Topology (H)

Xiaoshuo Lin, Fall 2024

https://xiaoshuo-lin.github.io

List of Problems

PSet 1, Part 1

Problem 1 (Fixed point theorems) Let $f\colon [a,b] \to \mathbb{R}$ be continuous.

(1) Prove: if $f([a,b]) \subset [a,b]$, then there exists $p \in [a,b]$ such that $f(p) = p$.

(2) Prove: if $f([a,b]) \supset [a,b]$, then there exists $p \in [a,b]$ such that $f(p) = p$.

(3) What if $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ is continuous with $f(\mathbb{D}^2) \supset \mathbb{D}^2$?

Proof (1) Let $g(x) = f(x) - x \in \mathcal{C}([a,b])$, then $g(a) \geqslant 0$ and $g(b) \leqslant 0$. By the intermediate value theorem, there exists $p \in [a,b]$ such that $g(p) = 0$, i.e., $f(p) = p$.

(2) Let $h(x) = f(x) - x \in \mathcal{C}([a,b])$. Suppose to the contrary that $h(x) \neq 0$ for all $x \in [a,b]$.

- If $h(a) > 0$, then $f(a) > a$ and there exists $x_0 \in (a,b]$ such that $f(x_0) = a$. Hence $h(x_0) = a - x_0 < 0$ and h has a zero in (a, x_0) by the intermediate value theorem, a contradiction.
- If $h(b) < 0$, then $f(b) < b$ and there exists $x_1 \in [a,b)$ such that $f(x_1) = b$. Hence $h(x_1) = b - x_1 > 0$ and h has a zero in (x_1, b) by the intermediate value theorem, a contradiction.

Therefore we have $h(a) < 0$ and $h(b) > 0$. Again the intermediate value theorem leads to a contradiction, so there exists $p \in [a,b]$ such that $f(p) = p$.

(3) In this case we don't have a fixed point theorem. A map without a fixed point is given in Figure 1.

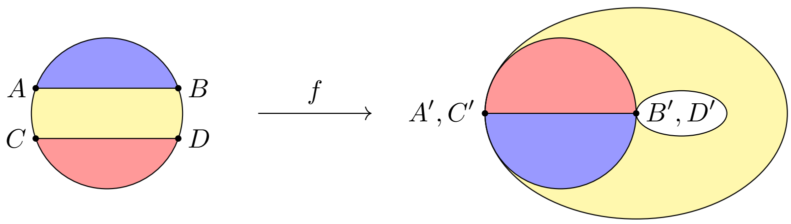


Figure 1: Attaching arc BA to arc DC

Here the upper third of the disk is mapped to the lower half, the lower third is mapped to the upper half, and the middle third is mapped to a band outside the disk that connects the two halves. □

Problem 2 (Escape) Look at Figure 2. Explain how.

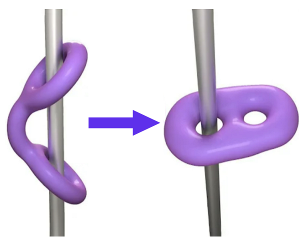


Figure 2: Two rings

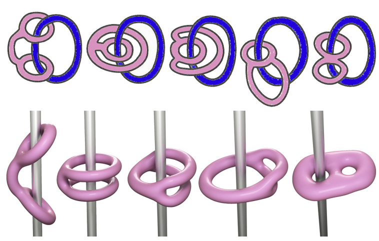


Figure 3: Undoing handcuffs

Solution Shown in Figure 3. □

Problem 3 (Inscribed square problem: a simple case) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Consider the simple closed curve C that consists of the graph of f and the line segment of the x-axis from $x = 0$ to $x = 1$. Prove: one can find four points on C that are the vertices of a square.

Proof We may assume $f(x) > 0$ for $x \in (0,1)$, since otherwise (as long as f is not identically zero) we can find two consecutive zeros of f such that f or $-f$ is positive in between, and then scale up this subgraph. By the extreme value theorem, there exist $x_1 \in (0,1)$ such that $f(x) \leqslant f(x_1)$ for $x \in [0,1]$. Now let $g(x) = x + f(x)$, then $g(0) = 0$ and $g(1) = 1$. Since $x_1 \in (0,1)$, there exists some $x_2 \in (0,1)$ such that $g(x_2) = x_1$ by the intermediate value theorem. Next we consider another function h defined by

$$h(x) = f(x) - f(g(x)).$$

For this function to be well-defined, we set $f(x) = 0$ for all $x \notin [0,1]$. Then

$$h(x_1) = f(x_1) - f(g(x_1)) \geqslant 0,$$
$$h(x_2) = f(x_2) - f(g(x_2)) = f(x_2) - f(x_1) \leqslant 0.$$

By the intermediate value theorem, there is some x_0 between x_1 and x_2 such that $h(x_0) = 0$, namely

$$f(x_0) = f(x_0 + f(x_0)).$$

Therefore, the points x_0 and $x_0 + f(x_0)$ on the x-axis are the base corners of the inscribed square, as illustrated in Figure 4.

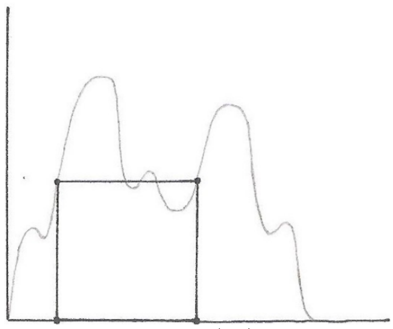


Figure 4: Inscribed square

□

Problem 4 (Weierstrass's counterexample to Dirichlet principle) For any u in

$$\mathcal{A} = \{u \in \mathcal{C}^1([-1,1]) : u(-1) = 0, u(1) = 1\},$$

define

$$F(u) = \int_{-1}^{1} |xu'(x)|^2 \, \mathrm{d}x.$$

(1) Prove: for each $n \in \mathbb{N}$, the function

$$u_n(x) := \left(\sin \frac{n\pi x}{2}\right)^2 \chi_{[0,1/n]}(x) + \chi_{(1/n,1]}(x)$$

is an element in $\mathcal{A}$.

(2) Prove: $\lim_{n\to\infty} F(u_n) = 0$.

(3) Prove: there is no function $u \in \mathcal{A}$ that attains the minimum of F.

Proof (1) First rewrite $u_n(x)$ as

$$u_n(x) = \begin{cases} 0, & -1 \leqslant x < 0, \\ \left(\sin \dfrac{n\pi x}{2}\right)^2, & 0 \leqslant x \leqslant \dfrac{1}{n}, \\ 1, & \dfrac{1}{n} < x \leqslant 1. \end{cases}$$

Since

$$\lim_{x\to 0^+} \frac{u_n(x)-u_n(0)}{x-0} = \lim_{x\to 0} \frac{\left(\sin \frac{n\pi x}{2}\right)^2}{x} = 0 = \lim_{x\to 0^-} \frac{u_n(x)-u_n(0)}{x-0}$$

and similarly

$$\lim_{x\to\left(\frac{1}{n}\right)^-} \frac{u_n(x)-u_n\left(\frac{1}{n}\right)}{x-\frac{1}{n}} = \lim_{x\to\frac{1}{n}} \frac{n\pi}{2}\sin(n\pi x) = 0 = \lim_{x\to\left(\frac{1}{n}\right)^+} \frac{u_n(x)-u_n\left(\frac{1}{n}\right)}{x-\frac{1}{n}},$$

we see the derivative of u_n is

$$u_n'(x) = \begin{cases} \dfrac{n\pi}{2}\sin(n\pi x), & 0 \leqslant x \leqslant \dfrac{1}{n}, \\ 0, & \text{else}. \end{cases} \tag{4–1}$$

From this we conclude that $u_n \in \mathcal{C}^1([-1,1])$, hence $u_n \in \mathcal{A}$.

(2) By (4–1) we have

$$F(u_n) = \int_{-1}^{1} |xu_n'(x)|^2 \,\mathrm{d}x = \frac{n^2\pi^2}{4}\int_0^{\frac{1}{n}} x^2 \sin^2(n\pi x)\,\mathrm{d}x = \frac{2\pi^2-3}{48n} \xrightarrow{n\to\infty} 0.$$

(3) If $u \in \mathcal{A}$ satisfies $F(u) = 0$, then from $xu'(x) \in \mathcal{C}([-1,1])$ we get $xu'(x) = 0$ for all $x \in [-1,1]$. Since $u'(x) \in \mathcal{C}([-1,1])$, we find $u'(x) = 0$ for all $x \in [-1,1]$, which is impossible for u is not constant. □

PSet 1, Part 2

Problem 5 (Examples of metrics) Check that the following are metrics.

(1) Let G be a group and S be a generating set, then the word metric

$$d_S(g_1,g_2) = \min\{n : \exists\, s_1,\cdots,s_n \in S \cup S^{-1} \text{ s.t. } g_1 s_1 \cdots s_n = g_2\}$$

is a metric on G. Moreover, if G is finitely generated and S_1, S_2 are two finite generating sets of G, then there exists $L_1, L_2 > 0$ so that

$$L_1 d_{S_1}(g_1,g_2) \leqslant d_{S_2}(g_1,g_2) \leqslant L_2 d_{S_1}(g_1,g_2).$$

(2) The Hausdorff metric on $X = \{\text{all bounded closed subsets in } \mathbb{R}^n\}$ given by

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon \geqslant 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$$

is a metric on X, where $A_\varepsilon = \bigcup_{x \in A} \mathbb{B}(x, \varepsilon)$. Moreover, describe the open ball centered at "$A =$ the closed unit disk" and with radius $\frac{1}{2}$.

Proof (1) First, the map $d_S \colon G \times G \to \mathbb{R}_{\geqslant 0}$ is well-defined, since S is a generating set for G.

- ◇ Clearly $d_S(g_1, g_2) \geqslant 0$ and $d_S(g_1, g_2) = 0$ if and only if $g_1 = g_2$.
- ◇ Let $g, h \in G$ and $d_S(g, h) = n$. Furthermore, let $s_1, \cdots, s_n \in S \cup S^{-1}$ be such that $g^{-1}h = s_1 \cdots s_n$. By taking inverses, we have $h^{-1}g = s_n^{-1} \cdots s_1^{-1}$, so $d_S(h, g) \leqslant n = d_S(g, h)$. Switching the roles of g and h in the above, we obtain the converse inequality, and hence equality.
- ◇ If $g^{-1}h = s_1 \cdots s_n$ and $h^{-1}k = r_1 \cdots r_m$, then

$$g^{-1}k = \left(g^{-1}h\right)\left(h^{-1}k\right) = s_1 \cdots s_n r_1 \cdots r_m.$$

It follows that $d_S(g, k) \leqslant d_S(g, h) + d_S(h, k)$.

If G is finitely generated and S_1, S_2 are two finite generating sets of G, then any $s_2 \in S_2 \cup S_2^{-1}$ can be written as

$$s_2 = s_{1,1} s_{1,2} \cdots s_{1,m_{s_2}},$$

where each $s_{1,i} \in S_1 \cup S_1^{-1}$. Since S_2 is finite, we can define

$$M_2 := \max_{s_2 \in S_2 \cup S_2^{-1}} m_k.$$

Similarly, for each $s_1 \in S_1 \cup S_1^{-1}$, we can write

$$s_1 = s_{2,1} s_{2,2} \cdots s_{2,n_{s_1}},$$

where each $s_{2,i} \in S_2 \cup S_2^{-1}$. Since S_1 is finite, we can define

$$M_1 := \max_{s_1 \in S_1 \cup S_1^{-1}} n_k.$$

Thus, each generator in S_1 can be expressed as a product of at most L_1 generators in S_2, and each generator in S_2 can be expressed as a product of at most L_2 generators in S_1. The desired inequality follows by setting $L_1 := \frac{1}{M_2}$ and $L_2 := M_1$.

(2) The map $d_{\mathrm{H}} \colon X \times X \to \mathbb{R}_{\geqslant 0}$ is well-defined by the definition of X.

① Clearly $d_{\mathrm{H}}(A, B) \geqslant 0$ and $d_{\mathrm{H}}(A, B) = 0 \overset{\star}{\Longleftrightarrow} A \subset B$ and $B \subset A \iff A = B$. (The assumption that A, B are bounded closed subsets in $\mathbb{R}^n$ is used in "$\overset{\star}{\Longleftrightarrow}$".)

② $d_{\mathrm{H}}(A, B) = d_{\mathrm{H}}(B, A)$ is immediate from the definition of d_{H}.

③ Suppose $A, B, C \in X$ satisfy

$$A \subset B_{\varepsilon_1}, \quad B \subset A_{\varepsilon_1}, \quad B \subset C_{\varepsilon_2}, \quad C \subset B_{\varepsilon_2}.$$

If $x \in B_{\varepsilon_1}$, then there exists $b \in B$ such that $d(x,b) < \varepsilon_1$. Since $b \in B \subset C_{\varepsilon_2}$, there exists $c \in C$ such that $d(b,c) < \varepsilon_2$. Then

$$d(x,c) \leqslant d(x,b) + d(b,c) < \varepsilon_1 + \varepsilon_2$$

and then $A \subset B_{\varepsilon_1} \subset C_{\varepsilon_1+\varepsilon_2}$. The same argument shows that $C \subset A_{\varepsilon_1+\varepsilon_2}$. Hence

$$d_{\mathrm{H}}(A,C) \leqslant \varepsilon_1 + \varepsilon_2.$$

Taking the infimum over all such ε_1 and ε_2 gives

$$d_{\mathrm{H}}(A,C) \leqslant d_{\mathrm{H}}(A,B) + d_{\mathrm{H}}(B,C).$$

Using the assumption that A, B are bounded and closed, one can show that

$$A \subset B_\varepsilon,\ \forall\, \varepsilon > \frac{1}{2} \iff A \subset \bigcap_{\varepsilon > \frac{1}{2}} B_\varepsilon = \overline{B_{\frac{1}{2}}},$$

and similarly

$$B \subset A_\varepsilon,\ \forall\, \varepsilon > \frac{1}{2} \iff B \subset \overline{A_{\frac{1}{2}}} = \overline{\mathbb{B}\big(0, \tfrac{3}{2}\big)}.$$

Hence, the open ball centered at "$A =$ the closed unit disk" and with radius $\frac{1}{2}$ can be expressed as

$$\mathbb{B}_{d_{\mathrm{H}}}\big(A, \tfrac{1}{2}\big) = \Big\{ B \in X : A \subset \overline{B_{\frac{1}{2}}} \text{ and } B \subset \overline{\mathbb{B}\big(0, \tfrac{3}{2}\big)} \Big\}. \qquad \square$$

Problem 6 (Metric-preserving functions) Let $f\colon [0,+\infty) \to [0,+\infty)$ be a function (which need not be continuous). We say f is a *metric-preserving function* if for any metric space (X,d), the map $\widetilde{d}\colon X \times X \to \mathbb{R}$ defined by $\widetilde{d}(x,y) := f(d(x,y))$ is a metric on X.

(1) Prove: $f(t) = \frac{t}{1+t}$ is a metric-preserving function.

(2) Prove: if f is a metric-preserving function, then $f^{-1}(\{0\}) = \{0\}$ and f is sub-additive:

$$f(\alpha+\beta) \leqslant f(\alpha) + f(\beta), \quad \forall \alpha, \beta \in [0,+\infty).$$

(3) Prove: a function $f\colon [0,+\infty) \to [0,+\infty)$ satisfying $f^{-1}(\{0\}) = \{0\}$ is metric-preserving if any one of the following conditions holds:

① f is non-decreasing and sub-additive.

② f is concave.

③ There exists a constant $c > 0$ so that for any $x > 0$, $f(x) \in [c, 2c]$.

Proof (1) ① Clearly $f(d(x,y)) \geqslant 0$ and $f(d(x,y)) = 0 \iff d(x,y) = 0 \iff x = y$.

② $d(x,y) = d(y,x)$ implies $f(d(x,y)) = f(d(y,x))$.

③ Since the function $f(t) = \frac{t}{1+t}$ is increasing on $[0,+\infty)$, we have

$$f(d(x,z)) \leqslant f(d(x,y) + d(y,z)) = \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)}$$

$$\leqslant \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$
$$\leqslant \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = f(d(x,y)) + f(d(y,z)).$$

(2) Consider $\mathbb{R}$ endowed with the standard metric $d(x,y) = |x-y|$. Clearly $f(0) = 0$, and if there exists $x > 0$ such that $f(x) = 0$, then $f(d(x,0)) = f(x) = 0$, which is impossible since $x \neq 0$. Thus $f^{-1}(\{0\}) = \{0\}$. For any $\alpha, \beta \in [0, +\infty)$, we have

$$f(\alpha+\beta) = f(d(\alpha, -\beta)) \leqslant f(d(\alpha, 0)) + f(d(0, -\beta)) = f(\alpha) + f(\beta).$$

(3) In each case we only need to show that $f \circ d$ satisfies the triangle inequality. Let $x, y, z \in X$.

① $f(d(x,z)) \leqslant f(d(x,y) + d(y,z)) \leqslant f(d(x,y)) + f(d(y,z))$.

② We first show that f is sub-additive. Suppose $0 \leqslant r \leqslant s$ and $t = r+s$. Let $p > 0$ be such that

$$s = pt + (1-p)r,$$

then

$$r = (1-p)s.$$

By the concavity of f, we have

$$f(s) \geqslant pf(t) + (1-p)f(r),$$
$$f(t) \geqslant (1-p)f(s).$$

Adding the above inequalities gives

$$f(r) + f(s) \geqslant pf(t) + (1-p)[f(r) + f(s)],$$

or equivalently

$$f(r+s) = f(t) \leqslant f(r) + f(s).$$

Next we shall show that f is non-decreasing. Suppose to the contrary that there exist $r < s$ such that $f(r) > f(s)$ (it follows that $r > 0$). Let

$$q = \frac{f(s)}{f(r)} \in (0,1), \quad u = \frac{sf(r) - rf(s)}{f(r) - f(s)} > 0, \quad v = r > 0,$$

then

$$s = (1-q)u + qv.$$

Since f is concave, we have

$$f(s) \geqslant (1-q)f(u) + qf(v) = (1-q)f(u) + f(s),$$

which implies $f(u) \leqslant 0$ and then $f(u) = 0$, a contradiction. Thus f is non-decreasing and sub-additive, and by ① it is metric-preserving.

③ Without loss of generality, we may assume x, y, z are distinct. Then

$$f(d(x,z)) \leqslant 2c = c + c \leqslant f(d(x,y)) + f(d(y,z)).$$

□

Problem 7 (Urysohn's lemma) Let (X, d) be a metric space. For any subset $A \subset X$, define

$$d_A\colon X \to [0, +\infty), \quad x \mapsto d_A(x) = \inf_{a \in A} d(x, a).$$

Prove:

(1) d_A is a continuous function on X.

(2) A is closed if and only if $d_A(x) = 0$ implies $x \in A$.

(3) **(Urysohn's lemma for metric spaces)** If A and B are closed subsets in (X, d) and $A \cap B = \varnothing$. Then there exists a continuous function $f\colon X \to [0, 1]$ such that

$$f \equiv 0 \text{ on } A \quad \text{and} \quad f \equiv 1 \text{ on } B.$$

Proof (1) For any $x_1, x_2 \in X$ and $a \in A$, we have

$$d_A(x_1) \leqslant d(x_1, a) \leqslant d(x_1, x_2) + d(x_2, a).$$

Taking the infimum over $a \in A$ gives

$$d_A(x_1) \leqslant d(x_1, x_2) + d_A(x_2).$$

Switching the roles of x_1 and x_2 in the above gives

$$d_A(x_2) \leqslant d(x_1, x_2) + d_A(x_1).$$

Hence

$$|d_A(x_1) - d_A(x_2)| \leqslant d(x_1, x_2),$$

which shows that d_A is Lipschitz continuous.

(2) ($\Leftarrow$) Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in A that converges to $x \in X$. Then $\lim\limits_{n\to\infty} d(x_n, x) = 0$, thus $d_A(x) = 0$. By assumption, $x \in A$, hence A is closed.

($\Rightarrow$) If A is closed and $d_A(x) = 0$, then for any $n \in \mathbb{N}$ there exists $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Now $a_n \to x$, and since A is closed, $x \in A$.

(3) Consider

$$f\colon X \to [0, 1], \quad x \mapsto \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

This function is well-defined since if the denominator $d_A(x) + d_B(x)$ is zero, then $d_A(x) = d_B(x) = 0$, which by (2) implies $x \in A \cap B = \varnothing$ as A and B are closed. It is clear that $f \equiv 0$ on A and $f \equiv 1$ on B, and by (1) f is continuous. □

Problem 8 (Uniform convergence as a metric convergence) Let X be a set, (Y, d_Y) be a metric space, $f_n\colon X \to (Y, d_Y)$ ($n \in \mathbb{N}$) and $f\colon X \to (Y, d_Y)$ be maps.

(1) Define "uniform convergence": f_n converge uniformly to f on X if...

(2) On the set $Y^X = \{f\colon X \to Y \mid f \text{ is any map}\}$, define

$$\bar{d}(f,g) \coloneqq \sup_{x\in X} \frac{d_Y(f(x),g(x))}{1+d_Y(f(x),g(x))}.$$

① Prove: $\bar{d}$ is a metric on Y^X.

② Prove: f_n converge to f uniformly if and only if as elements in the metric space $\left(Y^X,\bar{d}\right)$, f_n converge to f.

(3) Suppose (X,d_X) is also a metric space, and f_n are continuous maps that converge to f uniformly. Prove: f is continuous.

Proof (1) A sequence of functions (f_n) converges uniformly to a limiting function f on X if given any arbitrarily small positive number ε, a number N can be found such that each of the functions $f_N, f_{N+1}, f_{N+2}, \cdots$ differs from f by less than ε at every point $x \in X$, namely

$$d_Y(f_k(x), f(x)) < \varepsilon, \quad \forall k \geqslant N,\ x\in X.$$

(2) ① $\bar{d}$ is a metric on Y^X since

- $\bar{d}(f,g) \geqslant 0$ and $\bar{d}(f,g) = 0 \iff d_Y(f(x),g(x)) = 0$ for all $x\in X \iff f = g$.
- $\bar{d}(f,g) = \bar{d}(g,f)$ since $d_Y(f(x),g(x)) = d_Y(g(x),f(x))$ for all $x \in X$.
- Since $\varphi(t) = \frac{t}{1+t}$ is a metric-preserving function by Problem 6 (1), we have

$$\frac{d_Y(f(x),h(x))}{1+d_Y(f(x),h(x))} \leqslant \frac{d_Y(f(x),g(x))}{1+d_Y(f(x),g(x))} + \frac{d_Y(g(x),h(x))}{1+d_Y(g(x),h(x))}$$

for all $x\in X$. Taking the supremum over $x \in X$ gives

$$\bar{d}(f,h) \leqslant \sup_{x\in X}\left(\frac{d_Y(f(x),g(x))}{1+d_Y(f(x),g(x))} + \frac{d_Y(g(x),h(x))}{1+d_Y(g(x),h(x))}\right) \leqslant \bar{d}(f,g) + \bar{d}(g,h).$$

② ($\Rightarrow$) If $f_n \rightrightarrows f$, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in X} d_Y(f_n(x),f(x)) < \frac{\varepsilon}{2}, \quad \forall n > N.$$

Since the function $\varphi(t) = \frac{t}{1+t}$ is increasing on $[0,+\infty)$, we have

$$\frac{d_Y(f_n(x),f(x))}{1+d_Y(f_n(x),f(x))} \leqslant \frac{\frac{\varepsilon}{2}}{1+\frac{\varepsilon}{2}} < \frac{\varepsilon}{2}, \quad \forall n > N$$

for all $x \in X$. Then $\bar{d}(f_n,f) \leqslant \frac{\varepsilon}{2} < \varepsilon$ for all $n > N$, hence $f_n \to f$ in $\left(Y^X,\bar{d}\right)$.

($\Leftarrow$) If $f_n \to f$ in $\left(Y^X,\bar{d}\right)$, then for any $\varepsilon\in(0,1)$ there exists $N\in\mathbb{N}$ such that

$$\bar{d}(f_n,f) < \frac{\varepsilon}{2}, \quad \forall n > N.$$

Therefore

$$\frac{d_Y(f_n(x),f(x))}{1+d_Y(f_n(x),f(x))} < \frac{\varepsilon}{2}$$

for all $x \in X$ and $n > N$, which implies

$$d_Y(f_n(x), f(x)) < \frac{\frac{\varepsilon}{2}}{1 - \frac{\varepsilon}{2}} < \frac{\frac{\varepsilon}{2}}{1 - \frac{1}{2}} = \varepsilon$$

for all $x \in X$ and $n > N$. Hence $f_n \rightrightarrows f$ on X.

(3) For any $\varepsilon > 0$, since the sequence of functions (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that

$$d_Y(f_N(t), f(t)) < \frac{\varepsilon}{3}, \quad \forall t \in X.$$

Moreover, since f_N is continuous on X, for every $x \in X$ there exists an open neighborhood U such that

$$d_Y(f_N(x), f_N(y)) < \frac{\varepsilon}{3}, \quad \forall y \in U.$$

Now the triangle inequality gives

$$\begin{aligned} d_Y(f(x), f(y)) &\leqslant d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall y \in U. \end{aligned}$$

Hence f is continuous at every point $x \in X$. □

PSet 2, Part 1

Problem 9 (The Sorgenfrey line) On the set $X = \mathbb{R}$, define

$$\mathscr{T}_{\text{Sorgenfrey}} = \{U \subset \mathbb{R} : \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } [x, x+\varepsilon) \subset U\}.$$

(1) Check: $\mathscr{T}_{\text{Sorgenfrey}}$ is a topology.

(2) Prove: every left-closed-right-open interval $[a, b)$ is both open and closed.

(3) Prove: every open interval (a, b) is still open with respect to $\mathscr{T}_{\text{Sorgenfrey}}$.

(4) Show that there is no metric d on $\mathbb{R}$ such that $\mathscr{T}_{\text{Sorgenfrey}}$ is the metric topology $\mathscr{T}_d$.

Proof (1) ① Clearly $\varnothing, \mathbb{R} \in \mathscr{T}_{\text{Sorgenfrey}}$.

② If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}_{\text{Sorgenfrey}}$, then for any $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$, there exists $\lambda \in \Lambda$ such that $x \in U_\lambda$, and hence there exists $\varepsilon > 0$ such that $[x, x+\varepsilon) \subset U_\lambda \subset \bigcup_{\alpha \in \Lambda} U_\alpha$. Therefore $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}_{\text{Sorgenfrey}}$.

③ If $U_1, U_2 \in \mathscr{T}_{\text{Sorgenfrey}}$, then for any $x \in U_1 \cap U_2$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $[x, x+\varepsilon_1) \subset U_1$ and $[x, x+\varepsilon_2) \subset U_2$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then $[x, x+\varepsilon) \subset U_1 \cap U_2$. Therefore $U_1 \cap U_2 \in \mathscr{T}_{\text{Sorgenfrey}}$.

(2) ① For any $x \in [a, b)$, since $\varepsilon = b - x > 0$ satisfies $[x, x+\varepsilon) \subset [a, b)$, we see that $[a, b)$ is open.

② For any $x \notin [a, b)$, if $x < a$, then $\varepsilon = a - x > 0$ satisfies $[x, x+\varepsilon) \cap [a, b) = \varnothing$; if $x \geqslant b$, then $\varepsilon = 1 > 0$ satisfies $[x, x+\varepsilon) \cap [a, b) = \varnothing$. Therefore $[a, b)$ is closed.

(3) For any $x \in (a, b)$, since $\varepsilon = b - x > 0$ satisfies $[x, x+\varepsilon) \subset (a, b)$, we see that (a, b) is open.

(4) Suppose that $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is a metrizable space and let d be a metric on $\mathbb{R}$ inducing the topology $\mathscr{T}_{\text{Sorgenfrey}}$. For each $x \in \mathbb{R}$, since the interval $[x, x+1)$ is open by (2), we can choose $\varepsilon_x > 0$ such that $\mathbb{B}_d(x, \varepsilon_x) \subset [x, x+1)$. For each $n \in \mathbb{N}$, let $M_n = \{x \in \mathbb{R} : \varepsilon_x \geqslant \frac{1}{n}\}$. For distinct $x, y \in M_n$ with $x < y$, we have

$$\mathbb{B}_d(y, \tfrac{1}{n}) \subset \mathbb{B}_d(y, \varepsilon_y) \subset [y, y+1),$$

and since $x \notin [y, y+1)$, we get $x \notin \mathbb{B}_d(y, \frac{1}{n})$. Thus

$$d(x, y) \geqslant \frac{1}{n}. \tag{9–1}$$

On the other hand, by the definition of $\mathscr{T}_{\text{Sorgenfrey}}$, for each $x \in M_n$, there exists $\eta > 0$ such that $[x, x+\eta) \subset \mathbb{B}_d(x, \frac{1}{2n})$. Let $r_x \in \mathbb{Q} \cap [x, x+\eta)$, then

$$d(x, r_x) < \frac{1}{2n}. \tag{9–2}$$

From (9–1) and (9–2), we see that for distinct $x, y \in M_n$, the corresponding r_x, r_y are distinct. Now the coutability of $\mathbb{Q}$ implies that M_n is countable for each $n \in \mathbb{N}$, and hence $\mathbb{R} = \bigcup_{n \in \mathbb{N}} M_n$ is countable, which is a contradiction. $\square$

Problem 10 ("Uniform continuity" is not a topological conception) Let (X, d_X) and (Y, d_Y) be metric spaces. We say a map $f\colon (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t.}\ d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

(1) Prove: $d_0(x, y) := |\arctan(x) - \arctan(y)|$ is a metric on $\mathbb{R}$.

(2) Prove: the metric d_0 and the absolute value metric $d(x, y) = |x - y|$ on $\mathbb{R}$ are topologically equivalent. Are they strongly equivalent?

(3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the identity map, i.e., $f(x) = x$. Is $f\colon (\mathbb{R}, d) \to (\mathbb{R}, d_0)$ uniformly continuous? Is $f\colon (\mathbb{R}, d_0) \to (\mathbb{R}, d)$ uniformly continuous? Conclude that "uniform continuity" is not a topological conception.

(4) Is "uniform continuity" preserved if we replace metrics d_X, d_Y by strongly equivalent ones? Prove your conclusion.

Proof (1) ① Clearly $d_0(x, y) \geqslant 0$ for all $x, y \in \mathbb{R}$ and the injectivity of $\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ implies $d_0(x, y) = 0$ if and only if $x = y$.

② For all $x, y \in \mathbb{R}$, $d_0(x, y) = |\arctan(x) - \arctan(y)| = |\arctan(y) - \arctan(x)| = d_0(y, x)$.

③ For all $x, y, z \in \mathbb{R}$,

$$\begin{aligned} d_0(x, z) = |\arctan(x) - \arctan(z)| &\leqslant |\arctan(x) - \arctan(y)| + |\arctan(y) - \arctan(z)| \\ &= d_0(x, y) + d_0(y, z). \end{aligned}$$

(2) Since the map

$$\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is a homeomorphism, every open ball in $(\mathbb{R}, d_0)$ is contained in some open ball in $(\mathbb{R}, d)$ and vice versa. Therefore the metrics d_0 and d are topologically equivalent. However, since d_0 is bounded and d is not, they are not strongly equivalent.

(3) The mean value theorem implies that for all $x, y \in \mathbb{R}$,

$$|\arctan(x) - \arctan(y)| \leqslant |x - y|,$$

which shows that $f\colon (\mathbb{R}, d) \to (\mathbb{R}, d_0)$ is uniformly continuous. On the other hand, we also have

$$|\arctan(n+1) - \arctan(n)| \leqslant \frac{1}{1+n^2} \xrightarrow{n\to\infty} 0.$$

Hence for any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $d_0(n+1, n) = |\arctan(n+1) - \arctan(n)| < \delta$, but $d(n+1, n) = |(n+1) - n| = 1$. Therefore $f\colon (\mathbb{R}, d_0) \to (\mathbb{R}, d)$ is not uniformly continuous. From this fact and (2) we conclude that "uniform continuity" is not a topological conception.

(4) Suppose d_X and d'_X are two strongly equivalent metrics on X and d_Y and d'_Y are two strongly equivalent metrics on Y, namely

$$d_X \leqslant C d'_X, \quad d'_Y \leqslant K d_Y.$$

for some $C, K > 0$. If $f\colon (X, d_X) \to (Y, d_Y)$ is uniformly continuous, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$. Now for any $x_1, x_2 \in X$ with $d'_X(x_1, x_2) < \frac{\delta}{C}$, we have $d_X(x_1, x_2) \leqslant C d'_X(x_1, x_2) < \delta$, and then

$$d'_Y(f(x_1), f(x_2)) \leqslant K d_Y(f(x_1), f(x_2)) < K\varepsilon.$$

Therefore $f\colon (X, d'_X) \to (Y, d'_Y)$ is uniformly continuous, which shows that "uniform continuity" is preserved by strongly equivalent metrics. $\square$

Problem 11 (Equivalence of neighborhoods axioms and open sets axioms)

(1) Given a neighborhood structure $\mathscr{N}$ on X, one can define

$$\mathscr{T} = \{U \subset X : U \in \mathscr{N}(x) \text{ for any } x \in U\}.$$

Check: $\mathscr{T}$ is a topology on X, i.e., it satisfies (O1)-(O3).

(2) Given a topology $\mathscr{T}$ on X, one can define, for any $x \in X$,

$$\mathscr{N}(x) = \{N \subset X : \exists U \in \mathscr{T} \text{ s.t. } x \in U \text{ and } U \subset N\}.$$

Check: $\mathscr{N}$ is a neighborhood structure on X, i.e., it satisfies (N1)-(N4).

(3) You may have noticed that in part (1), you used only (N1)-(N3). Can we conclude that the set of axioms (N1)-(N3) is equivalent to the set of axioms (O1)-(O3)?

(4) Prove: the set of axioms (N1)-(N4) is equivalent to the set of axioms (O1)-(O3). Namely, the processes $\mathscr{T} \rightsquigarrow \mathscr{N}$ and $\mathscr{N} \rightsquigarrow \mathscr{T}$ described above are inverse to each other.

Proof (1) By definition, $U \subset X$ is open iff and only if $U = \mathrm{Int}(U)$.

(O1) The relation $\operatorname{Int}\varnothing \subset \varnothing$ implies that $\operatorname{Int}\varnothing = \varnothing$; thus $\varnothing \in \mathscr{T}$. If $x \in X$, then x has at least one neighborhood N; but $N \subset X$ and so X is a neighborhood of x by (N2). Thus $X \in \mathscr{T}$.

(O2) For $U, V \in \mathscr{T}$, if $U \cap V = \varnothing$, then it is open. If it is not empty, let $x \in U \cap V$. Then U and V are both neighborhoods of x, and hence $U \cap V$ is a neighborhood of x by (N3). Thus $U \cap V \in \mathscr{T}$.

(O3) Suppose $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}$ and let $U = \bigcup_{\alpha\in\Lambda} U_\alpha$. If U is empty, then it is open. If not, let $x \in U$. Then $x \in U_\lambda$ for some $\lambda \in \Lambda$, and U_λ, being open, is a neighborhood of x. But $U_\lambda \subset U$. So U is also a neighborhood of x by (N2).

(2) $X \in \mathscr{N}(x)$ for all $x \in X$, hence $\mathscr{N}(x) \neq \varnothing$.

(N1) If $N \in \mathscr{N}(x)$, then there exists $U \in \mathscr{T}$ such that $x \in U \subset N$.

(N2) If $M \supset N$ and $N \in \mathscr{N}(x)$, then there exists $U \in \mathscr{T}$ such that $x \in U \subset N \subset M$. Thus $M \in \mathscr{N}(x)$.

(N3) If $N_1, N_2 \in \mathscr{N}(x)$, then there exist $U_1, U_2 \in \mathscr{T}$ such that $x \in U_1 \subset N_1$ and $x \in U_2 \subset N_2$. Let $U = U_1 \cap U_2$, then $U \in \mathscr{T}$ by (O2) and $x \in U \subset N_1 \cap N_2$. Thus $N_1 \cap N_2 \in \mathscr{N}(x)$.

(N4) If $N \in \mathscr{N}(x)$, then there exists $U \in \mathscr{T}$ such that $x \in U \subset N$, and by definition $U \in \mathscr{N}(x)$. Moreover, for any $y \in U$, from $y \in U \subset N$ we see that $N \in \mathscr{N}(y)$.

(3) No, since we have not shown that the above two processes are inverse to each other. A counterexample is given below. Let $X = \{0, 1, 2\}$ and

$$\mathscr{N}(0) = \{\{0,1\}, X\}, \quad \mathscr{N}(1) = \{\{1,2\}, X\}, \quad \mathscr{N}(2) = \{\{1,2\}, X\}.$$

Then one can verify that (N1), (N2) and (N3) are satisfied, while (N4) is not, since $\{0,1\} \in \mathscr{N}(0)$ but there is no neighborhood M of 0 such that $M \subset \{0,1\}$ and $\{0,1\} \in \mathscr{N}(y)$ for all $y \in M$. In this example, the $\mathscr{N} \xrightarrow{(1)} \mathscr{T} \xrightarrow{(2)} \mathscr{N}'$ process yields

$$\mathscr{T} = \{\varnothing, \{1,2\}, X\},$$
$$\mathscr{N}'(0) = \{X\}, \quad \mathscr{N}'(1) = \{\{1,2\}, X\}, \quad \mathscr{N}'(2) = \{\{1,2\}, X\}.$$

Clearly $\mathscr{N}' \neq \mathscr{N}$, and hence (N1)-(N3) are not equivalent to (O1)-(O3).

(4) ① Let us show first that $\mathscr{N} \xrightarrow{(1)} \mathscr{T} \xrightarrow{(2)} \mathscr{N}$. Let $\mathscr{N}$ be a neighborhood structure on X.

- ⋄ Let us prove first the inclusion $\mathscr{N}(\mathscr{T}(\mathscr{N})) \subset \mathscr{N}$. Let $V \in \mathscr{N}_x(\mathscr{T}(\mathscr{N}))$ for some $x \in X$. By definition in (2), there exists $U \in \mathscr{T}(\mathscr{N})$ such that $x \in U \subset V$. By definition in (1), $U \in \mathscr{N}(y)$ for all $y \in U$. Since $V \supset U$, we have $V \in \mathscr{N}(y)$ by (N2) for each $y \in U$. In particular, $V \in \mathscr{N}(x)$. Therefore $\mathscr{N}(\mathscr{T}(\mathscr{N})) \subset \mathscr{N}$.
- ⋄ Conversely, let $V \in \mathscr{N}_x$ for some $x \in X$. Define

$$U = \{y \in X : V \in \mathscr{N}(y)\}.$$

Then $x \in U$ and by (N1) $U \subset V$. By (N4) for each $y \in U$ there exists a neighborhood W of y such that V is a neighborhood for each point of W. Hence $W \subset U$. In other words, each point of U has a neighborhood contained in U. Therefore $U \in \mathscr{T}(\mathscr{N})$ and then $V \in \mathscr{N}(\mathscr{T}(\mathscr{N}))$. Thus $\mathscr{N} \subset \mathscr{N}(\mathscr{T}(\mathscr{N}))$.

② Now we shall show that $\mathscr{T} \xrightarrow{(2)} \mathscr{N} \xrightarrow{(1)} \mathscr{T}$. Let $\mathscr{T}$ be a topology on X.

- ⋄ Let us prove first the inclusion $\mathscr{T} \subset \mathscr{T}(\mathscr{N}(\mathscr{T}))$. Let $U \in \mathscr{T}$, then by definition in (2) $U \in \mathscr{N}(x)$ for all $x \in U$, which means $U \in \mathscr{T}(\mathscr{N}(\mathscr{T}))$. Therefore $\mathscr{T} \subset \mathscr{T}(\mathscr{N}(\mathscr{T}))$.
- ⋄ Conversely, let $U \in \mathscr{T}(\mathscr{N}(\mathscr{T}))$, then U is the neighborhood of each point in $\mathscr{N}(\mathscr{T})$. This means that thee exists $V_x \in \mathscr{T}$ for each $x \in U$ such that $x \in V_x \subset U$. Then $U = \bigcup_{x\in U} V_x \in \mathscr{T}$ by (O3). Therefore $\mathscr{T}(\mathscr{N}(\mathscr{T})) \subset \mathscr{T}$. □

Problem 12 (Furstenberg's topological proof of the infinitude of primes) For any $a, b \in \mathbb{Z}$ with $b > 0$ we define

$$N_{a,b} \coloneqq \{a + nb : n \in \mathbb{Z}\}.$$

(1) Define a topology on $\mathbb{Z}$ by

$$\mathscr{T}_{\text{Furs}} = \{U \subset \mathbb{Z} : \text{ either } U = \varnothing, \text{ or } \forall a \in U, \exists\, b \in \mathbb{Z}_{>0} \text{ s.t. } N_{a,b} \subset U\}.$$

① Prove: $\mathscr{T}_{\text{Furs}}$ is a topology on $\mathbb{Z}$.

② Prove: each $N_{a,b}$ is open.

③ Prove: each $N_{a,b}$ is closed.

④ Let $\mathscr{P} = \{2, 3, \cdots\}$ be the set of all prime numbers. Prove:

$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p\in\mathscr{P}} N_{0,p}.$$

⑤ Conclude that $\mathscr{P}$ is not a finite set.

(2) Define a function $d\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(a,b) = \begin{cases} 0, & a = b, \\ 2^{-\tau(a-b)}, & a \neq b, \end{cases}$$

where $\tau(a-b)$ is the smallest positive integer that does not divide $a - b$.

① Prove: d is a metric on $\mathbb{Z}$.

② Describe the metric balls $\mathbb{B}(a, r)$.

③ Show that the metric topology generated by d is the topology $\mathscr{T}_{\text{Furs}}$ above.

Proof (1) ① It is clear that $\varnothing, \mathbb{Z} \in \mathscr{T}_{\text{Furs}}$. If $U, V \in \mathscr{T}_{\text{Furs}}$ (we may assume both U and V are nonempty), then for any $a \in U \cap V$, there exist $b_1, b_2 \in \mathbb{Z}_{>0}$ such that $N_{a,b_1} \subset U$ and $N_{a,b_2} \subset V$. Let $b = b_1 b_2 \in \mathbb{Z}_{>0}$, then $N_{a,b} \subset U \cap V$, which shows that $U \cap V \in \mathscr{T}_{\text{Furs}}$. If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}_{\text{Furs}}$, then for any $a \in \bigcup_{\alpha\in\Lambda} U_\alpha$, there exists $\lambda \in \Lambda$ such that $a \in U_\lambda$, and hence there exists $b \in \mathbb{Z}_{>0}$ such that $N_{a,b} \subset U_\lambda \subset \bigcup_{\alpha\in\Lambda} U_\alpha$. Therefore $\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}_{\text{Furs}}$.

② For any $a' = a + nb \in N_{a,b}$, set $b' = b \in \mathbb{Z}_{>0}$, then $N_{a',b'} = \{a + nb + mb : m \in \mathbb{Z}\} = N_{a,b}$. Therefore $N_{a,b}$ is open.

③ Note that $N_{a,b} = \mathbb{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$, then by ② we see that $N_{a,b}$ is closed.

④ $\bigcup_{p\in\mathscr{P}} N_{0,p} = \bigcup_{p\in\mathscr{P}} p\mathbb{Z} = \mathbb{Z}\setminus\{1,-1\}$.

⑤ If $\mathscr{P}$ is finite, then $\bigcup_{p\in\mathscr{P}} N_{0,p}$ is a finite union of closed sets, which is closed. Then $\{1,-1\} = \mathbb{Z}\setminus \bigcup_{p\in\mathscr{P}} N_{0,p}$ is open. However, for $1 \in \{1,-1\}$, there exists no $b \in \mathbb{Z}_{>0}$ such that $N_{1,b} = \{1+nb : n\in\mathbb{Z}\} \subset \{1,-1\}$, which is a contradiction. Therefore $\mathscr{P}$ is not a finite set.

