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Problem 1 (Pasting lemmas) Let X, Y be topological spaces. Consider a map $f\colon X \to Y$.

(1) Suppose $X = A \cup B$, where A, B are both closed subsets in X. Suppose $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous. Prove that $f\colon X \to Y$ is continuous.

(2) Prove that if $X = \bigcup\limits_{\alpha} U_\alpha$, where each U_α is open in X, and if each $f|_{U_\alpha}\colon U_\alpha \to Y$ is continuous, then f is continuous.

(3) Show that the same result fails for $X = \bigcup\limits_{n=1}^{\infty} A_n$, where each A_n is closed in X.

(4) Let A_α be a family of closed subsets in X with $X = \bigcup\limits_{\alpha} A_\alpha$, and suppose the family is **locally finite**, i.e., each point $p \in X$ has a neighborhood U_p that intersects finitely many A_α's. Prove that if each $f|_{A_\alpha}$ is continuous, then f is continuous.

Proof (1) It suffices to show that the preimage of each closed subset $K \subset Y$ is closed in X. Since $f^{-1}(K) \cap A = (f|_A)^{-1}(K)$ is closed in A and $f^{-1}(K) \cap B = (f|_B)^{-1}(K)$ is closed in B, and A, B are both closed in X, these two preimages are both closed in X. It follows that

$$f^{-1}(K) = f^{-1}(K) \cap (A \cup B) = (f|_A)^{-1}(K) \cup (f|_B)^{-1}(K)$$

is closed in X. Therefore f is continuous.

(2) Let $V \subset Y$ be any open subset. Any point $x \in f^{-1}(V)$ has an open neighborhood U_x on which f is continuous. Continuity of $f|_{U_x}$ implies, in particular, that $(f|_{U_x})^{-1}(V)$ is an open subset of U_x, and is therefore also an open subset of X. Then

$$(f|_{U_x})^{-1}(V) = \{y \in U_x : f(x) \in V\} = f^{-1}(V) \cap U_x$$

is an open neighborhood of x contained in $f^{-1}(V)$, hence $f^{-1}(V)$ is open in X. Therefore f is continuous.

(3) Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the subspace topology inherited from $\mathbb{R}$. Let $A_1 = \{0\}$ and $A_{n+1} = \left\{\frac{1}{n}\right\}$ for $n \geqslant 1$. Then $X = \bigcup\limits_{n=1}^{\infty} A_n$ and each A_n is closed in X. Take $f = \mathbb{1}_{\{0\}}$. Then f is not continuous at 0. However, $f|_{A_n}$ is continuous for all $n \geqslant 0$.

(4) Given $x \in X$ there is an open neighborhood U_x of x such that U_x intersects only finitely many A_α's, say $A_1, \ldots, A_n$. Note that for each $i = 1, \ldots, n$, $U_x \cap A_i$ is closed in U_x. Since $U_x = \bigcup\limits_{i=1}^{n} (U_x \cap A_i)$ and each $f|_{U_x \cap A_i}$ is continuous, an inductive version of (1) shows that $f|_{U_x}$ is continuous. Then $X = \bigcup\limits_{x \in X} U_x$ where each U_x is open in X and each $f|_{U_x}$ is continuous, so f is continuous by (2). $\square$

Problem 2 Prove that $\mathrm{O}(n)$ is homeomorphic to $\mathrm{SO}(n) \times \mathbb{Z}_2$. Are these two isomorphic as topological groups?

Proof Fix $R \in \mathrm{O}(n) \setminus \mathrm{SO}(n)$. Define the map

$$f\colon \mathrm{SO}(n) \times \mathbb{Z}_2 \to \mathrm{O}(n), \quad (A, b) \mapsto \begin{cases} A, & \text{if } b = 0, \\ RA, & \text{if } b = 1. \end{cases}$$

(Surjectivity) Let $M \in \mathrm{O}(n)$. If $\det(M) = 1$, then $M \in \mathrm{SO}(n)$ and $f(M, 0) = M$. If $\det(M) = -1$, then $\det(R^{-1}M) = 1$, so $R^{-1}M \in \mathrm{SO}(n)$ and $f(R^{-1}M, 1) = M$.

(Injectivity) Suppose $f(A, b) = f(C, d)$. If $b = d$, then $A = C$. If $b \neq d$, then without loss of generality assume $b = 0$, $d = 1$, so that $A = RC$. But then $\det(A) = \det(RC) = -\det(C)$, contradicting the fact that both $A, C \in \mathrm{SO}(n)$. Thus $b = d$ and $A = C$.

Since both $\mathrm{SO}(n) \times \{0\}$ and $\mathrm{SO}(n) \times \{1\}$ are closed in $\mathrm{SO}(n) \times \mathbb{Z}_2$, and $f|_{\mathrm{SO}(n)\times\{0\}}$ and $f|_{\mathrm{SO}(n)\times\{1\}}$ are continuous, by Problem 1 (1) f is continuous. Now, f is a continuous bijection from a compact space to a Hausdorff space, whence a homeomorphism.

