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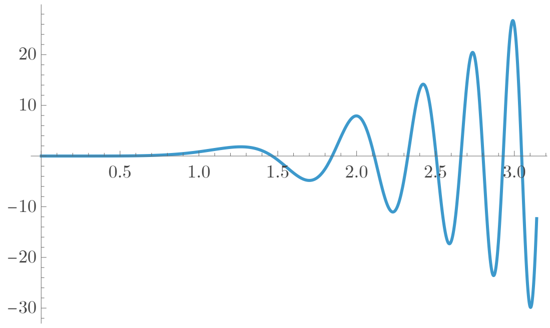
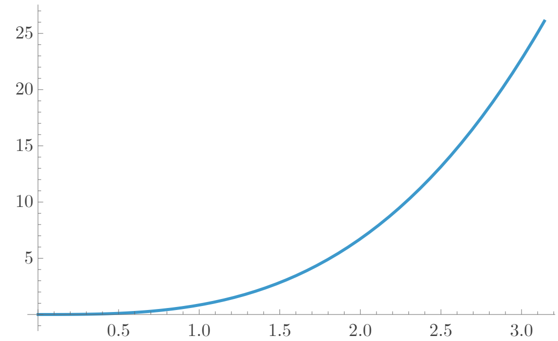
Problem 1 Define $\Gamma_{\alpha,\beta} = \left\{\left(x, x^\alpha \sin x^\beta\right) : x > 0\right\} \subset \mathbb{R}^2$ for $\alpha, \beta \in \mathbb{R}$. Determine the closure $\overline{\Gamma_{\alpha,\beta}}$.

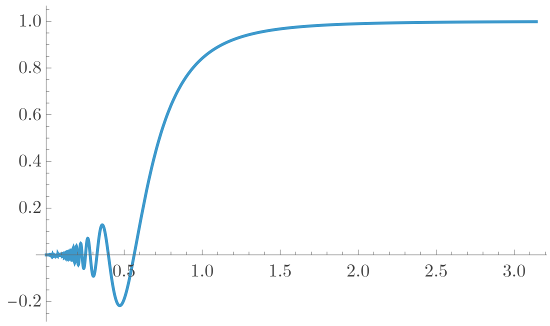
Solution Let us analyze the behavior of the curve $\Gamma_{\alpha,\beta}$ as x approaches 0. Note that the *amplitude* is scaled by α, and the *oscillation frequency* is determined by β.