(2) ① It is clear that $d(a,b) \geqslant 0$ for all $a,b\in\mathbb{Z}$ and $d(a,b)=0$ if and only if $a=b$. Since $\tau(a-b) = \tau(b-a)$ for any distinct $a,b\in\mathbb{Z}$, we have $d(a,b)=d(b,a)$ for all $a,b\in\mathbb{Z}$. For distinct $a,b,c\in\mathbb{Z}$, suppose $\tau(a-b)=m\geqslant 2$, $\tau(b-c)=n\geqslant 2$, and let $k=\min\{m,n\}\geqslant 2$. Then $1,\cdots,k-1$ all divide $a-b$ and $b-c$, and hence devide $a-c$. Therefore $\tau(a-c)\geqslant k$, and

$$d(a,c) = 2^{-\tau(a-c)} \leqslant 2^{-\min\{m,n\}} \leqslant 2^{-m}+2^{-n} = d(a,b)+d(b,c).$$

② Since $\tau(a-b)\geqslant 2$ for $a\neq b$, if $r>\frac{1}{4}$, then $\mathbb{B}(a,r)=\mathbb{Z}$; if $r\leqslant\frac{1}{4}$, then $\tau(a-b) > \dfrac{\log\frac{1}{r}}{\log 2}$ and

$$\mathbb{B}(a,r) = \left\{ b\in\mathbb{Z} : 1,\cdots,\left\lfloor \frac{\log\frac{1}{r}}{\log 2}\right\rfloor \text{ all devide } a-b \right\}. \tag{12–1}$$

③ $\boxed{\mathscr{T}_{\text{Furs}}\subset\mathscr{T}_d}$ For any $U\in\mathscr{T}_{\text{Furs}}\setminus\{\varnothing\}$ and $a\in U$, by the definition of $\mathscr{T}_{\text{Furs}}$, there exists $b\in\mathbb{Z}_{>0}$ such that $N_{a,b}\subset U$. Let $r=2^{-b}$, then

$$\mathbb{B}(a,r) = \{c\in\mathbb{Z} : 1,\cdots,b \text{ all devide } a-c\}.$$

In particular, $b\mid(c-a)$ for all $c\in\mathbb{B}(a,r)$, which implies $\mathbb{B}(a,r)\subset N_{a,b}\subset U$. Thus $U\in\mathscr{T}_d$, and hence $\mathscr{T}_{\text{Furs}}\subset\mathscr{T}_d$.

$\boxed{\mathscr{T}_d\subset\mathscr{T}_{\text{Furs}}}$ For any $V\in\mathscr{T}_d\setminus\{\varnothing\}$ and $a\in V$, there exists $r>0$ such that $\mathbb{B}(a,r)\subset V$. Set $k=\left\lfloor \dfrac{\log\frac{1}{r}}{\log 2}\right\rfloor$, then from (12–1) we see that $N_{a,k!}\subset\mathbb{B}(a,r)\subset V$. Hence $V\in\mathscr{T}_{\text{Furs}}$, and thus $\mathscr{T}_d\subset\mathscr{T}_{\text{Furs}}$. $\square$

PSet 2, Part 2

Problem 13 (Subspace topology) Given any topological space $(X,\mathcal{T})$ and any subspace $A\subset X$, define the subspace topology on Y to be

$$\mathcal{T}_A = \{U\cap A : U\in\mathcal{T}_X\}.$$

Prove:

(1) $\mathcal{T}_A$ is a topology on A.

(2) Suppose $B\subset A\subset X$, then "the subspace topology $\mathcal{T}_B$ on B (view B as a subset in $(X,\mathcal{T}_X)$)" coincides with "the subspace topology $\widetilde{\mathcal{T}_B}$ on B (view B as a subset in $(A,\mathcal{T}_A)$)".

(3) The inclusion map $\iota\colon (A,\mathcal{T}_A)\to(X,\mathcal{T}_X)$ is continuous. Moreover, the subspace topology $\mathcal{T}_A$ is the weakest topology on A so that the inclusion map is continuous.

(4) If $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous, then $f|_A\colon (A,\mathcal{T}_A)\to(Y,\mathcal{T}_Y)$ is continuous.

(5) A map $g\colon (Y,\mathcal{T}_Y) \to (A,\mathcal{T}_A)$ is continuous if and only if $\iota \circ g\colon (Y,\mathcal{T}_Y) \to (X,\mathcal{T}_X)$ is continuous.

Proof (1) ① $\varnothing \cap A = \varnothing$ and $X \cap A = A$ imply $\varnothing, A \in \mathcal{T}_A$.

② If $V_1, V_2 \in \mathcal{T}_A$, let $U_1, U_2 \in \mathcal{T}_X$ such that $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$. Then $U_1 \cap U_2 \in \mathcal{T}$ implies $V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A \in \mathcal{T}_A$.

③ If $\{V_\alpha : \alpha \in \Lambda\} \subset \mathcal{T}_A$, let $\{U_\alpha : \alpha \in \Lambda\} \subset \mathcal{T}_X$ such that $V_\alpha = U_\alpha \cap A$ for each $\alpha \in \Lambda$. Then $\displaystyle\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}$ implies $\displaystyle\bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda} (U_\alpha \cap A) = \left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) \cap A \in \mathcal{T}_A$.

(2) $\boxed{\mathcal{T}_B \subset \widetilde{\mathcal{T}_B}}$ For any $V \in \mathcal{T}_B$, there exists $U \in \mathcal{T}$ such that $V = U \cap B$. Since $B \subset A \subset X$, we have $V = (U \cap A) \cap B$, which implies $V \in \widetilde{\mathcal{T}_B}$ as $U \cap A \in \mathcal{T}_A$.

$\boxed{\widetilde{\mathcal{T}_B} \subset \mathcal{T}_B}$ For any $W \in \widetilde{\mathcal{T}_B}$, there exists $V \in \mathcal{T}_A$ such that $W = V \cap B$. Since $V = U \cap A$ for some $U \in \mathcal{T}$, we have $W = (U \cap A) \cap B = U \cap B \in \mathcal{T}_B$.

(3) For any open set $U \subset X$, we have $\iota^{-1}(U) = U \cap A \in \mathcal{T}_A$. Hence ι is continuous. For the inclusion map $A \hookrightarrow X$ to be continuous, $U \cap A$ must be open in A for each $U \in \mathcal{T}$, which means $\mathcal{T}_A$ is the weakest topology on A so that the inclusion map is continuous.

(4) Since $f|_A = f \circ \iota$ is the composition of continuous maps, it is continuous.

(5) ($\Leftarrow$) Suppose $\iota \circ g\colon (Y,\mathcal{T}_Y) \to (X,\mathcal{T}_X)$ is continuous. If U is any open subset of A, there is an open subset $V \subset X$ such that $U = A \cap V = \iota^{-1}(V)$. Thus

$$g^{-1}(U) = g^{-1}\big(\iota^{-1}(V)\big) = (\iota \circ g)^{-1}(V),$$

which is open in Y by our continuity assumption. This proves that f is continuous.

($\Rightarrow$) Suppose that $g\colon (Y,\mathcal{T}_Y) \to (A,\mathcal{T}_A)$ is continuous. For any open subset $V \subset X$, we have

$$(\iota \circ g)^{-1}(V) = g^{-1}\big(\iota^{-1}(V)\big) = g^{-1}(A \cap V),$$

which is open in Y since $A \cap V$ is open in A, so $\iota \circ g$ is continuous as well. □

Problem 14 (Convergence v.s. continuity) On the set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, define

$$\mathcal{T}_\infty = \{A : A \subset \mathbb{N} \text{ or } A^{\mathrm{c}} \text{ is a finite subset in } \mathbb{N}\}.$$

(1) Show that $\mathcal{T}_\infty$ is a topology on $\mathbb{N}_\infty$.

(2) Let (x_n) be a sequence in a topological space $(X,\mathcal{T})$, and $x_0 \in X$. Define a map $f\colon \mathbb{N}_\infty \to X$ by $f(n) = x_n$ and $f(\infty) = x_0$. Prove: $x_n \to x_0$ in $\mathcal{T}$ if and only if $f\colon (\mathbb{N}_\infty,\mathcal{T}_\infty) \to (X,\mathcal{T})$ is continuous.

Proof (1) ① Since $\varnothing \subset \mathbb{N}$ and $\mathbb{N}_\infty^{\mathrm{c}} = \varnothing$, we have $\varnothing, \mathbb{N}_\infty \in \mathcal{T}_\infty$.

② Suppose $U, V \in \mathcal{T}_\infty$. If either U or V is a subset of $\mathbb{N}$, then so is $U \cap V$. Otherwise, both U^{c} and V^{c} are finite subsets in $\mathbb{N}$, so $(U \cap V)^{\mathrm{c}} = U^{\mathrm{c}} \cup V^{\mathrm{c}}$ is also finite in $\mathbb{N}$. In either case we have $U \cap V \in \mathcal{T}_\infty$.

③ Suppose $\{U_\alpha : \alpha \in \Lambda\} \subset \mathcal{T}_\infty$. If all U_α are subsets of $\mathbb{N}$, then so is their union. Otherwise, there exists $\lambda \in \Lambda$ such that U_λ^{c} is a finite subset in $\mathbb{N}$. Then $\displaystyle\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right)^{\mathrm{c}} \subset U_\lambda^{\mathrm{c}}$ is also a finite subset in $\mathbb{N}$. In either case we have $\displaystyle\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}_\infty$.

(2) (⇒) Suppose $x_n \to x_0$ in $\mathcal{T}$ and U is any open subset of X. If $x_0 \notin U$, then $f^{-1}(U) \subset \mathbb{N}$ is open in $\mathbb{N}_\infty$. If $x_0 \in U$, from $x_n \to x_0$ we know there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. Thus $\left(f^{-1}(U)\right)^{\mathrm{c}}$ consists of at most N elements of $\mathbb{N}$ and is open in $\mathbb{N}_\infty$. In either case f is continuous.

(⇐) If $f\colon (\mathbb{N}_\infty, \mathcal{T}_\infty) \to (X, \mathcal{T})$ is continuous, then for any open subset $U \subset X$ containing x_0 we have $f^{-1}(U) \in \mathcal{T}_\infty$. Since $\infty \in f^{-1}(U)$, we have $f^{-1}(U)^{\mathrm{c}}$ is a finite subset in $\mathbb{N}$. This implies that $x_n \in U$ for all n sufficiently large, so $x_n \to x_0$. □

Problem 15 (Topologies for various continuity)

(1) **(Right continuity)** Endow $\mathbb{R}$ with the Sorgenfrey topology.

① Explore the meaning of convergence in $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$.

② Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *right continuous* if $\lim\limits_{x_n \to x_0^+} f(x_n) = f(x_0)$. Prove: a function $f\colon \mathbb{R} \to \mathbb{R}$ is right continuous if and only if the map $f\colon \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right) \to (\mathbb{R}, \mathscr{T}_{\text{usual}})$ is continuous. So people also call Sorgenfrey topology *the right continuous topology*.

(2) **(Upper semi-continuity)** Let $(X, \mathscr{T})$ be any topological space. We say a function $f\colon X \to \mathbb{R}$ us *upper semi-continuous* at a point $x_0 \in X$ if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \leqslant f(x_0) + \varepsilon$ holds for all $x \in U$, and we say f is an *upper semi-continuous* function if it is upper semi-continuous everywhere.

① Construct a topology $\mathscr{T}_{\text{u.s.c.}}$ on $\mathbb{R}$ so that a function $f\colon X \to \mathbb{R}$ is upper semi-continuous if and only if $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous.

② For which set $A \subset X$, the characteristic function $\chi_A(x)$ defined by "$\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$" is upper semi-continuous?

③ Extend $\mathscr{T}_{\text{u.s.c.}}$ to be a topology on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and prove: given any family of upper semi-continuous functions f_α, the infimum $f = \inf\limits_\alpha f_\alpha$ is upper semi-continuous.

Proof (1) ① By Problem 9 (2), every left-closed-right-open interval $[a, b)$ is open with respect to $\mathscr{T}_{\text{Sorgenfrey}}$. If $x_n \to x_0$ in $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$, then for any $\varepsilon > 0$, since $[x_0, x_0 + \varepsilon)$ is open and contains x_0, it must contain all but finitely many x_n. Conversely, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in [x_0, x_0 + \varepsilon)$ for all $n > N$, then for any open set U containing x_0, by the definition of $\mathscr{T}_{\text{Sorgenfrey}}$ we can choose $\varepsilon_0 > 0$ such that $[x_0, x_0 + \varepsilon_0) \subset U$. Thus $x_n \in U$ for all sufficiently large n. Therefore a sequence $\{x_n\}_{n=1}^\infty$ converges to x_0 in $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$ if and only if it "approaches x_0 from the right".

② (⇒) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is right continuous and U is any open subset of $(\mathbb{R}, \mathscr{T}_{\text{usual}})$. For any $x_0 \in f^{-1}(U)$, since $f\colon \mathbb{R} \to \mathbb{R}$ is right continuous at x_0, there exists $\varepsilon > 0$ such that $f(x) \in U$ for all $x \in [x_0, x_0 + \varepsilon)$, i.e., $[x_0, x_0 + \varepsilon) \subset f^{-1}(U)$. Therefore $f^{-1}(U) \in \mathscr{T}_{\text{Sorgenfrey}}$, which means f is continuous.

(⇐) Suppose the map $f\colon \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right) \to (\mathbb{R}, \mathscr{T}_{\text{usual}})$ is continuous. For any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, since $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ is open in $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$, there exists $\delta > 0$ such that $[x_0, x_0 + \delta) \subset f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$. Thus $x_n \to x_0^+$ implies $f(x_n) \to f(x_0)$ in $(\mathbb{R}, \mathscr{T}_{\text{usual}})$, i.e., $f\colon \mathbb{R} \to \mathbb{R}$ is right continuous.

(2) ① Define

$$\mathscr{T}_{\text{u.s.c.}} = \{\varnothing\} \cup \{\mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}.$$

Let us check that $\mathscr{T}_{\text{u.s.c.}}$ is a topology on $\mathbb{R}$:

- By definition $\varnothing, \mathbb{R} \in \mathscr{T}_{\text{u.s.c.}}$.
- Let $U, V \in \mathscr{T}_{\text{u.s.c.}}$. If one of U, V is $\varnothing$ or $\mathbb{R}$, then it is clear that $U \cap V \in \mathscr{T}_{\text{u.s.c.}}$. Otherwise, let $U = (-\infty, a)$ and $V = (-\infty, b)$ for some $a, b \in \mathbb{R}$. Then $U \cap V = (-\infty, \min\{a, b\}) \in \mathscr{T}_{\text{u.s.c.}}$.
- Suppose $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}_{\text{u.s.c.}}$. It suffices to consider the case where all U_α are of the form $(-\infty, a_\alpha)$ $(a_\alpha \in \mathbb{R})$. Let $a = \sup\{a_\alpha : \alpha \in \Lambda\}$. If $a = +\infty$, then $\bigcup\limits_{\alpha \in \Lambda} U_\alpha = \mathbb{R} \in \mathscr{T}_{\text{u.s.c.}}$. If $a \in \mathbb{R}$, then $\bigcup\limits_{\alpha \in \Lambda} U_\alpha = (-\infty, a) \in \mathscr{T}_{\text{u.s.c.}}$.

Now we shall show that a function $f\colon X \to \mathbb{R}$ is upper semi-continuous if and only if $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous.

($\Rightarrow$) Let $f\colon X \to \mathbb{R}$ be upper semi-continuous and U be any open subset of $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$. If $U = \varnothing$ or $\mathbb{R}$, then $f^{-1}(U)$ is open in X. Otherwise, let $U = (-\infty, a)$ for some $a \in \mathbb{R}$. For any $x_0 \in f^{-1}(U)$, we have $f(x_0) < a$. Since f is upper semi-continuous at x_0, for $\varepsilon = \dfrac{a - f(x_0)}{2} > 0$ there exists a neighborhood V of x_0 such that $f(x) \leqslant f(x_0) + \varepsilon < a$ for all $x \in V$. Hence $V \subset f^{-1}(U)$ and then $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous at x_0. Therefore $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous.

($\Leftarrow$) Suppose $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous. For any $x_0 \in X$ and $\varepsilon > 0$, since $f^{-1}((-\infty, f(x_0) + \varepsilon))$ is open in X and contains x_0, it follows by definition that f is upper semi-continuous at x_0.

② Let $x_0 \in X$. If $x_0 \in A$, then $\chi_A(x_0) = 1$ and obviously $f(x) \leqslant f(x_0) + \varepsilon$ for any $x \in X$ and $\varepsilon > 0$, which means $\chi_A(x)$ is upper semi-continuous at x_0. If $x_0 \notin A$, then $\chi_A(x_0) = 0$ and the upper semi-continuity of $\chi_A(x)$ at x_0 implies that there exists a neighborhood U of x_0 such that $\chi_A(x) = 0$ for all $x \in U$, i.e., $U \cap A = \varnothing$. This shows that A^{c} is open, so A is closed. Conversely, if A is closed, then the same argument shows that $\chi_A(x)$ is upper semi-continuous.

③ Let

$$\widetilde{\mathscr{T}_{\text{u.s.c.}}} = \{A \cup B : A \in \mathscr{T}_{\text{u.s.c.}}, B = \varnothing, \{+\infty\}, \{-\infty\}, \{\pm\infty\}\}.$$

Then it is immediate that $\widetilde{\mathscr{T}_{\text{u.s.c.}}}$ defines a topology on $\overline{\mathbb{R}}$ and is an extension of $\mathscr{T}_{\text{u.s.c.}}$. For any $x_0 \in x$ and $\varepsilon > 0$, by the definition of pointwise infimum, there exists $\lambda \in \Lambda$ such that $f_\lambda(x_0) < f(x_0) + \frac{\varepsilon}{2}$. Since f_λ is upper semi-continuous at x_0, there exists a neighborhood U of x_0 such that $f_\lambda(x) \leqslant f_\lambda(x_0) + \frac{\varepsilon}{2}$ for all $x \in U$. It follows that

$$f(x) \leqslant f_\lambda(x) \leqslant f_\lambda(x_0) + \frac{\varepsilon}{2} < f(x_0) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = f(x_0) + \varepsilon$$

for all $x \in U$. Therefore f is upper semi-continuous. $\square$

Problem 16 (Pasting lemma) Let X, Y be topological spaces. Consider a map $f\colon X \to Y$.

(1) Suppose $X = A \cup B$, where A, B are both closed subsets in X. Suppose $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous. Prove: $f\colon X \to Y$ is continuous.

(2) Show that the same result fails for $X = \bigcup\limits_{n=1}^{\infty} A_n$, where each A_n is closed in X.

(3) Let A_α be a family of closed subsets in X with $X = \bigcup_\alpha A_\alpha$, and suppose the family is *locally finite*, i.e., each point $p \in X$ has a neighborhood U_p that intersects finitely many A_α's. Prove: if each $f|_{A_\alpha}$ is continuous, then f is continuous.

(4) Prove: if $X = \bigcup_\alpha U_\alpha$, where each U_α is open in X, and if each $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then f is continuous.

Proof (1) It suffices to show that the preimage of each closed subset $K \subset Y$ is closed in X. Since $f^{-1}(K) \cap A = (f|_A)^{-1}(K)$ is closed in A and $f^{-1}(K) \cap B = (f|_B)^{-1}(K)$ is closed in B, and A, B are both closed in X, these two preimages are both closed in X. It follows that

$$f^{-1}(K) = f^{-1}(K) \cap (A \cup B) = (f|_A)^{-1}(K) \cup (f|_B)^{-1}(K)$$

is closed in X. Therefore f is continuous.

(2) Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the subspace topology inherited from $\mathbb{R}$. Let $A_1 = \{0\}$ and $A_{n+1} = \left\{\frac{1}{n}\right\}$ for $n \in \mathbb{N}$, then $X = \bigcup_{n=1}^{\infty} A_n$ and each A_n is closed in X. Take $f = \chi_{\{0\}}$, then f is not continuous at 0. However, $f|_{A_n}$ is continuous for each $n \in \mathbb{N} \cup \{0\}$.

(3) Given $x \in X$ there is a neighborhood U_x of x such that U_x (we can choose U_x to be open) intersects only finitely many A_α's, say $A_1, \cdots, A_n$. Note that for each $i = 1, \cdots, n$, $U_x \cap A_i$ is closed in U_x. Since $U_x = \bigcup_{i=1}^{n} (U_x \cap A_i)$ and each $f|_{U_x \cap A_i}$ is continuous, an inductive version of (1) shows that $f|_{U_x}$ is continuous. Then $X = \bigcup_{x \in X} U_x$ where each U_x is open in X and each $f|_{U_x}$ is continuous, so f is continuous by (4).

(4) Let $V \subset Y$ be any open subset. Any point $x \in f^{-1}(V)$ has an open neighborhood U_x on which f is continuous. Continuity of $f|_{U_x}$ implies, in particular, that $(f|_{U_x})^{-1}(V)$ is an open subset of U_x, and is therefore also an open subset of X. Then

$$(f|_{U_x})^{-1}(V) = \{y \in U_x : f(x) \in V\} = f^{-1}(V) \cap U_x$$

is an open neighborhood of x contained in $f^{-1}(V)$, hence $f^{-1}(V)$ is open in X. Therefore f is continuous. $\square$

PSet 3, Part 1

Problem 17 (Basis v.s. subset/product topologies)

(1) Let $(X, \mathcal{T}_X)$ be a topological space, and $\mathcal{B}$ a basis of $\mathcal{T}$. Let $A \subset X$ be a subset, with subspace topology $\mathcal{T}_A$. Prove: $\mathcal{B}_A \coloneqq \{A \cap B : B \in \mathcal{B}\}$ is a basis of $\mathcal{T}_A$.

(2) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, with basis $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Prove: $\mathcal{B}_{X \times Y} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology $\mathcal{T}_{X \times Y}$.

(3) Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X, and $\mathcal{B}_1, \mathcal{B}_2$ bases that generate $\mathcal{T}_1, \mathcal{T}_2$ respectively. Prove: $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ if and only if for any $x \in X$ and any $B \in \mathcal{B}_1$ that contains x, there exists $B' \in \mathcal{B}_2$ such that $x \in B' \subset B$.

Proof (1) $\mathcal{B} \subset \mathcal{T}$ implies $\mathcal{B}_A \subset \mathcal{T}_A$. For any $U \in \mathcal{T}_A$ and $x \in U$, there exists $V \in \mathcal{T}$ such that $U = V \cap A$. Since $x \in V$, there exists $B \in \mathcal{B}$ such that $x \in B \subset V$. Then $x \in A \cap B \subset A \cap V = U$. Therefore $\mathcal{B}_A$ is a basis of $\mathcal{T}_A$.

(2) $\mathcal{B}_X \subset \mathcal{T}_X$ and $\mathcal{B}_Y \subset \mathcal{T}_Y$ imply $\mathcal{B}_{X \times Y} \subset \mathcal{T}_{X \times Y}$. For any $U \in \mathcal{T}_{X \times Y}$ and $(x, y) \in U$, there exists $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$ such that $(x, y) \in V \times W \subset U$. Since $x \in U$, there exists $B_1 \in \mathcal{B}_X$ such that $x \in B_1 \subset V$. Similarly there exists $B_2 \in \mathcal{B}_Y$ such that $y \in B_2 \subset W$. Then $(x, y) \in B_1 \times B_2 \subset V \times W \subset U$. Therefore $\mathcal{B}_{X \times Y}$ is a basis of $\mathcal{T}_{X \times Y}$.

(3) ($\Leftarrow$) For any $U \in \mathcal{T}_1$ and $x \in U$, there exists $B \in \mathcal{B}_1$ such that $x \in B \subset U$. By assumption, there exists $B' \in \mathcal{B}_2$ such that $x \in B' \subset B \subset U$. Since $B' \in \mathcal{T}_2$, $U \in \mathcal{T}_2$. Therefore $\mathcal{T}_1 \subset \mathcal{T}_2$.

($\Rightarrow$) If $\mathcal{T}_1 \subset \mathcal{T}_2$, then for any $x \in X$ and any $B \in \mathcal{B}_1$ that contains x, since $B \in \mathcal{T}_1 \subset \mathcal{T}_2$, there exists $B' \in \mathcal{B}_2$ such that $x \in B' \subset B$. □

Problem 18 (Neighborhood basis) Let $(X, \mathcal{T})$ be a topological space. Like a basis, we can define a *neighborhood basis* of $(X, \mathcal{T})$ as follows: a family $\mathcal{B}(x) \subset \mathcal{N}(x)$ of neighborhoods of x is called a *neighborhood basis* at x if for any $A \in \mathcal{N}(x)$, there exists $B \in \mathcal{B}(x)$ such that $B \subset A$.

(1) Express $\mathcal{N}(x)$ in terms of a neighborhood basis $\mathcal{B}(x)$.

(2) Show that if $\mathcal{B}$ is a basis of $\mathcal{T}$, then $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$ is a neighborhood basis at x.

(3) Write down a theorem that characterizes the continuity of a map f at a point x via neighborhood basis, and prove your theorem.

(4) Define the concept "*neighborhood sub-basis*", and write/prove a statement that "characterizes the continuity of a map f at a point x via neighborhood sub-basis".

Proof (1) $\mathcal{N}(x) = \{V \subset X : V \supset B \text{ for some } B \in \mathcal{B}(x)\}$.

(2) Clearly $\mathcal{B}(x) \subset \mathcal{N}(x)$. For any $A \in \mathcal{N}(x)$, there exists $U \in \mathcal{T}$ such that $x \in U \subset A$. Since $\mathcal{B}$ is a basis of $\mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U \subset A$ (so $B \in \mathcal{B}(x)$). Therefore $\mathcal{B}(x)$ is a neighborhood basis at x.

(3) **Theorem** *A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at $x \in X$ if and only if for any $B \in \mathcal{B}(f(x))$, $f^{-1}(B)$ is a neighborhood of x, where $\mathcal{B}(f(x))$ is a neighborhood basis at $f(x)$.*

Proof The "only if" part is just the definition of continuity at x. For the "if" part, let N be a neighborhood of $f(x)$. By (1), there exists $B \in \mathcal{B}(f(x))$ such that $B \subset N$. Then $f^{-1}(N) \supset f^{-1}(B)$ is a neighborhood of x by assumption. Hence f is continuous at x.

(4) A "*neighborhood sub-basis*" at x is a family $\mathcal{S}(x) \subset \mathcal{P}(X)$ such that $x \in S$ for all $S \in \mathcal{S}(x)$ and the collection of all possible finite intersections of elements of $\mathcal{S}(x)$ forms a neighborhood basis at x, namely

$$\mathcal{N}(x) = \{N : N = S_1 \cap \cdots \cap S_m \text{ for some } S_1, \cdots, S_m \in \mathcal{S}(x)\}.$$

Theorem *A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at $x \in X$ if and only if for any $S \in \mathcal{S}(f(x))$, $f^{-1}(S)$ is a neighborhood of x, where $\mathcal{S}(f(x))$ is a neighborhood sub-basis at $f(x)$.*

Proof The "only if" part is trivial since $\mathcal{S}(f(x)) \subset \mathcal{N}(f(x))$. For the "if" part, let N be a neighborhood of $f(x)$, then $N = S_1 \cap \cdots \cap S_m$ for some $S_1, \cdots, S_m \in \mathcal{S}(f(x))$, and

$$f^{-1}(N) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_m),$$

where each $f^{-1}(S_i)$ is a neighborhood of x by assumption. Thus $f^{-1}(N)$ is a neighborhood of x, and f is continuous at x. $\square$

Problem 19 (Product topology and product metrics) Let (X, d_X) and (Y, d_Y) be metric spaces. Endow $X \times Y$ with the product metric

$$d^{(p)}_{X\times Y}((x_1, y_1), (x_2, y_2)) := \begin{cases} [d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p]^{\frac{1}{p}}, & 1 \leqslant p < \infty, \\ \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, & p = \infty. \end{cases}$$

Prove:

(1) If U is open in (X, d_X), V is open in (Y, d_Y), then $U \times V$ is open in $\left(X \times Y, d^{(\infty)}_{X\times Y}\right)$.

(2) W is an open set in $\left(X \times Y, d^{(\infty)}_{X\times Y}\right)$ if and only if for any $(x, y) \in W$, there exists $r > 0$ such that $\mathbb{B}(x, r) \times \mathbb{B}(y, r) \subset W$.

(3) Prove the same conclusion for $1 \leqslant p < \infty$.

So "the metric topology of the product metric" = "the product topology of the metric topologies".

Proof (1) For any $(x, y) \in U \times V$, there exists $r_1 > 0$ such that $\mathbb{B}(x, r_1) \subset U$ and $r_2 > 0$ such that $\mathbb{B}(y, r_2) \subset V$. Let $r = \min\{r_1, r_2\}$, then $\mathbb{B}^{(\infty)}_{X\times Y}((x, y), r) \subset \mathbb{B}(x, r_1) \times \mathbb{B}(y, r_2) \subset U \times V$. Therefore $U \times V$ is open in $\left(X \times Y, d^{(\infty)}_{X\times Y}\right)$.

(2) It suffices to note that $\mathbb{B}^{(\infty)}_{X\times Y}((x, y), r) = \mathbb{B}(x, r) \times \mathbb{B}(y, r)$.

(3) ① Suppose U is open in (X, d_X) and V is open in (Y, d_Y). For any $(x, y) \in U \times V$, there exists $r_1 > 0$ such that $\mathbb{B}(x, r_1) \subset U$ and $r_2 > 0$ such that $\mathbb{B}(y, r_2) \subset V$. Let $r = \min\{r_1, r_2\}$, then

$$d_X(x, \tilde{x}) \leqslant d^{(p)}_{X\times Y}((x, y), (\tilde{x}, \tilde{y})) < r \leqslant r_1, \quad d_Y(y, \tilde{y}) \leqslant d^{(p)}_{X\times Y}((x, y), (\tilde{x}, \tilde{y})) < r \leqslant r_2$$

for all $(\tilde{x}, \tilde{y}) \in \mathbb{B}^{(p)}_{X\times Y}((x, y), r)$. Hence $U \times V$ is open in $\left(X \times Y, d^{(p)}_{X\times Y}\right)$.

② If W is open in $\left(X \times Y, d^{(p)}_{X\times Y}\right)$, then for any $(x, y) \in W$, there exists $r_0 > 0$ such that

$$\mathbb{B}^{(p)}_{X\times Y}((x, y), r_0) \subset W.$$

Choose $r > 0$ such that $r < 2^{-\frac{1}{p}} r_0$, then for any $(\tilde{x}, \tilde{y}) \in \mathbb{B}(x, r) \times \mathbb{B}(y, r)$,

$$[d_X(x, \tilde{x})^p + d_Y(y, \tilde{y})^p]^{\frac{1}{p}} < (2r^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} r < r_0.$$

Therefore $\mathbb{B}(x, r) \times \mathbb{B}(y, r) \subset W$.

③ Suppose $W \subset X \times Y$ and for any $(x, y) \in W$, there exists $r > 0$ such that $\mathbb{B}(x, r) \times \mathbb{B}(y, r) \subset W$. Then by ① $\mathbb{B}(x, r) \times \mathbb{B}(y, r)$ is an open set in $\left(X \times Y, d^{(p)}_{X\times Y}\right)$ that contains (x, y), thus W is open in $\left(X \times Y, d^{(p)}_{X\times Y}\right)$. $\square$

Problem 20 (Various topologies on $\mathbb{R}^{\mathbb{N}}$) Consider the space of sequences of real numbers

$$X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \cdots) : x_n \in \mathbb{R}\}.$$

On X we have defined three topologies: the box topology $\mathscr{T}_{\text{box}}$, the product topology $\mathscr{T}_{\text{product}}$, and the "uniform topology" $\mathscr{T}_{\text{uniform}}$ induced from the uniform metric

$$d_{\text{uniform}}((x_n),(y_n)) = \sup_{n\in\mathbb{N}} \min\{|x_n - y_n|, 1\}.$$

(1) Prove: $\mathscr{T}_{\text{product}} \subset \mathscr{T}_{\text{uniform}} \subset \mathscr{T}_{\text{box}}$.

(2) One can also regard every element $(x_1, x_2, \cdots)$ in X as a map

$$f\colon \mathbb{N} \to \mathbb{R}, \quad n \mapsto x_n$$

and thus identify X with the space of maps $\mathcal{M}(\mathbb{N},\mathbb{R})$. By this way we get the pointwise convergence topology $\mathscr{T}_{\text{p.c.}}$ on X. Prove $\mathscr{T}_{\text{p.c.}} = \mathscr{T}_{\text{product}}$.

(3) Fix two elements $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ in X, and define a map

$$f\colon X \to X, \quad (x_1, x_2, \cdots) \mapsto (a_1x_1 + b_1, a_2x_2 + b_2, \cdots).$$

Prove that if we endow X with the product topology, then f is continuous.

(4) If we endow X with the box topology, is f continuous?

Proof (1) $\boxed{\mathscr{T}_{\text{product}} \subset \mathscr{T}_{\text{uniform}}}$ For any $U \in \mathscr{T}_{\text{product}}$ and any $(x_n) \in U$, by the definition of product topology, there exists open sets $U_1, \cdots, U_m \subset \mathbb{R}$ such that

$$(x_n) \in U_1 \times \cdots \times U_m \times \mathbb{R} \times \mathbb{R} \times \cdots \subset U.$$

For each U_i, there exists $r_i > 0$ such that $\mathbb{B}(x_i, r_i) \subset U_i$. Let $r = \min\{r_1, \cdots, r_m, 1\} > 0$, then

$$\mathbb{B}_{\text{uniform}}((x_n), r) \subset U_1 \times \cdots \times U_m \times \mathbb{R} \times \mathbb{R} \cdots \subset U.$$

Therefore $U \in \mathscr{T}_{\text{uniform}}$.

$\boxed{\mathscr{T}_{\text{uniform}} \subset \mathscr{T}_{\text{box}}}$ For any $U \in \mathscr{T}_{\text{uniform}}$ and $(x_n) \in U$, there exists $r \in (0,1)$ such that

$$\mathbb{B}_{\text{uniform}}((x_n), r) \subset U.$$

Then

$$\mathscr{T}_{\text{box}} \ni \prod_{i\in\mathbb{N}} \mathbb{B}\big(x_i, \tfrac{r}{2}\big) \subset \mathbb{B}_{\text{uniform}}((x_n), r) \subset U.$$

Hence $U \in \mathscr{T}_{\text{box}}$.

(2) $\boxed{\mathscr{T}_{\text{p.c.}} \subset \mathscr{T}_{\text{product}}}$ For any $U \in \mathscr{T}_{\text{p.c.}}$ and $(x_n) \in U$, by the definition of pointwise convergence topology, there exists $n_1, \cdots, n_m \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\omega((x_n); n_1, \cdots, n_m; \varepsilon) := \{(y_n) \in X : |y_{n_i} - x_{n_i}| < \varepsilon, \forall 1 \leqslant i \leqslant m\} \subset U.$$

Now let

$$U_k = \begin{cases} \mathbb{B}(x_k, \varepsilon), & \text{if } k = n_i \text{ for some } i \in \{1, \cdots, m\}, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

Then

$$\mathscr{T}_{\text{product}} \ni \prod_{k\in\mathbb{N}} U_k = \omega((x_n); n_1, \cdots, n_m; \varepsilon) \subset U.$$

Hence $U \in \mathscr{T}_{\text{product}}$.

$\boxed{\mathscr{T}_{\text{product}} \subset \mathscr{T}_{\text{p.c.}}}$ For any $U \in \mathscr{T}_{\text{product}}$ and $(x_n) \in U$, by the definition of product topology, there exists open sets $U_1, \cdots, U_m \subset \mathbb{R}$ such that

$$(x_n) \in U_1 \times \cdots \times U_m \times \mathbb{R} \times \mathbb{R} \times \cdots \subset U.$$

For each U_i, there exists $r_i > 0$ such that $\mathbb{B}(x_i, r_i) \subset U_i$. Let $\varepsilon = \min\{r_1, \cdots, r_m\} > 0$, then

$$\omega((x_n); 1, \cdots, m; \varepsilon) \subset U_1 \times \cdots \times U_m \times \mathbb{R} \times \mathbb{R} \cdots \subset U.$$

Therefore $U \in \mathscr{T}_{\text{p.c.}}$.

(3) For any $U \in \mathscr{T}_{\text{product}}$ and $(x_n) \in f^{-1}(U)$, let $(y_n) \coloneqq f((x_n))$. By (2), there exists $n_1, \cdots, n_k \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\omega((y_n); n_1, \cdots, n_k; \varepsilon) \coloneqq \{(z_n) \in X : |z_{n_i} - y_{n_i}| < \varepsilon, \forall i = 1, \cdots, k\} \subset U.$$

Let

$$V_k = \begin{cases} \mathbb{B}\left(x_{n_i}, \dfrac{\varepsilon}{|a_{n_i}|}\right), & \text{if } k = n_i \text{ for some } i \in \{1, \cdots, k\} \text{ and } a_{n_i} \neq 0, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

Then for any $(z_n) \in \prod_{k\in\mathbb{N}} V_k \in \mathscr{T}_{\text{product}}$, let $(w_n) \coloneqq f((z_n))$, we have

$$|w_{n_i} - y_{n_i}| = |(a_{n_i} z_{n_i} + b_{n_i}) - (a_{n_i} x_{n_i} + b_{n_i})| = |a_{n_i}(z_{n_i} - x_{n_i})| < \varepsilon, \quad \forall i = 1, \cdots, k.$$

This shows that $(x_n) \in \prod_{k\in\mathbb{N}} V_k \subset f^{-1}(\omega((y_n); n_1, \cdots, n_k; \varepsilon)) \subset f^{-1}(U)$. Therefore $f^{-1}(U) \in \mathscr{T}_{\text{product}}$, and thus f is continuous.

(4) For any $U \in \mathscr{T}_{\text{box}}$ and $(x_n) \in f^{-1}(U)$, there exists open sets $\{U_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that

$$(y_n) \coloneqq f((x_n)) \in \prod_{n\in\mathbb{N}} U_n \subset U.$$

For $i \in \mathbb{N}$, let

$$f_i \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto a_i x_i + b_i.$$

Then each f_i is continuous, and then $f_i^{-1}(U_i)$ is open in $\mathbb{R}$. Hence $\prod_{i\in\mathbb{N}} f_i^{-1}(U_i) \in \mathscr{T}_{\text{box}}$ and

$$(x_n) \in \prod_{i\in\mathbb{N}} f_i^{-1}(U_i) \subset f^{-1}(U).$$

Therefore $f^{-1}(U) \in \mathscr{T}_{\text{box}}$, and thus f is continuous. $\square$

PSet 3, Part 2

Problem 21 (Induced and co-induced topologies)

(1) Let $(Z, \mathcal{T}_Z)$ be a topological space, and $f: X \to Y$, $g: Y \to Z$ be maps. Let $\mathcal{T}_Y$ be the induced topology on Y by g. Prove: the induced topology on X by f (from $\mathcal{T}_Y$) is the same as the induced topology on X by $g \circ f$ (from $\mathcal{T}_Z$).

(2) State and prove a similar result on co-induced topology.

(3) Prove the universality for the co-induced topology.

Proof (1) $\mathscr{T}_{g\circ f} = \{(g \circ f)^{-1}(V) : V \in \mathcal{T}_Z\} = \{f^{-1}(g^{-1}(V)) : V \in \mathcal{T}_Z\} = \{f^{-1}(U) : U \in \mathcal{T}_Y\} = \mathscr{T}_f$.

(2) **Proposition** *Let $(X, \mathcal{T}_X)$ be a topological space, and $f: X \to Y$, $g: Y \to Z$ be maps. Let $\mathcal{T}_Y$ be the co-induced topology on Y by f. Prove: the co-induced topology on Z by g (from $\mathcal{T}_Y$) is the same as the co-induced topology on Z by $g \circ f$ (from $\mathcal{T}_X$).*

Proof Using the construction of $\mathcal{T}_Y$ we have

$$\begin{aligned}\mathscr{T}_g &= \{V \in Z : g^{-1}(V) \in \mathcal{T}_Y\} = \{V \in Z : f^{-1}(g^{-1}(V)) \in \mathcal{T}_Y\} \\ &= \{V \in Z : (g \circ f)^{-1}(V) \in \mathcal{T}_X\} = \mathscr{T}_{g\circ f}.\end{aligned}$$

(3) **Proposition** *Let $(X_\alpha, \mathscr{T}_\alpha)$ be a family of topological spaces, and $\mathcal{F} = \{f_\alpha : X_\alpha \to Y\}$ be a family of maps. Endow Y with the $\mathcal{F}$-induced topology. Then a map $f: Y \to Z$ is continuous if and only if each $f \circ f_\alpha : X_\alpha \to Z$ is continuous. Moreover, the co-induced topology on Y induced by $\mathcal{F}$ is the only topology satisfying this property.*

Proof If f is continuous then clearly all functions $f \circ f_\alpha$ are continuous. Conversely, suppose that all $f \circ f_\alpha$ are continuous. Let $U \subset Z$ be an open set. Then $(f \circ f_\alpha)^{-1}(U) = f_\alpha^{-1}(f^{-1}(U))$ is open in X_α for all α, which implies that $f^{-1}(U)$ is open in Y. Therefore f is continuous.

Denote the co-induced topology on Y by $\mathcal{T}_Y$. Suppose $\mathcal{T}'_Y$ is another topology on Y with the same property, we shall show that $\mathcal{T}_Y = \mathcal{T}'_Y$.

$\boxed{\mathcal{T}_Y \subset \mathcal{T}'_Y}$ Let $g: (Y, \mathcal{T}'_Y) \to (Y, \mathcal{T}_Y)$ be the identity map. Then $(g \circ f_\alpha)^{-1}(U) = f_\alpha^{-1}(U) \in \mathscr{T}_\alpha$ for all $U \in \mathcal{T}_Y$ and all α, which implies that g is continuous. Hence $\mathcal{T}_Y \subset \mathcal{T}'_Y$.

$\boxed{\mathcal{T}'_Y \subset \mathcal{T}_Y}$ Let $h: (Y, \mathcal{T}'_Y) \to (Y, \mathcal{T}'_Y)$ be the identity map, then h is automatically continuous. Hence $h \circ f_\alpha : (X_\alpha, \mathscr{T}_\alpha) \to (Y, \mathcal{T}'_Y)$ is continuous for all α. But $h \circ f_\alpha = f_\alpha$, by the definition of the co-induced topology, $\mathcal{T}'_Y \subset \mathcal{T}_Y$. ☐

Problem 22 (Hawaiian earring) Prove that the Hawaiian earring

$$E = \bigcup_{n=1}^{\infty} \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\}$$

is not homeomorphic to the wedge sum $\bigvee_{n=1}^{\infty} \mathbb{S}^1$ of countably many circles.

Proof The Hawaiian earring is compact since it is a bounded closed subset of $\mathbb{R}^2$. However, the wedge sum of countably many circles is not compact. For example, one can choose an open cover of the wedge sum where each circle $\mathbb{S}^1$ is covered by an open set that does not include points far away from the wedge point in other circles, then no finite number of these open sets can cover infinitely many circles. □

Problem 23 (Quotient map v.s. open/closed map)

(1) Suppose $p\colon X \to Y$ is a surjective continuous map. Prove: if p is either open or closed, then it is a quotient map.

(2) Construct a quotient map that is neither open nor closed.

(3) Let $\mathsf{SO}(n)$ be the special orthogonal group. Define a map

$$f\colon \mathsf{SO}(n) \to \mathbb{S}^{n-1}, \quad A \mapsto Ae_1,$$

where $e_1 = (0, \cdots, 0, 1)$ is the "north pole vector" on $\mathbb{S}^{n-1}$.

① Prove: f is surjective, continuous and open, and thus is a quotient map.

② Consider the natural (right) action of $\mathsf{SO}(n-1)$ on $\mathsf{SO}(n)$ by

$$B \cdot A := A\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall B \in \mathsf{SO}(n-1),\ A \in \mathsf{SO}(n).$$

Prove: the orbits of this action are the fibers of the quotient map f.

③ Conclude that $\mathsf{SO}(n)/\,\mathsf{SO}(n-1) \simeq \mathbb{S}^{n-1}$.

Proof (1) It suffices to show that "$A \subset Y$ is open/closed $\iff$ $p^{-1}(A)$ is open/closed in X".

($\Rightarrow$) Use the continuity of p.

($\Leftarrow$) Since p is open/closed and surjective, $A = p\big(p^{-1}(A)\big)$ is open/closed whenever $p^{-1}(A)$ is open/closed.

(2) Let $A = \big\{(x, y) \in \mathbb{R}^2 : x \geqslant 0 \text{ or } y = 0\big\}$ and $q\colon A \to \mathbb{R}$ be the projection on the first coordinate. We shall show that q is a quotient map but not open nor closed.

(Quotient map) Since q is continuous, it suffices to show that $C \subset \mathbb{R}$ is closed if $q^{-1}(C)$ is closed in A. Suppose $q^{-1}(C)$ is closed in A, and let $C_+ := C \cap \mathbb{R}_{\geqslant 0}$, $C_- := C \cap \mathbb{R}_{\leqslant 0}$. Then

$$q^{-1}(C_+) = q^{-1}(C) \cap \big\{(x, y) \in \mathbb{R}^2 : x \geqslant 0\big\}$$

is closed in $\mathbb{R}^2$, which implies that C_+ is closed in $\mathbb{R}$. Similarly,

$$q^{-1}(C_-) = q^{-1}(C) \cap \big\{(x, y) \in \mathbb{R}^2 : x \leqslant 0 \text{ and } y = 0\big\}$$

is closed in $\mathbb{R}^2$, which implies that C_- is closed in $\mathbb{R}$. Therefore $C = C_+ \cup C_-$ is closed in $\mathbb{R}$.