For the second question, we need to check whether there is a group homomorphism between these two groups that is also a homeomorphism.

- When n is odd, $\det(I_n) = (-1)^n = -1$. So we can take $R = I_n$ in the definition of f. Then f is clearly a group isomorphism, hence a topological group isomorphism.

- When n is even, suppose there is a group isomorphism $g\colon \mathrm{SO}(n) \times \mathbb{Z}_2 \to \mathrm{O}(n)$. Since $\{I_n\} \times \mathbb{Z}_2$ lies in the center[1] of $\mathrm{SO}(n) \times \mathbb{Z}_2$, its image $g(\{I_n\} \times \mathbb{Z}_2)$ lies in the center of $\mathrm{O}(n)$, which is $\{\pm I_n\}$[2]. Thus $g(I_n, 1) = -I_n \in \mathrm{SO}(n)$. Then for any $A \in \mathrm{SO}(n)$, we have

$$g(A, 1) = g(I_n, 1)g(A, 0) = -g(A, 0),$$

which implies that both $g(A, 0)$ and $g(A, 1)$ lie in $\mathrm{SO}(n)$, and hence $\mathrm{Im}(g) \subset \mathrm{SO}(n) \subsetneq \mathrm{O}(n)$. This contradicts the surjectivity of g.

Therefore, these two are isomorphic as topological groups if and only if n is odd. □

Remark There are examples of topological groups that are isomorphic as ordinary groups but not as topological groups. Indeed, any non-discrete topological group is also a topological group when considered with the discrete topology. The underlying groups are the same, but as topological groups there is not an isomorphism. //

Problem 3 Let $\{A_\alpha\}$ be a family of subsets of X that is **locally finite**, i.e., if for any $x \in X$, there exists an open neighborhood of U_x of x so that $A_\alpha \cap U_x \neq \varnothing$ for only finitely many α's.

(1) Prove that $\{\overline{A_\alpha}\}$ is a locally finite family.

(2) Prove that $\bigcup_\alpha \overline{A_\alpha} = \overline{\bigcup_\alpha A_\alpha}$.

Proof (1) For any $x \in X$, there exists an open neighborhood U_x of x so that $A_\alpha \cap U_x \neq \varnothing$ for only finitely many α's. Suppose $\overline{A_\beta} \cap U_x \neq \varnothing$ for some β, we shall show that $A_\beta \cap U_x \neq \varnothing$.

- If U_x contains a point of A_β, then we are done.

[1] In abstract algebra, the center of a group G is the set of elements that commute with every element of G.
[2] https://math.stackexchange.com/questions/554957

◇ If U_x contains a limit point of A_β, then U_x is a neighborhood of this limit point, and hence it must contain some point in A_β.

This shows that $\{\overline{A_\alpha}\}$ is locally finite.

(2) It suffices to prove that $\overline{\bigcup_\alpha A_\alpha} \subset \bigcup_\alpha \overline{A_\alpha}$, and note that it is enough to show that $\bigcup_\alpha \overline{A_\alpha}$ is closed. Take any $x \notin \bigcup_\alpha \overline{A_\alpha}$. By the local finiteness of $\{\overline{A_\alpha}\}$, there exists an open neighborhood U of x so that $\overline{A_\alpha} \cap U \neq \varnothing$ for only finitely many α's, say $\alpha_1, \ldots, \alpha_n$. Then

$$\widetilde{U} := U \setminus \left(\overline{A_{\alpha_1}} \cup \cdots \cup \overline{A_{\alpha_n}}\right)$$

is an open neighborhood of x that does not intersect any $\overline{A_\alpha}$ and contains x. Thus the complement of $\bigcup_\alpha \overline{A_\alpha}$ is open, i.e., $\bigcup_\alpha \overline{A_\alpha}$ is closed, and the proof is complete. □

Problem 4 Let G be a Hausdorff topological group, and let H be a discrete subgroup of G. Show that H is closed.

Proof We shall show that the family of closed sets $\mathcal{F} = \{\{h\} : h \in H\}$ is locally finite.

