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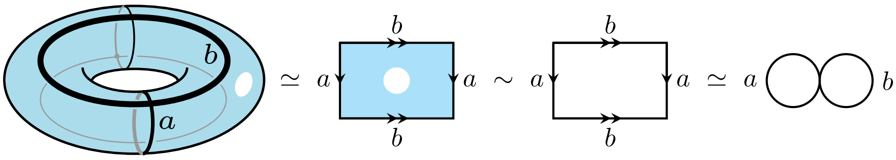
Problem 1 Show that the punctured torus deformation retracts onto the one-point union of two circles.

Proof If we use "$\simeq$" to denote homeomorphism and "$\sim$" to denote homotopy equivalence, then



□

Remark We can apply this result to compute the fundamental group of the torus using van Kampen's theorem (although we already know that $\pi_1(\mathbb{T}^2) = \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \simeq \mathbb{Z} \times \mathbb{Z}$ from the product property of fundamental groups). Write $\mathbb{T}^2$ as the union of two open sets $U_1 = \mathbb{T}^2 \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is the small disk removed above and $\widetilde{D}$ is a small open disk containing $\overline{D}$. Since U_1 deformation retracts onto $\mathbb{S}^1 \vee \mathbb{S}^1$ and U_2 is contractible, we have

$$\pi_1(U_1) \simeq \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \simeq \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle \quad \text{and} \quad \pi_1(U_2) \simeq \{e\}.$$

Furthermore, $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, so

$$\pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_* \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $aba^{-1}b^{-1}$. In other words,

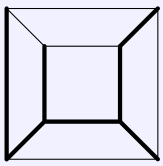
$$\iota_*(1) = aba^{-1}b^{-1}.$$

Hence by van Kampen's theorem,

$$\pi_1(\mathbb{T}^2) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \langle a, b \,|\, aba^{-1}b^{-1} = 1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}.$$

//

Problem 2 Let X be the graph shown below, consisting of the twelve edges of a cube. Find the fundamental group of X.



Solution The seven heavily shaded edges form a spanning tree T of X. For each edge e_α of $X \setminus T$, we choose an open neighborhood A_α of $T \cup e_\alpha$ in X that deformation retracts onto $T \cup e_\alpha$. The intersection of **two or more** A_α's retracts onto T, hence is contractible. The A_α's form an open cover of X satisfying the hypotheses of van Kampen's theorem (see Problem 4 from last time), and since the intersection of any two of them is simply connected we obtain an isomorphism $\pi_1(X) \simeq *_\alpha \pi_1(A_\alpha)$. Each A_α deformation retracts onto a circle, so $\pi_1(X)$ is free on five generators, i.e.,

$$\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}. \qquad \square$$

Remark As explicit generators we can choose for each edge e_α of $X \backslash T$ a loop f_α that starts at a basepoint in T, travels in T to one end of e_α, then across e_α, and then back to the basepoint along a path in T. //

One can generalize the idea in Problem 2 to compute the fundamental group of any connected graph:

Theorem 1 Let $G = (V, E)$ be a connected graph, and let T be a spanning tree[1] of G. If the edges in $G \setminus T$ are labeled by $\{e_\alpha\}_{\alpha \in \mathcal{A}}$, then

$$\pi_1(G) \simeq \ast_{\alpha \in \mathcal{A}} \mathbb{Z}.$$

In particular, for a finite connected graph with v vertices and e edges, its fundamental group is a free group on $(e - v + 1)$ generators[2].

Problem 3 (Wedge sum of circles)

(1) **(Finite wedge sum with applications)**

① Prove that $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \cdots \mathbb{S}^1) \simeq \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$.

② What is the fundamental group of $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$?

③ What is the fundamental group of $\mathbb{R}^2 \setminus \mathbb{Z}^2$?

④ Find the fundamental group of $\mathbb{R}^3 \setminus \{\text{finitely many lines passing through } 0\}$.

⑤ A group is called **finitely presented** if it has a presentation $G = \langle S \mid R \rangle$ where both S and R are finite sets. Prove that any finitely presented group is the fundamental group of some compact Hausdorff space.

