Stein 4.4.9 Here are further results similar to the Phragmén–Lindelöf theorem.

(1) Let F be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(\mathrm{i}y)| \leqslant 1$ for all $y \in \mathbb{R}$, and

$$|F(z)| \leqslant Ce^{c|z|^\gamma}$$

for some $c, C > 0$ and $\gamma < 1$. Prove that $|F(z)| \leqslant 1$ for all z in the right half-plane.

(2) More generally, let S be a sector whose vertex is the origin, and forming an angle of π/β. Let F be a holomorphic function in S that is continuous on the closure of S, so that $|F(z)| \leqslant 1$ on the boundary of S and

$$|F(z)| \leqslant Ce^{c|z|^\alpha} \quad \text{for all} \quad z \in S$$

for some $c, C > 0$ and $0 < \alpha < \beta$. Prove that $|F(z)| \leqslant 1$ for all $z \in S$.

Proof We shall prove (2) directly. Let $F_\varepsilon(z) = F(z)e^{-\varepsilon z^r}$, where $r \in (\alpha, \beta) \cap \mathbb{Q}$ and $\varepsilon > 0$. Then

$$|F_\varepsilon(z)| = |F(z)|e^{-\varepsilon|z|^r \cos(r \arg z)} \leqslant Ce^{c|z|^\alpha - \varepsilon|z|^r \cos(r \arg z)}.$$

By rotation, we may assume that

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{2\beta} < \arg z < \frac{\pi}{2\beta} \right\},$$

so that $r \arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cos(r \arg z) > 0$. Then $|F_\varepsilon(z)| \to 0$ as $|z| \to \infty$, and we conclude that $|F_\varepsilon(z)|$ achieves its maximum in $\overline{S}$ at some point $z_0 \neq \infty$. If $F_\varepsilon(z)$ is non-constant, then the maximum modulus principle shows that $z_0 \in \partial S$. That is, $|F_\varepsilon(z)| \leqslant |F_\varepsilon(z_0)| \leqslant 1$. The proof is complete by letting $\varepsilon \to 0^+$. $\square$

Stein 4.5.3 In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the three-lines lemma.

(1) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \operatorname{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leqslant 1$ on the boundary lines, then $|F(z)| \leqslant 1$ throughout the strip.

(2) For the more general F, let $\sup_{x\in\mathbb{R}} |F(x)| = M_0$ and $\sup_{x\in\mathbb{R}} |F(x + \mathrm{i})| = M_1$. Then,

$$\sup_{x\in\mathbb{R}} |F(x + \mathrm{i}y)| \leqslant M_0^{1-y} M_1^y, \quad \text{if } 0 \leqslant y \leqslant 1.$$

(3) As a consequence, prove that $\log \sup_{x\in\mathbb{R}} |F(x + \mathrm{i}y)|$ is a convex function of y when $0 \leqslant y \leqslant 1$.

Proof (1) Let $F_\varepsilon(z) = F(z)e^{-\varepsilon z^2}$ for some $\varepsilon > 0$. Then

$$|F_\varepsilon(z)| = |F(z)|e^{-\varepsilon(x^2 - y^2)} \to 0 \quad \text{as } x \to \infty.$$

Hence $|F_\varepsilon(z)|$ achieves its maximum in the strip at some point $z_0 \neq \infty$. If $F_\varepsilon(z)$ is non-constant, then the maximum modulus principle shows that z_0 must lie on the boundary of this strip. Thus $|F_\varepsilon(z)| \leqslant |F_\varepsilon(z_0)| \leqslant 1$, and by letting $\varepsilon \to 0^+$ we get $|F(z)| \leqslant 1$ throughout strip.

(2) Let $G(z) = M_0^{-\mathrm{i}z-1} M_1^{\mathrm{i}z} F(z)$. Then $G(z)$ satisfies the assumptions in (1), i.e.,

$$|G(x)| = \left|M_0^{-\mathrm{i}x-1}\right| \cdot \left|M_1^{\mathrm{i}x}\right| \cdot |F(x)| = M_0^{-1}|F(x)| \leqslant 1, \quad \forall x \in \mathbb{R},$$
$$|G(x+\mathrm{i})| = \left|M_0^{-\mathrm{i}(x+\mathrm{i})-1}\right| \cdot \left|M_1^{\mathrm{i}(x+\mathrm{i})}\right| \cdot |F(x+\mathrm{i})| = M_1^{-1}|F(x+\mathrm{i})| \leqslant 1, \quad \forall x \in \mathbb{R}.$$

Thus, we have $|G(z)| \leqslant 1$ throughout the strip, i.e.,

$$|G(z)| = \left|M_0^{-\mathrm{i}(x+\mathrm{i}y)-1}\right| \cdot \left|M_1^{\mathrm{i}(x+\mathrm{i}y)}\right| \cdot |F(z)| = M_0^{y-1} M_1^{-y} |F(x+\mathrm{i}y)| \leqslant 1,$$

which implies the desired result.