Suppose to the contrary that there exists $p \in G$ such that every open neighborhood of p contains infinitely many elements of H. Since H is discrete, there is an open subset $U \subset G$ such that $U \cap H = \{e\}$. Note that the map

$$\varphi\colon G \times G \to G, \quad (x, y) \mapsto x^{-1}y$$

is continuous by definition of topological group, so the preimage $\varphi^{-1}(U)$ is an open neighborhood of (e, e) in $G \times G$. In this product topology, we can find open neighborhoods V_1 and V_2 of e in G such that $V_1 \times V_2 \subset \varphi^{-1}(U)$. Then $V := V_1 \cap V_2$ is an open neighborhood of e in G such that $V \times V \subset \varphi^{-1}(U)$, i.e.,

$$V^{-1}V \subset U.$$

Now pV is an open neighborhood of p, so the hypothesis implies that there exist distinct $h_1, h_2 \in H$ such that $h_1, h_2 \in pV$. It follows that $h_1^{-1}h_2 \in V^{-1}V \subset U$ and then $e \neq h_1^{-1}h_2 \in U \cap H$, contradicting the choice of U.

Thus $\mathcal{F}$ is locally finite. Since G is Hausdorff, each singleton $\{h\}$ is closed in G. By Problem 3 (2), we have

$$H = \bigcup_{h \in H} \{h\} = \bigcup_{h \in H} \overline{\{h\}} = \overline{\bigcup_{h \in H} \{h\}} = \overline{H},$$

so H is closed. □

Problem 5 Let H be a nontrivial discrete subgroup of $\mathbb{R}$. Show that H is infinite cyclic.

Proof Since H is nontrivial, the set of all positive elements in H is nonempty, so it has an infimum h. Moreover, since H is discrete, there exists a neighborhood of 0 that contains no other points of H except 0 itself. Hence $h > 0$. Since H is closed by Problem 4, we have $h \in H$.

Now for any $a \in H$, we can write $a = qh + r$ where $q \in \mathbb{Z}$ and $0 \leqslant r < h$. Then $r = a - qh \in H$. By the minimality of h, we must have $r = 0$. Thus $a = qh \in \langle h \rangle$. Therefore $H = \langle h \rangle \cong \mathbb{Z}$. □

Problem 6 Let H be a nontrivial discrete subgroup of $\mathbb{S}^1$. Show that H is finite cyclic.

Proof Since H is nontrivial, the set of all angles $\theta \in [0, 2\pi)$ such that $e^{i\theta} \in H$ is nonempty, so it has an infimum θ_0. Moreover, since H is discrete, there exists a neighborhood of 1 in $\mathbb{S}^1$ that contains no other points of H except 1 itself. Hence $\theta_0 > 0$. Since H is closed by Problem 4, we have $e^{i\theta_0} \in H$.

Now for any $e^{i\theta} \in H$, we can write $\theta = q\theta_0 + r$ where $q \in \mathbb{Z}$ and $0 \leqslant r < \theta_0$. Then $e^{ir} = e^{i\theta}\left(e^{i\theta_0}\right)^{-q} \in H$. By the minimality of θ_0, we must have $r = 0$. Thus $\theta = q\theta_0$ and $e^{i\theta} = \left(e^{i\theta_0}\right)^q \in \left\langle e^{i\theta_0}\right\rangle$. Therefore $H = \left\langle e^{i\theta_0}\right\rangle$ is cyclic.

- ◇ If $\theta_0 \in 2\pi\mathbb{Q}$, then H is finite cyclic.
- ◇ If $\theta_0 \notin 2\pi\mathbb{Q}$, then by Dirichlet's approximation theorem, H is not discrete, contradicting the assumption.

Thus H is finite cyclic. □

Problem 7 Let $A, B \in \mathrm{O}(2)$ and suppose $\det(A) = 1$, $\det(B) = -1$. Show that $B^2 = I$ and $BAB^{-1} = A^{-1}$. Deduce that every discrete subgroup of $\mathrm{O}(2)$ is either cyclic or dihedral.

Proof Suppose that

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

for some $\theta, \varphi \in \mathbb{R}$. Then a direct computation shows that

$$B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad BAB^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A^{-1}.$$

Now let H be any discrete subgroup of $\mathrm{O}(2)$.

- ◇ If $H \subset \mathrm{SO}(2) \simeq \mathbb{S}^1$, then by Problem 6, H is finite cyclic.
- ◇ If $H \not\subset \mathrm{SO}(2)$, then pick any $B \in H \setminus \mathrm{SO}(2)$, and consider the subgroup $H_0 = H \cap \mathrm{SO}(2)$. By the first case, H_0 is cyclic, say $H_0 = \langle A\rangle$. For any $C \in H \setminus \mathrm{SO}(2)$, we have $B^{-1}C \in H_0 = \langle A\rangle$, i.e., $C \in B\langle A\rangle$. The relations in H are

  $$B^2 = I \quad \text{and} \quad BAB^{-1} = A^{-1},$$

  which are exactly the defining relations of a dihedral group[3]:

  $$D_n = \left\langle A, B \mid A^n = e, B^2 = e, BAB^{-1} = A^{-1}\right\rangle,$$

  where n is the order of A. □

[3]In abstract algebra, D_{2n} refers to this same dihedral group. Here we use the geometric convention, D_n.