(2) **(Infinite wedge sum)**

① Let $X = \bigcup_{n \geqslant 1} C_n$, where C_n is the circle in $\mathbb{R}^2$ of radius n centered at $(n, 0)$. Compute $\pi_1(X)$.

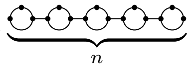
② Let $Y = \{(x, 0) : x \in \mathbb{R}\} \cup \bigcup_{n \geqslant 1} \widetilde{C}_n$, where $\widetilde{C}_n$ is the circle in $\mathbb{R}^2$ of radius $\frac{1}{3}$ centered at $\left(n, \frac{1}{3}\right)$. Compute $\pi_1(Y)$. Are X and Y homeomorphic?

Proof (1) ① View $\bigvee_{n=1}^{\infty} \mathbb{S}^1$ as a connected graph with $n + 1$ vertex and $2n$ edges. By Theorem 1,

$$\pi_1\left(\bigvee_{n=1}^{\infty} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-(n+1)+1=n}.$$

[1] The trees within a graph may be partially ordered by their subgraph relation, and any infinite chain in this partial order has an upper bound (the union of the trees in the chain). Zorn's lemma, one of many equivalent statements to the axiom of choice, requires that a partial order in which all chains are upper bounded have a maximal element; in the partial order on the trees of the graph, this maximal element must be a spanning tree. Therefore, if Zorn's lemma is assumed, every infinite connected graph has a spanning tree.

[2] This is because the number of vertices in a tree is one more than the number of edges.

② $\mathbb{R}^2 \setminus \{n \text{ points}\}$ is homotopy equivalent to the connected graph 

with $3n$ vertices and $4n-1$ edges. By Theorem 1, its fundamental group is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(4n-1)-3n+1=n}$.

③ $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is homotopy equivalent to an "infinite grid" (the union of all lines $x = m$ and $y = n$, $m, n \in \mathbb{Z}$), which admits a spanning tree consisting of all lines $x = m$ ($m \in \mathbb{Z}$) together with $y = 0$. The entire grid has countably many edges and the spanning tree misses infinitely many edges. By Theorem 1, its fundamental group is $*_{n\in\mathbb{N}}\mathbb{Z}$.

④ $\mathbb{R}^3 \setminus \{n \text{ lines passing through } 0\}$ is homotopy equivalent to $\mathbb{S}^2 \setminus \{2n \text{ points}\}$, which is homeomorphic to $\mathbb{R}^2 \setminus \{(2n-1) \text{ points}\}$. By ②, its fundamental group is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1}$.

⑤ Let $G = \langle S \mid R \rangle$ be a finitely presented group, where $S = \{s_1, \ldots, s_m\}$ and R is a finite set of relations. Begin with the wedge sum of m circles, whose fundamental group is the free group $\langle S \rangle$ by ①. Each circle C_i represents the generator s_i. Write each relation $r \in R$ as a word

$$r = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1.$$

For each such r, attach a disk to the wedge sum by identifying its boundary with the loop that successively traverses the circles $C_{i_1}, \ldots, C_{i_k}$, using the reverse orientation when $\varepsilon_j = -1$. Let X be the resulting space. Since X is obtained from finitely many circles and disks by gluing, it is compact and Hausdorff. Attaching a disk along a loop makes that loop null-homotopic, so by van Kampen's theorem the fundamental group of X is obtained from $\langle S \rangle$ by imposing exactly the relations in R, i.e., $\pi_1(X) \simeq G$.

(2) ① Let U be an open ball centered at $(0,0)$ with radius less than 1. Then $V := X \cap U$ is contractible. Let $A_n = V \cup (C_n \setminus \{0\})$. Then A_n is open in X since it is the union of two open sets, and $A_m \cap A_n = V$ for distinct m, n. Now $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a path-connected open set, and $A_k \cap A_m \cap A_n$ is path-connected for all k, m, n. By van Kampen's theorem, $\pi_1(X) \simeq *_{n\in\mathbb{N}}\mathbb{Z}$.