(Not open) Consider $U = \big\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x < 1, 1 < y < 2\big\}$, then U is open in A but $q(U) = [0, 1)$ is not open in $\mathbb{R}$.

(Not closed) Consider $C = \big\{(x, y) \in \mathbb{R}^2 : x > 0, xy = 1\big\}$, then C is closed in A but $q(C) = (0, +\infty)$ is not closed in $\mathbb{R}$.

(3) ① For each $A \in \mathrm{SO}(n)$, Ae_1 is the n-th column of A, which implies that f is continuous. Since for any $v \in \mathbb{S}^{n-1}$, there exists $A \in \mathrm{SO}(n)$ such that v is the n-th column of A, f is surjective. For any open set $U \subset \mathrm{SO}(n)$, $f(U)$ is just the image of U under the projection map on the n-th column, which is open in $\mathbb{S}^{n-1}$. Hence f is open.

② By expanding A in block form, we see that

$$A\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} A_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 B & a_2 \\ a_3 B & a_4 \end{pmatrix},$$

hence the n-th column of A is preserved under the action of $B \in \mathrm{SO}(n-1)$. Conversely, for any $\widetilde{A} \in \mathrm{SO}(n)$ with $\widetilde{A}e_1 = Ae_1$, there exists $B \in \mathrm{SO}(n-1)$ such that $\widetilde{A} = A\,\mathrm{diag}(B,1)$. Therefore

$$\mathcal{O}_A = f^{-1}(Ae_1)$$

for each $A \in \mathrm{SO}(n)$. In other words, the orbits of this action are the fibers of f.

③ The map $f\colon \mathrm{SO}(n) \to \mathbb{S}^{n-1}$ induces an equivalence relation on $\mathrm{SO}(n)$, where $A_1 \overset{f}{\sim} A_2$ if and only if $f(A_1) = f(A_2)$. From the surjectivity of f, we have the bijection

$$\left(\mathrm{SO}(n)/\overset{f}{\sim}\right) \simeq \mathrm{Im}(f) = \mathbb{S}^{n-1}.$$

But ② tells us that the fibers of f are just the orbits of the action of $\mathrm{SO}(n-1)$ on $\mathrm{SO}(n)$, namely

$$\left(\mathrm{SO}(n)/\overset{f}{\sim}\right) \simeq \mathrm{SO}(n)/\mathrm{SO}(n-1).$$

Therefore we find the bijection

$$\varphi\colon \mathrm{SO}(n)/\mathrm{SO}(n-1) \xrightarrow{\sim} \mathbb{S}^{n-1}, \quad [A] \mapsto Ae_1.$$

Breaking the bijection into two steps:

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \xrightarrow{g} \mathrm{SO}(n) \xrightarrow{f} \mathbb{S}^{n-1}.$$

Since $f\colon \mathrm{SO}(n) \to \mathbb{S}^{n-1}$ is continuous by ①, the universality of the quotient topology tells us that φ is continuous. Furthermore, the quotient topology on $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ implies that g is an open map. Also f is an open map by ①, hence $\varphi = f \circ g$ is an bijective continuous open map, which is a homeomorphism. □

Problem 24 (Covering space action) Let G be a group acting on a topological space X. Let $Y = X/G$ be the orbit space, and $p\colon X \to Y$ be the quotient map. Let $U \subset X$ be an open set, such that

$$g \cdot U \cap U = \varnothing, \quad \forall g \neq e \in G.$$

Prove:

(1) $V := p(U)$ is an open set in Y.

(2) For any $g \in G$, the map $p_g = p \circ \tau_g \colon g^{-1} \cdot U \to V$ is a homeomorphism.

Proof (1) We need to prove that $p^{-1}(p(U))$ is open in X. Note that the preimage of $p(U)$ under p is the union of the G-orbits of points in U:

$$p^{-1}(p(U)) = \bigcup_{g\in G} g \cdot U = \bigcup_{g\in G} \tau_g(U).$$

Since for each $g \in G$, τ_g is a homeomorphism between U and $\tau_g(U)$, the right-hand side is a union of open sets, hence open in X. Therefore $V := p(U)$ is open in Y.

(2) (Injectivity) If $p_g(x) = p_g(y)$ for some $x, y \in U$, then there exists $h \in G$ such that $g \cdot x = h \cdot (g \cdot y)$. This is equivalent to $x = \left(g^{-1}hg\right) \cdot y$, and then by our assumption $g^{-1}hg = e$, i.e., $hg = g$ and $h = e$. Hence $g \cdot x = g \cdot y$, which implies $x = y$.

(Surjectivity) Since τ_g and p are both surjective, $p_g = p \circ \tau_g$ is surjective.

(Continuity) Since τ_g and p are both continuous, $p_g = p \circ \tau_g$ is continuous.

(Inverse continuity) For any open set $W \subset U$, $\tau_g(U)$ is open in X since $\tau_g^{-1} = \tau_{g^{-1}}$ is continuous. Then by (1), $p(\tau_g(U))$ is open in Y. Therefore p_g^{-1} is continuous. □

PSet 4, Part 1

Problem 25 (G_δ sets) Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ us called G_δ set if there exists countably many open sets $U_1, U_2, \cdots$ so that $A = \bigcap_{n=1}^{\infty} U_n$.

(1) Show that $[0, 1)$ is a G_δ set in $\mathbb{R}$.

(2) Show that any closed set in a metric space is a G_δ set.

(3) Let (Y, d) be a metric space, and $f\colon (X, \mathcal{T}) \to (Y, d)$ a map. Prove: "the set of points where f is continuous" is a G_δ set in X.

Proof (1) $[0, 1) = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right)$ is a G_δ set.

(2) Suppose A is a closed set in a metric space X and let $G_n = \bigcup_{a\in A} \mathbb{B}\left(a, \frac{1}{n}\right)$, then each G_n is open. If $x \in \bigcap_{n=1}^{\infty} G_n$, then there is some $x_n \in A$ such that $d(x_n, x) < \frac{1}{n}$. Then $x_n \to x$, and since A is closed, $x \in A$. Therefore $A = \bigcap_{n=1}^{\infty} G_n$ is a G_δ set.

(3) The set of points where f is continuous is

$$\{x \in X : \omega_f(x) = 0\} = \bigcap_{n=1}^{\infty} \left\{x \in X : \omega_f(x) < \frac{1}{n}\right\},$$

where

$$\omega_f(x) := \lim_{\delta\searrow 0} \sup_{z,w\in\mathbb{B}(x,\delta)} |f(z) - f(w)|.$$

If $\omega_f(x) < \frac{1}{n}$, then for sufficiently small $\delta > 0$, we have

$$\sup_{z,w \in \mathbb{B}(x,2\delta)} |f(z) - f(w)| < \frac{1}{n}.$$

Since $\mathbb{B}(y, \delta) \subset \mathbb{B}(x, 2\delta)$ for $y \in \mathbb{B}(x, \delta)$, this implies

$$\omega_f(y) < \frac{1}{n}$$

for all $y \in \mathbb{B}(x, \delta)$. Therefore each $\left\{x \in X : \omega_f(x) < \frac{1}{n}\right\}$ is open, and the set of points where f is continuous is a G_δ set. □

Problem 26 ("Sequentially continuous = continuous" for (A1) spaces) Let X be an (A1) space, Y be any topological space. Prove: a map $f\colon X \to Y$ is continuous at x_0 if and only if it is *sequentially continuous* at x_0.

Proof The "only if" part holds for any topological space, so we only need to prove the "if" part. For any closed subset $C \subset Y$, to see that $f^{-1}(C)$ is closed in X, by the first countability of X, it suffices to show that for any sequence (x_n) in $f^{-1}(C)$ that converges to x_0, we have $x_0 \in f^{-1}(C)$. By sequential continuity, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$, and since $f(x_n) \in C$ and C is closed, we have $f(x_0) \in C$. Therefore $x_0 \in f^{-1}(C)$, as desired. □

Problem 27 (Closedness for the derived set)

(1) Consider a set $X = \{a, b, c\}$ of three elements. Let

$$\mathscr{T} = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}.$$

① Check: $\mathscr{T}$ is a topology on X.

② Denote $A = \{b\}$. Find A' and $(A')'$. Is A' closed?

(2) Let (X, d) be a metric space. Prove: for any $A \subset X$, the derived set A' is closed.

(3) For a general topological space $(X, \mathscr{T})$,

① For any subsets $A, B \subset X$, prove: $(A \cup B)' = A' \cup B'$.

② Prove: if $A \subset X$ is closed, then A' is closed.

③ For any subset $A \subset X$, prove: $(A')' \subset A \cup A'$.

Proof (1) ① We have $\varnothing, X \in \mathscr{T}$, $\{a\} \cup \{b, c\} = \{a, b, c\} \in \mathscr{T}$, and $\{a\} \cap \{b, c\} = \varnothing \in \mathscr{T}$ (unions or intersections with $\varnothing$ or X are trivial to check).

② $A' = \{c\}$ and $(A')' = \{b\}$. The derived set A' is not closed since $(A')^{\mathsf{c}} = \{a, b\} \notin \mathscr{T}$.

(2) It suffices to show that $(A')' \subset A'$. Let x be a limit point of A' and U be an open set containing x. Then there is some $y \in U \cap (A' \setminus \{x\})$. Now U is an open set containing $y \in A'$, and since X is a metric space, this means that U contains infinitely many points of A. To sum up, any open set containing x contains infinitely many points of A, so $x \in A'$. Therefore $(A')' \subset A'$, as desired.

(3) ① $\boxed{A' \cup B' \subset (A \cup B)'}$ $A \subset (A \cup B)$ implies $A' \subset (A \cup B)'$, and similarly for B'.

$\boxed{(A \cup B)' \subset A' \cup B'}$ Suppose $x \notin A' \cup B'$, then there exist open sets U_A, U_B containing x such that

$$U_A \cap (A \setminus \{x\}) = \varnothing, \quad U_B \cap (B \setminus \{x\}) = \varnothing.$$

Then

$$(U_A \cap U_B) \cap ((A \cup B) \setminus \{x\}) = \varnothing.$$

Since $U_A \cap U_B$ is an open set containing x, this means $x \notin (A \cup B)'$.

② Since A is closed, $A = A \cup A'$ and therefore $A' \subset A$. For any $x \in X \setminus A'$, since x in not a limit point of A, there is some open set U containing x such that $U \cap (A \setminus \{x\}) = \varnothing$. From $x \notin A'$ and $A' \subset A$ we have $A' \subset A \setminus \{x\}$, hence $U \cap A' = \varnothing$. This shows that A' is closed.

③ If $x \notin A \cup A'$, then there is an open set U containing x such that $U \cap (A \setminus \{x\}) = \varnothing$. And since $x \notin A$, this becomes $U \cap A = \varnothing$. Hence for any $y \in U$ we have $U \cap (A \setminus \{y\}) = U \cap A = \varnothing$, which implies $y \notin A'$. To sum up, x has an open neighborhood U such that $U \cap A' = \varnothing$, so $x \notin (A')'$. Therefore $(A')' \subset A \cup A'$. □

Problem 28 (Convergence by net) We call $(P, \preceq)$ a *directed set* if

- ◇ $(P, \preceq)$ is a partially ordered set.
- ◇ For any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

For a topological space X, a *net* is a map $f\colon (P, \preceq) \to X$ from a directed set $(P, \preceq)$ to X. We will use the notation (x_α) instead of a map "$f\colon \alpha \mapsto x_\alpha$" if there is no ambiguity. We say a net (x_α) *converges* to x_0, denoted by $x_\alpha \to x_0$, if for any neighborhood U of x_0, there is an $\alpha \in P$ such that $x_\beta \in U$ holds for any $\alpha \preceq \beta$.

(1) Realize $\mathcal{N}(x)$ as a directed set. You need to carefully choose the partial order relation so that it can be used in part (2) below.

(2) Prove: $x \in A \cup A'$ if and only if there exists a net (x_α) in A that converges to x.

(3) Prove: a map $f\colon X \to Y$ is continuous if and only if for any net (x_α) in X which converges to a limit x_0, the net $(f(x_\alpha))$ in Y converges in Y to $f(x_0)$.

Proof (1) $\mathcal{N}(x)$ can be partially ordered by reverse inclusion, i.e., $U \preceq V$ iff $V \subset U$. This is a directed set since for any $U, V \in \mathcal{N}(x)$, $U \cap V \in \mathcal{N}(x)$ and $U \preceq U \cap V$, $V \preceq U \cap V$.

(2) (⇐) Suppose (x_α) is a net in A that converges to x. Then for any open set U containing x, by the definition of convergence by net, it contains some $x_\alpha \in A$. Thus $x \in \overline{A}$.

(⇒) Assume $x \in \overline{A}$. For any $U \in \mathcal{N}(x)$, there exists $x_U \in U \cap A$ by assumption. Then $(x_U)_{U \in \mathcal{N}(x)}$ is a net in A. Moreover, for any neighborhood V of x, then $x_U \in V$ for all $U \subset V$, that is, for all U with $V \preceq U$. Thus $x_U \to x$.

(3) (⇒) Assume f is continuous and let (x_α) be a net in X converging to x_0. Let U be a neighborhood of $f(x_0)$. Then $f^{-1}(U)$ is a neighborhood of x, so there exists α such that $x_\beta \in f^{-1}(U)$ for all β with $\alpha \preceq \beta$, so that $f(x_\beta) \in U$ for all such β. Thus $f(x_\alpha) \to f(x_0)$.

(⇐) Suppose that f is not continuous. Then there exists U open in Y such that $f^{-1}(U)$ is not open in X. Then there exists $x_0 \in f^{-1}(U)$ such that no neighborhood of x_0 is contained in

$f^{-1}(U)$, so that $x_0 \in \overline{X \setminus f^{-1}(U)}$. It follows from part (2) that there is a net (x_α) in $X \setminus f^{-1}(U)$ converging to x_0. As no point $f(x_\alpha)$ belongs to U, it follows that the net $(f(x_\alpha))$ in Y does not converge to $f(x_0) \in U$. □

PSet 4, Part 2

Problem 29 (Points and sets in subspace topology) Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset, endowed with subspace topology. Let $A \subset Y$. We denote by $\overline{A}^Y$ the closure of A in Y etc. Find the relation between each pair below: if they are equal, prove it; if one is contained in another but not vice versa, prove the relation and provide a counterexample for the other.

(1) ${A'}^Y$ and ${A'}^X$.

(2) $\overline{A}^Y$ and $\overline{A}^X$.

(3) $\partial^Y A$ and $\partial^X A$.

(4) $\mathrm{Int}^Y(A)$ and $\mathrm{Int}^X(A)$.

Proof (1) We have ${A'}^Y \subset {A'}^X$, or more specifically, ${A'}^Y = {A'}^X \cap Y$.

$\boxed{{A'}^Y \subset \left({A'}^X \cap Y\right)}$ If $x \in {X'}^Y$, then $x \in Y$. Moreover, for any neighborhood U of x in X, since $U \cap Y$ is a neighborhood of x in Y, we have $(U \cap Y) \cap (A \setminus \{x\}) \neq \varnothing$. As $A \subset Y$, this is equivalent to $U \cap (A \setminus \{x\}) \neq \varnothing$, so $x \in {A'}^X \cap Y$.

$\boxed{\left({A'}^X \cap Y\right) \subset {A'}^Y}$ If $x \in {A'}^X \cap Y$, for any neighborhood U of x in Y, there is some neighborhood V of x in X with $U = V \cap Y$. Since $x \in {A'}^X$, $V \cap (A \setminus \{x\}) \neq \varnothing$, and therefore $U \cap (A \setminus \{x\}) \neq \varnothing$ as $A \subset Y$. Hence $x \in {A'}^Y$.

Take $X = \mathbb{R}$, $Y = (0, 1)$ and $A = (0, 1)$. Then ${A'}^X = [0, 1]$ and ${A'}^Y = (0, 1)$.

(2) From $\overline{A}^Y = A \cup {A'}^Y$, $\overline{A}^X = A \cup {A'}^X$, and results in part (1), we have $\overline{A}^Y = \overline{A}^X \cap Y$. In particular, $\overline{A}^Y \subset \overline{A}^X$. The reverse inclusion does not hold by the same counterexample in part (1), where $\overline{A}^Y = (0, 1)$ and $\overline{A}^X = [0, 1]$.

(3) It is always true that $\partial^Y A \subset \partial^X A$. To see this, let $x \in \partial^Y A$ and U be an open set in X containing x. Then $U \cap Y$ is open in Y, hence

$$\varnothing \neq (U \cap Y) \cap A = U \cap A,$$
$$\varnothing \neq (U \cap Y) \cap (Y \setminus A) \subset U \cap (X \setminus A).$$

Thus $x \in \partial^X A$, and $\partial^Y A \subset \partial^X A$. The reverse inclusion does not hold in general. Take $X = \mathbb{R}$ and $Y = A = (0, 1)$. Then $\partial^X A = \{0, 1\}$ and $\partial^Y A = \varnothing$.

(4) By the relation between the interior and the closure, and using part (2), we have

$$\begin{aligned}\mathrm{Int}^Y(A) &= Y \setminus \overline{X \setminus A}^X = Y \setminus \left(Y \cap \overline{Y \setminus A}^X\right) = Y \setminus \overline{Y \setminus A}^X \\ &= \left(X \setminus \overline{Y \setminus A}^X\right) \cap Y = \left(X \setminus \overline{X \setminus (A \cup (X \setminus Y))}^X\right) \cap Y\end{aligned}$$

$$
\begin{aligned}
&= \left(X \setminus \overline{X \setminus (A \cup Y^{c})}^{X}\right) \cap Y = \mathrm{Int}^X(A \cup Y^c) \cap Y \\
&\supset \left(\mathrm{Int}^X(A) \cup \mathrm{Int}^X(Y^c)\right) \cap Y \\
&= \mathrm{Int}^X(A) \cap Y = \mathrm{Int}^X(A).
\end{aligned}
$$

The reverse inclusion does not hold in general. Take $X = \mathbb{R}$, $Y = A = [0,1]$. Then $\mathrm{Int}^X(A) = (0,1)$ and $\mathrm{Int}^Y(A) = [0,1]$. □

Problem 30 (Closure and interior in box and product topology) Let X_α be topological spaces, and $A_\alpha \subset X_\alpha$. Consider the box topology $\mathscr{T}_{\text{box}}$ and the product topology $\mathscr{T}_{\text{product}}$ on $\prod_\alpha X_\alpha$.

(1) With respect to which topology, do we always have $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$?

(2) With respect to which topology, do we always have $\mathrm{Int}\left(\prod_\alpha A_\alpha\right) = \prod_\alpha \mathrm{Int}(A_\alpha)$?

Solution (1) The equality holds for both $\mathscr{T}_{\text{box}}$ and $\mathscr{T}_{\text{product}}$.

$\boxed{\prod_\alpha \overline{A_\alpha} \subset \overline{\prod_\alpha A_\alpha}}$ Let (x_α) be a point of $\prod_\alpha \overline{A_\alpha}$ and $U = \prod_\alpha U_\alpha$ be a basis element for either the box or product topology that contains (x_α). Since $x_\alpha \in \overline{A_\alpha}$ and U_α is an open neighborhood of x_α, we can choose a point $y_\alpha \in U_\alpha \cap (A_\alpha \setminus \{x_\alpha\})$ for each α. Then $(y_\alpha) \in U \cap \left(\prod_\alpha A_\alpha \setminus \{(x_\alpha)\}\right)$. Since U is arbitrary, it follows that $(x_\alpha) \in \overline{\prod_\alpha A_\alpha}$.

$\boxed{\overline{\prod_\alpha A_\alpha} \subset \prod_\alpha \overline{A_\alpha}}$ Suppose $(x_\alpha) \in \overline{\prod_\alpha A_\alpha}$ in either topology. For each given index β, let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_\alpha X_\alpha$ in either topology, if $x_\beta \notin A_\beta$, then $\pi_\beta^{-1}(V_\beta)$ contains a point $(y_\alpha) \in \prod_\alpha A_\alpha$. Then $y_\beta \in V_\beta \cap (A_\beta \setminus \{x_\beta\})$. Since V_β is arbitrary, it follows that $x_\beta \in \overline{A_\beta}$.

(2) The equality holds for box topology.

$\boxed{\prod_\alpha \mathrm{Int}(A_\alpha) \subset \mathrm{Int}\left(\prod_\alpha A_\alpha\right)}$ Since each $\mathrm{Int}(A_\alpha)$ is open in X_α, the inclusion holds.

$\boxed{\mathrm{Int}\left(\prod_\alpha A_\alpha\right) \subset \prod_\alpha \mathrm{Int}(A_\alpha)}$ Let $(x_\alpha) \in \mathrm{Int}\left(\prod_\alpha A_\alpha\right)$, then for each α there is an open set $U_\alpha \subset X_\alpha$ containing x_α with

$$
\prod_\alpha U_\alpha \subset \mathrm{Int}\left(\prod_\alpha A_\alpha\right) \subset \prod_\alpha A_\alpha.
$$

This implies $U_\alpha \subset A_\alpha$ for each α, hence $U_\alpha \subset \mathrm{Int}(A_\alpha)$ and $(x_\alpha) \in \prod_\alpha \mathrm{Int}(A_\alpha)$.

This equality fails for the product topology. Take $X_n = \mathbb{R}$ and $A_n = (0,1)$ for each $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty} \mathrm{Int}(A_n) = \prod_{n=1}^{\infty} (0,1)$. However, by the definition of product topology, this cannot be an open set in $\prod_{n=1}^{\infty} X_n$, and hence $\operatorname{Int}\left(\prod_{n=1}^{\infty} A_n\right) \neq \prod_{n=1}^{\infty} \operatorname{Int}(A_n)$. $\square$

Problem 31 (Closure of union of closed sets) Let $(X, \mathscr{T})$ be a topological space.

(1) Let A, B be subsets in X. Prove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(2) Let A_α be a family of subsets in X. Prove: $\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$.

(3) Find an example so that $\bigcup_\alpha \overline{A_\alpha} \neq \overline{\bigcup_\alpha A_\alpha}$ for a family of subsets $A_\alpha \subset \mathbb{R}$.

(4) We say a family $\{A_\alpha\}$ of subsets in X is *locally finite* if for any $x \in X$, there exists an open neighborhood of U_x of x so that $A_\alpha \cap U_x \neq \varnothing$ for only finitely many α's. Prove: if $\{A_\alpha\}$ is a *locally finite family*, then $\{\overline{A_\alpha}\}$ is a locally finite family, and $\bigcup_\alpha \overline{A_\alpha} = \overline{\bigcup_\alpha A_\alpha}$.

Proof (1) Since $\overline{A \cup B}$ is closed and contains both A and B, it contains $\overline{A}$ and $\overline{B}$, hence their union. The reverse inclusion follows since $\overline{A \cup B} \subset \overline{(\overline{A} \cup \overline{B})} = \overline{A} \cup \overline{B}$.

(2) If $x \in \bigcup_\alpha \overline{A_\alpha}$, then $x \in \overline{A_\beta}$ for some β. For any neighborhood U of x, $U \cap (A_\beta \setminus \{x\}) \neq \varnothing$, and hence $U \cap \left(\bigcup_\alpha A_\alpha \setminus \{x\}\right) \neq \varnothing$. Since U is arbitrary, this implies $x \in \overline{\bigcup_\alpha A_\alpha}$.

(3) Consider $\mathbb{R}$ with the standard topology and all singletons $\{r\}$ for $r \in \mathbb{Q}$. Then $\bigcup_{r \in \mathbb{Q}} \overline{\{r\}} = \bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$, while $\overline{\bigcup_{r \in \mathbb{Q}} \{r\}} = \overline{\mathbb{Q}} = \mathbb{R}$.

(4) For any $x \in X$, since $\{A_\alpha\}$ is locally finite, there exists an open neighborhood U_x of x so that $A_\alpha \cap U_x \neq \varnothing$ for only finitely many α's. Suppose $\overline{A_\beta} \cap U_x \neq \varnothing$ for some β, we shall show that $A_\beta \cap U_x \neq \varnothing$. If U_x contains a point of A_β, then we are done. If U_x contains a limit point of A_β, then U_x is a neighborhood of this limit point, and hence it must contain a point of A_β. This shows that $\{\overline{A_\alpha}\}$ is locally finite.

For the second part, by (2) it suffices to prove that $\overline{\bigcup_\alpha A_\alpha} \subset \bigcup_\alpha \overline{A_\alpha}$, and note that it is enough to show that $\bigcup_\alpha \overline{A_\alpha}$ is closed. Take any $x \notin \bigcup_\alpha \overline{A_\alpha}$. By the local finiteness of $\{\overline{A_\alpha}\}$, there exists an open neighborhood U of x so that $\overline{A_\alpha} \cap U \neq \varnothing$ for only finitely many α's, say $\alpha_1, \ldots, \alpha_n$. Then

$$\widetilde{U} \coloneqq U \setminus \left(\overline{A_{\alpha_1}} \cup \cdots \cup \overline{A_{\alpha_n}}\right)$$

is an open neighborhood of x that does not intersect any $\overline{A_\alpha}$ and contains x. Thus the complement of $\bigcup_\alpha \overline{A_\alpha}$ is open, i.e., $\bigcup_\alpha \overline{A_\alpha}$ is closed, and the proof is complete. $\square$

Problem 32 (Characterize continuity via interior)

(1) In class we proved

> *A map $f\colon X \to Y$ between two topological spaces is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ holds for any $A \subset X$.*

Apply the idea of "open-closed" duality, write down the corresponding characterization of continuity of f via the interior operation, and then prove it.

(2) Show that $f\colon X \to Y$ is a closed map if and only if $\overline{f(A)} \subset f\big(\overline{A}\big)$ holds for any $A \subset X$.

(3) Prove a similar property for open maps via interior.

Proof (1) We can characterize continuity of f via the interior operation as follows:

Proposition *A map $f\colon X \to Y$ between two topological spaces is continuous if and only if*

$$f^{-1}(\mathrm{Int}(A)) \subset \mathrm{Int}\big(f^{-1}(A)\big)$$

holds for any $A \subset Y$.

Proof

(⇒) Suppose f is continuous and let $p \in f^{-1}(\mathrm{Int}(A))$. Then $f(p) \in \mathrm{Int}(A)$, hence there exists an open neighborhood U of $f(p)$ contained in A. By the continuity of f, $f^{-1}(U) \subset f^{-1}(A)$ is open in X and contains p, so $p \in \mathrm{Int}\big(f^{-1}(A)\big)$.

(⇐) For any open set $U \subset Y$, we have

$$f^{-1}(U) = f^{-1}(\mathrm{Int}(U)) \subset \mathrm{Int}\big(f^{-1}(U)\big) \subset f^{-1}(U).$$

Therefore $f^{-1}(U) = \mathrm{Int}\big(f^{-1}(U)\big)$ is open in X. Hence f is continuous.

(2) (⇒) If f is a closed and $A \subset X$, then $f\big(\overline{A}\big)$ is closed in Y since $\overline{A}$ is closed in X. Therefore $f(A) \subset f\big(\overline{A}\big)$ implies $\overline{f(A)} \subset \overline{f\big(\overline{A}\big)} = f\big(\overline{A}\big)$.

(⇐) Suppose $A \subset X$ is closed. Then $\overline{f(A)} \subset f\big(\overline{A}\big) = f(A) \subset \overline{f(A)}$, hence $f(A) = \overline{f(A)}$ is closed. Therefore f is a closed map.

(3) We can characterize openness of a map in terms of interiors as follows:

Proposition *A map $f\colon X \to Y$ between two topological spaces is open if and only if*

$$f^{-1}(\mathrm{Int}(A)) \supset \mathrm{Int}\big(f^{-1}(A)\big)$$

holds for any $A \subset Y$.

Proof

(⇒) Suppose f is open. For any $A \subset Y$, $\mathrm{Int}\, f^{-1}(A)$ is open in X, hence $f\big(\mathrm{Int}\, f^{-1}(A)\big)$ is open in Y. Also note that $f\big(\mathrm{Int}\, f^{-1}(A)\big) \subset f\big(f^{-1}(A)\big) \subset A$, therefore $f\big(\mathrm{Int}\, f^{-1}(A)\big) \subset \mathrm{Int}(A)$, and thus

$$\mathrm{Int}\, f^{-1}(A) \subset f^{-1}\big(f\big(\mathrm{Int}\, f^{-1}(A)\big)\big) \subset f^{-1}(\mathrm{Int}(A)).$$

(⇐) Suppose $U \subset X$ is open, then take $A = f(U)$, so that

$$U = \mathrm{Int}(U) \subset \mathrm{Int}\, f^{-1}(f(U)) \subset f^{-1}(\mathrm{Int}\, f(U)).$$

Hence $f(U) \subset f\big(f^{-1}(\mathrm{Int}\, f(U))\big) \subset \mathrm{Int}\, f(U)$, which implies $f(U) = \mathrm{Int}\, f(U)$ is open in Y. Therefore f is an open map. □

PSet 5, Part 1

Problem 33 (Intersection of compact sets)

(1) Let X be a Hausdorff space. Prove: if K_α are compact subsets of X, then $\bigcap_\alpha K_\alpha$ is a compact subset of X.

(2) Find an example: A, B are compact subsets of a topological space X, while $A \cap B$ is non-compact.

Proof (1) Since every compact subset of a Hausdorff space is closed, each K_α is closed. Then $\bigcap_\alpha K_\alpha$ is closed, and it is compact since it is a closed subset of a compact set (any fixed K_β).

(2) Take $\mathbb{R}$ with the usual topology and add in two more points a and b. Declare the open sets to be the usual open sets in $\mathbb{R}$ together with $\mathbb{R} \cup \{a\}, \mathbb{R} \cup \{b\}$ and $\mathbb{R} \cup \{a, b\}$. Now $A := \mathbb{R} \cup \{a\}$ and $B := \mathbb{R} \cup \{b\}$ are both compact, but $A \cap B = \mathbb{R}$ is not compact. $\square$

Problem 34 (Compactness for the "upper semi-continuous" topology) In Problem 15 you are supposed to construct the upper semi-continuous topology on $\mathbb{R}$, and the solution is

$$\mathscr{T}_{\text{u.s.c.}} = \{\varnothing\} \cup \{\mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}.$$

(1) Is $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ compact / sequentially compact?

(2) Describe all compact subsets in $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$.

(3) State a theorem called "the extremal value theorem for upper semi-continuous functions" and prove it.

Proof (1) $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is non-compact since $\{(-\infty, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ with no finite subcover. Take the sequence $x_n = n$, then it has no convergent subsequence, which means $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is not sequentially compact.

(2) A subset $A \in (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is compact if and only if it is $\varnothing$ or has a maximum.

($\Leftarrow$) If A has a maximum M, then for any open cover of A, there must exist an open set $(-\infty, a)$ ($a \in \mathbb{R} \cup \{+\infty\}$) containing M, which does cover A.

($\Rightarrow$) Suppose $A \neq \varnothing$ does not have a maximum and let M be the supremum of A.

- ⋄ If $M < +\infty$, then $\left\{\left(-\infty, M - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is an open cover of A, but it has no finite subcover by the definition of supremum.
- ⋄ If $M = +\infty$, then $\{(-\infty, n) : n \in \mathbb{N}\}$ is an open cover of A, but it has no finite subcover.

(3) **Theorem** *If a function $f\colon (X, \mathscr{T}) \to \mathbb{R}$ is upper semi-continuous and $K \subset X$ is compact, then f attains its maximum on K.*

Proof In Problem 15 (2) ① we proved that a function f from a topological space $(X, \mathscr{T})$ to $\mathbb{R}$ is upper semi-continuous if and only if $f\colon (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous. So if $K \subset X$ is compact, then $f(K)$ is compact in $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$, and by (2) we know that $f(K)$ has a maximum. $\square$

Problem 35 (Limit point compact) Let X be a topological space. If for any infinite subset S of X one has $S' \neq \varnothing$, then we say X is *limit point compact*.

(1) Consider the cofinite topology $(X, \mathscr{T}_{\text{cofinite}})$. Is it limit point compact?

(2) Show that $X = (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}})$ is not compact, not sequentially compact, but is limit point compact.

(3) Prove: if X is compact or sequentially compact, then it is limit point compact.

(4) Show that any closed subset of a limit point compact space is limit point compact.

(5) Let X be limit point compact and $f\colon X \to Y$ be continuous. Is $f(X)$ limit point compact?

Proof (1) If $S \subset X$ is infinite, then for any $x \in X$, the set $S \setminus \{x\}$ is infinite, so $U \cap (S \setminus \{x\}) \neq \varnothing$ for any non-empty open set $U \subset X$. This means x is a limit point of S, and $S' = X$. Therefore $(X, \mathscr{T}_{\text{cofinite}})$ is limit point compact.

(2) ① $\{\{n\} \times \mathbb{N} : n \in \mathbb{N}\}$ is an open cover of X, but it has no finite subcover, so X is not compact.

② Take the sequence $x_n = (n, 1)$, then it has no convergent subsequence, which means X is not sequentially compact.

③ For any non-empty subset $S \subset X$, take some $(m_0, n_0) \in S$, then any (m_0, n_1) with $n_1 \neq n_0$ is a limit point of S. Thus X is limit point compact.

(3) ① Suppose X is compact, and $S \subset X$ is any subset. Suppose X has no limit point, then S is closed since $\overline{S} = S \cup S' = S$. For any $a \in S$, there exists an open set U_a such that $S \cap U_a = \{a\}$. Now $S^{\mathrm{c}} \cup \{U_a : a \in S\}$ is an open cover of X. By compactness, there exists $a_1, \cdots, a_k \in S$ such that

$$X = S^{\mathrm{c}} \cup \left(\bigcup_{i=1}^{k} U_{a_i} \right).$$

It follows that

$$S = S \cap X = S \cap \left(\bigcup_{i=1}^{k} U_{a_i} \right) = \bigcup_{i=1}^{k} (S \cap U_{a_i}) = \{a_1, \cdots, a_k\}$$

is a finite subset. This implies that X is limit point compact.

② Suppose X is sequentially compact, and $S \subset X$ is any infinite subset. Take any infinite sequence $(x_1, x_2, \cdots)$ in S such that $x_i \neq x_j$ for $i \neq j$. Then there exists a subsequence $(x_{n_1}, x_{n_2}, \cdots)$ converging to some $x_0 \in X$. It follows from definition that

$$x_0 \in \{x_{n_1}, x_{n_2}, \cdots\}' \subset \{x_1, x_2, \cdots\}' \subset S'.$$

Hence $S' \neq \varnothing$ and X is limit point compact.

(4) Suppose X is limit point compact, and $A \subset X$ is an infinite closed subset. For any infinite subset $S \subset A$, ${S'}^X \neq \varnothing$. By Problem 29 (1) we know that ${S'}^A = {S'}^X \cap A$. Since A is closed, ${S'}^X \subset {A'}^X \subset A$. Therefore ${S'}^A = {S'}^X \cap A = {S'}^X \neq \varnothing$, and then A is limit point compact.

(5) Not true in general. For example, $X = (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}})$ is limit point compact as shown in (2). Consider the projection onto the first factor:

$$\pi_1 \colon (\mathbb{N}, \mathscr{T}_{\text{discrete}}) \times (\mathbb{N}, \mathscr{T}_{\text{trivial}}) \to (\mathbb{N}, \mathscr{T}_{\text{discrete}}),$$

which is continuous. However, $\pi_1(X) = \mathbb{N}$ is not limit point compact since any subset in a discrete space has no limit point. $\square$

Problem 36 (One-point compactification) Given any topological space $(X, \mathscr{T})$, we say a compact topological space Y is a *compactification* of X if there exists a homeomorphism $f \colon X \to f(X) \subset Y$ such that $\overline{f(X)} = Y$.

(1) Prove: both $\mathbb{S}^1$ and $[0,1]$ are compactifications of $\mathbb{R}$.

(2) For any non-compact topological space $(X, \mathscr{T})$, define a topology $\mathscr{T}^*$ on the set $X^* = X \sqcup \{\infty\}$ by

$$\mathscr{T}^* = \mathscr{T} \cup \{X^*\} \cup \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\}.$$

Prove: $\mathscr{T}^*$ is a topology on X^*, and $(X^*, \mathscr{T}^*)$ is a compactification of $(X, \mathscr{T})$. This is called the *one-point compactification* of $(X, \mathscr{T})$.

(3) Prove: the one-point compactification of $\mathbb{N}$ is homeomorphic to $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (as a subset in $\mathbb{R}$).

(4) Construct a compact Hausdorff topology on any set X.

Proof (1) ① Let $\sigma \colon \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}$ be the stereographic projection that sends a point x other than the "north pole" N on $\mathbb{S}^1$ to the point $u \in \mathbb{R}$ chosen so that $U = (u, 0)$ is the point in $\mathbb{R}^2$ where the line through N and x meets the subspace $x_2 = 0$. It is easy to obtain a formula for σ:

$$\sigma \colon \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}, \quad (x_1, x_2) \mapsto \frac{x_1}{1 - x_2}.$$

Its inverse $f := \sigma^{-1}$ is given by

$$f \colon \mathbb{R} \to \mathbb{S}^1 \setminus \{N\}, \quad x \mapsto \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right).$$

Since both σ and f are continuous, f is a homeomorphism between $\mathbb{R}$ and $\mathbb{S}^1 \setminus \{N\}$. Moreover, $\mathbb{S}^1$ is compact in $\mathbb{R}^2$ and $\overline{f(\mathbb{R})} = \overline{\mathbb{S}^1 \setminus \{N\}} = \mathbb{S}^1$. Therefore $\mathbb{S}^1$ is a compactification of $\mathbb{R}$.

② Consider the homeomorphism

$$f \colon \mathbb{R} \to (0,1), \quad x \mapsto \frac{1}{\pi} \arctan x + \frac{1}{2}.$$

Since $\overline{f(\mathbb{R})} = \overline{(0,1)} = [0,1]$ is compact, $[0,1]$ is a compactification of $\mathbb{R}$.

(2) For conciseness, we regard $\varnothing$ as a closed and compact subset of X, then

$$\mathscr{T}^* = \mathscr{T} \cup \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\}.$$

① Clearly $\varnothing \in \mathscr{T} \subset \mathscr{T}^*$ and $X^* \in \mathscr{T}^*$.

② If $U_1, U_2 \in \mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T} \subset \mathscr{T}^*$. If $U_i = X^* \setminus C_i$ where C_i is closed and compact in X ($i = 1, 2$), then $C_i \cup C_2$ is again closed and compact in X, and

$$U_1 \cap U_2 = (X^* \setminus C_1) \cap (X^* \setminus C_2) = X^* \setminus (C_1 \cup C_2) \in \mathscr{T}^*.$$

If $U_1 \in \mathscr{T}$ and $U_2 = X^* \setminus C_2$ where C_2 is closed and compact in X, then

$$U_1 \cap U_2 = U_1 \cap (X^* \setminus C_2) = U_1 \cap (X \setminus C_2) \in \mathscr{T} \subset \mathscr{T}^*.$$

③ For any $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}^*$, if $\bigcup_{\alpha \in \Lambda} U_\alpha = \varnothing$ or X^*, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}^*$. Otherwise there are three cases:

- If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathscr{T}$, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}^*$.
- If $\{U_\alpha : \alpha \in \Lambda\} \subset \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\}$, then

$$X^* \setminus \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcap_{\alpha \in \Lambda} (X^* \setminus U_\alpha)$$

is closed in $(X, \mathscr{T})$, and for any fixed $\beta \in \Lambda$,

$$X^* \setminus \bigcup_{\alpha \in \Lambda} U_\alpha \subset X^* \setminus U_\beta.$$

Since any closed subset of a compact space is compact, $X^* \setminus \bigcup_{\alpha \in \Lambda} U_\alpha$ is compact in $X^* \setminus U_\beta$. Therefore

$$\bigcup_{\alpha \in \Lambda} U_\alpha \in \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\} \subset \mathscr{T}^*.$$

- If

$$\Lambda_1 \coloneqq \{\lambda \in \Lambda : U_\lambda \in \mathscr{T}\} \neq \varnothing$$

and

$$\Lambda_2 \coloneqq \{\lambda \in \Lambda : U_\lambda \in \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\}\} \neq \varnothing,$$

then by the above two cases we know that $\bigcup_{\alpha \in \Lambda_1} U_\alpha \in \mathscr{T}$ and

$$\bigcup_{\alpha \in \Lambda_2} U_\alpha \in \{K^{\mathrm{c}} \cup \{\infty\} : K \subset X \text{ is closed and compact}\}.$$

Now

$$\begin{aligned} X^* \setminus \bigcup_{\alpha \in \Lambda} U_\alpha &= X^* \setminus \left(\left(\bigcup_{\alpha \in \Lambda_1} U_\alpha \right) \cup \left(\bigcup_{\alpha \in \Lambda_2} U_\alpha \right) \right) \\ &= \left(X^* \setminus \bigcup_{\alpha \in \Lambda_1} U_\alpha \right) \cap \left(X^* \setminus \bigcup_{\alpha \in \Lambda_2} U_\alpha \right) \\ &\overset{\star}{=} \left(X \setminus \bigcup_{\alpha \in \Lambda_1} U_\alpha \right) \cap \left(X^* \setminus \bigcup_{\alpha \in \Lambda_2} U_\alpha \right). \end{aligned}$$

Note that in "$\star$" we used the fact that $\infty \notin X^* \setminus \bigcup_{\alpha\in\Lambda_2} U_\alpha$. Hence $X^* \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ is closed in the compact subspace $X^* \setminus \bigcup_{\alpha\in\Lambda_2} U_\alpha$ of $(X, \mathscr{T})$, so it is compact. Therefore

$$\bigcup_{\alpha\in\Lambda} U_\alpha \in \{K^c \cup \{\infty\} : K \subset X \text{ is closed and compact}\} \subset \mathscr{T}^*.$$

So $\mathscr{T}^*$ is a topology on X^*.

To see that $(X^*, \mathscr{T}^*)$ is a compact space, take any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X^*. Then there exists $\beta \in \Lambda$ such that $\infty \in U_\beta$. By the definition of $\mathscr{T}^*$, $X^* \setminus U_\beta$ is closed and compact in X. Since $\{U_\alpha : \alpha \in \Lambda\}$ is an open cover of $X^* \setminus U_\beta$, it has a finite subcover $\{U_1, \cdots, U_n\}$. Now $\{U_\beta, U_1, \cdots, U_n\}$ is a finite subcover of $\{U_\alpha : \alpha \in \Lambda\}$ that covers $U_\beta \cup (X^* \setminus U_\beta) = X^*$. Therefore $(X^*, \mathscr{T}^*)$ is compact.

The inclusion map $\iota\colon X \to X^*$ is clearly a homeomorphism between X and $\iota(X)$. Moreover, any open set containing ∞ is of the form $K^c \cup \{\infty\}$ for some closed and compact $K \subset X$, and $K^c \neq \varnothing$ since X is non-compact by assumption, this open set must intersect $\iota(X)$. Therefore $\infty \in \iota(X)'$ and $\overline{\iota(X)} = X^*$, which proves that $(X^*, \mathscr{T}^*)$ is a compactification of $(X, \mathscr{T})$.

(3) Consider the bijection

$$f\colon \mathbb{N} \cup \{\infty\} \to \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \quad n \mapsto \frac{1}{n}, \quad \infty \mapsto 0.$$

For any open set $U \subset \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, if $0 \notin U$, then $f^{-1}(U)$ consists of finitely many points and is open in $\mathbb{N} \cup \{\infty\}$. If $0 \in U$, then there exists $N \in \mathbb{N}$ such that $\frac{1}{n} \in U$ for all $n > N$, i.e., the complement of $f^{-1}(U)$ is a union of a finite set with $\{\infty\}$, which is closed and compact in $\mathbb{N}$. Hence in both cases one has $f^{-1}(U)$ is open in $\mathbb{N} \cup \{\infty\}$. Therefore f is a continuous bijection from the compact space $\mathbb{N} \cup \{\infty\}$ to the Hausdorff space $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, which is a homeomorphism.

(4) Pick a point $x_0 \in X$ and endow $X \setminus \{x_0\}$ with the discrete topology. Then let $(X \setminus \{x_0\}) \cup \{x_0\}$ be the one-point compactification of $X \setminus \{x_0\}$. Now X endowed with this new topology is compact by (2). To see that it is also Hausdorff, we only need to consider the case of separating x_0 from any other point $x_1 \in X \setminus \{x_0\}$ by disjoint open sets. Since $\{x_1\}$ is closed and compact in $X \setminus \{x_0\}$, $(X \setminus \{x_0, x_1\}) \cup \{x_0\} = X \setminus \{x_1\}$ is open in X. Then $\{x_1\}$ and $X \setminus \{x_1\}$ are two disjoint open sets separating x_1 and x_0. □

PSet 5, Part 2

Problem 37 (Compact subsets of $\mathbb{R}^n$)

(1) Prove Alexander subbasis theorem for $\mathbb{R}$ with $\mathcal{S} = \{(-\infty, a), (a, \infty) : a \in \mathbb{R}\}$.