(3) Set $M(y) = \sup_{x \in \mathbb{R}} |F(x+\mathrm{i}y)|$ for $y \in [0,1]$. For $0 \leqslant y_1 < y_2 \leqslant 1$, by scaling we see the result in (2) applies to the strip $y_1 < \operatorname{Im} z < y_2$, i.e., for all $y \in [y_1, y_2]$,

$$\log M(y) \leqslant \log\left(M(y_1)^{\frac{y_2-y}{y_2-y_1}} M(y_2)^{\frac{y-y_1}{y_2-y_1}}\right) = \frac{y_2-y}{y_2-y_1}\log M(y_1) + \frac{y-y_1}{y_2-y_1}\log M(y_2),$$

which implies the convexity of $\log M(y)$, since $\frac{y_2-y}{y_2-y_1}y_1 + \frac{y-y_1}{y_2-y_1}y_2 = y$. □

Stein 5.6.1 Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Proof Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, $f(0) \neq 0$, and f vanishes nowhere on the circle C_R. Let $z_1, \cdots, z_N$ denote the zeros of f inside the disc (counted with multiplicities), we want to show that

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta.$$

(1) First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then so does their product $f_1 f_2$.

(2) By setting $\tilde{f}(z) = f(Rz)$, what we want to prove can be reduced to the specific case when $R = 1$. Note that the function

$$g(z) = \frac{f(z)}{\psi_{z_1}(z) \cdots \psi_{z_N}(z)}$$

initially defined on $\Omega \setminus \{z_1, \cdots, z_N\}$, is bounded near each z_j. Therefore each z_j is a removable singularity, and hence we can write

$$f(z) = \psi_{z_1}(z) \cdots \psi_{z_N}(z) g(z)$$

where g is holomorphic in Ω and nowhere vanishing in $\overline{\mathbb{B}(0,1)}$. By (1) above, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for Blaschke factors.

(3) The case of functions that vanish nowhere follows from the mean value theorem for holomorphic functions. So it remains to show the result for Blaschke factors. In fact,

$$\log|\psi_\alpha(0)| = \log|\alpha| = \log|\alpha| + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\psi_\alpha\left(e^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta,$$

for $|\psi_\alpha(z)| = 1$ whenever $|z| = 1$ (see Exercise 1.4.7). $\square$

Stein 5.6.2 Find the order of growth of the following entire functions:

(1) $p(z)$ where p is a polynomial.

(2) e^{bz^n} for $b \neq 0$.

(3) e^{e^z}.

Solution (1) 0.

(2) n.

(3) ∞. $\square$

Stein 5.6.3 Show that if τ is fixed with $\mathrm{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z \mid \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 as a function of z. Further properties of Θ will be studied in Chapter 10.

Proof (**The order is at most** 2) We have

$$\begin{aligned}
|\Theta(z \mid \tau)| &\leqslant \sum_{n=-\infty}^{\infty} \left| e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \leqslant \sum_{n=-\infty}^{\infty} e^{-\pi n^2 \mathrm{Im}(\tau) + 2\pi n |z|} \\
&= \underbrace{\sum_{n < \frac{4|z|}{\mathrm{Im}(\tau)}} e^{-\pi n^2 \mathrm{Im}(\tau) + 2\pi n |z|}}_{2\pi n|z| \leqslant \frac{8\pi|z|^2}{\mathrm{Im}(\tau)} \text{ when } n < \frac{4|z|}{\mathrm{Im}(\tau)}} + \underbrace{\sum_{n \geqslant \frac{4|z|}{\mathrm{Im}(\tau)}} e^{-\pi n^2 \mathrm{Im}(\tau) + 2\pi n |z|}}_{-n^2 \mathrm{Im}(\tau) + 2n|z| \leqslant -\frac{n^2 \mathrm{Im}(\tau)}{2} \text{ when } n \geqslant \frac{4|z|}{\mathrm{Im}(\tau)}} \\
&\leqslant e^{\frac{8\pi|z|^2}{\mathrm{Im}(\tau)}} \sum_{n < \frac{4|z|}{\mathrm{Im}(\tau)}} e^{-\pi n^2 \mathrm{Im}(\tau)} + \sum_{n \geqslant \frac{4|z|}{\mathrm{Im}(\tau)}} e^{-\frac{\pi n^2 \mathrm{Im}(\tau)}{2}} \\
&\overset{e^{-x} \leqslant \frac{1}{x+1}}{\leqslant} e^{\frac{8\pi|z|^2}{\mathrm{Im}(\tau)}} \sum_{n < \frac{4|z|}{\mathrm{Im}(\tau)}} \frac{1}{\pi n^2 \mathrm{Im}(\tau) + 1} + \sum_{n \geqslant \frac{4|z|}{\mathrm{Im}(\tau)}} \frac{1}{\frac{\pi n^2 \mathrm{Im}(\tau)}{2} + 1} \\
&\leqslant C_1 e^{\frac{8\pi|z|^2}{\mathrm{Im}(\tau)}} + C_2.
\end{aligned}$$

(**The order is at least** 2) We repeatedly use Proposition 1.1 (iii) from Chapter 10, which concerns the quasi-periodicity of $\Theta(z \mid \tau)$, to obtain

$$\Theta(x + m\tau \mid \tau) = e^{-2\pi i m x - \pi i m^2 \tau} \Theta(x \mid \tau).$$

Setting $x = 0$ and taking the absolute value of both sides, we obtain

$$|\Theta(m\tau \mid \tau)| = e^{\pi m^2 \mathrm{Im}(\tau)} |\Theta(0 \mid \tau)| = A e^{B|m\tau|^2},$$

which shows that the order of $\Theta(z \mid \tau)$ is at least 2. $\square$