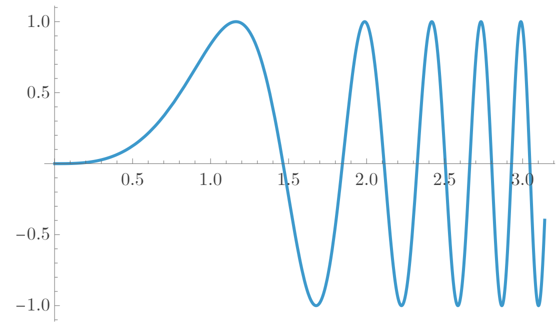
$\alpha > 0$ and $\beta > 0$



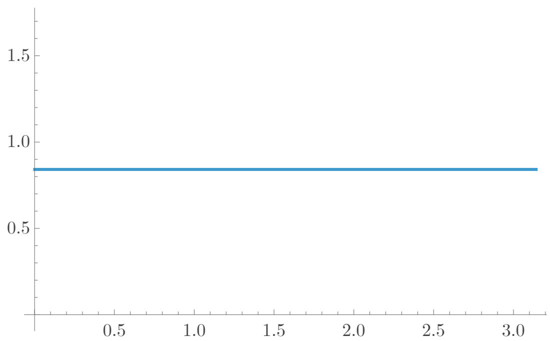
$\alpha > 0$ and $\beta = 0$



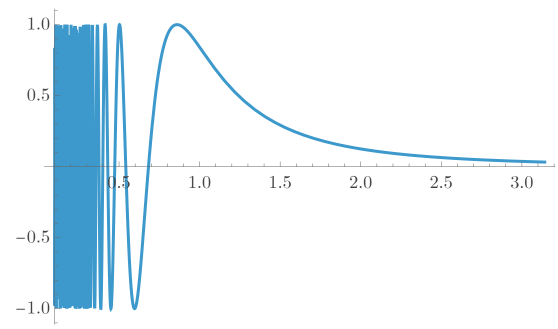
$\alpha > 0$ and $\beta < 0$



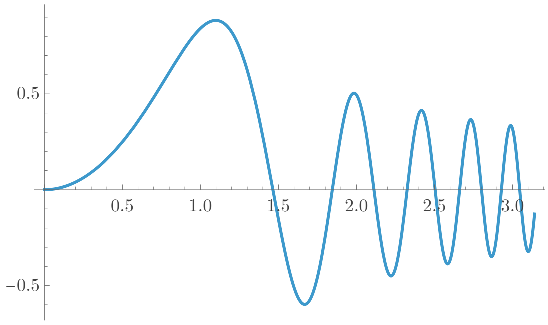
$\alpha = 0$ and $\beta > 0$



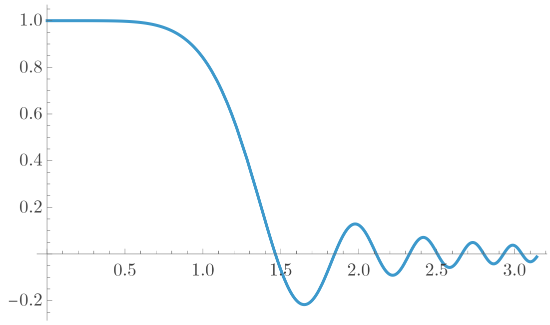
$\alpha = 0$ and $\beta = 0$



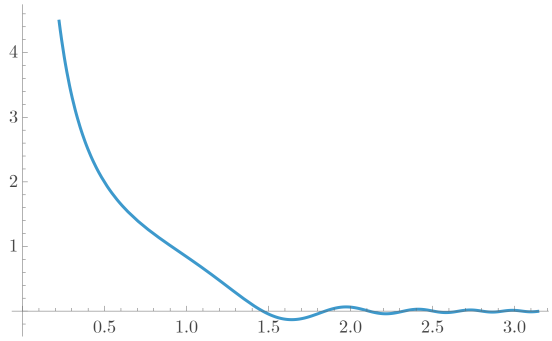
$\alpha = 0$ and $\beta < 0$



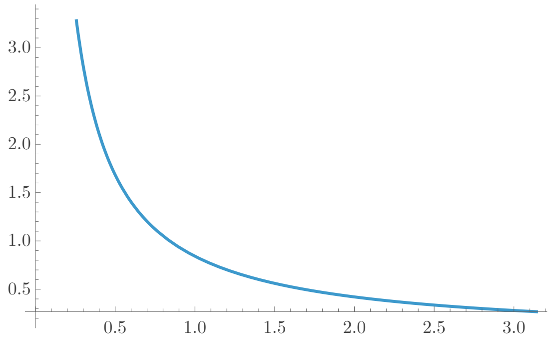
$\alpha < 0, \beta > 0$ and $\alpha + \beta > 0$



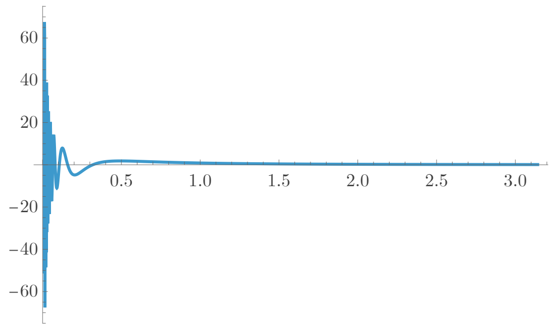
$\alpha < 0, \beta > 0$ and $\alpha + \beta = 0$