(2) Prove any finite closed interval $[a, b]$ in $\mathbb{R}$ is compact using (1).

(3) Show that a subset in $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof (1) Let us begin by showing that a subset $K \subset \mathbb{R}$ whose every $\mathcal{S}$-cover has a finite subcover is closed and bounded.

(Boundedness) If K has no upper bounded, then the $\mathcal{S}$-cover $\{(-\infty, n) : n \in \mathbb{N}\}$ has no finite subcover, a contradiction. Similarly, if K has no lower bound, then the $\mathcal{S}$-cover $\{(-n, \infty) : n \in \mathbb{N}\}$ has no finite subcover, again a contradiction. Thus, K is bounded.

(Closedness) For any sequence (x_n) in K that converges to $x \in \mathbb{R}$ (we may assume $x_n \neq x$ for all n), we can pick a subsequence (x_{n_k}) such that either $x_{n_k} > x$ for all k or $x_{n_k} < x$ for all k. In the former case, we must have $x \in K$ for otherwise the $\mathcal{S}$-cover $\{(x, \infty)\} \cup \{(-\infty, x - \frac{1}{n}) : n \in \mathbb{N}\}$ of K has no finite subcover. In the latter case, we still have $x \in K$, for otherwise the $\mathcal{S}$-cover $\{(-\infty, x)\} \cup \{(x + \frac{1}{n}, \infty) : n \in \mathbb{N}\}$ of K has no finite subcover. Thus, K is closed.

For any sequence (x_n) in K, since K is bonuded, (x_n) has a convergent subsequence (x_{n_k}). And since K is closed, the limit of (x_{n_k}) is in K. This implies that K is sequentially compact, which is equivalent to compactness in $\mathbb{R}$.

(2) Suppose $\{(-\infty, b_i)\}_{i \in I} \cup \{(a_j, \infty)\}_{j \in J}$ is a cover of $[a, b]$. It is reasonable to assume all $b_i > 0$ and $a_j < 0$. Since

$$\bigcup_{i \in I} (-\infty, b_i) = \left(-\infty, \sup_{i \in I} b_i\right), \quad \bigcup_{j \in J} (a_i, \infty) = \left(\inf_{j \in J} a_j, \infty\right),$$

we have $\sup_{i \in I} b_i > \inf_{j \in J} a_j$. Thus, there exists $i_0 \in I$ and $j_0 \in J$ such that $b_{i_0} > a_{j_0}$. Then there is a finite subcover $\{(-\infty, b_{i_0}), (a_{j_0}, \infty)\}$. By (1), $[a, b]$ is compact.

(3) ($\Rightarrow$) Suppose $K \subset \mathbb{R}$ is compact. Since a compact subset of a Hausdorff space is closed, K is closed. If K is unbounded, then the open cover $\{\mathbb{B}(0, n) : n \in \mathbb{N}\}$ has no finite subcover, a contradiction. Thus, K is bounded and closed.

($\Leftarrow$) Suppose $K \subset \mathbb{R}$ is bounded and closed. Since K is bounded, $K \subset [a, b]^n$ for some $a, b \in \mathbb{R}$. By (2), $[a, b]$ is compact, and then $[a, b]^n$ is compact by Tychonoff's theorem. So K is a closed subset of a compact set, hence compact. $\square$

Problem 38 (Topology of the Cantor set) Consider the Cantor set

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right).$$

(1) Prove: every point in the Cantor set is a limit point.

(2) Prove: as a subset of $[0, 1]$, the Cantor set is nowhere dense.

(3) For any closed subset $F \subset C$, construct a continuous map $f : C \to F$ so that $f(x) = x$ for all $x \in F$.

(4) Define a map

$$g : \{0, 1\}^{\mathbb{N}} \to [0, 1], \quad a = (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2a_k}{3^k}.$$

Prove: g induces a homeomorphism between $\left(\{0, 1\}^{\mathbb{N}}, \mathscr{T}_{\text{product}}\right)$ and C.

(5) Show that the map

$$h : \{0, 1\}^{\mathbb{N}} \to [0, 1]^2, \quad a = (a_1, a_2, \cdots) \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{2^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{2^k}\right)$$

is continuous and surjective. Is h injective?

Proof (1) Let $C_j = [0,1] \setminus \bigcup_{n=1}^{j} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$, then each C_j is a finite union of closed intervals. For any $x \in C$ and any $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $3^{-n} < \varepsilon$. Then the closed interval in C_n containing x is contained in $B(x, \varepsilon)$. Note that the two endpoints of this interval are both in C, and one of them is different from x. This shows that every open set containing x contains a point in C different from x. Thus, x is a limit point of C.

(2) Since $C = \bigcap_{n=1}^{\infty} C_n$ is closed, $\overline{C} = C$. Moreover, no open interval in $[0,1]$ is disjoint from all the deleted open intervals of $[0,1]$. Hence $\operatorname{Int}(\overline{C}) = \varnothing$.

(3) Let F be a nonempty closed subset of C. Then $[0,1] \setminus F$ is open in $[0,1]$, so it can be written as a countable disjoint union of open intervals in $[0,1]$:

$$[0,1] \setminus F = \bigsqcup_{n=1}^{\infty} J_n.$$

If J_n is of the form (a_n, b_n), then clearly $a_n, b_n \in F$. Since C does not contain any open interval, we can pick some $x_n \in (a_n, b_n) \setminus C$. If J_n is of the form $[0, b_n)$, then $b_n \in F$ and we take $x_n = 0$. If J_n is of the form $(a_n, 1]$, then $a_n \in F$ and we take $x_n = 1$. Now define

$$f\colon C \to F, \quad x \mapsto \begin{cases} x, & \text{if } x \in F, \\ a_n, & \text{if } x \in (a_n, x_n] \text{ for some } n, \\ b_n, & \text{if } x \in [x_n, b_n) \text{ for some } n. \end{cases}$$

For $x \in C \setminus F$, there is an open set containing x on which f is constant, so f is continuous at x. For $x \in F$, if x is a limit point of F on both sides, then since $f|_F = \mathrm{Id}_F$, we see that f is continuous at x; if x is an a_n, then f is left-continuous at x, and since f is constant on $(a_n, x_n]$, f is continuous at x; the same argument applies to the case where x is a b_n. Thus, f is a continuous function.

(4) The bijectivity of g is immediate from the ternary expansions of real numbers in C. To prove that $g^{-1}\colon C \to \{0,1\}^{\mathbb{N}}$ is continuous, by the property of product topology, it suffices to prove that each $\pi_n \circ g^{-1}\colon C \to \{0,1\}$ is continuous, where π_n is the projection onto the n-th component. Then we only need to show that the preimages of $\{0\}$ and $\{1\}$ under $\pi_n \circ g^{-1}$ are both closed in C. In fact, they are both intersections of C with finitely many closed intervals, which are closed in C as desired. Now g^{-1} is a continuous bijection from the compact space C to the Hausdorff space $\{0,1\}^{\mathbb{N}}$, which is a homeomorphism.

(5) The fact that h is surjective follows from the fact that every real number in $[0,1]$ has a binary expansion. To prove that h is continuous, by the property of product topology, it suffices to show that the map

$$\tilde{h}\colon \{0,1\}^{\mathbb{N}} \to [0,1], \quad (b_1, b_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

is continuous. By (4), we can regard $\{0,1\}^{\mathbb{N}}$ as the Cantor set C, and then $\tilde{h}$ can be viewed as a surjection from C to $[0,1]$ which is non-decreasing. Note that a non-decreasing function can only have jump discontinuities, and surjectivity implies there are no jumps. Thus, $\tilde{h}$ is continuous, and so is h. However, h is not injective. For example, both $(1,1,0,0,0,0,\cdots)$ and $(0,0,1,1,1,1,\cdots)$ are mapped to $\left(\frac{1}{2}, \frac{1}{2}\right)$. $\square$

Problem 39 (Sequential compactness for products)

(1) Let $X_1, \cdots, X_n$ be sequentially compact topological spaces. Prove: the product space $X = X_1 \times \cdots \times X_n$ is sequentially compact.

(2) Is $X = \{0,1\}^{\mathbb{N}}$ sequentially compact when equipped with the box topology $\mathscr{T}_{\text{box}}$?

(3) Let (X_n, d_n) be compact metric spaces. Define a metric on $X = \prod_{n=1}^{\infty} X_n$ via

$$d((x_n),(y_n)) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{[1 + \text{diam}(X_n)] \cdot 2^n}.$$

Prove: on X the metric topology $\mathscr{T}_d$ coincides with the product topology.

Proof (1) It suffices to prove the statement for $n = 2$. Let X and Y be two sequentially compact topological spaces. Let (x_k, y_k) be a sequence in $X \times Y$. Since X is sequentially compact, there is a subsequence $\left(x_{k_j}\right)$ converging to some $x \in X$. Since Y is sequentially compact, there is a subsequence $\left(y_{k_{j_l}}\right)$ converging to some $y \in Y$. Then $\left(x_{k_{j_l}}, y_{k_{j_l}}\right)$ converges to (x, y) in $X \times Y$. This shows that $X \times Y$ is sequentially compact.

(2) Consider $e_i \in X$ where the i-th coordinate is 1 and all other coordinates are 0. If the sequence $(e_1, e_2, \cdots)$ has a convergent subsequence, then the limit must be $(0, 0, \cdots)$, but $\{(0, 0, \cdots)\}$ itself is open in the box topology and no e_i is in this set. Thus, X is not sequentially compact.

(3) $\boxed{\mathscr{T}_d \subset \mathscr{T}_{\text{product}}}$ Suppose $U \in \mathscr{T}_d$. Then for any $(a_n) \in U$ there exists $r > 0$ such that $\mathbb{B}((a_n), r) \subset U$. Consider

$$V = \mathbb{B}_1\left(a_1, \tfrac{r}{2}\right) \times \cdots \times \mathbb{B}_N\left(a_N, \tfrac{r}{2}\right) \times X_{N+1} \times X_{N+2} \times \cdots \in \mathscr{T}_{\text{product}}.$$

Take $N \in \mathbb{N}$ such that $2^{-N} < \frac{r}{2}$, then for all $(b_n) \in V$ we have

$$\begin{aligned} d((a_n),(b_n)) &= \sum_{n=1}^{\infty} \frac{d_n(a_n, b_n)}{[1 + \text{diam}(X_n)] \cdot 2^n} \leqslant \sum_{n=1}^{N} \frac{\frac{r}{2}}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &= \frac{r}{2}\left(1 - \frac{1}{2^N}\right) + \frac{1}{2^N} < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Hence $(a_n) \in V \subset \mathbb{B}((a_n), r) \subset U$, which implies $U \in \mathscr{T}_{\text{product}}$.

$\boxed{\mathscr{T}_{\text{product}} \subset \mathscr{T}_d}$ Suppose $V \in \mathscr{T}_{\text{product}}$, which has the form

$$V = V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \cdots,$$

where each V_i is open in X_i. For any $(x_n) \in V$, there exists $r_1, \cdots, r_k > 0$ such that $\mathbb{B}(x_i, r_i) \subset V_i$ for $1 \leqslant i \leqslant k$. Let

$$r_0 = \min\left\{\frac{r_1}{[1 + \text{diam}(X_1)] \cdot 2^1}, \cdots, \frac{r_k}{[1 + \text{diam}(X_k)] \cdot 2^k}\right\} > 0,$$

then $(x_n) \in \mathbb{B}((x_n), r_0) \subset V$. Hence $V \in \mathscr{T}_d$. $\square$

Problem 40 (Interior of compact subsets)

(1) Let X_α be a family of topological spaces such that X_α is non-compact for infinitely many α's. Let K be a compact set in $\left(\prod_\alpha X_\alpha, \mathscr{T}_{\text{product}}\right)$. Prove: K has no interior point.

(2) Consider the space $\ell^2(\mathbb{R})$ defined by

$$\ell^2(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} : \|x\|_2 := \left(\sum_n |x_n|^2\right)^{\frac{1}{2}} < +\infty \right\} \subset \mathbb{R}^{\mathbb{N}},$$

endowed with the metric d_2. Is the closed unit ball compact? Can a compact subset have any interior point?

(3) A topological space $(X, \mathscr{T})$ is called *locally compact* if for every $x \in X$, there exists a compact set K_x and an open set U_x such that

$$x \in U_x \subset K_x.$$

Prove: the product $\left(\prod_\alpha X_\alpha, \mathscr{T}_{\text{product}}\right)$ of a family of topological spaces is locally compact if and only if there is a finite set of indices Λ_0 such that

$$X_\alpha \text{ is } \begin{cases} \text{compact for } \alpha \notin \Lambda_0, \\ \text{locally compact for } \alpha \in \Lambda_0. \end{cases}$$

Proof (1) Suppose $(x_\alpha) \in \text{Int}(K)$ and take an open neighborhood $U = \prod_\alpha U_\alpha$ of (x_α) contained in K. Then $U_\beta = X_\beta$ for some β where X_β is non-compact. Now fix an open cover $\{V_\lambda : \lambda \in \Lambda\}$ of X_β, then $\{W_\lambda : \lambda \in \Lambda\}$ is an open cover of K with no finite subcover, where $W_\lambda = \prod_\alpha W_{\lambda,\alpha}$ and

$$W_{\lambda,\alpha} = \begin{cases} X_\alpha, & \alpha \neq \beta, \\ V_\lambda, & \alpha = \beta. \end{cases}$$

(2) The closed unit ball in $\ell^2(\mathbb{R})$ is non-compact, for the sequence $(e_1, e_2, \cdots)$ has no convergent subsequence. Suppose $K \subset \ell^2(\mathbb{R})$ is compact and has an interior point $(x_n) \in K$. Then there exists $r > 0$ such that $\mathbb{B}((x_n), r) \subset K$. Let

$$y_n^{(i)} = \begin{cases} x_n, & n \neq i, \\ x_n + \dfrac{r}{2}, & n = i. \end{cases}$$

Then the sequence $\left(y_n^{(1)}\right), \left(y_n^{(2)}\right), \cdots$ lies in K but has no convergent subsequence since any two distinct points in this sequence have distance $\frac{r}{\sqrt{2}}$. This contradicts the compactness of K. Therefore any compact subset of $\ell^2(\mathbb{R})$ has no interior point.

(3) $(\Rightarrow)$ Suppose the product space is locally compact. Then for any X_α and $x \in X_\alpha$, we can pick a point in the product space whose α-th coordinate is x. This point has a compact neighborhood, and by projecting this neighborhood to X_α, we get a compact neighborhood of x. Thus,

each X_α is locally compact. Next, for any (x_α) in the product space, there exists a compact neighborhood of (x_α). By the definition of product topology, this neighborhood can be written as $\prod\limits_\alpha K_\alpha$, where $K_\alpha = X_\alpha$ for all but finitely many α's. And all these X_α's are compact since the projection maps are continuous. Thus, only finitely many X_α's can be non-compact.

($\Leftarrow$) Suppose each X_α is locally compact and all but finitely many X_α's are compact. Then for any (x_α) in the product space, there exists a compact neighborhood K_α of each x_α, and for $\alpha \notin \Lambda_0$ we can let $K_\alpha = X_\alpha$. Then $\prod\limits_\alpha K_\alpha$ is a compact neighborhood of (x_α) by Tychonoff's theorem and the definition of product topology. Thus, the product space is locally compact. $\square$

PSet 6, Part 1

Problem 41 (Totally bounded) Let (X, d) be a metric space, and $A \subset X$ (equipped with subspace metric).

(1) Suppose (X, d) is totally bounded. Show that (A, d) is totally bounded.

(2) Suppose (A, d) is totally bounded. Propose a condition on A so that (X, d) is totally bounded.

(3) Show that (X, d) is totally bounded if and only if any sequence in X has a subsequence that is Cauchy.

(4) Let (X, d) be complete. Prove: $\overline{A}$ is compact if and only if A is totally bounded.

Proof (1) For any $\varepsilon > 0$, there exists $x_1, \cdots, x_n \in X$ such that $A \subset X = \bigcup\limits_{i=1}^{n} \mathbb{B}\big(x_i, \frac{\varepsilon}{2}\big)$. After getting rid of the balls that do not intersect A, let us say $A \subset \bigcup\limits_{k=1}^{m} \mathbb{B}\big(x_{i_k}, \frac{\varepsilon}{2}\big)$. For each k, pick $a_k \in A \cap \mathbb{B}\big(x_{i_k}, \frac{\varepsilon}{2}\big)$. Note that $\mathbb{B}(a_k, \varepsilon) \supset \mathbb{B}\big(x_{i_k}, \frac{\varepsilon}{2}\big)$, hence $A \subset \bigcup\limits_{k=1}^{m} \mathbb{B}(a_k, \varepsilon)$. Finally, we see that $A = \bigcup\limits_{k=1}^{m} \mathbb{B}_A(a_k, \varepsilon)$, so (A, d) is totally bounded.

(2) **Condition:** A is dense in X.

Proof For any $\varepsilon > 0$, there exists $a_1, \cdots, a_n \in A$ such that $A \subset \bigcup\limits_{i=1}^{n} \mathbb{B}(a_i, \frac{\varepsilon}{2})$. Since A is dense in X, for any $x \in X$, there exists $a_x \in A$ such that $d(x, a_x) < \frac{\varepsilon}{2}$. Suppose $a_x \in \mathbb{B}\big(a_k, \frac{\varepsilon}{2}\big)$ for some k, then $d(x, a_k) \leqslant d(x, a_x) + d(a_x, a_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $x \in \mathbb{B}(a_k, \varepsilon)$ and thus $X = \bigcup\limits_{i=1}^{n} \mathbb{B}(a_i, \varepsilon)$. Therefore, (X, d) is totally bounded.

(3) ($\Rightarrow$) Suppose (X, d) is totally bounded. For any sequence $\{x_n\}$ and any $\varepsilon > 0$, since X can be covered by finitely many balls of radius $\frac{\varepsilon}{2}$, one of these balls must contain infinitely many terms of $\{x_n\}$. Let us denote this subsequence by $\{x_{n_k}\}$. Then for any $l_1, l_2 \in \mathbb{N}$, we have $d\big(x_{n_{l_1}}, x_{n_{l_2}}\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\{x_{n_k}\}$ is Cauchy.

($\Leftarrow$) Suppose any sequence in X has a subsequence that is Cauchy but (X, d) is not totally bounded. Then there exists $\varepsilon > 0$ such that X cannot be covered by finitely many balls of radius ε. Let us pick $x_1 \in X$. Since X cannot be covered by a ball of radius ε centered at x_1, there exists

$x_2 \in X \setminus \mathbb{B}(x_1, \varepsilon)$. Similarly, we can find $x_k \in X \setminus \bigcup_{i=1}^{k-1} \mathbb{B}(x_i, \varepsilon)$. Then the sequence $\{x_k\}$ has no Cauchy subsequence, since for any $n, m \in \mathbb{N}$, we have $d(x_n, x_m) \geqslant \varepsilon$, a contradiction.

(4) A subspace of a complete metric space is complete if and only if it is closed. Since $\overline{A}$ is already closed, $\overline{A}$ is compact if and only if it is totally bounded. Therefore, what we need to prove becomes:

A is totally bounded if and only if $\overline{A}$ is totally bounded.

($\Rightarrow$) Suppose A is totally bounded. For any $\varepsilon > 0$, there exists $a_1, \cdots, a_n \in A$ such that $A \subset \bigcup_{i=1}^{n} \mathbb{B}\big(a_i, \frac{\varepsilon}{2}\big)$. For any $x \in \overline{A} \setminus A$, the ball $\mathbb{B}\big(x, \frac{\varepsilon}{2}\big)$ must contain some $a_x \in A$. Suppose $a_x \in \mathbb{B}\big(a_k, \frac{\varepsilon}{2}\big)$ for some k, then $d(x, a_k) \leqslant d(x, a_x) + d(a_x, a_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\overline{A} \subset \bigcup_{i=1}^{n} \mathbb{B}(a_i, \varepsilon)$, which means $\overline{A}$ is totally bounded.

($\Leftarrow$) This is part (1). $\square$

Problem 42 (Isometric embedding on a compact metric space is a homeomorphism)
Let (X, d) be a compact metric space, and $f\colon X \to X$ be an isometric embedding. Prove: f is a homeomorphism. Can we remove compactness assumption on X?

Proof (Injectivity) If $f(x) = f(y)$, then $d(x, y) = d(f(x), f(y)) = 0$ and $x = y$.

(Continuity) Since f is an isometric embedding, $f^{-1}(\mathbb{B}(f(x), \varepsilon)) = \mathbb{B}(x, \varepsilon)$ for any $x \in X$ and $\varepsilon > 0$.

(Surjectivity) Suppose there exists $x \in X \setminus f(X)$. Since f is continuous, $f(X)$ is compact, hence closed. By Problem 7 (2), $d \coloneqq d_{f(X)}(x) > 0$. Now consider the recursively defined sequence

$$x_n = \begin{cases} x, & n = 0, \\ f(x_{n-1}), & n \geqslant 1. \end{cases}$$

Then $d(x_0, x_n) \geqslant d$ for all $n \geqslant 1$. This implies that $d(x_k, x_{k+n}) \geqslant d$ for all $k \geqslant 0$ and $n \geqslant 1$. Therefore $d(x_n, x_m) \geqslant d$ for all $n \neq m$, which violates the (sequential) compactness of X. Hence $f(X) = X$, i.e., f is surjective.

Now f is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. However, we cannot remove the compactness assumption on X, a counterexample is given by the right shift map on $[0, +\infty)$:

$$f\colon [0, +\infty) \to [0, +\infty), \quad x \mapsto x + 1.$$

It is obviously an isometric embedding, but not surjective, hence not a homeomorphism. $\square$

Problem 43 (Completion of metric spaces) Let X be a set, and (Y, d_Y) be a metric space. Consider the space of bounded maps,

$$\mathcal{B}(X, Y) = \{f\colon X \to Y : f(X) \text{ is bounded in } Y\}.$$

(1) Prove: the supremum metric $d_\infty(f, g) \coloneqq \sup_{x \in X} d_Y(f(x), g(x))$ is a metric on $\mathcal{B}(X, Y)$.

(2) Prove: if Y is complete, so is $(\mathcal{B}(X,Y), d_\infty)$.

➢ In what follows, suppose (X, d_X) is a metric space, and take $Y = \mathbb{R}$.

(3) Fix a point $x_0 \in X$. For any $a \in X$, define a function $f_a\colon X \to \mathbb{R}$ via $f_a(x) \coloneqq d_X(x,a) - d_X(x,x_0)$. Prove: $f_a \in \mathcal{B}(X, \mathbb{R})$.

(4) Prove: the map

$$\Phi\colon (X,d) \to (\mathcal{B}(X,\mathbb{R}), d_\infty), \quad a \mapsto f_a$$

is an isometric embedding, i.e., $d_X(a,b) = d_\infty(f_a, f_b)$ for any $a, b \in X$.

(5) Prove: any metric space (X, d_X) admits a completion.

(6) Prove: if (Y_1, d_1) and (Y_2, d_2) are two completions of (X, d_X), then (Y_1, d_1) and (Y_2, d_2) are isometric.

Proof (1) Clearly $d_\infty(f,g) \geqslant 0$ and $d_\infty(f,g) = 0$ if and only if $f = g$. For any $f, g \in \mathcal{B}(X,Y)$,

$$d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x)) = \sup_{x\in X} d_Y(g(x), f(x)) = d_\infty(g,f).$$

Finally, for any $f, g, h \in \mathcal{B}(X,Y)$, we have

$$\begin{aligned} d_\infty(f,h) &= \sup_{x\in X} d_Y(f(x), h(x)) \\ &\leqslant \sup_{x\in X}(d_Y(f(x), g(x)) + d_Y(g(x), h(x))) \\ &\leqslant \sup_{x\in X} d_Y(f(x), g(x)) + \sup_{x\in X} d_Y(g(x), h(x)) \\ &= d_\infty(f,g) + d_\infty(g,h). \end{aligned}$$

(2) Suppose $\{f_n\}$ is a Cauchy sequence in $(\mathcal{B}(X,Y), d_\infty)$. For any $x \in X$, $|f_n(x) - f_m(x)| \leqslant d_\infty(f_n, f_m)$, hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{f_n(x)\}$ converges to some $y_x \in Y$. Now define $f\colon X \to Y$ via $f(x) \coloneqq y_x$. By construction, $f(X)$ is bounded in Y, hence $f \in \mathcal{B}(X,Y)$. Moreover, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\infty(f_n, f_m) < \frac{\varepsilon}{2}$ for all $n, m \geqslant N$. Letting $m \to \infty$, we have $d_\infty(f_n, f) \leqslant \frac{\varepsilon}{2} < \varepsilon$ for all $n \geqslant N$. Therefore $\{f_n\}$ converges to f, and thus $(\mathcal{B}(X,Y), d_\infty)$ is complete.

(3) $|f_a(x)| = |d_X(x,a) - d_X(x,x_0)| \leqslant d_X(a,x_0) < \infty$ for all $x \in X$.

(4) The triangle inequality gives

$$\sup_{x\in X} |d_X(x,a) - d_X(x,b)| \leqslant d_X(a,b),$$

and equality holds when $x = a$ or $x = b$. Therefore

$$\begin{aligned} d_\infty(f_a, f_b) &= \sup_{x\in X} |[d_X(x,a) - d_X(x,x_0)] - [d_X(x,b) - d_X(x,x_0)]| \\ &= \sup_{x\in X} |d_X(x,a) - d_X(x,b)| \\ &= d_X(a,b). \end{aligned}$$

(5) There are two ways to construct the completion of (X, d_X).

(Method 1) By (2), (4) and Proposition 2.3.7, $\left(\overline{\Phi(X)}, d_\infty\right)$ is a completion of (X, d_X).

(Method 2) Let $\mathsf{C}(X)$ denote the set of all Cauchy sequences in X and define the equivalence relation $\sim$ on $\mathsf{C}(X)$ by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d_X(x_n, y_n) = 0.$$

Denote the equivalence class of $(x_n) \in \mathsf{C}(X)$ by $[x_n]$ and let $\widehat{X} = \mathsf{C}(X)/\sim$ be the set of all equivalence classes. Define a metric $\hat{d}\colon \widehat{X} \to \mathbb{R}_{\geqslant 0}$ by

$$\hat{d}([x_n], [y_n]) = \lim_{n\to\infty} d_X(x_n, y_n).$$

For $x \in X$, let $\hat{x} = (x, x, x, \cdots)$ be the constant sequence with value x and let $\phi\colon X \to \widehat{X}, x \mapsto [\hat{x}]$. If $\phi(x) = \phi(y)$, then by definition $\lim\limits_{n\to\infty} d(x, y) = 0$, which implies $x = y$. Hence ϕ is injective, so we can identify X with its isomorphic copy $\phi(X) \subset \widehat{X}$. Moreover, this also shows that $\hat{d}([\hat{x}], [\hat{y}]) = d_X(x, y)$. To show that $\overline{\phi(X)} = \widehat{X}$, let $[x_n] \in \widehat{X}$ and $\varepsilon > 0$ be arbitrary. Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \frac{\varepsilon}{2}, \quad \forall n, m \geqslant N.$$

Then we have

$$\hat{d}([\widehat{x_N}], [x_n]) = \lim_{n\to\infty} d_X(x_N, x_n) \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $\overline{\phi(X)} = \widehat{X}$. Finally we demonstrate that $\left(\widehat{X}, \hat{d}\right)$ is complete. By the completeness criterion, it suffices to show that every Cauchy sequence in $\phi(X)$ converges in $\widehat{X}$. Let $([\widehat{w_n}])$ be a Cauchy sequence in $\phi(X)$, so each $\widehat{w_n}$ has the form $(w_n, w_n, w_n, \cdots)$. Since ϕ is an isometry,

$$\hat{d}(\widehat{w_n}, \widehat{w_m}) = d_X(w_n, w_m), \quad \forall n, m \in \mathbb{N}.$$

Therefore, the sequence (w_n) is Cauchy in X. Let $\mathsf{w} = [(w_n)] \in \widehat{X}$. Then for any $\varepsilon > 0$, since there exists $M \in \mathbb{N}$ such that

$$d(w_n, w_m) < \frac{\varepsilon}{2}, \quad \forall n, m > M.$$

Thus for all $n > M$, we have

$$\hat{d}([\widehat{w_n}], \mathsf{w}) = \lim_{m\to\infty} d_X(w_n, w_m) \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $[\widehat{w_n}] \to \mathsf{w} \in \widehat{X}$ as $n \to \infty$, and $\widehat{X}$ is complete. So $\left(\widehat{X}, \hat{d}\right)$ is a completion of (X, d_X).

(6) Let $\phi_1\colon X \to Y_1$ and $\phi_2\colon X \to Y_2$ be the corresponding isometries. Then $\psi \coloneqq \phi_2 \circ \phi_1^{-1}$ gives an isometry from $\phi_1(X)$ to $\phi_2(X)$. Since $\phi_1(X)$ and $\phi_2(X)$ are dense in Y_1 and Y_2 respectively, we can extend ψ continuously to a map $\psi\colon Y_1 \to Y_2$. To be specific, for any $y \in Y_1$, we can find a Cauchy sequence (y_n) in Y_1 with limit y. Then we define

$$\psi(y) \coloneqq \lim_{n\to\infty} \psi(y_n),$$

which converges as Y_2 is complete. Next we show that ψ is surjective. For any $w \in Y_2$, let (w_n) be a Cauchy sequence in $\phi_2(X)$ with limit w. Let y_n be the preimage of w_n under ψ. Then $y := \lim\limits_{n\to\infty} y_n$ is well-defined since Y_1 is complete and satisfies

$$\psi(y) = \lim_{n\to\infty} \psi(y_n) = \lim_{n\to\infty} w_n = w.$$

Therefore ψ is surjective. To show that ψ is injective, suppose that

$$\lim_{n\to\infty} \psi(y_n) = \lim_{n\to\infty} \psi(y_n')$$

and

$$\lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} y_n' = y'.$$

For any $\varepsilon > 0$, pick $M \in \mathbb{N}$ such that $\psi(y_n)$ and $\psi(y_n')$ lie in $\mathbb{B}\big(\psi(y), \frac{\varepsilon}{3}\big)$ for all $n \geqslant M$. Then we have

$$d_1(y_n, y_n') = d_2(\psi(y_n), \psi(y_n')) \leqslant 2 \cdot \frac{\varepsilon}{3} < \varepsilon.$$

This implies $y = y'$, so ψ is injective. Since the distance function of a metric space is continuous, it follows that ψ is an isometry on all of Y_1, and Y_1 and Y_2 are isometric. □

Problem 44 (Lebesgue property) We say a metric space (X, d) has the Lebesgue property if any open covering of X has a positive Lebesgue number.

(1) Suppose (X, d_X) has the Lebesgue property. Prove:

① (X, d_X) is complete.

② For any metric space (Y, d_Y), any continuous map $f\colon X \to Y$ is uniformly continuous.

③ If A, B are non-empty disjoint closed subsets in (X, d_X), then $\operatorname{dist}(A, B) := \inf\{d_X(x, y) : x \in A, y \in B\} > 0$.

(2) Prove: if for any metric space (Y, d_Y), any continuous map $f\colon X \to Y$ is uniformly continuous, then (X, d_X) has the Lebesgue property.

Proof (1) ① If (X, d_X) is not complete, then there exists a Cauchy sequence $\{x_n\}$ that does not converge in X. By Problem 43 (5), (X, d_X) admits a completion $\big(\widehat{X}, \hat{d}\big)$. Let $x \in \widehat{X}$ be the limit of $\{x_n\}$ in $\widehat{X}$. Let $V_0 = \widehat{X} \setminus \overline{\mathbb{B}_{\widehat{X}}\big(x, \frac{1}{2}\big)}$, and for $k \geqslant 1$ let $V_k = \mathbb{B}_{\widehat{X}}\big(x, \frac{1}{2^{k-1}}\big) \setminus \overline{\mathbb{B}_{\widehat{X}}\big(x, \frac{1}{2^{k+1}}\big)}$. Then $\{U_k : k \geqslant 0\}$ where $U_k := V_k \cap X$ is an open cover of X. However, for any $\delta > 0$, the set $\mathbb{B}_{\widehat{X}}(x, \delta) \cap X$ cannot be contained in any U_k, which contradicts the Lebesgue property.

② Suppose $f\colon (X, d_X) \to (Y, d_Y)$ is continuous, and let $\varepsilon > 0$ be arbitrary. For each $x \in X$ choose an open neighborhood U_x of x with $f(U_x) \subset \mathbb{B}_Y\big(f(x), \frac{\varepsilon}{2}\big)$. Let δ be a Lebesgue number for the open cover $\{U_x : x \in X\}$. Then $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

③ Since A, B are non-empty disjoint closed subsets, by Problem 7 (2), $d_A(b) > 0$ and $d_B(a) > 0$ for all $a \in A$ and $b \in B$. Thus for every $a \in A$, we can find an open ball $\mathbb{B}(a, \varepsilon_a)$ that does not intersect B. Similarly, for every $b \in B$, we can find an open ball $\mathbb{B}(b, \varepsilon_b)$ that does not intersect A. Now $\{\mathbb{B}(a, \varepsilon_a) : a \in A\} \cup \{\mathbb{B}(b, \varepsilon_b) : b \in B\}$ is an open cover of $A \cup B$. By the Lebesgue property, there exists $\delta > 0$ such that any subset whose diameter is less than δ is contained in some $\mathbb{B}(a, \varepsilon_a)$ or $\mathbb{B}(b, \varepsilon_b)$. If $\operatorname{dist}(A, B) = 0$, then there exists $a_0 \in A$ and $b_0 \in B$

such that $d_X(a_0, b_0) < \delta$. So a_0 and b_0 must lie in the same open ball, which contradicts our construction.

(2) Suppose such metric space (X, d_X) does not have the Lebesgue property. Then there exists an open cover $\{U_\alpha : \alpha \in \Lambda\}$ such that for all $\delta > 0$, there exists a subset A_δ whose diameter is less than δ but is not contained in any U_α. Now for any $n \in \mathbb{N}$, choose $x_n, y_n \in A_{\frac{1}{n}}$ with $x_n \neq y_n$. We shall show that no subsequence of $\{x_n\}$ converges in X. If there is a subsequence $\{x_{n_k}\}$ converging to some $x \in X$, then $x \in U_\alpha$ for some α. Then there exists $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset U_\alpha$. Since $x_{n_k} \to x$, if we choose n_k large enough so that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $d(x_{n_k}, x) < \frac{\varepsilon}{2}$, then for any $a \in A_{n_k}$, one has

$$d(a, x) \leqslant d(a, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon.$$

Hence $A_{n_k} \subset \mathbb{B}(x, \varepsilon) \subset U_\alpha$, a contradiction. Therefore $\{x_n\}$ has no convergent subsequence, and similarly $\{y_n\}$ has no convergent subsequence. Therefore the set

$$E \coloneqq \{x_1, y_1, x_2, y_2, \cdots\}$$

has no limit point, which implies that it is discrete. So any function defined on E is continuous. By passing to a subsequence, we may assume that $x_n \neq y_m$ for all $n, m \in \mathbb{N}$. Now we can define a function $f\colon E \to \{0, 1\}$ by taking $f(x_n) = 0$ and $f(y_m) = 1$ for all $n, m \in \mathbb{N}$. Since $E = E \cup E' = \overline{E}$ is closed, by the Tietze extension theorem, f can be extended to a continuous function F on X. By assumption, $f\colon X \to \mathbb{R}$ is uniformly continuous. However, $|F(x_n) - F(y_n)| \equiv 1$ as $d(x_n, y_n) \to 0$, a contradiction. Therefore (X, d_X) has the Lebesgue property. $\square$

PSet 6, Part 2

Problem 45 (Uniform metric)

(1) Prove Proposition 2.4.3:

Suppose Y is a complete metric space. Then

$$d_u(f, g) \coloneqq \sup_{x \in X} \frac{d_Y(f(x), g(x))}{1 + d_Y(f(x), g(x))}$$

is a complete metric on $\mathcal{M}(X, Y)$.

(2) Here is another proof of Proposition 2.4.4 / Problem 8 (3) for the special case $Y = \mathbb{R}$:

> Suppose $f_n \in \mathcal{C}(X, \mathbb{R})$ and $f_n \to f$ in $(\mathcal{M}(X, \mathbb{R}), d_{\mathrm{u}})$. To prove f is continuous, it is enough to prove that for any $a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ and $f^{-1}((a, +\infty))$ are open. Let's prove the first one is open. Let's fix an x with $f(x) < a$. Take $\varepsilon = a - f(x)$. Then there exists N such that $d_{\mathrm{u}}(f_n, f) < \frac{\varepsilon}{3}$ for $n \geqslant N$. Since f_N is continuous, there exists an open neighborhood U of x such that $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$ for all $y \in U$. It follows that for all $y \in U$, $|f(x) - f(y)| < \varepsilon$. So $U \subset f^{-1}((-\infty, a))$.

It seems that this is not a good proof since it only works for $Y = \mathbb{R}$ and can't be easily adjusted to prove the general case (i.e., general Y). However, *what exists is reasonable.* Find out the advantage of this proof.

Proof (1) In Problem 8 (2) ① we have proved that d_u is a metric on $\mathcal{M}(X,Y)$. To show that d_u is complete, let $\{f_n\}$ be a Cauchy sequence in $\mathcal{M}(X,Y)$. Then for any $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that $d_u(f_n, f_m) < \varepsilon$ for all $n, m > N$. In particular, for any $x \in X$,

$$\frac{d_Y(f_n(x), f_m(x))}{1 + d_Y(f_n(x), f_m(x))} < \frac{\varepsilon}{1+\varepsilon}$$

for all $n, m > N$. This implies that

$$d_Y(f_n(x), f_m(x)) < \varepsilon \tag{45–1}$$

for all $n, m > N$, so $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $y_x \in Y$ such that $f_n(x) \to y_x$ as $n \to \infty$. Define $f(x) = y_x$, then $f \in \mathcal{M}(X,Y)$. Letting $m \to \infty$ in (45–1) we get $d_Y(f_n(x), f(x)) \leqslant \varepsilon$ for all $x \in X$ and $n > N$, i.e., $d_u(f_n, f) \leqslant \varepsilon$ for all $n > N$. Thus $f_n \to f$ in $\mathcal{M}(X,Y)$.

(2) This proof exploits the simplicity of the basic open sets in $\mathbb{R}$ and the order structure of $\mathbb{R}$. □

Problem 46 (More on LCH)

(1) **(Strucure of non-compact LCH)**

① Let K be a compact Hausdorff space, $p \in K$ and $X = K \setminus \{p\}$ is non-compact. Prove: X is LCH.

② Conversely, suppose X is an non-compact LCH. Let $X^* = X \sqcup \{\infty\}$ be the one-point compactification of X (see Problem 36). Prove: X^* is compact and Hausdorff.

(2) **(The evaluation map could fail to be continuous without local compactness)** Consider the evaluation map

$$e \colon \mathbb{Q} \times \mathcal{C}(\mathbb{Q}, [0,1]) \to [0,1], \quad (x, f) \mapsto e(x,f) = f(x).$$

① Prove: $\mathbb{Q}$ is not locally compact.

② Prove: for any $q_1 \in \mathbb{Q}$ and any closed subset $A \subset \mathbb{Q}$ with $q_1 \notin A$, there is a continuous function $f_1 \in \mathcal{C}(\mathbb{Q}, [0,1])$ such that $f_1(q_1) = 1$, $f_1(A) = \{0\}$.

③ Now let $f_0 \in \mathcal{C}(\mathbb{Q}, [0,1])$ be the zero map $f_0(\mathbb{Q}) = \{0\}$, and take any $q_0 \in \mathbb{Q}$. Prove: e is not continuous at (q_0, f_0) (where we endow $\mathcal{C}(\mathbb{Q}, [0,1])$ with the compact convergence topology).

Proof (1) ① Since (T2) property is hereditary, X is Hausdorff.

(Proof 1) Since K is Hausdorff, $\{p\}$ is closed, and then $X = K \setminus \{p\}$ is open. It suffices to show that *every open subspace X of a compact Hausdorff space K is LCH*. To show that X is locally compact, let $x \in X$ and let $U_x \subset X$ be an open neighborhood of x. Since X is assumed to be open, U_x is also open in K. Now K is compact Hausdorff, $\{x\}$ is a compact subset of K, and U_x is an open subset of K containing $\{x\}$. It follows that there exists an open subset $V_x \subset K$ such that

$$\{x\} \subset V_x \subset \overline{V_x} \subset U_x \subset X.$$

Since K is Hausdorff, $\overline{V_x}$ is a compact neighborhood of x in X. Thus X is LCH.

(Proof 2) Given $x \in X$, we show X is locally compact at x. Choose disjoint open sets U and V of K containing x and p, respectively. Then the set $C = K \setminus V$ is closed in K, so it is a compact subspace of K. Since C lies in X, it is also compact as a subspace of X. Moreover, C contains the neighborhood U of x, as desired.

② (X^* is compact) Let $\mathcal{A}$ be an open cover of X^*. The collection $\mathcal{A}$ must contain an open set of the form $X^* \setminus C$, where C is a compact subset of X. This is because otherwise the point ∞ would not be covered. Take all the members of $\mathcal{A}$ different from $X^* \setminus C$ and intersect them with X; they form a collection of open sets of X covering C. Since C is compact, finitely many of them cover C; the corresponding finite collection of elements of $\mathcal{A}$ will, along with the element $X^* \setminus C$, cover all of X^*.

(X^* is Hausdorff) Let x and y be two points of X^*. If both of them lie in X, there are disjoint open sets U and V open in X containing them, respectively. On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing an open neighborhood U of x since X is LCH. Then U and $X^* \setminus C$ are disjoint open neighborhoods of x and ∞, respectively, in X^*.

(2) ① Let U be an arbitrary neighborhood in $\mathbb{Q}$. Then U contains some open interval $(a, b) \cap \mathbb{Q}$ for some $a, b \in \mathbb{R}$. Because of the existence of irrational numbers, the set $(a, b) \cap \mathbb{Q}$ can be partitioned into infinitely many disjoint open intervals in $\mathbb{Q}$, which serves as an infinite cover with no finite subcover. Thus U cannot be compact. Therefore no neighborhood in $\mathbb{Q}$ is compact, which implies $\mathbb{Q}$ is not locally compact.

② This follows from the Urysohn's lemma for metric spaces (see Problem 7 (3)).

③ We have shown in ① that no neighborhood in $\mathbb{Q}$ is compact. Hence for any open neighborhood U of q_0 and any compact set K in $\mathbb{Q}$, there exists $q_1 \in U \setminus K$. By ② there is a continuous function $f_1 \in \mathcal{C}(\mathbb{Q}, [0, 1])$ such that $f_1(q_1) = 1$ and $f_1(K) = \{0\}$. Then $f_1 \in B(f_0; K, \varepsilon)$ for every $\varepsilon > 0$. This shows that any neighborhood of (q_0, f_0) contains some point (q_1, f_1) such that $e(q_1, f_1) = 1$, so e is not continuous at (q_0, f_0). $\square$

Problem 47 (More on compact-open topology)

(1) Prove Proposition 2.4.21, i.e., if (Y, d) is a metric space, then $\mathscr{T}_{\text{c.c.}} = \mathscr{T}_{\text{c.o.}}$ on $\mathcal{C}(X, Y)$.

(2) Let $(X, \mathscr{T}_{\text{discrete}})$ be discrete. What is $\mathscr{T}_{\text{c.o.}}$ on $\mathcal{M}(X, Y)$?

(3) Let Y be Hausdorff. Prove: $(\mathcal{C}(X, Y), \mathscr{T}_{\text{c.o.}})$ is Hausdorff.