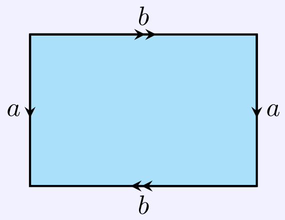
② Y is a strong deformation retract of $\mathbb{R}^2 \setminus \left\{\left(n, \frac{1}{3}\right) : n \geqslant 1\right\}$, and the latter is homotopy equivalent to the graph $\{(x, 0) : x \geqslant 0\} \cup \{(x, 1) : x \geqslant 0\} \cup \{(m, y) : m \in \mathbb{N} \cup \{0\}, 0 \leqslant y \leqslant 1\}$, which admits a spanning tree consisting of all the two horizontal rays together with $\{0\} \times [0, 1]$. The entire graph has countably many edges and the spanning tree misses infinitely many edges. By Theorem 1, its fundamental group is $*_{n\in\mathbb{N}}\mathbb{Z}$. However, $X \setminus \{(0, 0)\}$ has infinitely many path-components, while Y minus a point has at most 3 path-components. Therefore X and Y are not homeomorphic. $\square$

Problem 4 Use van Kampen's theorem to find the fundamental groups of the following spaces.

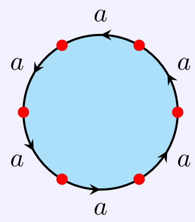
(1) The Klein bottle.

(2) $\Sigma_g = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$.

(3) The n-fold dunce cap. [Split the boundary circle of a closed disk into n parts (by n red dots), and identify the boundary segments according to the picture below (but keep the interior of the disk unchanged).]



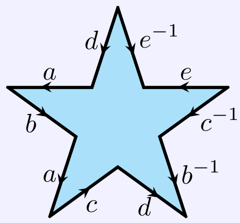
The Klein bottle



The n-fold dunce cap

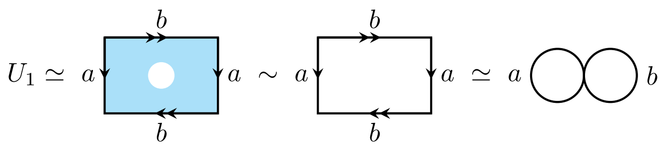
(4) $\mathbb{RP}^2$.

(5) The surface X obtained by gluing the sides of a star as shown below (a "letter edge" is glued counterclockwise, and an "inverse letter edge" is glued clockwise).



(6) $\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2$.

Solution (1) We first write the Klein bottle K as the union of two open sets $U_1 = K\setminus\overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disk and $\widetilde{D}$ is a small open disk containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\mathbb{S}^1 \wedge \mathbb{S}^1\right) \simeq \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1\left(\mathbb{S}^1\right) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

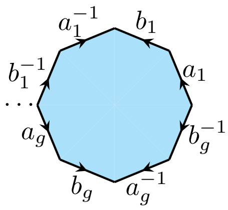
$$\iota_*\colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $baba^{-1}$. In other words,

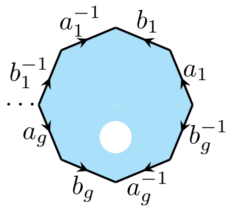
$$\iota_*(1) = baba^{-1}.$$

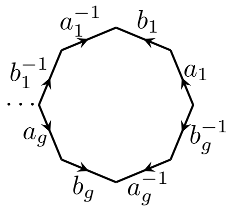
Hence by van Kampen's theorem,

$$\pi_1(K) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \big\langle a, b \,\big|\, baba^{-1} = 1 \big\rangle.$$

(2) Consider the polygonal presentation of Σ_g: 

We first write Σ_g as the union of two open sets $U_1 = \Sigma_g \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disk and $\widetilde{D}$ is a small open disk containing $\overline{D}$. Since