$\alpha < 0, \beta > 0$ and $\alpha + \beta < 0$



$\alpha < 0$ and $\beta = 0$



$\alpha < 0$ and $\beta < 0$

From the above plots, we can summarize the closure $\overline{\Gamma_{\alpha,\beta}}$ as follows:

$$\overline{\Gamma_{\alpha,\beta}} = \Gamma_{\alpha,\beta} \cup \begin{cases} \{(0,0)\}, & \text{if } \begin{cases} \alpha > 0, \\ \text{or } \alpha = 0, \beta > 0, \\ \text{or } \alpha < 0, \alpha + \beta > 0, \end{cases} \\ \{(0,y) : |y| \leqslant 1\}, & \text{if } \alpha = 0, \beta < 0, \\ \{(0, \sin 1)\}, & \text{if } \alpha = 0, \beta = 0, \\ \{(0,1)\}, & \text{if } \alpha < 0, \alpha + \beta = 0, \\ \{(0,y) : y \in \mathbb{R}\}, & \text{if } \alpha < 0, \beta < 0, \\ \varnothing, & \text{if } \begin{cases} \alpha < 0, \beta > 0, \alpha + \beta < 0, \\ \text{or } \alpha < 0, \beta = 0. \end{cases} \end{cases}$$

□

Problem 2 Construct a space-filling curve that fills out all of the unit cube $[0,1]^3$ in $\mathbb{R}^3$.

Solution 1 Denote $I=[0,1]$ and let $f\colon I\to I^2$ be a space-filling curve (cf. Section 2.3). By identifying I^4 with $I^2\times I^2$, we can construct a surjection

$$g\colon I^2\to I^4,\quad (x,y)\mapsto (f(x),f(y)).$$

Now let π be the projection from I^4 to I^3 by omitting the last coordinate. Then the composition

$$\pi\circ g\circ f\colon I\to I^3$$

is a space-filling curve filling out the entire unit cube I^3, since it is a composition of continuous surjections. $\square$

Solution 2 Let $f\colon I\to I^2$ be a space-filling curve. We claim that the map

$$(f\times \mathrm{Id})\circ f\colon I\to I^3=I^2\times I$$

is the desired space-filling curve. Indeed, for an arbitrary point $((a,b),c)\in I^2\times I$, there is $u\in I$ with $f(u)=(a,b)$. Again by surjectivity there is $v\in I$ with $f(v)=(u,c)$. Then

$$(f\times \mathrm{Id})\circ f(v)=(f\times \mathrm{Id})(u,c)=(f(u),c)=((a,b),c).$$ $\square$

Remark (1) There are also constructive methods to explicitly define such space-filling curves, e.g., the 3-dimensional Hilbert curve (cf. 常庚哲、史济怀《数学分析教程》15.8.2 节).

(2) Some students took the map

$$0.t_1t_2t_3\ldots\mapsto(0.t_1t_4t_7\ldots,0.t_2t_5t_8\ldots,0.t_3t_6t_9\ldots)\quad\text{in base 3}$$

as a space-filling curve from I to I^3. However, this map is *not* continuous at points like $1/3$, since the left limit at $1/3=0.100\ldots_{(3)}$ is $(0.022\ldots,0.222\ldots,0.222\ldots)_{(3)}=(1/3,1,1)$, while the right limit is $(0.100\ldots,0.000\ldots,0.000\ldots)_{(3)}=(1/3,0,0)$.

Problem 3 Show that the Hausdorff condition in the following statement cannot be omitted:

Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Proof Take $X=(\{0,1\},\mathscr{T}_{\text{discrete}})$ and $Y=(\{0,1\},\mathscr{T}_{\text{trivial}})$. Then X is compact, and Y is not Hausdorff. The identity map $\mathrm{Id}\colon X\to Y$ is a continuous bijection, but it is not an open map, hence not a homeomorphism. $\square$

Problem 4 Show that any space-filling curve $f\colon I\to I^n$ ($n\geqslant 2$) cannot be injective. Here $I=[0,1]$.