(4) Prove: if X is locally compact and Hausdorff, then on $\mathcal{C}(X, Y)$,

$$S(\{x\}, U) = \bigcup_{\text{compact neighborhood } K \text{ of } x} S(K, U).$$

Proof (1) $\boxed{\mathscr{T}_{\text{c.o.}} \subset \mathscr{T}_{\text{c.c.}}}$ Let $S(K, V) = \{f \in \mathcal{C}(X, Y) : f(K) \subset V\}$ be arbitrary, where K is compact in X and V is open in Y. It suffices to show that $S(K, V) \in \mathscr{T}_{\text{c.c.}}$. For any $f \in S(K, V)$, we have $f(K) \subset V$. Since f is continuous, $f(K)$ is compact and then closed in Y. Since the closed set V^{c} and the compact set $f(K)$ are disjoint, the distance $d_f := \text{dist}(f(K), V^{\text{c}})$ is positive. This implies that $B(f;K,d_f) \subset S(K,V)$. Thus

$$S(K,V) = \bigcup_{f \in S(K,V)} B(f;K,d_f) \in \mathscr{T}_{\text{c.c.}}.$$

$\boxed{\mathscr{T}_{\text{c.c.}} \subset \mathscr{T}_{\text{c.o.}}}$ Let $B(f;K,\varepsilon)$ be arbitrary, where $f \in \mathcal{C}(X,Y)$, K is compact in X, and $\varepsilon > 0$. It suffices to find a basis element for $\mathscr{T}_{\text{c.o.}}$ that contains f and lies in $B(f;K,\varepsilon)$. Each point of X has a neighborhood V_x such that $f\big(\overline{V_x}\big)$ lies in an open set U_x of Y having diameter less than ε. (For example, choose V_x so that $f(V_x)$ lies in the $\frac{\varepsilon}{4}$-neighborhood of $f(x)$. Then $f\big(\overline{V_x}\big)$ lies in the $\frac{\varepsilon}{3}$-neighborhood of $f(x)$, which has diameter at most $\frac{2\varepsilon}{3}$.) Since K is compact, we can cover K by finitely many such sets V_x, say for $x = x_1, \cdots, x_n$. Let $K_{x_i} = \overline{V_{x_i}} \cap K$. Then each K_{x_i} is closed in the compact set K, so it is compact. Now the basis element

$$S(K_{x_1}, U_{x_1}) \cap \cdots \cap S(K_{x_n}, U_{x_n}) \in \mathscr{T}_{\text{c.o.}}$$

contains f and lies in $B(f;K,\varepsilon)$, as desired.

(2) If X is a discrete space, then the compact subsets of X are the finite subsets. Therefore the compact-open topology on $\mathcal{M}(X,Y)$ is the product topology, i.e., the pointwise convergence topology.

(3) For any $f \neq g$ in $\mathcal{C}(X,Y)$, there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Since Y is Hausdorff, there exist disjoint open neighborhoods U and V of $f(x_0)$ and $g(x_0)$, respectively. Now $f \in S(\{x_0\},U)$ and $g \in S(\{x_0\},V)$. Also, $S(\{x_0\},U)$ and $S(\{x_0\},V)$ are disjoint open sets in $(\mathcal{C}(X,Y), \mathscr{T}_{\text{c.o.}})$. Therefore $(\mathcal{C}(X,Y), \mathscr{T}_{\text{c.o.}})$ is Hausdorff.

(4) $\boxed{\text{LHS} \subset \text{RHS}}$ For any $f \in S(\{x\},U)$, since f is continuous, $f^{-1}(U)$ is open in X and contains the compact set $\{x\}$. Sicne X is LCH, there exists an open set $V \subset X$ such that $\overline{V}$ is compact and

$$\{x\} \subset V \subset \overline{V} \subset f^{-1}(U).$$

This shows that $\overline{V}$ is a compact neighborhood of x and $f \in S\big(\overline{V},U\big) \subset \text{RHS}$.

$\boxed{\text{RHS} \subset \text{LHS}}$ This is trivial. □

Problem 48 (Compactly generated spaces)

(1) Read the materials on compactly generated spaces (page 103), and prove: any locally compact space is compactly generated.

(2) Prove: any first countable space is compactly generated.

(3) Find a compactly generated space that is not locally compact.

Proof (1) Suppose that X is locally compact. Let $A \cap K$ be open in K for every compact subset K of X. We show A is open in X. Given $x \in A$, choose an open neighborhood U of x that lies in a compact subset K of X. Since $A \cap K$ is open in K by hypothesis, $A \cap U = (A \cap K) \cap U$ is open in U, and hence open in X. Then $A \cap U$ is an open neighborhood of x contained in A, so A is open in X. Therefore X is compactly generated.

(2) Suppose that X is first countable. If $A \cap K$ is closed in K for every compact subset K of X, we show that A is closed in X. For any $x \in \overline{A}$, we show that $x \in A$. Since X has a countable neighborhood

basis at x, there is a sequence $\{x_n\}_{n=1}^\infty$ of points of A converging to x. The subspace

$$K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$$

is compact, so that $A \cap K$ is by assumption closed in K. Since $A \cap K$ contains x_n for every n, it contains x as well. Therefore, $x \in A$, as desired. Hence $A = \overline{A}$ is closed in X, and X is compactly generated.

(3) Similar to Problem 46 (2) ①, the space $\mathbb{Q} \cap [0,1]$ is not locally compact. However, $\mathbb{Q} \cap [0,1]$ is first countable, so it is compactly generated by part (2). □

PSet 7, Part 1

Problem 49 (Uniformly equicontinuous) Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F} \in \mathcal{C}(X, Y)$ is called *uniformly equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ (which depends only on ε) such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ holds for any $f \in \mathcal{F}$ and any pair of points $x_1, x_2 \in X$ satisfying $d_X(x_1, x_2) < \delta$.

(1) Prove: if $\mathcal{F}$ is a finite set consisting of uniformly continuous functions, then $\mathcal{F}$ is uniformly equicontinuous.

(2) Show that if $\mathcal{F}$ is a family of Lipschitz continuous functions with a common Lipschitz constant, then $\mathcal{F}$ is uniformly equicontinuous.

(3) Show that if X is compact, then $\mathcal{F}$ is uniformly equicontinuous if and only if it is equicontinuous.

Proof (1) Suppose $\mathcal{F} = \{f_1, \cdots, f_n\}$. Since each f_i is uniformly continuous, for each $\varepsilon > 0$, there exists $\delta_i > 0$ such that $d_X(x_1, x_2) < \delta_i$ implies $d_Y(f_i(x_1), f_i(x_2)) < \varepsilon$. Let $\delta = \min\{\delta_1, \cdots, \delta_n\}$. Then $d_X(x_1, x_2) < \delta$ implies $d_Y(f_i(x_1), f_i(x_2)) < \varepsilon$ for all $i = 1, \cdots, n$.

(2) Suppose $d_Y(f(x_1), f(x_2)) \leqslant L \cdot d_X(x_1, x_2)$ for all $f \in \mathcal{F}$ and $x_1, x_2 \in X$. Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{L}$, then $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$.

(3) The "only if" part is trivial. For the "if" part, suppose $\mathcal{F}$ is equicontinuous. Given $\varepsilon > 0$, for any $x_0 \in X$, there exists $\delta_{x_0} > 0$ such that $d_X(x, x_0) < \delta_{x_0}$ implies $d_Y(f(x), f(x_0)) < \frac{\varepsilon}{2}$ for all $f \in \mathcal{F}$. Since X is a compact metric space, the open cover $\{\mathbb{B}(x_0, \delta_{x_0}) : x_0 \in X\}$ has a Lebesgue number $\delta > 0$. Then $d_X(x_1, x_2) < \delta$ implies $x_1, x_2 \in \mathbb{B}(x_0, \delta_{x_0})$ for some $x_0 \in X$, and hence

$$d_Y(f(x_1), f(x_2)) \leqslant d_Y(f(x_1), f(x_0)) + d_Y(f(x_2), f(x_0)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $f \in \mathcal{F}$. □

Problem 50 (Applications of Arzelà–Ascoli)

(1) Suppose $k = k(x, y) \in \mathcal{C}([0,1] \times [0,1], \mathbb{R})$. For any $f \in \mathcal{C}([0,1], \mathbb{R})$, define

$$Kf(x) = \int_0^1 k(x, y) f(y) \, \mathrm{d}y.$$

Prove: K is a *compact operator*, i.e., it maps any bounded set in $(\mathcal{C}([0,1], \mathbb{R}), d_\infty)$ into a compact subset in the same space.

(2) We want to minimize the functional $\Phi[f] := \displaystyle\int_{-1}^{1} f(t)\,\mathrm{d}t$. Consider the set

$$\mathcal{F} = \{f \in \mathcal{C}([-1,1],[0,1]) : f(-1) = f(1) = 1\}.$$

① What is $\inf\limits_{f\in\mathcal{F}} \Phi[f]$? Is the infimum attained?

② For any constant $C > 0$, let

$$\mathcal{F}_C = \{f \in \mathcal{F} : |f(x) - f(y)| \leqslant C|x-y|\}.$$

Prove: the infimum $\inf\limits_{f\in\mathcal{F}_C} \Phi[f]$ is attained. Can you find the function?

Proof (1) Suppose $\mathcal{F} \subset (\mathcal{C}([0,1],\mathbb{R}), d_\infty)$ and $\sup\limits_{x\in[0,1]} |f(x)| < M$ for all $f \in \mathcal{F}$. Since $k(x,y) \in \mathcal{C}([0,1]\times[0,1],\mathbb{R})$, it is uniformly continuous. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|k(x,y) - k(x_0,y)| < \frac{\varepsilon}{M}$ for all $x, x_0, y \in [0,1]$ satisfying $|x - x_0| < \delta$. So when $|x - x_0| < \delta$, we have

$$\begin{aligned} |Kf(x) - Kf(x_0)| &= \left|\int_0^1 [k(x,y) - k(x_0,y)] f(y)\,\mathrm{d}y\right| \\ &\leqslant \int_0^1 |k(x,y) - k(x_0,y)| \cdot |f(y)|\,\mathrm{d}y \\ &< \frac{\varepsilon}{M} \cdot M = \varepsilon. \end{aligned}$$

This shows that $\mathcal{G} := \{Kf : f \in \mathcal{F}\} \subset \mathcal{C}([0,1],\mathbb{R})$ is equicontinuous. For any $a \in [0,1]$, the set

$$\mathcal{G}_a := \{g(a) : g \in \mathcal{G}\} = \{Kf(a) : f \in \mathcal{F}\}$$

is bounded by $M \max\limits_{(x,y)\in[0,1]^2} |k(x,y)|$, so $\mathcal{G}$ is pointwise bounded and then pointwise precompact. By Theorem 2.5.8 (1), $\overline{\mathcal{G}}$ is compact in $(\mathcal{C}([0,1],\mathbb{R}), \mathscr{T}_{\text{c.c.}})$. Since $[0,1]$ is compact, $(\mathcal{C}([0,1],\mathbb{R}), \mathscr{T}_{\text{c.c.}}) = (\mathcal{C}([0,1],\mathbb{R}), d_\infty)$. Therefore, K maps $\mathcal{F}$ into a compact subset in $(\mathcal{C}([0,1],\mathbb{R}), d_\infty)$.

(2) ① Define

$$f_n(x) = \begin{cases} -nx - n + 1, & x \in \left[-1, -1 + \frac{1}{n}\right], \\ 0, & x \in \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right), \\ nx - n + 1, & x \in \left[1 - \frac{1}{n}, 1\right]. \end{cases}$$

Then $f_n \in \mathcal{F}$ and $\Phi[f_n] = \frac{1}{n} \to 0$ as $n \to \infty$. So $\inf\limits_{f\in\mathcal{F}} \Phi[f] = 0$. However, the infimum is not attained since $\Phi[f] = 0$ would imply $f \equiv 0$ on $[-1,1]$.

② For any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{C}$. Then $|x - y| < \delta$ implies $|f(x) - f(y)| \leqslant C|x-y| < \varepsilon$. Therefore, $\mathcal{F}_C$ is equicontinuous. Since $(\mathcal{F}_C)_a \subset [0,1]$ for all $a \in [0,1]$, $\mathcal{F}_C$ is pointwise precompact. By Arzelà–Ascoli theorem (compact space version), the infimum $\inf\limits_{f\in\mathcal{F}_C} \Phi[f]$ is attained.

- If $C \leqslant 1$, then the function

$$f(x) = \begin{cases} -Cx + 1 - C, & x \in [-1,0], \\ Cx + 1 - C, & x \in (0,1] \end{cases}$$

attains the infimum.

◇ If $C > 1$, then the function

$$f(x) = \begin{cases} -Cx + 1 - C, & x \in \left[-1, \frac{1-C}{C}\right], \\ 0, & x \in \left(\frac{1-C}{C}, \frac{C-1}{C}\right], \\ Cx + 1 - C, & x \in \left(\frac{C-1}{C}, 1\right] \end{cases}$$

attains the infimum. □

Problem 51 (Arzelà–Ascoli for locally compact + σ-compact spaces) Prove Theorem 2.5.12:

Let X be locally compact and σ-compact, and (Y, d) be a metric space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a subset which is equicontinuous and pointwise precompact. Then any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact sets of X to a limit function $f \in \mathcal{C}(X, Y)$.

Proof Let us first construct a sequence of compact sets $\{K_i\}_{i=1}^{\infty}$ such that $X = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$. Since X is σ-compact, we can write $X = \bigcup_{i=1}^{\infty} C_i$, with each C_i compact. Take $K_1 = C_1$. Now given K_i, we define K_{i+1}. Since X is locally compact, for each point $x \in K_i$, there is an open set U_x and a compact set V_x such that $x \in U_x \subset V_x$. Then $\{U_x : x \in K_i\}$ is an open cover of K_i. Since K_i is compact, there exists $x_1, \cdots, x_n \in K_i$ such that $K_i \subset \bigcup_{j=1}^{n} U_{x_j} =: U$. Let $K_{i+1} = C_{i+1} \cup \bigcup_{j=i}^{n} V_{x_j}$, then K_{i+1} is a finite union of compact sets, so it is compact. Since U is open and $U \subset K_{i+1}$, we get $K_i \subset U \subset \operatorname{Int} K_{i+1}$, as desired.

Since $\mathcal{F}$ is equicontinuous and pointwise precompact, so is its restriction to each K_i. Then we can pick a sequence $\left\{f_n^{(1)}\right\}_{n=1}^{\infty}$ in $\mathcal{F}$ which converges uniformly on K_1. Next, we pick a subsequence $\left\{f_n^{(2)}\right\}_{n=1}^{\infty}$ of $\left\{f_n^{(1)}\right\}_{n=1}^{\infty}$ which converges uniformly on K_2. We continue this process to get a sequence of subsequences, and then apply Cantor's trick: define a sequence by $f_n = f_n^{(n)}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on all K_i.

Finally, for any compact set $K \subset X$, we claim that $K \subset K_i$ for some i. If not, then $K \setminus K_i \neq \varnothing$ for all i. Take $x_i \in K \setminus K_i$ such that all x_i are distinct. Then the sequence $\{x_i\}_{i=1}^{\infty}$ lies in the compact set K, so it has a limit point $x_0 \in K$. Suppose $x_0 \in K_n$, then the neighborhood $\operatorname{Int} K_{n+1}$ of x_0 must contain infinitely many x_i, which is a contradiction. Therefore any compact set $K \subset X$ is contained in some K_i, and the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on K to a limit function $f \in \mathcal{C}(X, Y)$. □

Problem 52 (σ compactness) A topolotical space is called σ-compact if it can be written as the union of countably many compact subsets.

(1) Is the Sorgenfrey line σ-compact?

(2) Show that the product of two σ-compact spaces is σ-compact.

(3) What about the product of countably many σ-compact spaces?

(4) Prove: if X is σ-compact and locally compact Hausdorff, then X has the following exhaustion property: there exist open sets $\{U_n\}$ such that

◇ Each $\overline{U_n}$ is compact.

◇ $\overline{U_n} \subset U_{n+1}$ for each n.

◇ $X = \bigcup_n U_n$.

Proof (1) The Sorgenfrey line is not σ-compact. We shall prove that any compact subset of the Sorgenfrey line is at most countable. As a consequence, countable union of compact sets is at most countable, which cannot cover $\mathbb{R}$. To see this, consider a non-empty compact subset K of the Sorgenfrey line. Fix an $x \in K$, consider the following open cover of K:

$$\{[x, +\infty)\} \cup \left\{\left(-\infty, x - \tfrac{1}{n}\right) : n \in \mathbb{N}\right\}.$$

Since K is compact, this cover has a finite subcover, and hence there exists $r_x \in \mathbb{R}$ such that the interval $(r_x, x]$ contains no point of K apart from x. This is true for all $x \in K$. Now choose a rational number $q_x \in (r_x, x] \cap \mathbb{Q}$. Since the intervals $(r_x, x]$, parametrized by $x \in K$, are pairwise disjoint, the function $q\colon K \to \mathbb{Q}$ is injective. Therefore K is at most countable.

(2) Suppose $X = \bigcup_{i=1}^{\infty} K_i$ and $Y = \bigcup_{j=1}^{\infty} L_j$, where all K_i and L_j are compact. Then

$$X \times Y = \bigcup_{i,j=1}^{\infty} (K_i \times L_j).$$

Each $K_i \times L_j$ is compact since K_i and L_j are compact. Therefore $X \times Y$ is σ-compact.

(3) The product of countably many σ-compact spaces may fail to be σ-compact. For example, $\prod_{k=1}^{\infty} \mathbb{R}$ is not σ-compact. To see this, suppose to the contrary that $\prod_{k=1}^{\infty} \mathbb{R} = \bigcup_{i=1}^{\infty} C_i$, where all C_i are compact. Take $K_n = \bigcup_{i=1}^{n} C_i$, then $\{K_n\}_{n=1}^{\infty}$ is a sequence of increasing compact sets whose union is $\prod_{k=1}^{\infty} \mathbb{R}$. Since each projection map π_k is continuous, $\pi_k(K_n) \subset \mathbb{R}$ is compact for all k. So there exists $M_{n,k} \in \mathbb{R}_{>0}$ such that $\pi_k(K_n) \subset [-M_{n,k}, M_{n,k}]$. Then

$$K_n \subset \prod_{k=1}^{\infty} [-M_{n,k}, M_{n,k}].$$

Since $\{K_n\}_{n=1}^{\infty}$ is increasing, we can assume that $\{M_{n,k}\}_{n=1}^{\infty}$ is increasing for each fixed k. Now consider the element $x = (M_{1,1} + 1, M_{2,2} + 1, \cdots) \in \prod_{k=1}^{\infty} \mathbb{R}$. Then $x \notin K_n$ for all n, which is a contradiction. Therefore $\prod_{k=1}^{\infty} \mathbb{R}$ is not σ-compact.

(4) In the first paragraph of the proof of Problem 51, we have constructed a sequence of compact sets $\{K_i\}_{i=1}^{\infty}$ such that $X = \bigcup_{i=1}^{\infty} K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$. Since X is LCH, by Proposition 2.4.16, for each i, there exists an open set U_i such that $\overline{U_i}$ is compact and

$$K_i \subset U_i \subset \overline{U_i} \subset \operatorname{Int} K_{i+1}.$$

Now the open sets $\{U_i\}$ satisfy all the required properties. □

PSet 7, Part 2

Problem 53 (Topological algebra) Let X be a topological space. Endow $\mathcal{C}(X,\mathbb{R})$ with the compact convergence topology.

(1) Prove: the addition, multiplication and the scalar multiplication

$$\begin{aligned} a\colon \mathcal{C}(X,\mathbb{R})\times\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R}), &\quad (f,g)\mapsto a(f,g)=f+g,\\ m\colon \mathcal{C}(X,\mathbb{R})\times\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R}), &\quad (f,g)\mapsto m(f,g)=fg,\\ s\colon \mathbb{R}\times\mathcal{C}(X,\mathbb{R})\to\mathcal{C}(X,\mathbb{R}), &\quad (\lambda,g)\mapsto s(\lambda,g)=\lambda g \end{aligned}$$

are continuous. As a consequence, $\mathcal{C}(X,\mathbb{R})$ is a topological algebra.

(2) Prove Proposition 2.6.4:

Let $\mathcal{A}$ be a topological algebra, and $\mathcal{A}_1\subset\mathcal{A}$ a subalgebra. Then the closure $\overline{\mathcal{A}_1}$ is a (closed) subalgebra of $\mathcal{A}$.

Proof (1) ① For any $(f,g)\in\mathcal{C}(X,\mathbb{R})\times\mathcal{C}(X,\mathbb{R})$, consider the basis element for $\mathscr{T}_{\text{c.c.}}$ of the form $B(f+g;K,\varepsilon)$, where $K\subset X$ is compact and $\varepsilon>0$. Then $B\big(f;K,\frac{\varepsilon}{2}\big)\times B\big(g;K,\frac{\varepsilon}{2}\big)\subset a^{-1}(B(f+g;K,\varepsilon))$ is an open neighborhood of (f,g). Therefore, a is continuous.

② For any $(f,g)\in\mathcal{C}(X,\mathbb{R})\times\mathcal{C}(X,\mathbb{R})$, consider the basis element for $\mathscr{T}_{\text{c.c.}}$ of the form $B(fg;K,\varepsilon)$, where $K\subset X$ is compact and $\varepsilon>0$. Let $F\coloneqq\sup\limits_{x\in K}|f(x)|$ and $G\coloneqq\sup\limits_{x\in K}|g(x)|$.

◇ If $F\neq 0$, then for any $\big(\tilde f,\tilde g\big)\in B\Big(f;K,\frac{\varepsilon}{\frac{\varepsilon}{F}+2G}\Big)\times B\big(g;K,\frac{\varepsilon}{2F}\big)$, we have

$$\begin{aligned} \Big|f(x)g(x)-\tilde f(x)\tilde g(x)\Big| &\leqslant |f(x)|\cdot|g(x)-\tilde g(x)|+|\tilde g(x)|\cdot\Big|f(x)-\tilde f(x)\Big|\\ &\leqslant |f(x)|\cdot|g(x)-\tilde g(x)|+(|\tilde g(x)-g(x)|+|g(x)|)\cdot\Big|f(x)-\tilde f(x)\Big|\\ &< F\cdot\frac{\varepsilon}{2F}+\left(\frac{\varepsilon}{2F}+G\right)\frac{\varepsilon}{\frac{\varepsilon}{F}+2G}\\ &=\varepsilon \end{aligned}$$

for all $x\in K$. Therefore $(f,g)\in B\Big(f;K,\frac{\varepsilon}{\frac{\varepsilon}{F}+2G}\Big)\times B\big(g;K,\frac{\varepsilon}{2F}\big)\subset m^{-1}(B(fg;K,\varepsilon))$.

◇ If $F=0$, then $f(x)=0$ for all $x\in K$. For any $\big(\tilde f,\tilde g\big)\in B\big(f;K,\frac{\varepsilon}{2G}\big)\times B(g;K,G)$, we have

$$\Big|\tilde f(x)\tilde g(x)\Big|\leqslant\Big|\tilde f(x)\Big|\cdot(|\tilde g(x)-g(x)|+|g(x)|)<\frac{\varepsilon}{2G}\cdot 2G=\varepsilon.$$

Therefore $(f,g)\in B\big(f;K,\frac{\varepsilon}{2G}\big)\times B(g;K,G)\subset m^{-1}(B(fg;K,\varepsilon))$.

Therefore, m is continuous.

③ For any $(\lambda,g)\in\mathbb{R}\times\mathcal{C}(X,\mathbb{R})$, consider the basis element for $\mathscr{T}_{\text{c.c.}}$ of the form $B(\lambda g;K,\varepsilon)$, where $K\subset X$ is compact and $\varepsilon>0$. Let $M\coloneqq\sup\limits_{x\in K}|g(x)|$.

◇ If $M>0$, then for any $(\mu,h)\in\mathbb{B}\big(\lambda,\frac{\varepsilon}{2M}\big)\times B\Big(g;K,\frac{\varepsilon}{\frac{\varepsilon}{M}+2|\lambda|}\Big)$, we have

$$|\mu h(x)-\lambda g(x)|=|[\mu h(x)-\mu g(x)]+|[\mu g(x)-\lambda g(x)]|$$

$$
\begin{aligned}
&\leqslant |\mu| \cdot |h(x) - g(x)| + |\mu - \lambda| \cdot |g(x)| \\
&\leqslant (|\mu - \lambda| + |\lambda|) \cdot |h(x) - g(x)| + |\mu - \lambda| \cdot |g(x)| \\
&< \left(\frac{\varepsilon}{2M} + |\lambda|\right) \frac{\varepsilon}{\frac{\varepsilon}{M} + 2|\lambda|} + \frac{\varepsilon}{2M} \cdot M \\
&= \varepsilon.
\end{aligned}
$$

for all $x \in K$. Therefore $(\lambda, g) \in \mathbb{B}\big(\lambda, \frac{\varepsilon}{2M}\big) \times B\Big(g; K, \frac{\varepsilon}{\frac{\varepsilon}{M}+2|\lambda|}\Big) \subset s^{-1}(B(\lambda g; K, \varepsilon))$.

◇ If $M = 0$ and $\lambda \neq 0$, then $g(x) = 0$ for all $x \in K$. For any $(\mu, h) \in \mathbb{B}(\lambda, |\lambda|) \times B\Big(g; K, \frac{\varepsilon}{2|\lambda|}\Big)$,

$$|\mu h(x) - \lambda g(x)| \leqslant (|\mu - \lambda| + |\lambda|) \cdot |h(x)| < 2|\lambda| \cdot \frac{\varepsilon}{2|\lambda|} = \varepsilon$$

for all $x \in K$. Therefore $(\lambda, g) \in \mathbb{B}(\lambda, |\lambda|) \times B\Big(g; K, \frac{\varepsilon}{2|\lambda|}\Big) \subset s^{-1}(B(\lambda g; K, \varepsilon))$.

◇ If $M = 0$ and $\lambda = 0$, then $(\lambda, g) \in \mathbb{B}(0, 1) \times B(g; K, \varepsilon) \subset s^{-1}(B(\lambda g; K, \varepsilon))$.

Therefore, s is continuous.

(2) For the topological algebra $(\mathcal{A}, +, \cdot)$ and its subalgebra $\mathcal{A}_1$, since addition $+\colon \mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_1$ is continuous, we have

$$\overline{\mathcal{A}_1} + \overline{\mathcal{A}_1} \subset \overline{\mathcal{A}_1 + \mathcal{A}_1} = \overline{\mathcal{A}_1}.$$

Similarly, since multiplication $\cdot\colon \mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_1$ is continuous, we have

$$\overline{\mathcal{A}_1} \cdot \overline{\mathcal{A}_1} \subset \overline{\mathcal{A}_1 \cdot \mathcal{A}_1} \subset \overline{\mathcal{A}_1}.$$

Likewise, the fact that $\overline{\mathcal{A}_1}$ is a vector subspace of $\mathcal{A}$ follows from the continuity of scalar multiplication. Therefore, $\overline{\mathcal{A}_1}$ is a (closed) subalgebra of $\mathcal{A}$. □

Problem 54 (Applications of Stone–Weierstrass)

(1) Prove: any continuous function on $[0, 1]$ can be approximated uniformly by functions of the form $a_0 + a_1 \mathrm{e}^{x} + a_2 \mathrm{e}^{2x} + \cdots + a_n \mathrm{e}^{nx}$, $n \in \mathbb{N}$.

① As a consequence, prove if f is a continuous function on $[0, 1]$ satisfying

$$\int_0^1 f(x) \mathrm{e}^{nx} \, \mathrm{d}x = 0, \quad n = 0, 1, 2, \cdots, \tag{54–1}$$

then $f = 0$.

② What if (54–1) holds only for even n?

(2) Let X, Y be compact Hausdorff spaces. Prove: any $f \in \mathcal{C}(X \times Y, \mathbb{R})$ can be approximated uniformly by functions of the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_n(x)g_n(y), \quad n \in \mathbb{N},$$

where $f_k \in \mathcal{C}(X, \mathbb{R})$, $g_k \in \mathcal{C}(Y, \mathbb{R})$.

(3) Let $\mathcal{A}$ be the set of (rational) functions of the form $\frac{p(x)}{q(x)}$, where p, q are polynomials with $\deg(p) \leqslant \deg(q)$, and $q(x) \neq 0$ for all $x \in \mathbb{R}$. Prove: if $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x)$, then f can be approximated uniformly by functions in $\mathcal{A}$.

Proof (1) Since $[0,1]$ is compact and Hausdorff, $\mathcal{A} := \langle 1, \mathrm{e}^x \rangle$ is a subalgebra of $\mathcal{C}([0,1],\mathbb{R})$ that vanishes at no point and separates points, by Stone–Weierstrass theorem, $\mathcal{A}$ is dense in $(\mathcal{C}([0,1],\mathbb{R}), d_\infty)$.

① For any $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that $d_\infty(f,g) < \varepsilon$. By assumption,

$$\begin{aligned}\int_0^1 f^2(x)\,\mathrm{d}x &= \int_0^1 f^2(x)\,\mathrm{d}x - \int_0^1 f(x)g(x)\,\mathrm{d}x = \int_0^1 f(x)[f(x)-g(x)]\,\mathrm{d}x\\ &\leqslant \int_0^1 |f(x)||f(x)-g(x)|\,\mathrm{d}x \leqslant \varepsilon \int_0^1 |f(x)|\,\mathrm{d}x.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $f = 0$.

② Note that $\mathcal{A}' := \langle 1, \mathrm{e}^{2x} \rangle$ is a subalgebra of $\mathcal{C}([0,1],\mathbb{R})$ that vanishes at no point and separates points. By Stone–Weierstrass theorem, $\mathcal{A}'$ is dense in $(\mathcal{C}([0,1],\mathbb{R}), d_\infty)$. Therefore, the conclusion still holds by the same argument as above.

(2) Since X, Y are compact and Hausdorff, so is $X \times Y$. Moreover, both $\mathcal{C}(X,\mathbb{R})$ and $\mathcal{C}(Y,\mathbb{R})$ separate points. Since $\mathcal{A} := \langle fg : f \in \mathcal{C}(X,\mathbb{R}), g \in \mathcal{C}(Y,\mathbb{R}) \rangle$ is a subalgebra of $\mathcal{C}(X \times Y,\mathbb{R})$ that vanishes at no point and separates points, by Stone–Weierstrass theorem, $\mathcal{A}$ is dense in $(\mathcal{C}(X \times Y,\mathbb{R}), d_\infty)$.

(3) Let $\mathcal{A}^*$ be the set of functions of the form $\frac{p(x)}{q(x)}$ where p, q are polynomials with $\deg(p) < \deg(q)$ and $q(x) \neq 0$ for all $x \in \mathbb{R}$. Then $\mathcal{A}^*$ is a subalgebra of $\mathcal{C}_0(\mathbb{R},\mathbb{R})$.

- $\dfrac{1}{x^2+1} \in \mathcal{A}^*$, so $\mathcal{A}^*$ vanishes at no point.
- For any $a, b \in \mathbb{R}$ with $a \neq b$, the rational function $\dfrac{x}{\left(x - \frac{a+b}{2}\right)^2 + 1} \in \mathcal{A}^*$ separates a and b.

Since $\mathbb{R}$ is locally compact and Hausdorff, by Problem 56 (2), $\mathcal{A}^*$ is dense in $\mathcal{C}_0(\mathbb{R},\mathbb{R})$. Now for any $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ with $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) =: L$, the function $f - L$ lies in $\mathcal{C}_0(\mathbb{R},\mathbb{R})$, so $f - L$ can be approximated uniformly by functions in $\mathcal{A}^* \subset \mathcal{A}$. And since the constant function L is in $\mathcal{A}$, $f = (f - L) + L$ can be approximated uniformly by functions in $\mathcal{A}$. □

Problem 55 (Stone–Weierstrass for complex/quaternion-valued functions)

(1) Prove Theorem 2.6.16:

Let X be compact Hausdorff, and $\mathcal{A} \subset \mathcal{C}(X,\mathbb{C})$ be a complex subalgebra which separates points and vanishes at no point. Moreover, assume $\mathcal{A}$ is self adjoint, then $\mathcal{A}$ is dense in $\mathcal{C}(X,\mathbb{C})$.

(2) For any quaternion $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \in \mathbb{H}$, check: $a = \dfrac{q - \mathrm{i}q\mathrm{i} - \mathrm{j}q\mathrm{j} - \mathrm{k}q\mathrm{k}}{4}$. Then prove Theorem 2.6.12 for $\mathcal{C}(X,\mathbb{H})$.

Proof (1) Since $\operatorname{Re} f = \frac{f+\bar{f}}{2}$ and $\operatorname{Im} f = \frac{f-\bar{f}}{2\mathrm{i}}$, the set $\mathcal{A}_\mathbb{R}$ of real and imaginary parts of functions in $\mathcal{A}$ is a real subalgebra of $\mathcal{C}(X,\mathbb{R})$ to which the Stone–Weierstrass theorem applies. Since $\mathcal{A} = \{f + \mathrm{i}g : f, g \in \mathcal{A}_\mathbb{R}\}$, the desired result follows.

(2) For $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \in \mathbb{H}$, we have

$$\begin{aligned}&q - \mathrm{i}q\mathrm{i} - \mathrm{j}q\mathrm{j} - \mathrm{k}q\mathrm{k}\\ =&(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}) - \mathrm{i}(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\mathrm{i} - \mathrm{j}(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\mathrm{j} - \mathrm{k}(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\mathrm{k}\\ =&a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} - a\mathrm{i}^2 - b\mathrm{i}^3 - c\mathrm{i}\mathrm{j}\mathrm{i} - d\mathrm{i}\mathrm{k}\mathrm{i} - a\mathrm{j}^2 - b\mathrm{j}\mathrm{i}\mathrm{j} - c\mathrm{j}^3 - d\mathrm{j}\mathrm{k}\mathrm{j} - a\mathrm{k}^2 - b\mathrm{k}\mathrm{i}\mathrm{k} - c\mathrm{k}\mathrm{j}\mathrm{k} - d\mathrm{k}^3\end{aligned}$$

$$=a+bi+cj+dk-a+bi-cj-dk+a-bi+cj-dk+a-bi-cj+dk$$
$$=4a.$$

Therefore, the scalar part a is the real number $\dfrac{q-iqi-jqj-kqk}{4}$. Likewise,

- ◇ the scalar part of $-qi$ is b which is the real number $\dfrac{-qi-iq+jqk+qj}{4}$.
- ◇ the scalar part of $-qj$ is c which is the real number $\dfrac{-qj-iqk-jq+kqi}{4}$.
- ◇ the scalar part of $-qk$ is d which is the real number $\dfrac{-qk+iqj-jqk-kq}{4}$.

Now the theorem follows by similar arguments as in part (1). □

Problem 56 (Stone–Weierstrass on LCH)

(1) Let X be LCH. Prove: $\mathcal{C}_0(X,\mathbb{R})$ is an algebra.

(2) Prove Theorem 2.6.17 (Stone–Weierstrass theorem on LCH):

Suppose X is an non-compact LCH. Let $\mathcal{A}\subset\mathcal{C}_0(X,\mathbb{R})$ be a subalgebra which vanishes at no point and separates points. Then $\mathcal{A}$ is dense in $\mathcal{C}_0(X,\mathbb{R})$.

(3) Prove: any $f\in\mathcal{C}_0([0,+\infty),\mathbb{R})$ can be approximated uniformly by functions of the form

$$\sum_{k=-n}^{n} a_k e^{-kx}, \quad n\in\mathbb{N}.$$

Proof (1) It suffices to show that $\mathcal{C}_0(X,\mathbb{R})$ is closed under addition, multiplication and scalar multiplication.

- ◇ For any $f,g\in\mathcal{C}_0(X,\mathbb{R})$ and $\varepsilon>0$, there exists compact $K_f,K_g\subset X$ such that
$$|f(x)|<\frac{\varepsilon}{2}\text{ on }(K_f)^c \quad\text{and}\quad |g(x)|<\frac{\varepsilon}{2}\text{ on }(K_g)^c.$$
Then $|f(x)+g(x)|\leqslant|f(x)|+|g(x)|<\varepsilon$ outside the compact set $K_f\cup K_g$, so $f+g\in\mathcal{C}_0(X,\mathbb{R})$.
- ◇ For any $f,g\in\mathcal{C}_0(X,\mathbb{R})$ and $\varepsilon>0$, there exists compact $K_f,K_g\subset X$ such that
$$|f(x)|<\varepsilon\text{ on }(K_f)^c \quad\text{and}\quad |g(x)|<1\text{ on }(K_g)^c.$$
Then $|f(x)g(x)|<\varepsilon\cdot 1$ outside the compact set $K_f\cup K_g$, so $fg\in\mathcal{C}_0(X,\mathbb{R})$.
- ◇ For any $f\in\mathcal{C}_0(X,\mathbb{R})$, $\lambda\in\mathbb{R}$ and $\varepsilon>0$, if $\lambda=0$, then $\lambda f=0\in\mathcal{C}_0(X,\mathbb{R})$; if $\lambda\neq0$, there exists compact $K\subset X$ such that
$$|f(x)|<\frac{\varepsilon}{|\lambda|}\text{ on }K^c.$$
Then $|\lambda f(x)|<\varepsilon$ on K^c, so $\lambda f\in\mathcal{C}_0(X,\mathbb{R})$.

Therefore, $\mathcal{C}_0(X,\mathbb{R})$ is an algebra.

(2) Consider the one-point compactification $X^*:=X\sqcup\{\infty\}$ of X (see Problem 36). By Problem 46 (1) ②, X^* is compact and Hausdorff. Any $f\in\mathcal{C}_0(X,\mathbb{R})$ can be extended continuously to $f^*\in\mathcal{C}(X^*,\mathbb{R})$

by $f^*(x) = f(x)$ for all $x \in X$ and $f^*(\infty) = 0$. To see this, it suffices to check the continuity of f^* at ∞: for any neighborhood U of $f^*(\infty) = 0$, there is some $\varepsilon > 0$ such that $\mathbb{B}(0, \varepsilon) \subset U$; and there exists compact $K \subset X$ such that $|f(x)| < \varepsilon$ on K^c, i.e., $K^c \cup \{\infty\} \subset (f^*)^{-1}(\mathbb{B}(0, \varepsilon))$. Now $\mathcal{A}$ corresponds to a subalgebra $\mathcal{A}^* \subset \mathcal{C}(X^*, \mathbb{R})$ that vanishes only at ∞ and separates points. Since $\mathcal{A}^*$ is not dense in $\mathcal{C}(X^*, \mathbb{R})$, by Theorem 2.6.13,

$$\overline{\mathcal{A}^*} = \{f^* \in \mathcal{C}(X^*, \mathbb{R}) : f^*(\infty) = 0\}.$$

Note that the right-hand side restricts to $\mathcal{C}_0(X, \mathbb{R})$, so $\mathcal{A}$ is dense in $\mathcal{C}_0(X, \mathbb{R})$.

(3) The set $\mathcal{A}$ of functions of the form $\sum_{k=-n}^{n} a_k e^{-kx}$ ($n \in \mathbb{N}$) is a subalgebra of $\mathcal{C}_0([0, +\infty), \mathbb{R})$ that vanishes at no point and separates points. Since $[0, +\infty)$ is LCH, $\mathcal{A}$ is dense in $\mathcal{C}_0([0, +\infty), \mathbb{R})$ by part (2). □

PSet 8, Part 1

Problem 57 (Closedness of graph) Let X, Y be topological spaces, define the *graph* of a map $f: X \to Y$ to be the set

$$G_f := \{(x, f(x)) : x \in X\} \subset X \times Y.$$

(1) Prove: Y is Hausdorff $\iff$ for any X and $f \in \mathcal{C}(X, Y)$, G_f is closed in $X \times Y$.

(2) Construct a discontinuous function $f: \mathbb{R} \to \mathbb{R}$ whose graph is closed.

(3) **(Closed graph theroem)** Prove: if Y is a compact Hausdorff space, then f is continuous if and only if G_f is closed.

Proof (1) ($\Rightarrow$) Let $(x, y) \in (X \times Y) \setminus G_f$, so that $f(x) \neq y$. Since Y is Hausdorff, there exist disjoint open sets U and V in Y such that $f(x) \in U$ and $y \in V$. Then $f^{-1}(U) \times V$ is an open set in $X \times Y$ containing (x, y) but disjoint from G_f, so G_f is closed.

($\Leftarrow$) Take $X = Y$, and consider the identity map $\mathrm{Id}_Y : Y \to Y$. Then $G_f = \{(y, y) : y \in Y\}$ is closed in $Y \times Y$ by assumption, so Y is Hausdorff by Proposition 2.7.19 (2).

(2) $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(3) The "only if" part is already proved in (1). Now assume G_f is closed in $X \times Y$. Let $x \in X$ and let V be an open neighborhood of $f(x)$ in Y. Then $C := G_f \cap (X \times (Y \setminus V))$ is closed in $X \times Y$. Denote by π_X the projection map $X \times Y \to X$. We claim that $\pi_X(C)$ is closed in X. Indeed, let $x_0 \in X \setminus \pi_X(C)$ be arbitrary. Then the slice $\{x_0\} \times Y$ is contained in the open set $(X \times Y) \setminus C$. Since both $\{x_0\}$ and Y are compact, by the tube lemma, there exists open neighborhood W of x_0 in X such that $\{x_0\} \times Y \subset W \times Y \subset (X \times Y) \setminus C$. This implies $x_0 \in W \subset X \setminus \pi_X(C)$. Since x_0 is arbitrary, $X \setminus \pi_X(C)$ is open, so $\pi_X(C)$ is closed in X. Now $U := X \setminus \pi_1(C)$ is a neighborhood of x, and we claim that $f(U) \subset V$. Suppose to the contrary that there exists $x_1 \in U$ with $f(x_1) \notin V$. Then $(x_1, f(x_1)) \in C$, so $\pi_X((x_1, f(x_1))) = x_1 \in \pi_X(C)$, a contradiction. Therefore $x \in U \subset f^{-1}(V)$, so f is continuous. □

Problem 58 (Lindelöf property) A topological space $(X, \mathscr{T})$ is called *Lindelöf* if any open covering of X admits a countable subcovering.

(1) Prove Proposition 2.7.14:

- ⋄ *Any second countable space is Lindelöf.*
- ⋄ *Any σ-compact space is Lindelöf.*

(2) Suppose $(X, \mathscr{T})$ is second countable. Prove: any basis $\mathcal{B}$ of $\mathscr{T}$ has a countable sub-family $\mathcal{B}_0 \subset \mathcal{B}$ that is still a basis.

(3) Prove Proposition 2.7.15:

- ⋄ *Any closed subspace of a Lindelöf space is Lindelöf.*
- ⋄ *The continuous image of a Lindelöf space is Lindelöf.*
- ⋄ *A metric space is Lindelöf if and only if it is second countable.*

(4) Check: $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is Lindelöf but not σ-compact.

(5) Check: the Sorgenfrey line $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is Lindelöf.

Proof (1) Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X.

- ⋄ Suppose $(X, \mathscr{T})$ is second countable and $\{B_n\}_{n=1}^\infty$ is a countable basis of $\mathscr{T}$. For each B_n, if there is some U_α containing B_n, then choose such U_α and denote it by U_n. Otherwise, let $U_n = \varnothing$. Since each point $x \in X$ has a neighborhood V_x contained in some U_α, and V_x contains some B_n about x, it follows that $\{U_n\}_{n=1}^\infty$ is a countable subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$.
- ⋄ Suppose X is σ-compact and $X = \bigcup\limits_{n=1}^\infty K_n$ where each K_n is compact. For each n, there exists a finite subcover $\{U_{\alpha_n} : \alpha_n \in \Lambda_n\}$ of $\{U_\alpha : \alpha \in \Lambda\}$ for K_n. Then $\bigcup\limits_{n=1}^\infty \{U_{\alpha_n} : \alpha_n \in \Lambda_n\}$ is a countable subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$.

(2) Let $\{U_n\}_{n=1}^\infty$ be a countable basis of $\mathscr{T}$. Let $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ be an arbitrary basis of $\mathscr{T}$. For each n, there exists $\Lambda_n \subset \Lambda$ such that $U_n = \bigcup\limits_{\alpha\in\Lambda_n} B_\alpha$. Since X is Lindelöf by (1), there exists a countable subset $\Lambda_n' \subset \Lambda_n$ such that $U_n = \bigcup\limits_{\alpha\in\Lambda_n'} B_\alpha$. Then $\mathcal{B}_0 := \bigcup\limits_{n=1}^\infty \{B_\alpha : \alpha \in \Lambda_n'\}$ is a countable sub-family of $\mathcal{B}$ that is still a basis.

(3)
- ⋄ Suppose X is Lindelöf and A is a closed subspace of X. For any open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of A, let $U_\alpha = V_\alpha \cap A$ for each $\alpha \in \Lambda$, where V_α is open in X. Since A is closed in X, $\{V_\alpha : \alpha \in \Lambda\} \cup A^{\mathrm{c}}$ is an open cover of X. By Lindelöf property, there exists a countable subcover $\{V_n\}_{n=1}^\infty \cup A^{\mathrm{c}}$ of X. Then $\{V_n \cap A\}_{n=1}^\infty$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$.
- ⋄ Suppose X is Lindelöf and $f\colon X \to Y$ is continuous. For any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of $f(X)$, $\left\{f^{-1}(U_\alpha) : \alpha \in \Lambda\right\}$ is an open cover of X. By Lindelöf property, there exists a countable subset $\Lambda' \subset \Lambda$ such that $\left\{f^{-1}(U_\alpha) : \alpha \in \Lambda'\right\}$ is still a cover of X. Then $\{U_\alpha : \alpha \in \Lambda'\}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$.

