For latest updates of this note, visit https://xiaoshuo-lin.github.io/001707E.

Problem 1 Cut a Möbius strip along its central circle. What is the result?

Solution One long strip with **four half-twists** (or, equivalently, **two full twists**) in it (relative to an untwisted annulus or cylinder). Two of the half-twists come from the fact that this thinner strip goes two times through the half-twist in the original Möbius strip, and the other two come from the way the two halves of the thinner strip wrap around each other. □

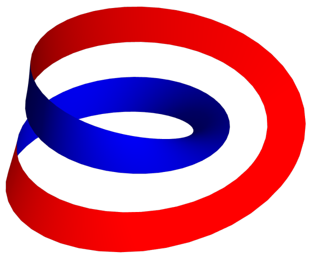

Figure 1: Möbius strip cut along centerline; sides colored differently

Remark (1) A half-twist is a twist of 180 degrees, and a full twist is a twist of 360 degrees. The ordinary Möbius strip has one half-twist in it.

(2) The result is not a Möbius strip, but instead is topologically equivalent to a cylinder.

(3) The result is an orientable surface, so it has an even number of half-twists.

(4) For those whose answer was "two half-twists": the other two half-twists come from the fact that just after you have made the cut, the resulting half-width strip goes two times around the cut, so it will turn an extra time when you unfold it to a large circle. To see this, first try making an ordinary strip that goes two times around a cylinder and then meets itself, without a Möbius twist. If you remove the cylinder and try to unfold your strip to a circle, it will have one full twist. This twist arises from the fact that the strip's centerline must wind around itself when it goes around the cylinder twice.

(5) Anyway, the best way to see what happens is to try it with a strip of paper. You can also run the following Mathematica code to visualize the result.

```
twist[{f_, g_}, a_, b_, u_] := {
   Cos[u] (a + f Cos[b u] - g Sin[b u]),
   Sin[u] (a + f Cos[b u] - g Sin[b u]),
   g Cos[b u] + f Sin[b u]
};

ParametricPlot3D[
  twist[{1/2 - v, 0}, 1, 1/2, u],
  {u, 0, 4 Pi}, {v, 0, 1/3},
  Axes -> None, Boxed -> False, Mesh -> None,
  PerformanceGoal -> "Quality", PlotPoints -> 100,
  PlotStyle -> FaceForm[Red, Blue]
]
```

Problem 2 Cut a Möbius strip along the circle which lies halfway between the boundary of the strip and the central circle. Do the same for the circle which lies one-third of the way in from the boundary. What are the resulting spaces?

Solution Two linked strips: one is a Möbius strip (the transparent one in Figure 1), and the other is a strip with four half-twists (the colored one in Figure 1). □

Problem 3 Take a strip which has one full twist in it and cut along its central circle. What is the result?

Solution Two linked strips, each with one full twist in it. □

Remark For further exploration, you may consult resources on **knot theory**. //

For topics related to **polygonal presentation** and **symbolic expression** of a surface, please refer to Professor Zuoqin Wang's lecture notes:

- `http://staff.ustc.edu.cn/~wangzuoq/Courses/21S-Topology/Notes/Lec28.pdf`
- `http://staff.ustc.edu.cn/~wangzuoq/Courses/21S-Topology/Notes/Lec29.pdf`

These two ways of representing surfaces are very useful in the classification of surfaces and the computation of their simplicial homology groups. Moreover, they are also helpful in the study of **surfaces with boundary**.

Fact 1 Every compact surface admits a polygonal presentation.

Fact 2 The polygonal presentation of a surface may not be unique.

Example 3 Here are some examples of gluing surfaces with boundary along their boundary circles.

(1) Gluing a Möbius strip to a disk along their boundary circles yields the projective plane $\mathbb{RP}^2$.

(2) Gluing two Möbius strips along their boundary circles yields the Klein bottle.

> *A mathematician named Klein*
> *Thought the Möbius band was divine.*
> *Said he: "If you glue*
> *The edges of two,*
> *You'll get a weird bottle like mine."*
>
> — Leo Moser, mathematician

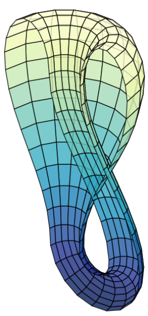

Figure 2: Dissecting the Klein bottle results in two Möbius strips