Proof Since I is compact and I^n is Hausdorff, the continuous bijection $f\colon I\to I^n$ is a homeomorphism if it is injective. But I would be disconnected if we remove three distinct points from it, while I^n ($n\geqslant 2$) remains connected after removing any three points. This is a contradiction. $\square$

Problem 5 (The Sorgenfrey line) On the set $X = \mathbb{R}$, define

$$\mathscr{T}_{\text{Sorgenfrey}} = \{U \subset \mathbb{R} : \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } [x, x+\varepsilon) \subset U\}.$$

(1) Check that $\mathscr{T}_{\text{Sorgenfrey}}$ is a topology.

(2) Prove that every left-closed-right-open interval $[a, b)$ is both open and closed. Then show that $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is Hausdorff.

(3) Prove that every open interval (a, b) is still open with respect to $\mathscr{T}_{\text{Sorgenfrey}}$.

(4) What is the meaning of convergence in $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$?

(5) Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is *right continuous* if $\lim\limits_{x_n \to x_0^+} f(x_n) = f(x_0)$. Prove that a function $f: \mathbb{R} \to \mathbb{R}$ is right continuous if and only if the map $f: (\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}) \to (\mathbb{R}, \mathscr{T}_{\text{usual}})$ is continuous. So people also call $\mathscr{T}_{\text{Sorgenfrey}}$ the *right continuous topology*.

(6) Show that there is no metric d on $\mathbb{R}$ such that $\mathscr{T}_{\text{Sorgenfrey}}$ is the metric topology $\mathscr{T}_d$.

Proof (1) ① Clearly $\varnothing, \mathbb{R} \in \mathscr{T}_{\text{Sorgenfrey}}$.

② If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}_{\text{Sorgenfrey}}$, then for any $x \in \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, there exists $\lambda \in \Lambda$ such that $x \in U_\lambda$, and hence there exists $\varepsilon > 0$ such that $[x, x+\varepsilon) \subset U_\lambda \subset \bigcup\limits_{\alpha \in \Lambda} U_\alpha$. Therefore $\bigcup\limits_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}_{\text{Sorgenfrey}}$.

③ If $U_1, U_2 \in \mathscr{T}_{\text{Sorgenfrey}}$, then for any $x \in U_1 \cap U_2$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $[x, x+\varepsilon_1) \subset U_1$ and $[x, x+\varepsilon_2) \subset U_2$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then $[x, x+\varepsilon) \subset U_1 \cap U_2$. Therefore $U_1 \cap U_2 \in \mathscr{T}_{\text{Sorgenfrey}}$.

(2) ① For any $x \in [a, b)$, since $\varepsilon = b - x > 0$ satisfies $[x, x+\varepsilon) \subset [a, b)$, we see that $[a, b)$ is open.

② For any $x \notin [a, b)$, if $x < a$, then $\varepsilon = a - x > 0$ satisfies $[x, x+\varepsilon) \cap [a, b) = \varnothing$; if $x \geqslant b$, then $\varepsilon = 1 > 0$ satisfies $[x, x+\varepsilon) \cap [a, b) = \varnothing$. Therefore $[a, b)$ is closed.

For any $x, y \in (\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ with $x < y$, the open sets $[x, y)$ and $[y, y+1)$ are disjoint and contain x and y respectively. Hence $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is Hausdorff.

(3) For any $x \in (a, b)$, since $\varepsilon = b - x > 0$ satisfies $[x, x+\varepsilon) \subset (a, b)$, we see that (a, b) is open.

(4) If $x_n \to x_0$ in $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$, then for any $\varepsilon > 0$, since $[x_0, x_0+\varepsilon)$ is open and contains x_0, it must contain all but finitely many x_n. Conversely, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in [x_0, x_0+\varepsilon)$ for all $n > N$, then for any open set U containing x_0, by the definition of $\mathscr{T}_{\text{Sorgenfrey}}$ we can choose $\varepsilon_0 > 0$ such that $[x_0, x_0+\varepsilon_0) \subset U$. Thus $x_n \in U$ for all sufficiently large n. Therefore a sequence $\{x_n\}_{n=1}^\infty$ converges to x_0 in $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ if and only if it "approaches x_0 from the right".