◇ The "if" part is already proved in (1). Now suppose X is a Lindelöf metric space. For each $n \in \mathbb{N}$, the collection $\{\mathbb{B}(x, \frac{1}{n}) : x \in X\}$ is an open cover of X. By Lindelöf property, there exists a countable subcover $\{U_{n,k}\}_{k=1}^{\infty}$. For any open set U in X and any $x \in U$, there is some $n \in \mathbb{N}$ such that $x \in \mathbb{B}(x, \frac{1}{n}) \subset U$. Since x must be contained in some $U_{2n,k}$, which is a ball of radius $\frac{1}{2n}$, it follows by the triangle inequality that $x \in U_{2n,k} \subset \mathbb{B}(x, \frac{1}{n})$. Hence $\{U_{2n,k}\}_{n,k=1}^{\infty}$ is a countable basis of X.

(4) For any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$, pick any $\alpha_1 \in \Lambda$ such that $U_{\alpha_1} \neq \varnothing$. Then $\mathbb{R} \setminus U_{\alpha_1}$ is countable, denote it by $\{x_n\}_{n=2}^{\infty}$. For each $n \geqslant 2$, choose $\alpha_n \in \Lambda$ such that $x_n \in U_{\alpha_n}$. Then $\{U_{\alpha_n}\}_{n=1}^{\infty}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$. Therefore $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is Lindelöf.

However, compact sets in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ must be finite (any infinite subset A contains some countable subset $\{x_n\}_{n=1}^{\infty}$, and the open cover $\left\{\left(\bigcup_{k\neq n}\{x_k\}\right)^{\mathrm{c}}\right\}_{n=1}^{\infty}$ of A has no finite subcover), so countable union of compact sets is still countable, which cannot cover $\mathbb{R}$. Therefore $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is not σ-compact.

(5) Since $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ is a basis of $\mathscr{T}_{\text{Sorgenfrey}}$, it suffices to show that each $\mathcal{B}$-cover of $\mathbb{R}$ admits a countable subcover. Suppose $\mathbb{R} = \bigcup_{\alpha\in\Lambda}[a_\alpha, b_\alpha)$. We claim that $\mathbb{R} \setminus \bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha)$ is at most countable. In fact, for any $x \notin \bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha)$, there exists $\beta_x \in \Lambda$ such that $x = a_{\beta_x}$. Note that such open intervals $(a_{\beta_x}, b_{\beta_x})$ are disjoint for different x, so the claim follows. Now the set

$$\Lambda_0 := \left\{\beta_x \in \Lambda : x \notin \bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha)\right\}$$

is at most countable. Since $\{(a_\alpha, b_\alpha) : \alpha \in \Lambda\}$ is an open cover of $\bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha)$ in the usual topology of $\mathbb{R}$, and $\bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha)$ is a second countable metric space as $\mathbb{R}$ is (A2), by (3) it is Lindelöf. Therefore there exists a countable subcover $\Lambda' \subset \Lambda$ such that $\bigcup_{\alpha\in\Lambda}(a_\alpha, b_\alpha) = \bigcup_{\alpha\in\Lambda'}(a_\alpha, b_\alpha)$. Then

$$\{[a_\alpha, b_\alpha) : \alpha \in \Lambda'\} \cup \{[a_\beta, b_\beta) : \beta \in \Lambda_0\}$$

is a countable subcover of $\{[a_\alpha, b_\alpha) : \alpha \in \Lambda\}$. Therefore $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is Lindelöf. ☐

Problem 59 (Hereditary properties) A topological property P is called *hereditary* if

$$(X, \mathscr{T}) \text{ satisfies } P \implies \text{any subspace } Y \text{ of } X \text{ satisfies } P.$$

(1) Prove: (A1) and (A2) are hereditary, but (T4) is not hereditary.

(2) Which of the following properties are hereditary:

compact / sequentially compact / locally compact / separable / Lindelöf / (T1) / (T2) / (T3)

(3) A topological property P is called *closed hereditary* if

$$(X, \mathscr{T}) \text{ satisfies } P \implies \text{any closed subspace } Y \text{ of } X \text{ satisfies } P.$$

For the non-hereditary properties above, determine whether they are closed hereditary.

Proof (1) Since the intersection of a basis / neighborhood basis of the total space with a subspace is still a basis / neighborhood basis of the subspace, (A1) and (A2) are hereditary. However, (T4) is not hereditary. Consider $\mathbb{R}$ endowed with the cofinite topology, then any two nonempty open sets in $\mathbb{R}$ intersect, so $\mathbb{R}$ is not (T4). Let $X := \mathbb{R} \sqcup \{\infty\}$, whose open sets are those of $(\mathbb{R}, \mathscr{T}_{\text{cofinite}})$ together with X. Now any two nonempty closed sets in X intersect (they both contain ∞), so X is (T4). This shows that (T4) is not hereditary.

(2) ① Compactness is not hereditary. For example, $[0,1]$ is compact, but $(0,1)$ is not.

② Sequential compactness is not hereditary. One can take the same example as above.

③ Local compactness is not hereditary. For example, $[0,1]$ is locally compact, but $[0,1] \cap \mathbb{Q}$ is not (see Problem 46 (2) ①).

④ Separability is not hereditary. Take any non-separable space Y and consider $X := Y \sqcup \{\infty\}$, whose open sets are given by

$$\{\varnothing\} \cup \{U \cup \{\infty\} : U \text{ is open in } Y\}.$$

Then X is separable since $\overline{\{\infty\}} = X$. However, $Y \subset X$ is not separable.

⑤ Lindelöf property is not hereditary. Take any non-Lindelöf space Y and consider $X := Y \sqcup \{\infty\}$, whose open sets are given by

$$\{U : U \text{ is open in } Y\} \cup \{V \cup \{\infty\} : V \subset Y \text{ and } Y \setminus V \text{ is countable}\}.$$

Then for any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, one can first pick $\alpha_1 \in \Lambda$ such that $\infty \in U_{\alpha_1}$. By construction $U_{\alpha_1} = V \cup \{\infty\}$, where V cocountable in Y. Let $Y \setminus V = \{x_n\}_{n=2}^{\infty}$. For each $n \geqslant 2$, choose $\alpha_n \in \Lambda$ such that $x_n \in U_{\alpha_n}$. Then $\{U_{\alpha_n}\}_{n=1}^{\infty}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$, so X is Lindelöf. However, $Y \subset X$ is not Lindelöf.

⑥ (T1) is hereditary since (T1) is equivalent to "every singleton is closed".

⑦ (T2) is hereditary since the intersections of any two disjoint open sets with a subspace are disjoint open sets in the subspace.

⑧ (T3) is hereditary. Let X be (T3) and Y be a subspace of X. For any closed subset A of Y and any $x \in Y \setminus A$, there exists a closed subset B of X such that $A = B \cap Y$. Since X is (T3) and $x \notin B$, there exist disjoint open sets U and V in X such that $B \subset U$ and $x \in V$. Then $A \subset U \cap Y$ and $x \in V \cap Y$. Therefore Y is (T3).

(3) ① (T4) is closed hereditary. Suppose X is (T4) and Y is a closed subspace of X. For any two disjoint closed sets A and B in Y, since Y is closed in X, A and B are closed in X. Since X is (T4), there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$. Then $A \subset U \cap Y$ and $B \subset V \cap Y$, so Y is (T4).

② Compactness is closed hereditary. This is Proposition 2.1.16 (1).

③ Sequential compactness is closed hereditary. This is Proposition 2.1.16 (2).

④ Local compactness is closed hereditary. Suppose X is locally compact and Y is a closed subspace of X. For any $x \in Y$, since X is locally compact, there exists a compact neighborhood K of x in X. Then $K \cap Y$ is a neighborhood of x in Y. Moreover, it is compact. In fact, for any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of $K \cap Y$ in X, $\{Y^{\mathrm{c}}\} \cup \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of K

in X. Since K is compact, there exists a finite subcover $\{Y^c\} \cup \{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ of K in X. Then $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ is a finite subcover of $\{U_\alpha : \alpha \in \Lambda\}$. Therefore $K \cap Y$ is a compact neighborhood of x in Y and Y is locally compact.

⑤ Separability is not closed hereditary. One can take the same example as in (2) ④, where $Y = X \setminus \{\infty\}$ is closed in X but not separable.

⑥ Lindelöf property is closed hereditary. This is Problem 58 (3). □

Problem 60 (The Sorgenfrey plane) Consider the product of two Sorgenfrey lines,

$$\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right) := \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right) \times \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right),$$

which is known as the *Sorgenfrey plane*.

(1) Prove: it is first countable, separable but not second countable.

(2) Is it Hausdorff?

(3) Consider the subspace $A = \{(x, -x) : x \in \mathbb{R}\}$. Is it closed? What is the induced subspace topology on A?

(4) Prove: it is not Lindelöf.

Proof (1) For any $(x, y) \in \left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$, the collection $\left\{\left[x, \frac{1}{n}\right) \times \left[y, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is a countable neighborhood basis of (x, y). Hence $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is first countable. Since $\mathbb{Q}^2$ is a countable dense subset of $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$, it is separable. However, $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is not second countable. In fact, if it were second countable, by Problem 59 (1), (A2) is hereditary, so $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$ would be second countable, which contradicts Example 2.7.8 (2).

(2) For any $x, y \in \left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$ with $x < y$, the open sets $[x, y)$ and $[y, y+1)$ are disjoint and contain x and y respectively. Hence $\left(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is Hausdorff. By Problem 61 (1), (T2) is productive, so $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is also Hausdorff.

(3) By Problem 9 (3), the topology $\mathscr{T}_{\text{Sorgenfrey}}$ is finer than the usual topology on $\mathbb{R}$. It follows that $\mathscr{T}_{\text{Sorgenfrey}}$ is finer than the usual topology on $\mathbb{R}^2$. Since A is closed in the usual topology of $\mathbb{R}^2$, it is also closed in $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$. For any $(x, -x) \in A$, the open set $[x, x+1) \times [-x, -x+1)$ of $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ intersects A only at $(x, -x)$, so $\{(x, -x)\}$ is open in A. Therefore the induced subspace topology on A is discrete.

(4) The subspace $A \subset \left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is not Lindelöf since the open cover $\{(x, -x) : x \in \mathbb{R}\}$ has no countable subcover. By Problem 59 (3), Lindelöf property is closed hereditary, so $\left(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}\right)$ is not Lindelöf. □

PSet 8, Part 2

Problem 61 (Productive properties) A topological property P is called *productive* if

$$\text{each } (X_\alpha, \mathscr{T}_\alpha) \text{ satisfies } P \implies \left(\prod_\alpha X_\alpha, \mathscr{T}_{\text{product}}\right) \text{ satisfies } P.$$

(1) Prove: (T1), (T2) and (T3) are productive.

(2) Conversely, if $\left(\prod_\alpha X_\alpha, \mathscr{T}_{\text{product}}\right)$ is (T1), (T2) or (T3), can we conclude that each $(X_\alpha, \mathscr{T}_\alpha)$ is (T1), (T2) or (T3)?

(3) Is (T4) productive? Is Lindelöf productive?

(4) Prove: *separability* and *metrizability* are not productive. What about (A1), (A2)?

(5) Can you introduce a weaker version of productivity, so that those non-productive properties in part (4) satisfy the weaker one?

Proof (1) If each $(X_\alpha, \mathscr{T}_\alpha)$ is (T1), then for any distinct $x, y \in \prod_\alpha X_\alpha$, there exists β such that $x_\beta \neq y_\beta$. Since X_β is (T1), there exist open sets U_β, V_β in X_β such that $x_\beta \in U_\beta \setminus V_\beta$ and $y_\beta \in V_\beta \setminus U_\beta$. Let $U_\alpha = X_\alpha$ and $V_\alpha = Y_\alpha$ for all $\alpha \neq \beta$. Then $U = \prod_\alpha U_\alpha$ and $V = \prod_\alpha V_\alpha$ are open sets in $\prod_\alpha X_\alpha$ such that $x \in U \setminus V$ and $y \in V \setminus U$. Hence $\prod_\alpha X_\alpha$ is (T1).

If each $(X_\alpha, \mathscr{T}_\alpha)$ is (T2), then for any distinct $x, y \in \prod_\alpha X_\alpha$, there exists β such that $x_\beta \neq y_\beta$. Since X_β is (T2), there exist disjoint open sets U_β, V_β in X_β such that $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$. Let $U_\alpha = X_\alpha$ and $V_\alpha = Y_\alpha$ for all $\alpha \neq \beta$. Then $U = \prod_\alpha U_\alpha$ and $V = \prod_\alpha V_\alpha$ are disjoint open sets in $\prod_\alpha X_\alpha$ such that $x \in U$ and $y \in V$. Hence $\prod_\alpha X_\alpha$ is (T2).

If each $(X_\alpha, \mathscr{T}_\alpha)$ is (T3), then for any $x \in \prod_\alpha X_\alpha$ and any open neighborhood $U = \prod_\alpha U_\alpha$ of x, we can always find an open neighborhood $V = \prod_\alpha V_\alpha$ of x such that $\overline{V} \subset U$. Indeed, when $U_\alpha = X_\alpha$ let $V_\alpha = X_\alpha$ too. For the finitely many α's such that $U_\alpha \neq X_\alpha$, since each X_α is (T3), we can find open sets V_α such that $\overline{V_\alpha} \subset U_\alpha$. Then by Problem 30 (1),

$$\overline{V} = \overline{\prod_\alpha V_\alpha} = \prod_\alpha \overline{V_\alpha} \subset \prod_\alpha U_\alpha = U.$$

By Proposition 2.7.19 (3), this implies that $\prod_\alpha X_\alpha$ is (T3).

(2) Each $(X_\alpha, \mathscr{T}_\alpha)$ can be viewed as a subspace of $\prod_\alpha X_\alpha$, so by Problem 59 (2), if $\left(\prod_\alpha X_\alpha, \mathscr{T}_{\text{product}}\right)$ is (T1) / (T2) / (T3), then each $(X_\alpha, \mathscr{T}_\alpha)$ is (T1) / (T2) / (T3).

(3) By Problem 58 (5), the Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$ is Lindelöf. However, $(\mathbb{R}^2, \mathscr{T}_{\text{product}})$ is not Lindelöf by Problem 60 (4). So Lindelöf property is not productive.

The above example can also be used to show that (T4) is not productive:

$\boxed{(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}}) \text{ is (T4)}}$ Let A and B be disjoint closed sets in $(\mathbb{R}, \mathscr{T}_{\text{Sorgenfrey}})$. For any $a \in A$, we have $a \in B^c$. Subce B^c is open, we can take $\varepsilon_a > 0$ such that $[a, a+\varepsilon_a) \cap B = \varnothing$. Similarly, for any $b \in B$, we can take $\varepsilon_b > 0$ such that $[b, b+\varepsilon_b) \cap A = \varnothing$. Note that we always have

$$[a, a+\varepsilon_a) \cap [b, b+\varepsilon_b) = \varnothing, \quad \forall a \in A \text{ and } b \in B,$$

for otherwise we would have $b \in [a, a+\varepsilon_a)$ or $a \in [b, b+\varepsilon_b)$, which is a contradiction. It follows that

$$U_A := \bigcup_{a \in A} [a, a+\varepsilon_a) \quad \text{and} \quad U_B := \bigcup_{b \in B} [b, b+\varepsilon_b)$$

are disjoint open sets separating A and B.

$\boxed{(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}) \text{ is not (T4)}}$ In Problem 60 (3) we have shown that $\Delta = \{(x, -x) : x \in \mathbb{R}\}$ is closed in $(\mathbb{R}^2, \mathscr{T}_{\text{product}})$, and the subspace topology on A is discrete. Thus any subset of Δ is closed in $(\mathbb{R}^2, \mathscr{T}_{\text{product}})$. If $(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}})$ is (T4), then by Urysohn's lemma, for any $A \subset \Delta$, since A and $B := \Delta \setminus A$ are both closed in the Sorgenfrey plane, there exists a continuous function $f\colon (\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}) \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Therefore

$$\left|\mathcal{C}\big((\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}), [0,1]\big)\right| \geqslant \left|2^{\Delta}\right| = \left|2^{\mathbb{R}}\right| = 2^{\aleph_1} = \aleph_2.$$

On the other hand, by Problem 60 (1), the Sorgenfrey plane is separable, so any continuous function on it is determined by its values on a countable dense subset, which means

$$\left|\mathcal{C}\big((\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}}), [0,1]\big)\right| \leqslant |[0,1]|^{\aleph_0} = \aleph_1^{\aleph_0} = \aleph_1.$$

This is a contradiction. Therefore $(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}})$ is not (T4).

(4) All these properties may not be preserved by products. Let each X_α be the discrete space $\{0,1\}$, and let $X = \prod_{\alpha \in \Lambda} X_\alpha$, where $|\Lambda| > \aleph_1$. Then each X_α is separable, metrizable, (A1) and (A2).

$\boxed{X \text{ is not separable}}$ Suppose D were a countable dense subset of X. Then for distinct $\alpha, \beta \in \Lambda$, the sets $\pi_\alpha^{-1}(0)$, $\pi_\beta^{-1}(0)$ and $\pi_\beta^{-1}(1)$ are all nonempty open sets in X. Then $\pi_\alpha^{-1}(0) \cap \pi_\beta^{-1}(1)$ and $\pi_\beta^{-1}(0)$ are both nonempty open sets in X, and they are disjoint. Since D is dense,

$$D \cap \left(\pi_\alpha^{-1}(0) \cap \pi_\beta^{-1}(1)\right) \quad \text{and} \quad D \cap \pi_\beta^{-1}(0)$$

are disjoint nonempty subsets of D. It follows that

$$D \cap \pi_\alpha^{-1}(0) \quad \text{and} \quad D \cap \pi_\beta^{-1}(0)$$

are distinct subsets of D. Thus the map

$$\Phi\colon \Lambda \to 2^D, \quad \alpha \mapsto D \cap \pi_\alpha^{-1}(0)$$

is injective, which implies $|\Lambda| \leqslant \left|2^D\right| = 2^{\aleph_0} = \aleph_1$, a contradiction.

$\boxed{X \text{ is not (A1)/(A2)/metrizable}}$ Assume $\{B_n\}_{n=1}^\infty$ is a countable neighborhood basis at the point $p \in X$. For each n, $\pi_\alpha(B_n) = \{0,1\}$ for all but finitely many α. Since there are uncountably many α, we can select one, say α_0, such that $\pi_{\alpha_0}(B_n) = \{0,1\}$ for all n. Then $\pi_{\alpha_0}^{-1}(p_{\alpha_0}) =$

$\{x \in X : x_{\alpha_0} = p_{\alpha_0}\}$ is an open neighborhood of p which contains no B_n, a contradiction. Therefore X is not (A1), and then not (A2). Since each X_α is compact Hausdorff, by Tychnoff's theorem and then productivity of (T2), X is compact Hausdorff. So by Proposition 2.8.13, X is metrizable if and only if it is (A2). Therefore X is not metrizable.

(5) The four properties in (4) are all preserved by countable products.

$\boxed{\text{Separability}}$ Suppose each X_n is separable and let $X = \prod_{n=1}^{\infty} X_n$. For each n, let D_n be a countable dense subset of X_n and fix a point $x_n \in D_n$. Then consider

$$E_m = \left\{ y \in \prod_{n=1}^{\infty} D_n : y_n = x_n \text{ for all } n \geqslant m \right\}$$

and let $E = \bigcup_{m=1}^{\infty} E_m$. Since each E_m is countable, it follows that E is countable. Note that any nonempty open set in X is of the form

$$V = \prod_{n=1}^{m-1} U_n \times \prod_{n=m}^{\infty} X_n,$$

where U_n is a nonempty open set in X_n for $1 \leqslant n < m$. Since

$$V \cap E_m = \prod_{n=1}^{m-1} (V_n \cap D_n) \times \prod_{n=m}^{\infty} \{x_n\} \neq \varnothing,$$

we see that $V \cap E \neq \varnothing$. Therefore E is a countable dense subset of X.

$\boxed{\text{Metrizability}}$ Suppose each X_n is metrizable. Moreover, by replacing the original metric with its corresponding uniform metric d_n, we can assume that each X_n is bounded. Then by Problem 39 (3), the topology induced by the metric

$$d((x_n), (y_n)) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{[1 + \text{diam}(X_n)] \cdot 2^n}$$

coinicides with the product topology on X. Therefore X is metrizable.

$\boxed{\text{(A1)}}$ Suppose each X_n is first countable and let $X = \prod_{n=1}^{\infty} X_n$. Take any $x = (x_n) \in X$. For each n, let $\{B_{n,k}\}_{k=1}^{\infty}$ be a countable neighborhood basis at x_n. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \prod_{m=1}^{n} B_{m,k} \times \prod_{m=n+1}^{\infty} X_m \right\}$$

is a countable neighborhood basis at x.

$\boxed{\text{(A2)}}$ Suppose each X_n is second countable and let $X = \prod_{n=1}^{\infty} X_n$. For each n, let $\{B_{n,k}\}_{k=1}^{\infty}$ be a countable basis for X_n. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \prod_{m=1}^{n} B_{m,k} \times \prod_{m=n+1}^{\infty} X_m \right\}$$

is a countable basis for X. $\square$

Problem 62 (Baire space) A topological space is called a *Baire space* if every intersection of countable collection of open dense sets in the space is dense.

(1) Use "open-closed" duality to give an equivalent characterization of Baire space.

(2) Prove: any complete metric space is a Baire space.

(3) Prove: any compact Hausdorff space is a Baire space.

(4) Prove: any locally compact Hausdorff space is a Baire space.

Proof (1) A topological space is a Baire space if and only if every countable union of closed sets in the space with empty interior has empty interior.

(2) & (3) By Problem 7 (2) and Proposition 2.7.20, whether X is complete metric or compact Hausdorff, it is regular. Given a countable collection $\{A_n\}_{n=1}^{\infty}$ of closed sets of X having empty interiors, we want to show that their union $\bigcup\limits_{n=1}^{\infty} A_n$ also has empty interior in X. So, given the nonempty open set U_0 of X, we must find a point x of U_0 that does not lie in any of the sets A_n.

Consider the first set A_1. By hypothesis, A_1 does not contain U_0. Therefore, we may choose a point y of U_0 that is not in A_1. Since X is regular and $U_0 \setminus A_1$ is open in X, by Proposition 2.7.19 (3), we can choose an open neighborhood U_1 of y such that $x \in U_1 \subset \overline{U_1} \subset U_0 \setminus A_1$, that is,

$$\overline{U_1} \cap A_1 = \varnothing \quad \text{and} \quad \overline{U_1} \subset U_0.$$

If X is metric, we also choose U_1 small enough that its diameter is less than 1.

In general, given the nonempty open set U_{n-1}, we choose a point of U_{n-1} that is not in the closed set A_n, and then we choose U_n to be an open neighborhood of this point such that

$$\overline{U_n} \cap A_n = \varnothing \quad \text{and} \quad \overline{U_n} \subset U_{n-1},$$

and also $\operatorname{diam} U_n < \frac{1}{n}$ in the metric case.

We assert that the intersection $\bigcap\limits_{n=1}^{\infty} \overline{U_n}$ is nonempty. From this fact, the desired result will follow. For if $x \in \bigcap\limits_{n=1}^{\infty} \overline{U_n}$, then x is in U_0 because $\overline{U_1} \subset U_0$. And for each n, the point x is not in A_n because $\overline{U_n}$ is disjoint from A_n.

- If X is complete metric, since $\overline{U_1} \supset \overline{U_2} \supset \cdots$ is a nested sequence of nonempty closed sets in the complete metric space X, and $\operatorname{diam} \overline{U_n} \to 0$ as $n \to \infty$, our assertion follows from the Cantor's intersection theorem.
- If X is compact Hausdorff, since the collection $\{\overline{U_n}\}_{n=1}^{\infty}$ has the finite intersection property, our assertion follows from Proposition 2.1.6.

(4) Suppose X is a non-compact LCH space. Let $X^* = X \sqcup \{\infty\}$ be the one-point compactification of X. Then by Problem 46 (1) ②, X^* is a compact Hausdorff space. So by (3) X^* is a Baire space. Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of open dense sets in X. Since X is non-compact, $\{\infty\} \notin \mathscr{T}^*$ by

the construction of $\mathscr{T}^*$ (see Problem 36 (2)). Therefore any open neighborhood of ∞ is of the form $V \cup \{\infty\}$ where V is a nonempty open subset of X. Then $V \cap U_n \neq \varnothing$ for all n, which implies that ∞ is a limit point of each U_n. So $\{U_n\}_{n=1}^{\infty}$ is also a collection of open dense sets in X^*, and since X^* is a Baire space, $\displaystyle\bigcap_{n=1}^{\infty} U_n$ is dense in X^*. It follows that $\displaystyle\bigcap_{n=1}^{\infty} U_n$ is dense in X. Therefore X is a Baire space. □

Problem 63 (Applications of Urysohn lemma)

(1) Let X be a compact Hausdorff space, $x_0 \in X$, and U is an open neighborhood of x_0. Prove: for any $\varepsilon > 0$ and any continuous function $f\colon X \to \mathbb{R}$, there exists a continuous function $g\colon X \to \mathbb{R}$ satisfying all of the following three conditions:

- ◇ $\sup\limits_{x \in X} |g(x) - f(x)| < \varepsilon$.
- ◇ $g = f$ on U^{c}.
- ◇ there exists a neighborhood V of x_0 such that $g(x) \equiv f(x_0)$ on V.

(2) Let X be LCH. Recall

- ◇ $\mathcal{C}_{\mathrm{b}}(X, \mathbb{R}) = \{f\colon X \to \mathbb{R} : f \text{ is continuous and bounded}\}$.
- ◇ $\mathcal{C}_{\mathrm{c}}(X, \mathbb{R}) = \{f\colon X \to \mathbb{R} : f \text{ is continuous and compactly supported}\}$.
- ◇ $\mathcal{C}_0(X, \mathbb{R}) = \{f\colon X \to \mathbb{R} : f \text{ is continuous and vanishes at infinity}\}$.

On $\mathcal{C}_{\mathrm{b}}(X, \mathbb{R})$ we have a metric $d_\infty(f, g) \coloneqq \sup\limits_{x \in X} |f(x) - g(x)|$. Prove: the closure of $\mathcal{C}_{\mathrm{c}}(X, \mathbb{R})$ in $\mathcal{C}_{\mathrm{b}}(X, \mathbb{R})$ is $\mathcal{C}_0(X, \mathbb{R})$.

Proof (1) After translating f by $f(x_0)$, we may assume $f(x_0) = 0$. Since $f\colon X \to \mathbb{R}$ is continuous, the set $W = f^{-1}\big(\mathbb{B}(0, \frac{\varepsilon}{2})\big)$ is an open neighborhood of x_0. Then $U_0 \coloneqq U \cap W$ is an open neighborhood of x_0. Since X is Hausdorff, $\{x_0\}$ is compact. By Proposition 2.4.16, there exists an open set V such that $\overline{V}$ is compact and $\{x_0\} \subset V \subset \overline{V} \subset U_0$. By Proposition 2.7.22, X is normal. So by the Urysohn lemma, there exists a continuous function $h\colon X \to [0, 1]$ such that $h\big(\overline{V}\big) = 0$ and $h(U_0^{\mathrm{c}}) = 1$. Now take $g = fh$, then we have

- ◇ $\sup\limits_{x \in X} |g(x) - f(x)| = \sup\limits_{x \in X} |f(x)||h(x) - 1| = \sup\limits_{x \in U_0} |f(x)||h(x) - 1| \leqslant \sup\limits_{x \in U_0} |f(x)| \leqslant \frac{\varepsilon}{2} < \varepsilon$.
- ◇ $g(x) = f(x)h(x) = f(x)$ for $x \in U_0^{\mathrm{c}}$, thus $g = f$ on U^{c}.
- ◇ $g(x) = f(x)h(x) = 0 = f(x_0)$ on V.

(2) By Theorem 2.6.17 / Problem 56 (2), it suffices to show that $\mathcal{C}_{\mathrm{c}}(X, \mathbb{R})$ is a subalgebra of $\mathcal{C}_0(X, \mathbb{R})$ which vanishes at no point and separates points.

- ◇ Since the union of two compact sets are compact, the support of the sum/product of two compactly supported functions is a closed subset of a compact set, hence compact. Also it is obvious that scalar multiples of compactly supported functions are compactly supported.
- ◇ For any $x \in X$, choose an open neighborhood U of x. Since the compact set $\{x\}$ and the closed set U^{c} are disjoint, by Theorem 2.8.9, there exists a compactly supported continuous function $f\colon X \to [0, 1]$ such that $f(x) = 1$ and $f(U^{\mathrm{c}}) = 0$. Hence $\mathcal{C}_{\mathrm{c}}(X, \mathbb{R})$ vanishes at no point.

- $\diamond$ For any $x, y \in X$ with $x \neq y$, since X is Hausdorff, the singleton $\{y\}$ is closed. And since $\{x\}$ is compact, by Theorem 2.8.9, there exists a compactly supported continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$. Hence $\mathcal{C}_c(X, \mathbb{R})$ separates points. $\square$

Problem 64 (Metrizability for compact Hausdorff spaces) Let $(X, \mathscr{T})$ be a compact Hausdorff space. Show that the following are equivalent:

(1) X is metrizable.

(2) The diagonal $\Delta \subset X \times X$ is a G_δ set.

(3) There exists a continuous function $f\colon X \times X \to \mathbb{R}$ such that $f^{-1}(0) = \Delta$.

Proof $\boxed{(3) \Rightarrow (2)}$ $\Delta = f^{-1}(0) = \bigcap_{n=1}^{\infty} f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is a G_δ set.

$\boxed{(2) \Rightarrow (1)}$ Since X is compact Hausdorff, by Corollary 2.8.13, it suffices to show that X is second countable. Suppose $\Delta = \bigcap_{n=1}^{\infty} G_n$, where each G_n is open in $X \times X$. For each fixed n and for any $x \in X$, since $(x, x) \in \Delta \subset G_n$, we can find some open neighborhood U_x^n of x such that $U_x^n \times U_x^n \subset G_n$. Since X is regular, there exists an open neighborhood V_x^n of x such that $x \in V_x^n \subset \overline{V_x^n} \subset U_x^n$. Since X is compact, finitely many of the V_x^n cover X, say $X = V_{x_1}^n \cup \cdots \cup V_{x_{k_n}}^n$. Now consider

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{k_n} \left\{ V_{x_m}^n, \overline{V_{x_m}^n}^{\mathrm{c}} \right\},$$

which is countable. Moreover, for any distinct $x, y \in X$, since $(x, y) \notin \Delta$, there is some n such that $(x, y) \notin G_n$. Choose some $V_{x_m}^n$ containing x. Note that $\overline{V_{x_m}^n} \times \overline{V_{x_m}^n} \subset G_n$, so $y \notin \overline{V_{x_m}^n}$. Hence $V_{x_m}^n$ and $\overline{V_{x_m}^n}^{\mathrm{c}}$ are disjoint open sets separating x and y.

Now consider the topology $\mathscr{T}_{\mathcal{S}}$ generated by $\mathcal{S}$. The above observation shows that $(X, \mathscr{T}_{\mathcal{S}})$ Hausdorff. We claim that the corresponding basis

$$\mathcal{B} = \{B : B = S_1 \cap \cdots \cap S_m \text{ for some } S_1, \cdots, S_m \in \mathcal{S}\}$$

is a countable basis for $\mathscr{T}_{\mathcal{S}}$. Indeed, since $\mathcal{S}$ is countable,

$$|\mathcal{B}| \leqslant |\text{finite subsets of } \mathcal{S}| \leqslant \aleph_0^1 + \aleph_0^2 + \aleph_0^3 + \cdots = \aleph_0 + \aleph_0 + \aleph_0 + \cdots = \aleph_0.$$

Since $\mathscr{T}_{\mathcal{S}} \subset \mathscr{T}$, the identity map

$$\mathrm{Id}_X \colon (X, \mathscr{T}) \to (X, \mathscr{T}_{\mathcal{S}})$$

is continuous. Moreover, it is a bijection from a compact space to a Hausdorff space, so it is a homeomorphism. Therefore these two topologies are the same. Now $\mathcal{B}$ is a countable basis of $(X, \mathscr{T})$, so X is second countable as we needed.

$\boxed{(1) \Rightarrow (3)}$ For the metric space (X, d), the function $d\colon X \times X \to \mathbb{R}$ is continuous and $d^{-1}(0) = \Delta$. $\square$

PSet 9, Part 1

Problem 65 (Uniqueness of extension)

(1) Prove Lemma 2.9.2:

> *Let X, Y be topological spaces, $A \subset X$ be a dense subset, and $f\colon A \to Y$ be a continuous map. If Y is a (T2) space, then f admits at most one continuous extension.*

(2) Can we replace (T2) by (T1)? If yes, prove it; if not, give a counterexample.

Proof (1) Suppose that f admits two continuous extensions $g_1, g_2\colon X \to Y$. Then there exists $x \in X$ such that $g_1(x) \neq g_2(x)$. Since Y is Hausdorff, there exist disjoint open sets U_1, U_2 such that $g_1(x) \in U_1$ and $g_2(x) \in U_2$. Now $g_1^{-1}(U_1) \cap g_2^{-1}(U_2)$ is an open neighborhood of x, so it contains some $a \in A$ by density. But this contradicts the fact that $g_1(a) = f(a) = g_2(a)$.

(2) The conclusion does not hold if Y is a (T1) space. Consider $X = Y = (\mathbb{R}, \mathscr{T}_{\text{cofinite}})$. Then Y is a (T1) space, and $A = \mathbb{Z}$ is dense in X. For the inclusion map $f\colon A \hookrightarrow Y$, any function of the form

$$f_t(x) = \begin{cases} x, & x \neq t, \\ 0, & x = t, \end{cases} \quad \text{where} \quad t \notin \mathbb{Z}$$

is a continuous extension of f. □

Problem 66 (Tietze extensions with restrictions) Let $(X, \mathscr{T})$ be a (T4) space, $A \subset X$ be closed.

(1) Let C be a convex compact subset of $\mathbb{R}^m$. Prove: any continuous map $f\colon A \to C$ can be extended to a continuous map $f\colon X \to C$. In particular, any complex-valued continuous function on A can be extended to X while keeping the norm.

(2) Let $f\colon A \to \mathbb{R}$ and $g_1, g_2\colon X \to \mathbb{R}$ be continuous functions. Suppose

$$g_1(x) \leqslant f(x) \leqslant g_2(x), \forall x \in A \quad \text{and} \quad g_1(x) \leqslant g_2(x), \forall x \in X.$$

Prove: f can be extended to a continuous function $\tilde{f}\colon X \to \mathbb{R}$ such that

$$g_1(x) \leqslant \tilde{f}(x) \leqslant g_2(x), \quad \forall x \in X.$$

Proof (1) As stated in Remark 2.9.7, f can be extended to a continuous function $g\colon X \to \mathbb{R}^m$. Since C is a convex compact subset of $\mathbb{R}^m$, it is a strong deformation retract of $\mathbb{R}^m$, so there exists a continuous retraction $r\colon \mathbb{R}^m \to C$. Then $r \circ g\colon X \to C$ is the desired extension of f.

(2) Since X is normal and A is closed, by Tietze extension theorem, f can be extended to a continuous function $g_0\colon X \to \mathbb{R}$. Then $\tilde{f} = \min\{\max\{g_0, g_1\}, g_2\}$ is a continuous extension of f that satisfies $g_1(x) \leqslant \tilde{f}(x) \leqslant g_2(x)$ for all $x \in X$. □

Problem 67 (Retraction) Let X be a topological space, $A \subset X$ be a subspace. We say A is a *retract* of X if there exists a continuous map $r\colon X \to A$ such that

$$r(x) = x, \quad \forall x \in A.$$

Such a map r is called a *retraction*.

(1) Prove: if X is Hausdorff, A is a retract of X, then A is closed.

(2) Prove: A is a retract of X if and only if for any topological space Y, any continuous map $f\colon A \to Y$ has an extension $\tilde{f}\colon X \to Y$.

(3) Suppose X is normal and A is closed. Prove: if Y is a retract of $\mathbb{R}^J$ (with product topology, where J is any set), then any continuous map $f\colon A \to Y$ admits a continuous extension $\tilde{f}\colon X \to Y$.

Proof (1) Fix any $a \notin A$. Suppose $r\colon X \to A$ is a retraction and $r(a) = b = r(b) \in A$. Since X is Hausdorff, there exist disjoint open sets U, V such that $a \in U$ and $b \in V$. Then $r^{-1}(U \cap A)$ and $r^{-1}(V \cap A)$ are disjoint open sets. Note that $a \in r^{-1}(b) \subset r^{-1}(V \cap A)$, so $U \cap r^{-1}(V \cap A)$ is an open neighborhood of a disjoint from A. This shows that A^c is open, so A is closed.

(2) ($\Rightarrow$) Just take $\tilde{f} = f \circ r$ where $r\colon X \to A$ is a retraction.

($\Leftarrow$) Take $Y = A$ and $f = \mathrm{Id}_A$. Then the extension $\tilde{f}\colon X \to A$ is a retraction.

(3) Let $\iota\colon Y \to \mathbb{R}^J$ be the inclusion map. Then $\iota \circ f\colon A \to \mathbb{R}^J$ is continuous and so is each $\pi_j \circ \iota \circ f\colon A \to \mathbb{R}$ for $j \in J$. Since X is normal and A is closed, by Tietze extension theorem, each $\pi_j \circ \iota \circ f$ can be extended to a continuous function $g_j\colon X \to \mathbb{R}$. With these we obtain a continuous function $g\colon X \to \mathbb{R}^J$, and then $\tilde{f} = r \circ g\colon X \to Y$ is the desired extension of f. $\square$

Problem 68 (Different compactifications) Let X, Y, Z be LCH spaces.

(1) Prove that the Stone–Čech compactification βX is the largest compactification of X: for any compact Hausdorff compactification K of X (with an embedding $\varphi\colon X \to K$), there is a surjective continuous closed map $F\colon \beta X \to K$ which extends the embedding $\varphi\colon X \to K$.

(2) Prove that the one-point compactification X^* is the smallest compactification of X.

(3) Given any continuous map $\varphi\colon X \to Y$, we constructed a continuous map $\beta\varphi\colon \beta X \to \beta Y$. Prove that the "lifting" $\varphi \rightsquigarrow \beta\varphi$ is "functorial" in the following sense:

① If Id_X is the identity map, then $\beta\mathrm{Id}_X = \mathrm{Id}_{\beta X}$.

② If $\varphi\colon X \to Y$, $\psi\colon Y \to Z$ are continuous maps, then $\beta(\psi \circ \varphi) = \beta\psi \circ \beta\varphi$.

Proof (1) By the universal property of Stone–Čech compactification, there exists a continuous map $\tilde{\varphi}\colon \beta X \to K$ such that $\tilde{\varphi} \circ \beta = \varphi$. Since $\tilde{\varphi}$ is a continuous map from a compact space to a Hausdorff space, it is closed. By Problem 32 (2), $\tilde{\varphi}(\beta X) = \tilde{\varphi}\left(\overline{\beta(X)}\right) \supset \overline{\tilde{\varphi}(\beta X)} = \overline{\varphi(X)} = K$, so $\tilde{\varphi}$ is surjective.

(2) **Lemma 1** *If A is dense in X, then for every open $U \subset X$ we have $\overline{U} = \overline{U \cap A}$.*

Proof For every $x \in \overline{U}$ and any neighborhood W of x, the intersection $W \cap U$ is open and nonempty. Since A is dense, we have $W \cap U \cap A \neq \varnothing$, and it follows that $x \in \overline{U \cap A}$. Thus the inclusion $\overline{U} \subset \overline{U \cap A}$ holds. The reverse inclusion is obvious.

Lemma 2 *A locally compact dense subspace M of a Hausdorff space X is open in X.*

Proof Every point $x \in M$ has a neighborhood U in the subspace M such that the set $\overline{U}^M = \overline{U} \cap M$ (see Problem 29 (2)) is compact and thus closed in X. Since $U \subset \overline{U} \cap M$, we have $\overline{U} \subset \overline{\overline{U} \cap M} = \overline{U} \cap M \subset M$. Let W be an open subset of X satisfying $U = M \cap W$. Then by Lemma 1

$$x \in W \subset \overline{W} = \overline{M \cap W} = \overline{U} \subset M,$$

which shows that every point $x \in M$ has a neighborhood W in the space X contained in the subspace M, i.e., that M is open in X.

Note that Lemma 2 shows that any Hausdorff compactification $\varphi\colon X \to Y$ of the LCH space X satisfies $\varphi(X)$ is open in Y, and it follows that φ is an open map. Now consider the map

$$F\colon Y \to X^* = X \sqcup \{\infty\}, \quad y \mapsto \begin{cases} x, & \text{if } y = \varphi(x) \text{ for some } x \in X, \\ \infty, & \text{if } y \notin \varphi(X). \end{cases}$$

It is obvious that F is surjective. Let $U \subset X^*$ be open.

- ◇ If $\infty \notin U$, then $F^{-1}(U) = \varphi(U)$ is open in $\varphi(X)$, and then open in Y by Lemma 2.
- ◇ If $\infty \in U$, let $U = (X \setminus K) \cup \{\infty\}$ where K is compact in X. Then $F^{-1}(K) = \varphi(K)$ is compact in Y, so it is closed in Y. Hence $F^{-1}(U) = Y \setminus F^{-1}(K)$ is open in Y.

Therefore F is a continuous map from a compact space to a Hausdorff space, which is closed. This shows that X^* is the smallest compactification of X in the sense of Remark 2.9.22.

(3) ① Since $\mathrm{Id}_{\beta X}$ satisfies $\mathrm{Id}_{\beta X} \circ \beta = \beta \circ \mathrm{Id}_X$, by Proposition 2.9.20, $\beta\mathrm{Id}_X = \mathrm{Id}_{\beta X}$.

② Since $(\beta\psi \circ \beta\varphi) \circ \beta = \beta\psi \circ (\beta\varphi \circ \beta) = \beta\psi \circ (\beta \circ \varphi) = (\beta\psi \circ \beta) \circ \varphi = (\beta \circ \psi) \circ \varphi = \beta \circ (\psi \circ \varphi)$, by Proposition 2.9.20, $\beta(\psi \circ \varphi) = \beta\psi \circ \beta\varphi$. □

PSet 9, Part 2

Problem 69 (Local finiteness)

(1) Prove: if $\mathcal{F} = \{A\}$ is a locally finite family of subsets, so is $\widetilde{\mathcal{F}} = \{\overline{A}\}$.

(2) Prove: if X is countably compact, then any locally finite family of subsets $\mathcal{F}$ (whose elements need not be open) is indeed a finite family.

(3) Prove: countably compact paracompact space is compact.

(4) Prove: X is compact if and only if every open cover of X has a locally finite subcover.

(5) Prove: if every locally finite open covering of X has a finite subcover, then X is pseudocompact.

Proof (1) This has been proved in Problem 31 (4).

(2) **Lemma / Exercise 2.1.3 (3)** *X is countably compact if and only if for every nested sequence $F_1 \supset F_2 \supset \cdots$ of nonempty closed subsets of X, the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty.*

Proof

($\Rightarrow$) If $\bigcap_{n=1}^{\infty} F_n = \varnothing$, then $\bigcup_{n=1}^{\infty} F_n^{\mathrm{c}} = X$, i.e., $\{F_n^{\mathrm{c}}\}_{n=1}^{\infty}$ is a countable open cover of X. Since X is countably compact, there exists a finite subcover $\{F_{n_i}^{\mathrm{c}}\}_{i=1}^{k}$, which implies $\bigcap_{i=1}^{k} F_{n_i} = \varnothing$, a contradiction.

(⇐) Let $\{U_n\}_{n=1}^{\infty}$ be a countable open cover of X. For each n, let $V_n = U_1 \cup \cdots \cup U_n$ and $F_n = V_n^{\mathrm{c}}$. Suppose that no finite subcollection of $\{U_n\}_{n=1}^{\infty}$ covers X. Then each F_n is nonempty, and $F_1 \supset F_2 \supset \cdots$. By the assumption, $\bigcap\limits_{n=1}^{\infty} F_n \neq \varnothing$, which implies $\bigcup\limits_{n=1}^{\infty} V_n \subsetneq X$, a contradiction.

Now suppose X is countably compact and there exists a locally finite family $\{A_n\}_{n=1}^{\infty}$ of nonempty subsets of X. Let $F_n = \bigcup\limits_{k=n}^{\infty} \overline{A_k}$. Then $F_1 \supset F_2 \supset \cdots$ and by (1), $\{\overline{A_n}\}_{n=1}^{\infty}$ is also a locally finite family, so

$$\bigcap_{n=1}^{\infty} F_n = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} \overline{A_k} = \varnothing.$$

This is a contradiction by the lemma.