$U_1 \simeq$ 

$\sim$ 

$\simeq \bigvee_{k=1}^{2g} \mathbb{S}^1$

we have

$$\pi_1(U_1) \simeq \pi_1\left(\bigvee_{k=1}^{2g} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g} = \langle a_1, b_1, \cdots, a_g, b_g \rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1\big(\mathbb{S}^1\big) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_* \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. In other words,

$$\iota_*(1) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Hence by van Kampen's theorem,

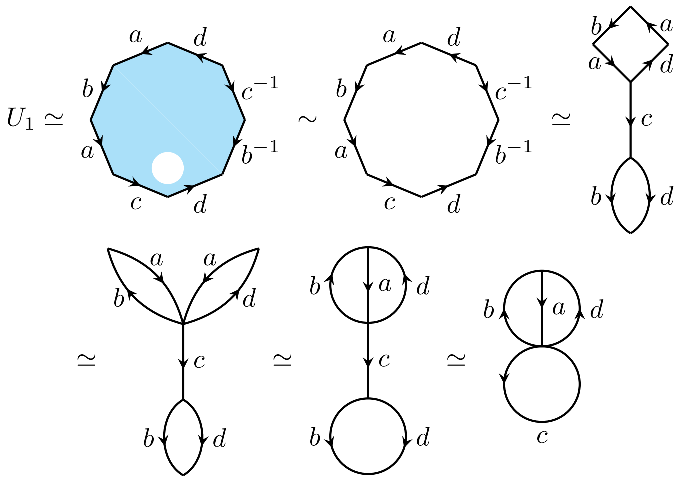
$$\pi_1(\Sigma_g) \simeq (\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g}) *_{\mathbb{Z}} \{e\} = \big\langle a_1, b_1, \cdots, a_g, b_g \,\big|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \big\rangle.$$

(3) This is exactly the way we used to construct a presentation complex in Problem 3 (1) ⑤, so the fundamental group of the n-fold dunce cap is $\langle a \,|\, a^n = 1 \rangle \simeq \mathbb{Z}_n$.

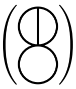
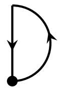
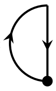
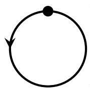
(4) If we form $\mathbb{RP}^2$ by identifying antipodal points of $\mathbb{S}^2$, and obtain a hemisphere with antipodal points on the equator identified, then it reduces to the case in (3) where $n = 2$. So $\pi_1\big(\mathbb{RP}^2\big) \simeq \mathbb{Z}_2$.

(5) We first write X as the union of two open sets $U_1 = X \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disk

and $\widetilde{D}$ is a small open disk containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\right.$$  $$\left.\right) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \left\langle \alpha := \right.$$  $$, \beta :=$$  $$, \gamma :=$$  $$\left.\right\rangle$$

by Theorem 1, for the last graph has 2 vertices and 4 edges (why do we allow loops as edges here?).

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_* \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

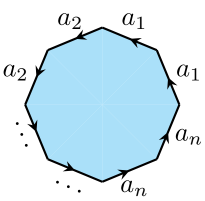
The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $dabacdb^{-1}c^{-1}$, which is represented by the loop $\alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1}$ in the last graph. In other words,

$$\iota_*(1) = \alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1}.$$

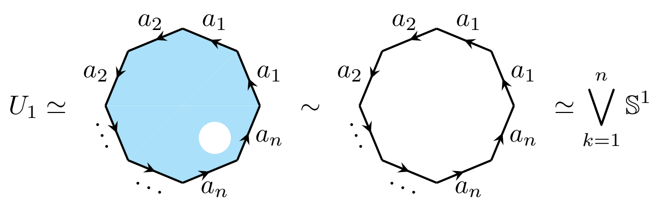
Hence by van Kampen's theorem,

$$\pi_1(X) \simeq (\mathbb{Z} * \mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \left\langle \alpha, \beta, \gamma \,\middle|\, \alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1} = 1 \right\rangle.$$

(6) Consider the polygonal presentation of $\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$:



We first write $\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$ as the union of two open sets $U_1 = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n} \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disk and $\widetilde{D}$ is a small open disk containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\bigvee_{k=1}^{n} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n} = \langle a_1, \cdots, a_n \rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_*\colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $a_1^2 a_2^2 \cdots a_n^2$. In other words,

$$\iota_*(1) = a_1^2 a_2^2 \cdots a_n^2.$$

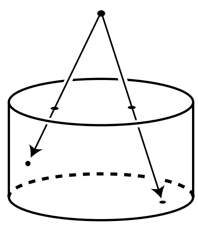
Hence by van Kampen's theorem,

$$\pi_1\big(\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}\big) \simeq \big(\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}\big) *_{\mathbb{Z}} \{e\} = \big\langle a_1, \cdots, a_n \,\big|\, a_1^2 a_2^2 \cdots a_n^2 = 1 \big\rangle.$$

$\square$

Problem 5 Prove that if $f, g\colon \mathbb{S}^{n-1} \to X$ are homotopic maps, then the spaces $X \cup_f \mathbb{D}^n$ and $X \cup_g \mathbb{D}^n$ obtained by attaching an n-disk to X via f and g, respectively, are homotopy equivalent.

Proof Let $I = [0, 1]$. We first show that $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$ is a deformation retract of $\mathbb{D}^n \times I$.



There is a retraction $r\colon \mathbb{D}^n \times I \to \mathbb{D}^n \times \{0\} \cup \partial\mathbb{D}^n \times I$, for example the radial projection from the point $(0, 2) \in \mathbb{D}^n \times \mathbb{R}$ as shown in the figure above. Then setting $r_t = tr + (1-t)\mathrm{Id}$ gives a deformation retraction of $\mathbb{D}^n \times I$ onto $\mathbb{D}^n \times \{0\} \cup \partial\mathbb{D}^n \times I$.

Now let $H\colon \mathbb{S}^{n-1} \times I \to X$ be a homotopy from f to g, and consider the space $X \cup_H (\mathbb{D}^n \times I)$. This contains both $X \cup_f \mathbb{D}^n$ and $X \cup_g \mathbb{D}^n$ as subspaces. The deformation retraction above induces a deformation retraction of $X \cup_H (\mathbb{D}^n \times I)$ onto $X \cup_f \mathbb{D}^n$.[3] Similarly, $X \cup_H (\mathbb{D}^n \times I)$ deformation retracts onto $X \cup_g \mathbb{D}^n$. Hence $X \cup_f \mathbb{D}^n$ and $X \cup_g \mathbb{D}^n$ are homotopy equivalent. $\square$

[3] Note that $X \cup_H (\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I) \simeq X \cup_H (\mathbb{D}^n \times \{0\}) \simeq X \cup_f \mathbb{D}^n$.

Problem 6 Show that the open unit disk in $\mathbb{R}^2$ does not have the fixed point property.

Proof The map

$$f\colon \mathbb{B}(0,1) \to \mathbb{R}^2, \quad (r,\theta) \mapsto \left(\frac{r}{1-r}, \theta\right)$$

is a homeomorphism, and since $\mathbb{R}^2$ does not have the fixed point property (e.g. the translation map $(x,y) \mapsto (x+1,y)$ has no fixed point), neither does $\mathbb{B}(0,1)$. $\square$

Problem 7 Show that the figure-eight space $\mathbb{S}^1 \vee \mathbb{S}^1$ does not have the fixed point property.

Proof Denote the two circles by C_1 and C_2, and their common point by p. Then the map that rotates C_1 by π and sends C_2 to the image of p has no fixed point. $\square$

Problem 8 Let K be a compact subset of $\mathbb{R}^n$. Show that $\mathbb{R}^n \setminus K$ has exactly one unbounded component.

Proof Since K is bounded, there exists some $R > 0$ such that $K \subset \mathbb{B}(0,R)$. Then the connected set $\mathbb{R}^n \setminus \mathbb{B}(0,R)$ is contained in one component of $\mathbb{R}^n \setminus K$, which is unbounded. $\square$