\circ(1\bullet g)=\boxed{h\circ g}. \end{aligned}$$

② Suppose γ_i $(i=1,2,3,4)$ are loops in G based at e such that $\gamma_1\underset{p}{\sim}\gamma_2$ and $\gamma_3\underset{p}{\sim}\gamma_4$. Let F,G be their respective path-homotopies. Then $F\bullet H$ is a path-homotopy between $\gamma_1\bullet\gamma_3$ and $\gamma_2\bullet\gamma_4$.

③ Now $(\pi_1(G,e),*,\gamma_e)$ and $(\pi_1(G,e),\bullet,\gamma_e)$ are two "semigroup with unitary" structures on $\pi_1(G,e)$. Moreover, for any $\langle\gamma_i\rangle\in\pi_1(G,e)$ $(i=1,2,3,4)$, we have

$$(\langle\gamma_1\rangle*\langle\gamma_2\rangle)\bullet(\langle\gamma_3\rangle*\langle\gamma_4\rangle)=(\langle\gamma_1\rangle\bullet\langle\gamma_3\rangle)*(\langle\gamma_2\rangle\bullet\langle\gamma_4\rangle).$$

By ①, $\pi_1(G)\simeq(\pi_1(G,e),*,\gamma_e)$ is abelian. □