(3) For any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, since X is paracompact, it admits a locally finite open refinement $\{V_\beta : \beta \in \Lambda'\}$. By (2), $|\Lambda'| < \infty$. Then for each $\beta \in \Lambda'$, there exists $\alpha_\beta \in \Lambda$ such that $V_\beta \subset U_{\alpha_\beta}$. Thus $\{U_{\alpha_\beta} : \beta \in \Lambda'\}$ is a finite subcover of $\{U_\alpha : \alpha \in \Lambda\}$. Hence X is compact.

(4) The "only if" part is trivial. Now suppose every open cover of X has a locally finite subcover. By (2), it suffices to show that X is countably compact. Let $\{U_n\}_{n=1}^{\infty}$ be a countable open cover of X. Suppose it has no finite subcover. For each n, let $V_n = \bigcup\limits_{k=1}^{n} U_k$. Then $\{V_n\}_{n=1}^{\infty}$ has no finite subcover and $V_1 \subset V_2 \subset \cdots$. It follows that $\{V_n\}_{n=1}^{\infty}$ is an open cover of X with no locally finite subcover (for any subcover $V_{n_1} \subset V_{n_2} \subset \cdots$, any point $x \in V_{n_1}$ is contained all V_{n_i} for $i \geqslant 1$), a contradiction. Therefore X is countably compact and hence compact by (2).

(5) Let f be a continuous real-valued function defined on X. Note that $\{f^{-1}(n-1, n+1)\}_{n\in\mathbb{Z}}$ is a locally finite open covering of X, the existence of a finite subcover implies that f is bounded. □

Problem 70 (Products of paracompact spaces)

(1) Prove: the Sorgenfrey line is paracompact, while the Sorgenfrey plane is not.

(2) Is paracompactness productive? Is it preserved under continuous maps?

(3) Prove: if X is compact, Y is paracompact, then $X \times Y$ is paracompact.

Proof (1) ① The Sorgenfrey line is Lindelöf (Problem 58 (5)), (T1) and (T4) (Example 2.7.17), hence (T3). So it is paracompact by Proposition 2.10.7.

② By the proof of Problem 61 (3), $(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}})$ is not normal. Since a paracompact Hausdorff space is normal by Proposition 2.10.11, $(\mathbb{R}^2, \mathscr{T}_{\text{Sorgenfrey}})$ cannot be paracompact.

(2) ① By (1) we know that paracompactness is not productive.

② By Problem 15 (2) the identity map from $\mathbb{R}$ to $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is continuous. The usual real line is paracompact since it is Lindelöf and (T3). However, $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is not paracompact. For example, the open cover $\{(-\infty, n)\}_{n=1}^{\infty}$ has no locally finite refinement, since any nonempty open set in $(\mathbb{R}, \mathscr{T}_{\text{u.s.c.}})$ is of the form $(-\infty, a)$ where $a \in \mathbb{R} \cup \{\infty\}$.

(3) Let $\mathcal{U}$ be an open cover of $X \times Y$. For any $y \in Y$, the slice $X \times \{y\}$ is compact, so there exists a finite subcover $\{U_{y,k}\}_{k=1}^{n_y}$ of $\mathcal{U}$. Let $N_y = U_{y,1} \cup \cdots \cup U_{u,n_y}$. Then N_y is an open set containing $X \times \{y\}$, and by the tube lemma, there exists an open neighborhood W_y of y such that $X \times \{y\} \subset X \times W_y \subset N_y$.

Now $\{W_y : y \in Y\}$ forms an open cover of Y. Since Y is paracompact, there exists a locally finite open refinement $\{W_{y_\alpha} : \alpha \in \Lambda\}$. Let us show that

$$\bigcup_{\alpha\in\Lambda}\bigcup_{k=1}^{n_{x_\alpha}}\{U_{y_\alpha,k}\cap(X\times W_{y_\alpha})\}$$

is a locally finite open refinement of $\mathcal{U}$. Take any point $(x,y)\in X\times Y$. First y is in some W_{y_α}, and $(x,y)\in X\times W_{y_\alpha}\subset N_{y_\alpha}$ is covered by $\{U_{y_\alpha,k}\cap(X\times W_{y_\alpha})\}_{k=1}^{n_{x_\alpha}}$. Since Y is paracompact, there exists an open neighborhood V of y in Y such that V intersects only finitely many W_{y_α}. Then $X\times V$ is an open neighborhood of (x,y) that intersects only finitely many $U_{y_\alpha,k}$. Therefore $X\times Y$ is paracompact. □

Problem 71 (LCH version of P.O.U.) Let X be a σ-compact LCH space, and $\mathcal{U}=\{U_\alpha\}$ be an open cover of X.

(1) There exist two locally finite open coverings $\mathcal{V}=\{V_n\}$ and $\mathcal{W}=\{W_n\}$ such that

- ◇ $W_n\subset\overline{W_n}\subset V_n\subset\overline{V_n}$, and $\overline{V_n}$ is compact,
- ◇ for each n, there exists $U_\alpha\in\mathcal{U}$ such that $\overline{V_n}\subset U_\alpha$.

(2) Prove Theorem 2.10.15 (LCH version of P.O.U.):

Let X be a σ-compact LCH space. Then for any open covering $\mathcal{U}=\{U_\alpha\}$ of X, there exists a partition of unity $\{\rho_n\}$ such that

- ◇ *each* $\operatorname{supp}(\rho_n)$ *is compact,*
- ◇ *for each n, there exists $U_\alpha\in\mathcal{U}$ such that* $\operatorname{supp}(\rho_n)\subset U_\alpha$.

Proof (1) By Problem 52 (4), there exists an open cover $\{G_n\}_{n=1}^\infty$ of X such that $G_n\subset\overline{G_n}\subset G_{n+1}$ and each $\overline{G_n}$ is compact. Let $G_{-1}=G_0=\varnothing$ and fix $n\geqslant 1$. For each $x\in\overline{G_n}\setminus G_{n-1}$, choose $U_x\in\mathcal{U}$ so that $x\in U_x$. Since

- ◇ the compact set $\overline{G_n}\setminus G_{n-1}$ is contained in the open set $G_{n+1}\setminus\overline{G_{n-2}}$,
- ◇ the compact set $\{x_0\}$ is contained in the open set U_x,

by Proposition 2.4.16 we can choose an open neighborhood N_x of x such that

$$\overline{N_x}\subset U_x\cap\left(G_{n+1}\setminus\overline{G_{n-2}}\right)\quad\text{and}\quad\overline{N_x}\text{ is compact.}$$

Since $\{N_x\}_{x\in\overline{G_n}\setminus G_{n-1}}$ is an open cover of the compact set $\overline{G_n}\setminus G_{n-1}$, it admits a finite subcover $\Gamma_n\subset\{N_x\}_{x\in\overline{G_n}\setminus G_{n-1}}$. By construction, for each $W\in\Gamma_n$, there is some $U\in\mathcal{U}$ such that $\overline{W}\subset U\cap\left(G_{n+1}\setminus\overline{G_{n-2}}\right)$. Apply Proposition 2.4.16 again to find for each $W\in\Gamma_n$, an open set V_W such that

$$\overline{W}\subset V_W\subset\overline{V_W}\subset U\cap\left(G_{n+1}\setminus\overline{G_{n-2}}\right)\quad\text{and}\quad\overline{V_W}\text{ is compact.}$$

Now enumerate $\{W_n\}_{n=1}^\infty=\bigcup_{n=1}^\infty\Gamma_n$ and define $V_n=V_{W_n}$. The collections $\mathcal{V}=\{V_n\}$ and $\mathcal{W}=\{W_n\}$ satisfy the desired properties.

(2) Let $\mathcal{V}=\{V_n\}$ and $\mathcal{W}=\{W_n\}$ be as in (1). By the LCH version of Urysohn lemma (Theorem 2.8.9), for each n, there exists $f_n\in\mathcal{C}_c(X,[0,1])$ such that $f_n\left(\overline{W_n}\right)=\{1\}$ and $f_n(V_n^c)=\{0\}$. Since $\mathcal{V}=\{V_n\}$

is locally finite, each $x \in X$ has an open neighborhood on which $f := \sum_{n=1}^{\infty} f_n$ is well-defined and continuous. Note that $f \geqslant 1$ because $\mathcal{W} = \{W_n\}$ covers X, so $\rho_n := \frac{f_n}{f}$ is a well-defined member of $\mathcal{C}_c(X, [0,1])$ for each n and satisfies $\operatorname{supp}(\rho_n) \subset \overline{V_n} \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. ☐

Problem 72 (Examples of non-examples of topological manifolds)

(1) Prove: every topological manifold is σ-compact.

(2) Prove: $\mathbb{RP}^n$ is a topological manifold.

(3) **(Line with doubled point)** Let $X = (\mathbb{R} \times \{0,1\})/\sim$, where $(x,0) \sim (x,1)$ for all $x \neq 0$. Prove: X is (A2) and locally Euclidean, but not (T2).

(4) **(Long line)** Let Ω be the smallest uncountable well-ordered set. That is, Ω is an uncountable set, and there is a well-order $<$ on Ω such that for any $a \in \Omega$, the set $\{b \in \Omega : b < a\}$ is countable. Let $L = \Omega \times (0,1]$. Define an order on L via

$$(a,t) \prec (b,s) \text{ if and only if “} a < b \text{” or “} a = b \text{ and } t < s\text{”}.$$

For any $x \prec y$ in L, consider the interval $(x,y) = \{z \in L : x \prec z \prec y\}$.

① Prove: there “intervals” (x,y) form a basis for a topology on L.

② Prove: with respect to this topology, L is (T2), locally Euclidean, but not (A2). It is called the *long line*.

Proof (1) Let M be a topological manifold and $\mathcal{B}$ be a countable basis of M. For each $x \in M$, there exists an open neighborhood U_x of x homeomorphic to an open subset V_x of $\mathbb{R}^n$, denoted by $\varphi_x \colon U_x \xrightarrow{\sim} V_x$. Find an open ball $\mathbb{B}(\varphi(x), \varepsilon)$ in V_x. Then there exists $B_x \in \mathcal{B}$ such that $B_x \subset \varphi^{-1}\big(\mathbb{B}\big(\varphi(x), \frac{\varepsilon}{2}\big)\big)$. Since $\overline{\varphi(B_x)}$ is compact, it follows that $\overline{B_x}$ is compact. Since $\mathcal{B}$ is countable, the collection $\big\{\overline{B_x} : x \in M\big\}$ is a countable cover of M by compact sets, so M is σ-compact.

(2) For each $0 \leqslant i \leqslant n$, consider $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}$ and the map

$$\varphi_i \colon U_i \to \mathbb{R}^n, \quad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i} : \cdots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right).$$

Then φ_i is a homeomorphism between U_i and $\mathbb{R}^n$. Since $\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i$, it is locally Euclidean. For any two points in $\mathbb{RP}^n$, either they are in different U_i and we are done, or they are in the same U_i and by the homeomorphism φ_i, we can find disjoint open neighborhoods of them. Hence $\mathbb{RP}^n$ is (T2). Since $\mathbb{R}^n$ is (A2) and there are $n+1$ charts in total, $\mathbb{RP}^n$ is (A2).

(3) The collection

$$\bigcup_{q \in \mathbb{Q} \setminus \{0\}} \bigcup_{r \in (0,|q|) \cap \mathbb{Q}} \{[(q-r, q+r) \times \{0\}]\} \cup \bigcup_{r \in \mathbb{Q}_{>0}} \{[(-r,r) \times \{0\}]\} \cup \bigcup_{r \in \mathbb{Q}_{>0}} \{[(-r,r) \times \{1\}]\}$$

is a countable basis of X, so X is (A2). For $x \neq 0$, the open neighborhood $\left[\left(x - \frac{|x|}{2}, x + \frac{|x|}{2}\right) \times \{0\}\right]$ is homeomorphic to $\left(x - \frac{|x|}{2}, x + \frac{|x|}{2}\right)$. For the two origins, both $[(-1,1) \times \{0\}]$ and $[(-1,1) \times \{1\}]$

are homeomorphic to $(-1, 1)$. Hence X is locally Euclidean. By construction, the two origins $[(0,0)]$ and $[(0,1)]$ cannot be separated by disjoint open sets, so X is not (T2).

(4) ① For any $z = (a,t) \in \Omega \times (0,1]$, take $x = \left(a, \frac{t}{2}\right)$ and $y = (b, 1)$ with $a < b$. (Since Ω is the smallest uncountable well-ordered set, such b exists.) Then $z \in (x, y)$, so the intervals (x, y) cover L. For any two intervals (x_1, y_1) and (x_2, y_2), if they intersect, then $(x_1, y_1) \cap (x_2, y_2)$ is still an interval. Therefore the intervals (x, y) form a basis for a topology on L.

② Let us denote the successor of $a \in \Omega$ by a^+. (Note that there is no maximal element in Ω, and then the set $\{b \in \Omega : a < b\}$ is nonempty and has a least element by well-order.) For any two distinct points $x = (a,t)$ and $y = (b,s)$ in L, without loss of generality assume $a < b$ or $a = b$.

- ◇ If $a < b$, the disjoint open sets $\left(\left(a, \frac{t}{2}\right), \left(a^+, \frac{s}{2}\right)\right)$ and $\left(\left(a^+, \frac{s}{2}\right), (b^+, 1)\right)$ separate x and y.
- ◇ If $a = b$, we may assume $t < s$. Then $\left(\left(a, \frac{t}{2}\right), \left(a, \frac{t+s}{2}\right)\right)$ and $\left(\left(a, \frac{t+s}{2}\right), (a^+, 1)\right)$ are disjoint open sets separating x and y.

Hence L is (T2). For any $a \in \Omega$, consider the map φ_a from the open set $\{a\} \times (0,1] \cup \{a^+\} \times (0,1)$ to $(-1,1)$ defined by

$$\varphi_a(a,t) = t - 1 \quad \text{and} \quad \varphi_a(a^+, t) = t.$$

Note that this is a homeomorphism, so L is locally Euclidean. Since $\{\{a\} \times (0,1) : a \in \Omega\}$ is an uncountable collection of disjoint open sets, L admits no countable basis so is not (A2). □

PSet 10, Part 1

Problem 73 (Connectedness) Clarify whether the following spaces are connected, totally disconnected or neither.

(1) $(X, \mathscr{T}_{\text{cofinite}})$.

(2) $(\{0,1\}, \mathscr{T})$, where $\mathscr{T} = \{\varnothing, \{1\}, \{0,1\}\}$.

(3) (X, d), where d is an *ultrametric*, i.e., the triangle inequality in Definition 1.1.1 is strengthened to $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$. The p-adic numbers (Example 1.1.6 (8)) form an ultrametric space.

(4) $\left(\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{\text{uniform}}\right)$ (see Problem 20).

Solution (1) If X is finite, then $(X, \mathscr{T}_{\text{cofinite}})$ is discrete, so X is totally disconnected. If X is infinite, then any two nonempty open sets must intersect, so X is connected.

(2) $(\{0,1\}, \mathscr{T})$ is connected since the only two nonempty open sets intersect.

(3) Any open ball $\mathbb{B}(x,r)$ in (X,d) is clopen. Indeed, for any $y \in \mathbb{B}(x,r)$ and any $z \in \mathbb{B}(y,r)$, we have $d(x,z) \leqslant d(y,z) < r$, so $y \in \mathbb{B}(y,r) \subset \mathbb{B}(x,r)$. Let S be any subset of X that contains more than one point, say x and y, set $r = \frac{1}{2}d(x,y)$. Then S is covered by two disjoint open sets $\mathbb{B}(x,r)$ and $\mathbb{B}(x,r)^{\mathrm{c}}$ both having nonempty intersection with S. Hence S is disconnected. Thus (X,d) is totally disconnected.

(4) $\left(\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{\text{uniform}}\right)$ is neither connected nor totally disconnected. See Problem 79 (1) ③ for a classification of its connected components. □

Problem 74 (Connectedness of subspaces) Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$ a subspace. Which of the following statements are equivalent to the fact that Y is disconnected? Prove the correct ones and give counterexamples for the wrong ones.

(1) There exist nonempty sets $A, B \subset X$ with $A \cap \overline{B} = \overline{A} \cap B = \varnothing$, such that $Y = A \cup B$, where the closures are taken in X.

(2) There exist disjoint open sets A, B in X with $A \cap Y \neq \varnothing$, $B \cap Y \neq \varnothing$, such that $Y \subset A \cup B$.

(3) There exist disjoint closed sets A, B in X with $A \cap Y \neq \varnothing$, $B \cap Y \neq \varnothing$, such that $Y \subset A \cup B$.

(4) There exists a clopen set A in X such that $A \cap Y \neq \varnothing$ and $A \cap Y \neq Y$.

(5) There is a surjective continuous map $f \colon Y \to \{0, 1\}$.

Solution (1) ✔ By Problem 29 (2), $A \cap \overline{B}^Y = A \cap (\overline{B} \cap Y) = (A \cap Y) \cap \overline{B} = A \cap \overline{B}$, and similarly $\overline{A}^Y \cap B = \overline{A} \cap B$. Hence the conditions are equivalent to Y being disconnected.

(2) ✘ Consider $\mathbb{R}$ where the open sets are $\varnothing$ and all subsets that contain 0. Then $\mathbb{R} \setminus \{0\}$ is discrete, so it is disconnected. However, any two nonempty open sets in $\mathbb{R}$ intersect.

(3) ✘ Consider $\mathbb{R}$ where the open sets are $\mathbb{R}$ and all subsets that does not contain 0. Then $\mathbb{R} \setminus \{0\}$ is discrete, so it is disconnected. However, any two nonempty closed sets in $\mathbb{R}$ intersect.

(4) ✘ Consider the example in (2), where $\varnothing$ and $\mathbb{R}$ are the only clopen sets.

(5) ✔ This is Proposition 3.1.2 (5). □

Problem 75 (Connected components) Let X be a topological space. The *connected component* containing $x \in X$ is defined to be the maximal connected subsets of X containing x.

(1) Prove: the connected component containing x is the union of all connected subsets of X that contain x.

(2) Prove: each connected component is a closed subset.

(3) Give an example showing that the connected component need not be open.

(4) Prove: if $f \colon X \to Y$ is continuous, then for any subset A of X, the cardinality of connected components of $f(A)$ is no more than the cardinality of connected components of A.

(5) Denote the connected component of X_α containing x_α to be $C(x_\alpha)$. Prove: the connected component of $\prod\limits_\alpha X_\alpha$ containing the point (x_α) is $\prod\limits_\alpha C(x_\alpha)$.

(6) Let X be a compact Hausdorff space. Prove: for any x, the connected component $C(x)$ is the intersection of all clopen sets that contain x.

Proof (1) By Proposition 3.1.14, the union of all connected subsets of X containing x is connected and contains x. Its maximality is clear.

(2) Let C be a connected component of X. Then $\overline{C}$ is connected by Proposition 3.1.12. The maximality of C implies $C = \overline{C}$, so C is closed.

(3) By Problem 79 (1) ①, each point on the Sorgenfrey line is a connected component, while it is not open.

(4) The image of any connected component of A under f is connected, so the cardinality of connected components of $f(A)$ is no more than that of A.

(5) Denote by C the connected component of $\prod_\alpha X_\alpha$ containing (x_α). Then each $\pi_\alpha(C)$ is connected and contains x_α, so $\pi_\alpha(C) \subset C(x_\alpha)$ and $C \subset \prod_\alpha \pi_\alpha(C) \subset \prod_\alpha C(x_\alpha)$. On the other hand, by Proposition 3.1.18, $\prod_\alpha C(x_\alpha)$ is connected and contains (x_α), so $\prod_\alpha C(x_\alpha) \subset C$. Hence $C = \prod_\alpha C(x_\alpha)$.

(6) Let $A(x)$ be the intersection of all clopen sets that contain x.

$\boxed{C(x) \subset A(x)}$ Any clopen set containing x also contains $C(x)$, for otherwise $C(x)$ would be disconnected by this clopen set and its complement.

$\boxed{A(x) \subset C(x)}$ It suffices to show that $A(x)$ is connected, and we prove this by contradiction. If A is disconnected, then there exist disjoint nonempty closed sets E, F in $A(x)$ such that $A(x) = E \cup F$. Since $A(x)$ is the intersection of closed sets, it is closed. Hence both E and F are closed in X. Since X is compact Hausdorff, X is normal, so there exist disjoint open sets U, V in X such that $E \subset U$ and $F \subset V$. Note that $(U \cup V)^{\mathrm{c}}$ is closed and then compact for X is compact, and it is covered by the open sets

$$\mathcal{F} := \{F : F^{\mathrm{c}} \text{ is a clopen set containing } x\}.$$

Hence there exist finitely many $F_1, \ldots, F_n \in \mathcal{F}$ such that $(U \cup V)^{\mathrm{c}} \subset F_1 \cup \cdots \cup F_n$, or equivalently,

$$K := F_1^{\mathrm{c}} \cap \cdots \cap F_n^{\mathrm{c}} \subset U \cup V.$$

We can assume $x \in U$. Note that K is clopen, and $K \cap U = K \setminus V$ since $K \subset U \cup V$ and $U \cap V = \varnothing$. It follows that $K \cap U$ is clopen and contains x, but it does not contain all of $A(x)$ for U does not contain all of $A(x)$. This contradicts the definition of $A(x)$, so $A(x)$ is connected and then $A(x) \subset C(x)$. □

Problem 76 (Non-products)

(1) Let X, Y be topological spaces, and $A \subsetneq X$, $B \subsetneq Y$. Prove: if X, Y are connected, so is $(X \times Y) \setminus (A \times B)$.

(2) Suppose $\mathbb{R} \simeq X \times Y$. Prove: either X or Y is a single point set.

(3) Prove the same conclusion for $\mathbb{S}^1$.

Proof (1) Suppose $f\colon (X \times Y) \setminus (A \times B) \to \{0, 1\}$ is continuous. Fix some $x_0 \in X \setminus A$ and $y_0 \in Y \setminus B$. For any $(x, y) \in (X \times Y) \setminus (A \times B)$, without loss of generality, assume $x \notin A$. Then $\{x\} \times Y$ is homeomorphic to Y and contained in $(X \times Y) \setminus (A \times B)$, so the restriction $f|_{\{x\} \times Y}$ is constant. Similarly, $f|_{X \times \{y_0\}}$ is constant. Hence

$$f(x, y) = f(x, y_0) = f(x_0, y_0).$$

Thus f is constant, and by Proposition 3.1.2 (5), $(X \times Y) \setminus (A \times B)$ is connected.

(2) Since $\mathbb{R}$ is connected, both X and Y are connected. If both X and Y contain more than one point, then for any $x \in X$ and $y \in Y$ the set $(X \times Y) \setminus \{(x, y)\}$ is connected by (1). However, $\mathbb{R}$ would no longer be connected if we remove some point from it (if it misses r, then $\mathbb{R} \setminus \{r\}$ can be covered by $(-\infty, r)$ and $(r, +\infty)$), a contradiction.

(3) Suppose $\mathbb{S}^1 \simeq X \times Y$. Since $\mathbb{S}^1$ is connected, both X and Y are connected. Suppose $|X| \geqslant 2$ and $|Y| \geqslant 2$. The bijection between $\mathbb{S}^1$ and $X \times Y$ implies that either $|X| \geqslant 3$ or $|Y| \geqslant 3$. Without loss of generality, assume $|X| \geqslant 3$. Then we can pick two distinct points $x_1, x_2 \in X$ and $y \in Y$. The set $(X \times Y) \setminus \{(x_1, y), (x_2, y)\}$ is connected by (1), but $\mathbb{S}^1$ minus two points is disconnected ($\mathbb{S}^1$ minus one point is homeomorphic to $\mathbb{R}$, and $\mathbb{R}$ minus one point is disconnected), a contradiction. $\square$

PSet 10, Part 2

Problem 77 (Path-connectedness: examples)

(1) Although looks quite non-obvious, the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected. We give two proofs here:

First proof Since $\mathbb{Q}^2$ is a countable set, for any $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, there exist uncountably many lines l such that

$$x \in l \subset \mathbb{R}^2 \setminus \mathbb{Q}^2.$$

Now for $x \neq y \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, pick two such lines, one contains x and the other contains y, such that they are not parallel. Now you can connect x to the intersection point through the first line, then to y through the second line.

Second proof Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. If $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$, then we pick $y_0 \in \mathbb{R} \setminus \mathbb{Q}$, and connect (x_1, y_1) to (x_1, y_0) through the line $x = x_1$, and connect (x_1, y_0) to (x_2, y_0) through the line $y = y_0$, and finally connect (x_2, y_0) to (x_2, y_2) through the line $x = x_2$. Similar arguments hold if $x_1, y_2 \in \mathbb{R} \setminus \mathbb{Q}$ or $y_1, y_2 \in \mathbb{R} \setminus \mathbb{Q}$ or $x_2, y_1 \in \mathbb{R} \setminus \mathbb{Q}$.

It turns out that each proof can be extended to prove a more general result on path-connectedness:

Proposition 1 *Let S be ... then $\mathbb{R}^n \setminus S$ is path-connected.*

Proposition 2 *Let X, Y be path-connected, and ...*

Complete the full statements and prove.

(2) Show that the topological space $(X = \{v, s\}, \mathscr{T} = \{\varnothing, \{s\}, \{v, s\}\})$ is path-connected.

Proof (1) ① *Let S be a countable subset of $\mathbb{R}^n$ ($n \geqslant 2$). Then $\mathbb{R}^n \setminus S$ is path-connected.*

Proof Since S is countable, for any $x \in \mathbb{R}^n \setminus S$, there exist uncountably many lines l such that $x \in l \subset \mathbb{R}^n \setminus S$. Now for $x \neq y \in \mathbb{R}^n \setminus S$, pick two such lines, one contains x and the other contains y, such that they intersect at a point z. Then we can connect x to z through the first line, then to y through the second line.

② *Let X, Y be path-connected, then for any $U \subset X$ and $V \subset Y$, the set $(U \times Y) \cup (X \times V)$ is path-connected.*

Proof Suppose $(x_1, y_1), (x_2, y_2) \in (U \times Y) \cup (X \times V)$. If $x_1, x_2 \in U$, then we pick $y_0 \in V$, and connect (x_1, y_1) to (x_1, y_0) through the line $x = x_1$, and connect (x_1, y_0) to (x_2, y_0) through the line $y = y_0$, and finally connect (x_2, y_0) to (x_2, y_2) through the line $x = x_2$. Similar arguments hold if $x_1 \in U, y_2 \in V$ or $y_1, y_2 \in V$ or $x_2 \in U, y_1 \in V$.

(2) It suffices to construct a path from v to s:

$$\gamma\colon [0,1] \to X, \quad t \mapsto \begin{cases} v, & 0 \leqslant t \leqslant \frac{1}{2}, \\ s, & \frac{1}{2} < t \leqslant 1. \end{cases}$$

□

Problem 78 (Local connectedness)

(1) Define the concept of local connectedness:

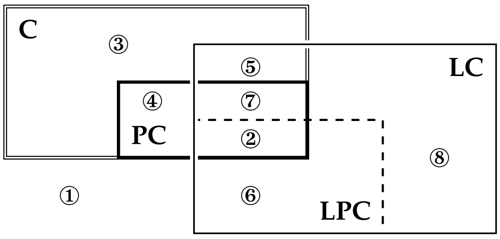
Definition We say a topological space X is *locally connected* if ...

(2) Is $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ connected/locally connected/path-connected/locally path-connected?

(3) For simplicity, let us denote

C = connected, LC = locally connected,
PC = path-connected, LPC = locally path-connected.

Give examples for regions ① to ⑧ in the following picture:



(4) Prove: if X is compact and locally connected, then X has finitely many connected components. Can we remove the local connectedness condition?

(5) Prove: X is locally connected if and only if for any open set U in X, any connected component of U is open. In particular, any connected component of a locally connected space is open.

(6) Suppose X is locally connected, $f\colon X \to Y$ is continuous. Prove: if f is either open or closed, then $f(X)$ is locally connected. Can we remove the assumption "f is either open or closed"?

Proof (1) A topological space X is locally connected if every point admits a neighbourhood basis consisting of open connected sets. In other words, for any $x \in X$, every neighborhood of x contains a connected open neighborhood of x.

(2) ① Since $\mathbb{R}$ is uncountable, there are no disjoint non-empty open subsets in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$, so it is connected.

② For any open neighbourhood U of x and any two open sets $A, B \subset \mathbb{R}$ such that $U \cap A \neq \varnothing$ and $U \cap B \neq \varnothing$, we have $U \cap A, U \cap B \in \mathscr{T}_{\text{cocountable}}$. Hence $A \cap B \cap U = (U \cap A) \cap (U \cap B)$ is the intersection of two nonempty open sets, which is nonempty for the same reason as in ①. Therefore any open neighbourhood of x is connected, so $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is locally connected.

③ For any $f \in \mathcal{C}([0,1], (\mathbb{R}, \mathscr{T}_{\text{cocountable}}))$, $f([0,1])$ is compact. However, as we have shown in the proof of Problem 58 (4), compact sets in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ must be finite. Since the topology on any finite set in $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is discrete, $f([0,1])$ would be totally disconnected if it has more than one point. But now it is the continuous image of $[0,1]$, so $f([0,1])$ must be a singleton. Therefore f must be constant, which means $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is not path-connected.

④ Since $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is connected but not path-connected, it is not locally path-connected by Proposition 3.2.10.

(3) ① $\mathbb{Q}$ is neither **C** nor **LC**.

② $\mathbb{R}$ is both **PC** and **LPC**.

③ The topologist's sine curve is **C** but neither **PC** nor **LC**.

④ The topologist's sine curve with an additional path from $(0,0)$ to $(1,0)$ is **PC** but not **LC**.

⑤ $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is **C** and **LC** but not **PC**, as shown in (2).

⑥ $(0,1) \cup (1,2)$ is **LPC** but not **PC**.

⑦ Let L^* be the one-point compactification of the Long line L defined similarly as in Problem 72 (4) with $(0,1]$ replaced by $[0,1)$. Equivalently, L^* is the space $\Omega \times [0,1) \cup \{(\omega_1, 0)\}$ with the lexicographic order topology, where $\Omega = [0, \omega_1)$ is the minimal uncountable well-ordered set. Denote by C the quotient of L^* identifying its initial point $(0,0)$ and final point $(\omega_1, 0)$. Then C is **PC**, **LC**, but not **LPC**. (It is not locally path-connected at $[(\omega_1, 0)]$.)

⑧ The disjoint union of two copies of ⑤-type space is **LC** but neither **C** nor **LPC**.

(4) Since X is compact and locally connected, it can be covered by finitely many connected open sets. Therefore X has finitely many connected components. The local connectedness condition is necessary. For example, the Cantor set is compact, but it has uncountably many connected components.

(5) $(\Rightarrow)$ Let C be a connected component of U. Since any point in C has a connected open neighbourhood, which must lie in C, C is open.

$(\Leftarrow)$ For any $x \in X$ and any neighbourhood V of x, V contains an open neighbourhood U of x. Then the connected component of U containing x is the desired connected open neighbourhood of x which is contained in V.

(6) We shall use the characterization of local connectedness in (5). By Problem 23 (1), $f\colon X \to f(X)$ is a quotient map. Let V be an open set in $f(X)$ and C be a connected component of V. We want to show that C is open, i.e., $f^{-1}(C)$ is open in X. For any $x \in f^{-1}(C)$, we have $x \in f^{-1}(V)$, which is open in X. Since X is locally connected, there exists a connected open neighbourhood U of x such that $U \subset f^{-1}(V)$. Then $f(U)$ is also connected and $f(U) \cap C \ni f(x)$, so $f(U) \cup C$ is connected and contained in V. Since C is a connected component of V, we must have $f(U) \cup C = C$, i.e., $f(U) \subset C$. Therefore $x \in U \subset f^{-1}(C)$, which implies $f^{-1}(C)$ is open. It follows that C is connected and by (5) we see that $f(X)$ is locally connected.

The assumption "f is either open or closed" is necessary. Let $(X, \mathscr{T}_X)$ be any non-locally connected space. Then $(X, \mathscr{T}_{\text{discrete}})$ is locally connected and the identity map $\mathrm{Id}_X \colon (X, \mathscr{T}_{\text{discrete}}) \to (X, \mathscr{T}_X)$ is continuous. However, the image is not locally connected. □

Problem 79 (Components and path components)

(1) Find the components and path components for the following space:

① The Sorgenfrey line.

② $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$.

③ $(\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{\text{uniform}})$ (see Problem 20).

(2) Prove Proposition 3.2.22 and Proposition 3.2.23 (π_0 and π_c are functors).

Proof (1) ① We have shown in Example 3.1.4 (4) that the Sorgenfrey line is totally disconnected. Therefore each point is a component and a path component.

② By Problem 78 (2), $(\mathbb{R}, \mathscr{T}_{\text{cocountable}})$ is connected. In part ③ of the proof of Problem 78 (2), we have seen that only singletons are path-connected. Therefore each point is a path component.

③ Consider the equivalence relation $\sim$ on X:

$$(x_n) \sim (y_n) \iff \text{the sequence } (x_n - y_n) \text{ is bounded.}$$

We shall show that the (path) component of $(x_n) \in X$ is $[x_n] := \{(y_n) \in X : (y_n) \sim (x_n)\}$, and it suffices to prove that each such $[x_n]$ is clopen and path-connected.

- For any $(y_n) \in [x_n]$, we have $\mathbb{B}((y_n), 1) \subset [x_n]$, so $[x_n]$ is open. The complement of $[x_n]$ is the union of all other equivalence classes, which is open as well. Therefore $[x_n]$ is clopen.
- For any $(y_n), (z_n) \in [x_n]$, there exists $M > 0$ such that $|y_n - z_n| < M$ for all n. Consider

$$\gamma\colon [0,1] \to \mathbb{R}^{\mathbb{N}}, \quad t \mapsto (\gamma_n(t)) := (tz_n + (1-t)y_n).$$

Then $\gamma(0) = (y_n)$ and $\gamma(1) = (z_n)$. We are left to show that γ is continuous. It suffices to show that the preimage of any basic open set $\mathbb{B}((w_n), \varepsilon)$ $(0 < \varepsilon < 1)$ is open. Suppose $t \in \gamma^{-1}(\mathbb{B}((w_n), \varepsilon))$. Then $\delta := \sup\limits_{n\in\mathbb{N}} |\gamma_n(t) - (w_n)| < \varepsilon$. Note that each γ_n is linear, hence

$$|\gamma_n(t') - \gamma_n(t)| = |z_n - y_n| \cdot |t' - t| \leqslant M \cdot \frac{\varepsilon - \delta}{2M} < \varepsilon - \delta$$

for all $t' \in \mathbb{B}\big(t, \frac{\varepsilon-\delta}{2M}\big)$ and $n \in \mathbb{N}$. It follows that

$$d(\gamma(t'), (w_n)) \leqslant d(\gamma(t'), \gamma(t)) + d(\gamma(t), (w_n)) < (\varepsilon - \delta) + \delta = \varepsilon.$$

Therefore $t \in \mathbb{B}\big(t, \frac{\varepsilon-\delta}{2M}\big) \subset \gamma^{-1}(\mathbb{B}((w_n), \varepsilon))$. This shows that γ is continuous, so $[x_n]$ is path-connected.

(2) $\boxed{\pi_c(f) \in \mathcal{C}(\pi_c(X), \pi_c(Y))}$ Let V be open in $\pi_c(Y)$. Then $\pi_c^{-1}(V)$ is open in Y. Since f is continuous, $f^{-1}\big(\pi_c^{-1}(V)\big)$ is open in X. Note that $f^{-1}\big(\pi_c^{-1}(V)\big)$ is the union of all connected components of X whose images under f are contained in $\pi_c^{-1}(V)$, hence $\pi_c\big(f^{-1}\big(\pi_c^{-1}(V)\big)\big) = \pi_c(f)^{-1}(V)$ is open in $\pi_c(X)$. Therefore $\pi_c(f)$ is continuous.

$\boxed{\pi_c(\mathrm{Id}_X) = \mathrm{Id}_{\pi_c(X)}}$ The identity map preserves connected components.

$\boxed{\pi_c(g \circ f) = \pi_c(g) \circ \pi_c(f)}$ Suppose $\pi_c(f)([u]) = [v]$ and $\pi_c(g)([v]) = [w]$. Then there exist connected components $U \subset X$, $V \subset Y$ and $W \subset Z$, such that $u \in U$, $v \in V$, $w \in W$ and $f(U) \subset V$, $g(V) \subset W$. Since $g(f(U)) \subset g(V) \subset W$, we have $\pi_c(g \circ f)([u]) = [w] = \pi_c(g) \circ \pi_c(f)([u])$.

$\boxed{\pi_0(\mathrm{Id}_X) = \mathrm{Id}_{\pi_0(X)}}$ The identity map preserves path components.

$\boxed{\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)}$ Suppose $\pi_c(f)([u]) = [v]$ and $\pi_c(g)([v]) = [w]$. Then there exist path components $U \subset X$, $V \subset Y$, $W \subset Z$, such that $u \in U$, $v \in V$, $w \in W$ and $f(U) \subset V$, $g(V) \subset W$. Since $g(f(U)) \subset g(V) \subset W$, we have $\pi_c(g \circ f)([u]) = [w] = \pi_c(g) \circ \pi_c(f)([u])$. $\square$

Problem 80 (Divisible properties) We say a topological property (P) is a *divisible property* if

$$X \text{ satisfies (P)}, Y \text{ is a quotient of } X \implies Y \text{ satisfies (P)}.$$

(1) Prove: compactness, connectedness, path-connectedness are divisible.

(2) Is (T1), (T2), (T3), (T4) divisible? Is local compactness divisible?

(3) Is (A1), (A2) divisible? Is separability, Lindelöf property divisible?

Proof (1) Quotient maps are continuous, hence they preserve compactness / connectedness / path-connectedness.

(2) Consider the map

$$p\colon X = [0,2] \to Y = \{0,1,2\}, \quad x \mapsto \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x < 2, \\ 2, & \text{if } x = 2. \end{cases}$$

Endow Y with the quotient topology induced by p so that

$$\mathscr{T}_Y = \{\varnothing, \{0,1\}, \{1\}, \{1,2\}, \{0,1,2\}\}.$$

$\boxed{Y \text{ is not (T1)/(T2)}}$ Consider the points 1 and 2. Any open neighbourhood of 2 must contain 1, so Y is not (T1) and then not (T2).

$\boxed{Y \text{ is not (T3)}}$ Y is the only open set containing $\{0,2\}$, so $\{0,2\}$ and 1 cannot be separated by disjoint open sets.

$\boxed{Y \text{ is not (T4)}}$ The closed sets $\{0\}$ and $\{2\}$ cannot be separated by disjoint open sets.

$\boxed{\text{Local compactness is not divisible}}$ Consider the map

$$q\colon X = \mathbb{R} \to Y = (\mathbb{R} \setminus \mathbb{Q}) \sqcup \{Q\}, \quad x \mapsto \begin{cases} Q, & \text{if } x \in \mathbb{Q}, \\ x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Endow Y with the quotient topology induced by q. We shall show that the point $Q \in Y$ has no compact neighbourhood. Let U be any open set containing Q. Then $(U \setminus \{Q\}) \cup \mathbb{Q}$ is open in X. Let K be any compact set containing U. Since $(K \setminus \{Q\}) \cup \mathbb{Q}$ contains the unbounded open set $(U \setminus \{Q\}) \cup \mathbb{Q}$, we can choose an increasing sequence (x_n) in $(K \setminus \{Q\}) \cup \mathbb{Q}$ consisting of irrational numbers that tends to infinity. For each $n \in \mathbb{N}$, let $U_n = \mathbb{R} \setminus \bigcup_{k=n}^{\infty} \{x_k\}$. Then U_n is open and contains $\mathbb{Q}$, so $q(U_n)$ is open in Y. Now $\{q(U_n)\}_{n=1}^{\infty}$ is an open cover of K with no finite subcover for $(K \setminus \{Q\}) \cup \mathbb{Q}$ cannot be covered by finitely many U_n. This contradicts the assumption that K is compact. Therefore Q has no compact neighbourhood and Y is not locally compact.

(3) $\boxed{\text{(A1)/(A2) is not divisible}}$ Let $X = \mathbb{R}$ and consider the equivalence relation $\sim$ defined by

$$x \sim y \iff x = y \text{ or } x, y \in \mathbb{Z}.$$

Denote by q the quotient map $X \to Y$. We shall show that the quotient space $Y = X/\sim$ is admits no countable neighbourhood basis at $q(0)$. Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of open neighbourhoods of $q(0)$. For each n, $q^{-1}(U_n)$ is open in $\mathbb{R}$ and contains $\mathbb{Z}$, so there exists $\{\varepsilon_{n,k}\}_{k\in\mathbb{Z}} \subset (0,1)$ such that

$$U_n \supset q\left(\bigcup_{k\in\mathbb{Z}} \mathbb{B}(k,\varepsilon_{n,k})\right).$$

Let $\delta_k = \frac{1}{2}\varepsilon_{k,k}$ for each $k \in \mathbb{Z}$ and consider

$$V = q\left(\bigcup_{k\in\mathbb{Z}} \mathbb{B}(k,\delta_k)\right).$$

Then V is an open neighborhood of $q(0)$ by the construction of q. Moreover, $U_n \not\subset V$ for all n since $q(n+\delta_n) \in U_n \setminus V$. Therefore any countable collection of open neighbourhoods of $q(0)$ cannot form a neighbourhood basis, so Y is not (A1) and then not (A2).

Separability is divisible It suffices to show that the continuous image of a separable space is separable. Let X be separable and A be a countable dense subset of X. For any continuous map $f\colon X \to Y$, by Proposition 1.6.23,

$$\overline{f(A)} \supset f(\overline{A}) = f(X).$$

Therefore $f(A)$ is a countable dense subset of $f(X)$ and $f(X)$ is separable.

Lindelöf property is divisible By Proposition 2.7.15 (2), the continuous image of a Lindelöf space is Lindelöf. □

PSet 11, Part 1

Problem 81 (Constructing homotopies)

(1) Prove Proposition 3.3.3:

① *If $f_i \in \mathcal{C}(X,Y)$, $g_i \in \mathcal{C}(Y,Z)$ $(i=0,1)$, and $f_0 \sim f_1$, $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

② *If $\varphi \in \mathcal{C}(X_0,X_1)$, $f_i \in \mathcal{C}(X_1,Y)$ $(i=0,1)$, and $f_0 \sim f_1$, then $f_0 \circ \varphi \sim f_1 \circ \varphi$.*

③ *If $\psi \in \mathcal{C}(Y_0,Y_1)$, $f_i \in \mathcal{C}(X,Y_0)$ $(i=0,1)$, and $f_0 \sim f_1$, then $\psi \circ f_0 \sim \psi \circ f_1$.*

(2) Prove that "homotopy equivalence between topological spaces" is an equivalence relation.

Proof (1) ① Take $F \in \mathcal{C}([0,1]\times X, Y)$ and $G \in \mathcal{C}([0,1]\times Y, Z)$ such that

$$F(0,x) = f_0(x), \quad F(1,x) = f_1(x), \quad G(0,y) = g_0(y), \quad G(1,y) = g_1(y).$$

Then $H(t,x) \in \mathcal{C}([0,1]\times X, Z)$ defined by $H(t,x) = G(t,F(t,x))$ satisfies

$$H(0,x) = G(0,F(0,x)) = g_0(f_0(x)), \quad H(1,x) = g_1(f_1(x)).$$

② Since $\varphi \sim \varphi$, this follows directly from ①.

③ Since $\psi \sim \psi$, this follows directly from ①.

(2) (Reflexivity) The identity map gives the homotopy equivalence.

(Symmetry) This follows from the definition of homotopy equivalence.

(Transitivity) Suppose $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,X)$, $h \in \mathcal{C}(Y,Z)$ and $k \in \mathcal{C}(Z,Y)$ satisfy

$$g \circ f \sim \mathrm{Id}_X, \quad f \circ g \sim \mathrm{Id}_Y, \quad k \circ h \sim \mathrm{Id}_Y, \quad h \circ k \sim \mathrm{Id}_Z.$$

Then by ② and ③ of (1),

$$\begin{aligned} g \circ k \circ h \circ f &\sim g \circ \mathrm{Id}_Y \circ f = g \circ f \sim \mathrm{Id}_X, \\ h \circ f \circ g \circ k &\sim h \circ \mathrm{Id}_Y \circ k = h \circ k \sim \mathrm{Id}_Z. \end{aligned}$$

□

Problem 82 (Homotopy v.s. subspace/product)

(1) Prove: if $f_0, f_1 \in \mathcal{C}(X,Y)$ are homotopic, and $A \subset X$, then $f_0|_A, f_1|_A \in \mathcal{C}(A,Y)$ are homotopic.