(5) ($\Rightarrow$) Suppose $f: \mathbb{R} \to \mathbb{R}$ is right continuous and U is any open subset of $(\mathbb{R}, \mathscr{T}_{\text{usual}})$. For any $x_0 \in f^{-1}(U)$, since $f: \mathbb{R} \to \mathbb{R}$ is right continuous at x_0, there exists $\varepsilon > 0$ such that $f(x) \in U$ for all $x \in [x_0, x_0+\varepsilon)$, i.e., $[x_0, x_0+\varepsilon) \subset f^{-1}(U)$. Therefore $f^{-1}(U) \in \mathscr{T}_{\text{Sorgenfrey}}$, which means f is continuous.

($\Leftarrow$) Suppose the map $f: (\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}) \to (\mathbb{R}, \mathscr{T}_{\text{usual}})$ is continuous. For any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, since $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ is open in $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$, there exists $\delta > 0$ such that

$[x_0, x_0+\delta) \subset f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$. Thus $x_n \to x_0^+$ implies $f(x_n) \to f(x_0)$ in $(\mathbb{R}, \mathscr{T}_{\text{usual}})$, i.e., $f: \mathbb{R} \to \mathbb{R}$ is right continuous.

(6) Suppose that $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is a metrizable space and let d be a metric on $\mathbb{R}$ inducing the topology $\mathscr{T}_{\text{Sorgenfrey}}$. For each $x \in \mathbb{R}$, since the interval $[x, x+1)$ is open by (2), we can choose $\varepsilon_x > 0$ such that $\mathbb{B}_d(x, \varepsilon_x) \subset [x, x+1)$. For each $n \in \mathbb{N}$, let $M_n = \left\{x \in \mathbb{R} : \varepsilon_x \geqslant \frac{1}{n}\right\}$. For distinct $x, y \in M_n$ with $x < y$, we have

$$\mathbb{B}_d\left(y, \tfrac{1}{n}\right) \subset \mathbb{B}_d(y, \varepsilon_y) \subset [y, y+1),$$

and since $x \notin [y, y+1)$, we get $x \notin \mathbb{B}_d\left(y, \frac{1}{n}\right)$. Thus

$$d(x, y) \geqslant \frac{1}{n}. \tag{P5–1}$$

On the other hand, by the definition of $\mathscr{T}_{\text{Sorgenfrey}}$, for each $x \in M_n$, there exists $\eta > 0$ such that $[x, x+\eta) \subset \mathbb{B}_d\left(x, \frac{1}{2n}\right)$. Let $r_x \in \mathbb{Q} \cap [x, x+\eta)$, then

$$d(x, r_x) < \frac{1}{2n}. \tag{P5–2}$$

From (P5–1) and (P5–2), we see that for distinct $x, y \in M_n$, the corresponding r_x, r_y are distinct. Now the coutability of $\mathbb{Q}$ implies that M_n is countable for each $n \in \mathbb{N}$, and hence $\mathbb{R} = \bigcup\limits_{n \in \mathbb{N}} M_n$ is countable, which is a contradiction. □

Problem 6 (The Sorgenfrey plane) Consider the product of two Sorgenfrey lines,

$$\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right) := \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right) \times \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right),$$

which is known as the *Sorgenfrey plane*.

(1) Is it Hausdorff?

(2) Consider the subspace $A = \{(x, -x) : x \in \mathbb{R}\}$. Is it closed? What is the induced subspace topology on A?

Proof (1) By Problem 5 (2) and Theorem 3.14, $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is also Hausdorff.

(2) By Problem 5 (3), the topology $\mathscr{T}_{\text{Sorgenfrey}}$ is finer than the usual topology on $\mathbb{R}$. It follows that $\mathscr{T}_{\text{Sorgenfrey}}$ is finer than the usual topology on $\mathbb{R}^2$. Since A is closed in the usual topology of $\mathbb{R}^2$, it is also closed in $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$. For any $(x, -x) \in A$, the open set $[x, x+1) \times [-x, -x+1)$ of $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ intersects A only at $(x, -x)$, so $\{(x, -x)\}$ is open in A. Therefore the induced subspace topology on A is discrete. □