(2) Let $Y = \prod_\alpha Y_\alpha$. Prove: $f_0, f_1 \in \mathcal{C}(X,Y)$ are homotopic if and only if for each α, the maps $\pi_\alpha \circ f_0, \pi_\alpha \circ f_1 \in \mathcal{C}(X, Y_\alpha)$ are homotopic.

Proof (1) If F is a homotopy between f_0 and f_1, then $F|_{[0,1]\times A}$ is a homotopy between $f_0|_A$ and $f_1|_A$.

(2) By the universal mapping property of the product topology, we have

($\Rightarrow$) If F is a homotopy between f_0 and f_1, then $F_\alpha(t,x) = \pi_\alpha \circ F(t,x)$ is a homotopy between $\pi_\alpha \circ f_0$ and $\pi_\alpha \circ f_1$.

($\Leftarrow$) If F_α is a homotopy between $\pi_\alpha \circ f_0$ and $\pi_\alpha \circ f_1$ for each α, then $F(t,x) = (F_\alpha(t,x))_\alpha$ is a homotopy between f_0 and f_1. □

Problem 83 (Maps to $\mathbb{S}^n$)

(1) Prove: any non-surjective continuous map $f\colon X \to \mathbb{S}^n$ is null-homotopic.

(2) Let $f, g\colon X \to \mathbb{S}^n$ be continuous maps. Suppose they are never anti-podal, i.e., $g(x) \neq -f(x)$ holds for all x. Prove: f is homotopic to g.

(3) Prove: $f \in \mathcal{C}(X,Y)$ is null-homotopic if and only if f has a continuous extension $F \in \mathcal{C}(C(X),Y)$, where $C(X)$ is the cone over X.

(4) Let $\mathbb{D}^{n+1}$ be the closed unit ball in $\mathbb{R}^{n+1}$. Prove: there exists a retraction $f \in \mathcal{C}\big(\mathbb{D}^{n+1}, \mathbb{S}^n\big)$ if and only if $\mathrm{Id}_{\mathbb{S}^n}$ is null-homotopic.

Proof (1) Without loss of generality, assume $N = (0, \cdots, 0, 1) \notin f(X)$ and consider the stereographic projection

$$\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad (x_1, \cdots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \cdots, x_n).$$

It is a homeomorphism with inverse

$$\sigma^{-1}\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}, \quad (u_1, \cdots, u_n) \mapsto \frac{1}{|u|^2 + 1}\big(2u_1, \cdots, 2u_n, |u|^2 - 1\big).$$

Then $F(t,x) = \sigma^{-1}((1-t)\sigma(f(x)))$ is a homotopy from f to the constant map $x \mapsto (0, \cdots, 0, -1)$.

(2) Since $f(x)$ and $g(x)$ are never anti-podal, their convex combination $(1-t)f(x)+tg(x)$ is never zero. Thus the map $H(t,x) = \dfrac{(1-t)f(x)+tg(x)}{\|(1-t)f(x)+tg(x)\|}$ is well-defined and is a homotopy from f to g.

(3) For convenience, let $C(X) = ([0,1]\times X)/(\{1\}\times X)$.

($\Rightarrow$) Let $H\colon [0,1]\times X \to Y$ be a homotopy from f to a constant map c_{y_0}, for some $y_0 \in Y$. Then $H(0,x) = f(x)$ and $H(1,x) = y_0$ for all $x \in X$. Since H is constant on the subspace $\{1\}\times X$, it induces a continuous map $\widetilde{H}\colon C(X) \to Y$ which agrees with f on $\{0\}\times X$. Thus $\widetilde{H}$ is an extension of f to $C(X)$.

($\Leftarrow$) Suppose $F \in \mathcal{C}(C(X), Y)$ is an extension of f. Then $H\colon [0,1]\times X \to Y$ defined by $H(t,x) = F([(t,x)])$ is a homotopy from f to the constant map $x \mapsto F([(1,x)])$.

(4) ($\Rightarrow$) Suppose $f \in \mathcal{C}\left(\mathbb{D}^{n+1}, \mathbb{S}^n\right)$ is a retraction. Then $F\colon [0,1]\times\mathbb{S}^n \to \mathbb{S}^n$ defined by $F(t,x) = f((1-t)x)$ is a homotopy from $\mathrm{Id}_{\mathbb{S}^n}$ to the constant map $x \mapsto f(0) \in \mathbb{S}^n$.

($\Leftarrow$) Suppose $F\colon [0,1]\times\mathbb{S}^n \to \mathbb{S}^n$ is a homotopy from $\mathrm{Id}_{\mathbb{S}^n}$ to a constant map. Fix any point $p_0 \in \mathbb{S}^n$. Then $f \in \mathcal{C}\left(\mathbb{D}^{n+1}, \mathbb{S}^n\right)$ defined by

$$f(x) = \begin{cases} F(1, p_0), & x = 0, \\ F\Big(1 - \|x\|, \frac{x}{\|x\|}\Big), & x \neq 0 \end{cases}$$

is a retraction. It is continuous since $F(1,x)$ is constant for all $x \in \mathbb{S}^n$. $\square$

Problem 84 (Relative homotopy) Let X, Y be topological spaces, and $A \subset X$. Let $f_1, f_2 \in \mathcal{C}(X,Y)$ be continuous maps such that $f_1 = f_2$ on A. We say f_1, f_2 are *homotopic relative* to A, denoted as $f_1 \overset{A}{\sim} f_2$, if there exists a continuous map $F\colon [0,1]\times X \to Y$ such that

$$F(0,x) = f_1(x), \quad F(1,x) = f_2(x), \quad \forall x \in X,$$
$$F(t,x) = f_1(x), \quad \forall x \in A.$$

(1) Prove: relative homotopy is an equivalence relation.

(2) Let X, Y, Z be topological spaces, $A \subset X$, $f_1, f_2 \in \mathcal{C}(X,Y)$ and $g_1, g_2 \in \mathcal{C}(Y,Z)$. Prove: if $f_1 \overset{A}{\sim} f_2$ and $g_1 \overset{f_1(A)}{\sim} g_2$, then $g_1 \circ f_1 \overset{A}{\sim} g_2 \circ f_2$.

(3) Define "pull-back" and "push-forward" for relative homotopy classes, and check the well-definedness.

Proof (1) (Reflexivity) The identity map gives the relative homotopy.

(Symmetry) If $F(t,x)$ is a relative homotopy from f_1 to f_2, then $F(1-t,x)$ is a relative homotopy from f_2 to f_1.

(Transitivity) Suppose $f_1 \overset{A}{\sim} f_2$ and $f_2 \overset{A}{\sim} f_3$. If $G(t,x)$ is a relative homotopy from f_1 to f_2, and $H(t,x)$ is a relative homotopy from f_2 to f_3, then

$$F\colon [0,1]\times X \to Y, \quad (t,x) \mapsto \begin{cases} G(2t,x), & 0 \leqslant t \leqslant \frac{1}{2}, \\ H(2t-1,x), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is a relative homotopy from f_1 to f_3.

(2) Suppose $F(t,x)$ and $G(t,y)$ are the corresponding relative homotopies for $f_1 \overset{A}{\sim} f_2$ and $g_1 \overset{f_1(A)}{\sim} g_2$. Then $G(t, F(t,x))$ is a homotopy from $g_1 \circ f_1$ to $g_2 \circ f_2$ relative to A.

(3) Any map $\varphi \in \mathcal{C}(X_0, X_1)$ induces a pull-back defined by

$$\varphi^* \colon [X_1, Y]_A \to [X_0, Y]_{\varphi^{-1}(A)}, \quad [f] \mapsto [f \circ \varphi].$$

Any map $\psi \in \mathcal{C}(Y_0, Y_1)$ induces a push-forward defined by

$$\psi_* \colon [X, Y_0]_A \to [X, Y_1]_A, \quad [f] \mapsto [\psi \circ f].$$

For the well-definedness, as in the proof of Proposition 3.3.3 / Problem 81 (1), we only need to verify that *if* $f_i \in \mathcal{C}(X,Y)$, $g_i \in \mathcal{C}(Y,Z)$ $(i = 1,2)$, *and for* $A \subset X$, $f_1 \overset{A}{\sim} f_2$, $g_1 \overset{f_1(A)}{\sim} g_2$, *then*

$$g_1 \circ f_1 \overset{A}{\sim} g_2 \circ f_2.$$

This is exactly the statement in (2). □

PSet 11, Part 2

Problem 85 (Simple connectedness)

(1) Let X be path-connected. Prove that the following statements are equivalent:

① X is simply connected, i.e., $\pi_1(X) = \{e\}$.

② Any loop in X can be continuously deformed to a point in X.

③ For any $x_0, x_1 \in X$, any paths $\gamma_1, \gamma_2 \in \Omega(X; x_0, x_1)$ are path-homotopic.

(2) Show that "simple connectedness" is a topological property. Is it multiplicative / preserved under continuous maps / hereditary?

Proof (1) $\boxed{① \Rightarrow ②}$ For any loop in X, fix a point x_0 on it. Since $\pi_0(\Omega(X, x_0)) = \pi_1(X, x_0) = \{e\}$, this loop can be continuously deformed to x_0 in X.

$\boxed{② \Rightarrow ③}$ Since $\gamma_1 * \overline{\gamma_2} \in \Omega(X, x_0)$, it can be continuously deformed to x_0, i.e., $\gamma_1 * \overline{\gamma_2} \underset{p}{\sim} \gamma_{x_0}$. Then

$$\gamma_1 \underset{p}{\sim} \gamma_1 * \overline{\gamma_2} * \gamma_2 \underset{p}{\sim} \gamma_{x_0} * \gamma_2 \underset{p}{\sim} \gamma_2.$$

$\boxed{③ \Rightarrow ①}$ Take $x_0 = x_1$, then $\pi_1(X, x_0) = \Omega(X, x_0)/\underset{p}{\sim} = \{e\}$.

(2) ① Suppose $f \colon X_1 \to X_2$ is a homeomorphism and $H \colon [0,1] \times [0,1] \to X_1$ is a path-homotopy. Then $f \circ H$ is continuous and thus a path-homotopy in X_2. Therefore, any loop in X_2 with base point $f(x_0)$ (where $x_0 \in X_1$) are path-homotopic, and X_2 is simply connected.

② Suppose X_α is simply connected for each α and let $X = \prod_\alpha X_\alpha$. Then for any $\gamma \in \Omega(X, (x_\alpha))$, $\pi_\alpha \circ \gamma_\alpha$ is a loop in X_α for each α. Since X_α is simply connected, $\pi_\alpha \circ \gamma_\alpha \underset{p}{\sim} \gamma_{x_\alpha}$. By Problem 82 (2), $\gamma \underset{p}{\sim} \gamma_{(x_\alpha)}$. Thus X is simply connected. Hence simple connectedness is multiplicative.

③ Simple connectedness may not be preserved under continuous maps. For example, the map $f\colon [0,1] \to \mathbb{S}^1 \subset \mathbb{C}$, $t \mapsto \mathrm{e}^{2\pi i t}$ turns a simply connected space into a non-simply connected one.

④ Simple connectedness is not hereditary. For example, $\mathbb{R}^2$ is simply connected, but the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is not simply connected. □

Problem 86 (Fundamental group of a product space)

(1) Prove: $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(2) Write down a formula for the fundamental group of an arbitrary product, $\pi_1\left(\prod_\alpha X_\alpha, (x_\alpha)\right)$, and prove your formula.

Proof We shall prove

$$\pi_1\left(\prod_\alpha X_\alpha, (x_\alpha)\right) \simeq \bigotimes_\alpha \pi_1(X_\alpha, x_\alpha).$$

Let p_α denote projection on the α-th factor so as not to create confusion with the notation π_1 for the fundamental group. Choosing base points $x_\alpha \in X_\alpha$, we get maps

$$p_{\alpha *}\colon \pi_1\left(\prod_\alpha X_\alpha, (x_\alpha)\right) \to \pi_1(X_\alpha, x_\alpha).$$

Putting these together, we define a map

$$P\colon \pi_1\left(\prod_\alpha X_\alpha, (x_\alpha)\right) \to \bigotimes_\alpha \pi_1(X_\alpha, x_\alpha), \quad [\gamma] \mapsto (p_{\alpha *}[\gamma])_\alpha.$$

This is a well-defined map by Problem 82 (2), and we claim it is an isomorphism.

$\boxed{P \text{ is injective}}$ Suppose γ is a loop in $\prod_\alpha X_\alpha$ such that $P[\gamma]$ is the identity element of $\bigotimes_\alpha \pi_1(X_\alpha, x_\alpha)$. Writing γ in terms of its component functions as $\gamma(t) = (\gamma_\alpha(t))_\alpha$, the assumption means that $[\gamma_{x_\alpha}]_p = p_{\alpha *}[\gamma]_p = [p_\alpha \circ \gamma]_p = [\gamma_\alpha]_p$ for each α. If we choose homotopies $H_\alpha\colon \gamma_\alpha \underset{p}{\sim} \gamma_{x_\alpha}$, it follows that the map

$$H\colon [0,1] \times [0,1] \to \prod_\alpha X_\alpha, \quad (t, x) \mapsto (H_\alpha(t, x))_\alpha$$

is a homotopy from γ to the constant loop $\gamma_{(x_\alpha)}$.

$\boxed{P \text{ is surjective}}$ Let $[\gamma_\alpha]_p \in \pi_1(X_\alpha, x_\alpha)$ be arbitrary for each α. By the universal mapping property of the product space, we can define a loop γ in $\prod_\alpha X_\alpha$ by $\gamma(t) = (\gamma_\alpha(t))_\alpha$. Then

$$P[\gamma]_p = (p_{\alpha *}[\gamma]_p)_\alpha = ([p_\alpha \circ \gamma]_p)_\alpha = ([\gamma_\alpha]_p)_\alpha.$$

$\boxed{P \text{ is a group homomorphism}}$ For any $[\gamma_1]_p, [\gamma_2]_p \in \pi_1\left(\prod_\alpha X_\alpha, (x_\alpha)\right)$, we have

$$P([\gamma_1]_p * [\gamma_2]_p) = (p_{\alpha *}([\gamma_1]_p * [\gamma_2]_p))_\alpha = ([p_\alpha \circ (\gamma_1 * \gamma_2)]_p)_\alpha = ([\gamma_{1,\alpha}]_p * [\gamma_{2,\alpha}]_p)_\alpha = P[\gamma_1]_p P[\gamma_2]_p.$$

□

Problem 87 (Base point change isomorphism) Let X be path-connected, $x_0, x_1 \in X$. We have seen in Proposition 3.4.9 that any path λ from x_0 to x_1 induces a group isomorphism $\Gamma_\lambda\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$.

(1) Suppose λ_1 is a path from x_0 to x_1, and λ_2 is a path from x_1 to x_2. Prove: $\Gamma_{\lambda_1 * \lambda_2} = \Gamma_{\lambda_2} \circ \Gamma_{\lambda_1}$.

(2) Prove: $\pi_1(X, x_0)$ is abelian if and only if for any two paths λ_1, λ_2 from x_0 to x_1, we have $\Gamma_{\lambda_1} = \Gamma_{\lambda_2}$.

(3) Suppose X, Y are path-connected, and $f \in \mathcal{C}(X, Y)$. I have a vague intuition that "the group homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is independent of the choice of x_0". Please write down an explicit formula/rigorous statement and prove it.

Proof (1) $\Gamma_{\lambda_1 * \lambda_2}([\gamma]_p) = \left[\overline{\lambda_1 * \lambda_2} * \gamma * \lambda_1 * \lambda_2\right]_p = \left[\overline{\lambda_2} * \overline{\lambda_1} * \gamma * \lambda_1 * \lambda_2\right]_p = \Gamma_{\lambda_2} \circ \Gamma_{\lambda_1}([\gamma]_p)$.

(2) $(\Rightarrow)$ If there exist two paths λ_1, λ_2 from x_0 to x_1 such that $\Gamma_{\lambda_1}([\gamma]_p) \neq \Gamma_{\lambda_2}([\gamma]_p)$ for some $[\gamma]_p \in \pi_1(X, x_0)$, i.e., $\left[\overline{\lambda_1} * \gamma * \lambda_1\right]_p \neq \left[\overline{\lambda_2} * \gamma * \lambda_2\right]_p$, then $\left[\lambda_2 * \overline{\lambda_1} * \gamma\right]_p \neq \left[\gamma * \lambda_2 * \overline{\lambda_1}\right]_p$. This shows that $\left[\lambda_2 * \overline{\lambda_1}\right]_p, [\gamma]_p \in \pi_1(X, x_0)$ do not commute, a contradiction.

$(\Leftarrow)$ For any two loops γ_1, γ_2 based at x_0, by assumption $\Gamma_{\gamma_1}([\gamma_2]_p) = \Gamma_{\gamma_2}([\gamma_2]_p)$, that is,

$$\left[\overline{\gamma_1} * \gamma_2 * \gamma_1\right]_p = \left[\overline{\gamma_2} * \gamma_2 * \gamma_2\right]_p = [\gamma_2]_p.$$

Then $[\gamma_2]_p * [\gamma_1]_p = [\gamma_1]_p * [\gamma_2]_p$, so $\pi_1(X, x_0)$ is abelian.

(3) Suppose λ is a path from x_0 to x_1. Then the following diagram commutes:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] \arrow[d, "\Gamma_\lambda"] & \pi_1(Y, f(x_0)) \arrow[d, "\Gamma_{f \circ \lambda}"] \\
\pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

Proof Note that $\left[\overline{f \circ \lambda} * (f \circ \gamma) * (f \circ \lambda)\right]_p = \left[f \circ \left(\overline{\lambda} * \gamma * \lambda\right)\right]_p$ for any loop γ in X based at x_0, so

$$\begin{tikzcd}
{[\gamma]_p} \arrow[r, mapsto, "(f_{x_0})_*"] \arrow[d, "\Gamma_\lambda"] & {[f \circ \gamma]_p} \arrow[d, "\Gamma_{f \circ \lambda}"] \\
{\left[\overline{\lambda} * \gamma * \lambda\right]_p} \arrow[r, mapsto, "(f_{x_1})_*"] & {\left[f \circ \left(\overline{\lambda} * \gamma * \lambda\right)\right]_p}
\end{tikzcd}$$

□

Problem 88 (Fundamental group of a topological group) Let $(G, \bullet)$ be a path-connected topological group. We want to prove that $\pi_1(G, e)$ is an abelian group. Let γ_1, γ_2 be two loops in G based at e.

(1) (First proof) Denote by γ_e the constant loop at e. Check:

$$F(s, t) = (\gamma_1 * \gamma_e)\big(\max\{0, t - \tfrac{s}{2}\}\big) \bullet (\gamma_e * \gamma_2)\big(\min\{1, t + \tfrac{s}{2}\}\big)$$

is a path-homotopy between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$, where $\bullet$ is the group multiplication.

(2) (Second proof) Construct explicit path-homotopies to verify

① $\gamma_1(t) \bullet \gamma_2(t) \underset{p}{\sim} \gamma_2(t) \bullet \gamma_1(t)$.

② $(\gamma_1 * \gamma_2)(t) \underset{p}{\sim} \gamma_1(t) \bullet \gamma_2(t)$.

(3) (Third proof, the **Eckmann–Hilton argument**)

① Let S be a set on which there are two "semigroup with unitary" structures, $(S, \circ, 1_\circ)$ and $(S, \bullet, 1_\bullet)$. Moreover, suppose

$$(g \circ h) \bullet (g' \circ h') = (g \bullet g') \circ (h \bullet h'), \quad \forall g, g', h, h' \in S.$$

Prove: $1_\circ = 1_\bullet$, $g \bullet h = g \circ h$, and $g \circ h = h \circ g$.

② Define $[\gamma_1]_p \bullet [\gamma_2]_p = [\gamma_1 \bullet \gamma_2]_p$. Show that $\bullet$ is well-defined on $\pi_1(G, e)$.

③ Use ① to prove that $\pi_1(G)$ is abelian.

Proof (1) Clearly $F(s,t) \in \mathcal{C}([0,1] \times [0,1], G)$. When $s = 0$, we have

$$\begin{aligned} F(0,t) &= (\gamma_1 * \gamma_e)(t) \bullet (\gamma_e * \gamma_2)(t) \\ &= \begin{cases} \gamma_1(2t) \bullet \gamma_e(2t) = \gamma_1(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_e(2t-1) \bullet \gamma_2(2t-1) = \gamma_2(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\ &= \gamma_1 * \gamma_2(t). \end{aligned}$$

When $s = 1$, we have

$$\begin{aligned} F(1,t) &= (\gamma_1 * \gamma_e)\big(\max\{0, t - \tfrac{1}{2}\}\big) \bullet (\gamma_e * \gamma_2)\big(\min\{1, t + \tfrac{1}{2}\}\big) \\ &= \begin{cases} (\gamma_1 * \gamma_e)(0) \bullet (\gamma_e * \gamma_2)\big(t + \frac{1}{2}\big) = \gamma_2(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ (\gamma_1 * \gamma_e)\big(t - \frac{1}{2}\big) \bullet (\gamma_e * \gamma_2)(1) = \gamma_1(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\ &= \gamma_2 * \gamma_1(t). \end{aligned}$$

(2) ① Consider $F(s,t) = (\gamma_1(ts))^{-1} \bullet \gamma_1(t) \bullet \gamma_2(t) \bullet \gamma_1(ts)$. We have

$$F(0,t) = \gamma_1(t) \bullet \gamma_2(t), \quad F(1,t) = \gamma_2(t) \bullet \gamma_1(t).$$

② Consider $F(s,t) = (\gamma_1 * \gamma_e)\big(t(1 - \frac{s}{2})\big) \bullet (\gamma_e * \gamma_2)\big(t(1 - \frac{s}{2}) + \frac{s}{2}\big)$. We have

$$\begin{aligned} F(0,t) &= (\gamma_1 * \gamma_e)(t) \bullet (\gamma_e * \gamma_2)(t) \overset{(1)}{=\!=} \gamma_1 * \gamma_2(t), \\ F(1,t) &= (\gamma_1 * \gamma_e)\big(\tfrac{t}{2}\big) \bullet (\gamma_e * \gamma_2)\big(\tfrac{t+1}{2}\big) = \gamma_1(t) \bullet \gamma_2(t). \end{aligned}$$

(3) ① The units of the two operations coincide:

$$1_\circ = 1_\circ \circ 1_\circ = (1_\bullet \bullet 1_\circ) \circ (1_\circ \bullet 1_\bullet) = (1_\bullet \circ 1_\circ) \bullet (1_\circ \circ 1_\bullet) = 1_\bullet \bullet 1_\bullet = 1_\bullet.$$

For any $g, h \in S$, we have

$$\begin{aligned} \boxed{g \circ h} &= (1 \bullet g) \circ (h \bullet 1) = (1 \circ h) \bullet (g \circ 1) = \boxed{h \bullet g} \\ &= (h \circ 1) \bullet (1 \circ g) = (h \bullet 1) \circ (1 \bullet g) = \boxed{h \circ g}. \end{aligned}$$

② Suppose γ_i $(i = 1, 2, 3, 4)$ are loops in G based at e such that $\gamma_1 \underset{p}{\sim} \gamma_2$ and $\gamma_3 \underset{p}{\sim} \gamma_4$. Let F, G be their respective path-homotopies. Then $F \bullet H$ is a path-homotopy between $\gamma_1 \bullet \gamma_3$ and $\gamma_2 \bullet \gamma_4$.

③ Now $(\pi_1(G, e), *, \gamma_e)$ and $(\pi_1(G, e), \bullet, \gamma_e)$ are two "semigroup with unitary" structures on $\pi_1(G, e)$. Moreover, for any $[\gamma_i]_p \in \pi_1(G, e)$ $(i = 1, 2, 3, 4)$, we have

$$([\gamma_1]_p * [\gamma_2]_p) \bullet ([\gamma_3]_p * [\gamma_4]_p) = ([\gamma_1]_p \bullet [\gamma_3]_p) * ([\gamma_2]_p \bullet [\gamma_4]_p).$$

By ①, $\pi_1(G) \simeq (\pi_1(G, e), *, \gamma_e)$ is abelian. □

PSet 12, Part 1

Problem 89 (Fundamental group of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$) Repeat the proof of $\pi_1(\mathbb{S}^1, x_0) \simeq \mathbb{Z}$ to show that $\pi_1(\mathbb{T}^2, x_0) \simeq \mathbb{Z} \times \mathbb{Z}$, and explicitly write down the generators.

Proof Fix $x_0 = (1, 1) \in \mathbb{T}^2$. For $m, n \in \mathbb{Z}$, consider the loop

$$\gamma_{(m,n)} \colon [0, 1] \to \mathbb{T}^2, \quad t \mapsto \left(\mathrm{e}^{2\pi \mathrm{i} m t}, \mathrm{e}^{2\pi \mathrm{i} n t}\right).$$

Define the map

$$\Phi \colon \mathbb{Z} \times \mathbb{Z} \to \pi_1\left(\mathbb{T}^2, x_0\right), \quad (m, n) \mapsto \left[\gamma_{(m,n)}\right]_p.$$

Note that $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$, where the quotient map is given by

$$p \colon \mathbb{R}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto \left(\mathrm{e}^{2\pi \mathrm{i} x}, \mathrm{e}^{2\pi \mathrm{i} y}\right).$$

The lifted map $\widetilde{\gamma}_{(m,n)}$ is a path $\widetilde{\gamma}_{(m,n)} \colon [0, 1] \to \mathbb{R}^2$ with $\widetilde{\gamma}_{(m,n)}(0) = (0, 0)$ such that the following diagram commutes (i.e., $p \circ \widetilde{\gamma}_{(m,n)} = \gamma_{(m,n)}$):

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[d, "p"] \\
{[0,1]} \arrow[ur, dashed, "\widetilde{\gamma}_{(m,n)}"] \arrow[r, "\gamma_{(m,n)}"] & \mathbb{T}^2
\end{tikzcd}$$

Comparing the expressions of $\gamma_{(m,n)}$ and p, we see that $\widetilde{\gamma}_{(m,n)}(t) = (mt, nt)$.

$\boxed{\Phi \text{ is a group homomorphism}}$ By defintion, $\Phi((m_1, n_1) + (m_2, n_2))$ is represented by the loop

$$\gamma_{(m_1+m_2, n_1+n_2)}(t) = \left(\mathrm{e}^{2\pi \mathrm{i}(m_1+m_2)t}, \mathrm{e}^{2\pi \mathrm{i}(n_1+n_2)t}\right).$$

To relate the lifting of $\gamma_{(m_1+m_2, n_1+n_2)}$ with the liftings of $\gamma_{(m_1,n_1)}$ and $\gamma_{(m_2,n_2)}$, we introduce the translation map

$$T_{(m_1,n_1)} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x + m_1, y + n_1).$$

Then the paths $\widetilde{\gamma}_{(m_1+m_2, n_1+n_2)}$ and $\widetilde{\gamma}_{(m_1,n_1)} * \left(T_{(m_1,n_1)} \circ \widetilde{\gamma}_{(m_2,n_2)}\right)$ are path-homotopic since $\mathbb{R}^2$ is simply connected. It follows that

$$\gamma_{(m_1+m_2, n_1+n_2)} = p \circ \widetilde{\gamma}_{(m_1+m_2, n_1+n_2)} \underset{p}{\sim} p \circ \left(\widetilde{\gamma}_{(m_1,n_1)} * \left(T_{(m_1,n_1)} \circ \widetilde{\gamma}_{(m_2,n_2)}\right)\right) = \gamma_{(m_1,n_1)} * \gamma_{(m_2,n_2)}.$$

In other words,

$$\Phi((m_1,n_1)+(m_2,n_2)) = \left[\gamma_{(m_1+m_2,n_1+n_2)}\right]_p = \left[\gamma_{(m_1,n_1)} * \gamma_{(m_2,n_2)}\right]_p = \left[\gamma_{(m_1,n_1)}\right]_p \cdot \left[\gamma_{(m_2,n_2)}\right]_p$$
$$= \Phi((m_1,n_1)) \cdot \Phi((m_2,n_2)).$$

$\boxed{\Phi \text{ is surjective}}$ Let $\gamma\colon [0,1] \to \mathbb{T}^2$ be any loop with $\gamma(0) = \gamma(1) = (1,1)$. We need the following lemma.

> **(Path lifting)** *For any path $\gamma\colon [0,1] \to \mathbb{T}^2$ with $\gamma(0) = (1,1)$, there exists a unique path $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^2$ with $\widetilde{\gamma}(0) = (0,0)$ such that $p \circ \widetilde{\gamma} = \gamma$.*

Assume the lemma for now. Then the fact that $(1,1) = \gamma(1) = p \circ \widetilde{\gamma}(1)$ implies $\widetilde{\gamma}(1) \in \mathbb{Z}^2$. In other words, there exist $m,n \in \mathbb{Z}$ such that

$$\widetilde{\gamma}(0) = (0,0), \quad \widetilde{\gamma}(1,1) = (m,n).$$

Since $\mathbb{R}^2$ is conctractible, we must have $\widetilde{\gamma} \underset{p}{\sim} \widetilde{\gamma}_{(m,n)}$. Let $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}^2$ be a path-homotopy connecting $\widetilde{\gamma}$ and $\widetilde{\gamma}_{(m,n)}$. Then

$$F = p \circ \widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{T}^2$$

is a path-homotopy connecting γ and $\gamma_{(m,n)}$. So $[\gamma]_p = \left[\gamma_{(m,n)}\right]_p = \Phi((m,n))$.

$\boxed{\Phi \text{ is injective}}$ Suppose $\Phi((m_1,n_1)) = \Phi((m_2,n_2))$, i.e., there exists a path-homotopy $F\colon [0,1] \times [0,1] \to \mathbb{T}^2$ connecting $\gamma_{(m_1,n_1)}$ and $\gamma_{(m_2,n_2)}$. We need the following lemma.

> **(Homotopy lifting)** *For any homotopy $F\colon [0,1] \times [0,1] \to \mathbb{T}^2$ with fixed starting point $F(s,0) \equiv (1,1)$, there exists a unique homotopy $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}^2$ with fixed starting point $\widetilde{F}(s,0) \equiv (0,0)$ such that $p \circ \widetilde{F} = F$. If further F is a path-homotopy, i.e., it has a fixed end point $F(s,1) \equiv (x_0,y_0) \in \mathbb{T}^2$, then $\widetilde{F}$ is also a path-homotopy, i.e., there exists $(x,y) \in p^{-1}((x_0,y_0))$ such that $\widetilde{F}(s,1) \equiv (x,y)$.*

Assume the lemma for now. Then there exists a homotopy $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}^2$ such that $p \circ \widetilde{F} = F$. It follows that

$$p \circ \widetilde{F}(0,t) = \gamma_{(m_1,n_1)}(t), \quad p \circ \widetilde{F}(1,t) = \gamma_{(m_2,n_2)}(t).$$

Since $\widetilde{F}(0,0) = (0,0) = \widetilde{F}(1,0)$, by uniqueness of path lifting above, we must have

$$\widetilde{F}(0,t) = \widetilde{\gamma}_{(m_1,n_1)}(t), \quad \widetilde{F}(1,t) = \widetilde{\gamma}_{(m_2,n_2)}(t).$$

So by the second part of the homotopy lifting lemma, we have

$$(m_1,n_1) = \widetilde{\gamma}_{(m_1,n_1)}(1) = \widetilde{F}(0,1) = \widetilde{F}(1,1) = \widetilde{\gamma}_{(m_2,n_2)}(1) = (m_2,n_2).$$

The proof is complete, and we see that the generators of $\pi_1(\mathbb{T}^2, x_0)$ are $[\gamma_{(1,0)}]_p$ and $[\gamma_{(0,1)}]_p$. $\square$

Problem 90 (Fundamental group of $X = U \cup V$) Suppose U, V are open subsets of X and $X = U \cup V$. Suppose $U \cap V$ is path-connected and $x_0 \in U \cap V$. Let $\iota\colon U \hookrightarrow X$ and $\jmath\colon V \hookrightarrow X$ be inclusion maps. Prove: $\pi_1(X,x_0)$ is generated by $\mathrm{Im}(\iota_*\colon \pi_1(U,x_0) \to \pi_1(X,x_0))$ and $\mathrm{Im}(\jmath_*\colon \pi_1(V,x_0) \to \pi_1(X,x_0))$. (We don't require U or V to be path-connected.)

Proof We begin by showing that there is a subdivision $a_0 < a_1 < \cdots < a_n$ of $[0,1]$ such that $\gamma(a_i) \in U \cap V$ and $\gamma([a_{i-1}, a_i])$ is contained either in U or in V, for each i. First, choose a subdivision $b_0, \cdots, b_m$ of $[0,1]$ such that for each i, the set $\gamma([b_{i-1}, b_i])$ is contained in either U or V. (Use the Lebesgue number lemma, as in the proof of Proposition 3.5.1.) If $\gamma(b_i)$ belongs to $U \cap V$ for each i, we are finished. If not, let i be an index such that $\gamma(b_i) \notin U \cap V$. Each of the sets $\gamma([b_{i-1}, b_i])$, $\gamma([b_i, b_{i+1}])$ lies either in U or in V. If $\gamma(b_i) \in U$, then both of these sets must lie in U; and if $\gamma(b_i) \in V$, both of them must lie in V. In either case, we may delete b_i, obtaining a new subdivision $c_0, \cdots, c_{m-1}$ that still satisfies the condition that $\gamma([c_{i-1}, c_i])$ is contained either in U or V, for each i. A finite number of repetitions of this process leads to the desired subdivision.

Given any loop γ in X based at x_0, let $a_0, \cdots, a_n$ be the subdivision constructed above. Define $\gamma_i = f \circ k_i$, where

$$k_i \colon [0,1] \to [a_{i-1}, a_i], \quad t \mapsto (a_i - a_{i-1})t + a_{i-1}.$$

Then γ_i is a path that lies either in U or in V, and

$$[\gamma]_p = [\gamma_1]_p * [\gamma_2]_p * \cdots * [\gamma_n]_p.$$

For each i, since $U \cap V$ is path-connected, we can choose a path λ_i in $U \cap V$ from x_0 to $\gamma(a_i)$. Since $\gamma(a_0) = \gamma(a_n) = x_0$, we can set $\lambda_0 = \lambda_n = \gamma_{x_0}$.

Now we set

$$\beta_i = (\lambda_{i-1} * \gamma_i) * \overline{\lambda_i}$$

for each i. Then β_i is a loop in X based at x_0 whose image lies either in U or in V. Direct computation shows that

$$[\beta_1]_p * [\beta_2]_p * \cdots * [\beta_n]_p = [\gamma_1]_p * [\gamma_2]_p * \cdots * [\gamma_n]_p.$$

This shows that any loop in X based at x_0 is path-homotopic to a product of the form $\beta_1 * \beta_2 * \cdots * \beta_n$, where each β_i is a loop in X based at x_0 that lies either in U or in V. This leads to the desired conclusion. □

Problem 91 (Induced group homomorphisms) For each of the following maps, compute f_* on the corresponding fundamental groups (with base points 1 or $(1,1)$).

(1) $f \colon \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^n$.

(2) $f \colon \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, $z \mapsto (z^m, z^n)$.

(3) $f \colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$, $(z_1, z_2) \mapsto z_1^m z_2^n$.

Solution (1) $\pi_1\big(\mathbb{S}^1, 1\big) = \mathbb{Z}$ is generated by $[\gamma_1]_p$ where $\gamma_1 \colon [0,1] \to \mathbb{S}^1$, $t \mapsto \mathrm{e}^{2\pi \mathrm{i} t}$, and

$$f_*([\gamma_1]_p) = [f \circ \gamma_1]_p,$$

which is represented by the loop $\gamma_n \colon [0,1] \to \mathbb{S}^1$, $t \mapsto \mathrm{e}^{2\pi \mathrm{i} n t}$. When considering the group $\mathbb{Z}$, this shows f_* is multiplication by n, i.e., $f_* \colon \mathbb{Z} \to \mathbb{Z}$, $k \mapsto nk$.

(2) Similar to (1), we have $f_* \colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $k \mapsto (mk, nk)$.

(3) In Problem 89 we showed that $\pi_1\big(\mathbb{S}^1 \times \mathbb{S}^1, (1,1)\big) = \mathbb{Z} \times \mathbb{Z}$ is generated by $[\gamma_{(1,0)}]_p$ and $[\gamma_{(0,1)}]_p$. The map f_* take them to $[\gamma_m]_p$ and $[\gamma_n]_p$, respectively. Hence $f_* \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $(k,l) \mapsto mk + nl$. □

Problem 92 (Not-so-fundamental group) Let X be a path-connected topological space, and $x_0 \in X$ be a base point. Given any two loops γ_0, γ_1 based at x_0, we define a *pseudo-homotopy* between γ_0 and γ_1 to be a map (not necessarily continuous) $F\colon [0,1] \times [0,1] \to X$ such that

- ◇ For any fixed t, the map $\gamma_t(s) := F(t,s)$ is continuous in s.
- ◇ For any fixed s, the map $\lambda_s(t) := F(t,s)$ is continuous in t.
- ◇ For any s, $F(0,s) = \gamma_0(s)$, $F(1,s) = \gamma_1(s)$.
- ◇ For any t, $F(t,0) = F(t,1) = x_0$.

We define the "NOT-SO-Fundamental group" of X at x_0 to be the pseudo-homotopy classes.

(1) Show that the "NOT-SO-Fundamental group" of $\mathbb{S}^1$ is the trivial group $\{e\}$.

(2) Show that the "NOT-SO-Fundamental group" is not so interesting, since it is always the trivial group $\{e\}$.

(3) In proving $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$, where did we use the continuity of the homotopy?

Proof (1) Fix $x_0 = 1 \in \mathbb{S}^1$. Since $\pi_1(\mathbb{S}^1, x_0) = \mathbb{Z}$ is generated by the loop $\gamma_1\colon [0,1] \to \mathbb{S}^1$, $t \mapsto \mathrm{e}^{2\pi \mathrm{i} t}$, it suffices to show that γ_1 is pseudo-homotopic to the constant loop γ_{x_0}. Consider the map

$$F\colon [0,1]\times[0,1] \to \mathbb{S}^1, \quad (t,s) \mapsto \begin{cases} \gamma_1(s^{1-t}), & (t,s) \neq (1,0), \\ 1, & (t,s) = (1,0). \end{cases} \tag{92–1}$$

Then $F(t,s)$ is continuous in s for any fixed t, continuous in t for any fixed s, and satisfies

$$F(0,s) = \gamma_1(s), \quad F(1,s) = \gamma_1(1) = \gamma_{x_0}(s).$$

Hence γ_1 is pseudo-homotopic to γ_{x_0}.

(2) For any loop γ in X based at x_0, the F defined in (92–1) (replace $\mathbb{S}^1$ with X and γ_1 with γ) is always a pseudo-homotopy between γ and the constant loop γ_{x_0}.

(3) The continuity of F is crucial in the proof of the lifting lemma (Lemma 3.5.8). □

PSet 12, Part 2

Problem 93 (More fundamental groups) Find the fundamental groups of the following spaces:

(1) $\mathbb{R}^{n+k} \setminus (\mathbb{R}^n \times \{(0,\cdots,0)\})$ $(k \geqslant 2)$.

(2) $\mathbb{R}^3 \setminus \mathbb{Z}^3$.

(3) $\mathbb{S}^2 \vee \mathbb{S}^2$.

(4) $\mathbb{S}^1 \vee \mathbb{S}^2$.

(5) $\{(x,y,0) : x,y \in \mathbb{R}\} \cup \{(0,y,z) : y^2 + z^2 = 1, z \geqslant 0\}$.

(6) $\mathbb{R}^3 \setminus (\{(0,0,z) : z \in \mathbb{R}\} \cup \{(x,y,0) : x^2 + y^2 = 1\})$.

(7) $\mathbb{R}^3 \setminus \{(x, y, 0) : x^2 + y^2 = 1\}$.

(8) $\mathbb{R}^3 \setminus (\{(0,0,0)\} \cup \{(1,1,z) : z \in \mathbb{R}\})$.

Solution Let X denote the space in question.

(1) Let $A = (\{0, \cdots, 0\} \times \mathbb{R}^k) \setminus \{0\}$. Consider the map

$$F\colon [0,1] \times X \to X, \quad (x,t) \mapsto ((1-t)x_1, \cdots, (1-t)x_n, x_{n+1}, \cdots, x_{n+k}).$$

We have

$$F(0,x) = x, \quad F(1,x) \in A, \quad \forall x \in X$$

and

$$F(t,a) = a, \quad \forall a \in A, \forall t \in [0,1].$$

That is, A is a strong deformation retract of X. So $X \sim A \sim \mathbb{R}^k \setminus \{0\} \sim \mathbb{S}^{k-1}$. Thus

$$\pi_1(X) \simeq \pi_1(\mathbb{S}^{k-1}) \simeq \begin{cases} \mathbb{Z}, & k = 2, \\ \{e\}, & k \geqslant 3. \end{cases}$$

(2) We first show that $\mathbb{R}^3$ minus finitely many (say, n) points is simply connected. When $n = 1$, this follows from $\mathbb{R}^3 \setminus \{0\} \sim \mathbb{S}^2$. Assume the statement holds for $n-1$. For $Y = \mathbb{R}^3 \setminus \{x_1, \cdots, x_n\}$, without loss of generality, assume the third coordinates of these n points are not all equal, and let m denote the smallest and M the largest. Take $\varepsilon = \frac{M-m}{3} > 0$, and let

$$U = \{(x,y,z) \in Y : z > m + \varepsilon\}, \quad V = \{(x,y,z) \in Y : z < M - \varepsilon\}.$$

Then $U \cap V$ is path-connected, and U, V are both simply connected by induction. Hence $Y = U \cup V$ is simply connected by Proposition 3.5.1.

Now for any loop $\gamma\colon [0,1] \to X$, since $\gamma([0,1])$ is compact in $\mathbb{R}^3$, there exists $R > 0$ such that $\gamma([0,1]) \subset \mathbb{B}(0,R) \setminus \mathbb{Z}^3$. Note that only finitely many points are removed from $\mathbb{B}(0,R)$, so $\mathbb{B}(0,R) \setminus \mathbb{Z}^3$ is homeomorphic to $\mathbb{R}^3$ minus finitely many points, which is simply connected. Thus γ is path-homotopic to a constant loop, and $\pi_1(X) = \{e\}$.

(3) Regard X as $\partial \mathbb{B}((0,0,1),1) \cup \partial \mathbb{B}((0,0,-1),1)$ and let

$$\begin{aligned} U &= \{(x,y,z) \in X : z > \tfrac{1}{3}\}, \\ V &= \{(x,y,z) \in X : -\tfrac{2}{3} < z < \tfrac{2}{3}\}, \\ W &= \{(x,y,z) \in X : z < -\tfrac{1}{3}\}. \end{aligned}$$

Both U and V are homeomorphic to the open disk, hence simply connected. Note that $\{0\}$ is the strong deformation retract of V, so V is also simply connected. Since $U \cap V$ is homeomorphic to a ring, which is path-connected, by Proposition 3.5.1, $U \cup V$ is simply connected. Now $(U \cup V) \cap W$ is again path-connected, by the same proposition, $X = U \cup V \cup W$ is simply connected. Hence $\pi_1(X) = \{e\}$.

(4) Regard X as $\partial\mathbb{B}((0,0,-1),1) \cup \{(0,y,z) \in \mathbb{R}^3 : y^2+(z-1)^2=1\}$ and let

$$U = X \setminus \{(0,0,2)\}, \quad V = X \setminus \{(0,0,-2)\}.$$

Then both U and V are open in X, and $U \cap V$ is path-connected. Since $0 \in U \cap V$, by Problem 90, any loop in X based at 0 is path-homotopic to a product of the form $\gamma_1 * \cdots * \gamma_n$, where each γ_i is a loop in X based at 0 that lies either in U or in V. Note that $\mathbb{S}^2$ is a strong deformation retract of U and $\mathbb{S}^1$ is a strong deformation retract of V, and since $\pi_1(\mathbb{S}^2) \simeq \{e\}$, the loop $\gamma_1 * \cdots * \gamma_n$ is path-homotopic to a loop in $\mathbb{S}^1 = \{(0,y,z) \in \mathbb{R}^3 : y^2+(z-1)^2=1\}$. Hence $\pi_1(X) \simeq \mathbb{Z}$.

(5) Let $A = \{(0,y,z) \in \mathbb{R}^3 : y^2+z^2=1, z \geqslant 0\} \cup \{(0,y,0) \in \mathbb{R}^3 : -1 \leqslant y \leqslant 1\}$. Consider the map

$$F\colon [0,1] \times X \to X, \quad (t,x,y,z) \mapsto ((1-t)x, (1-t)y + \operatorname{sgn}(y)t\min\{|y|,1\}, z).$$

We have

$$F(0,x,y,z) = (x,y,z), \quad F(1,x,y,z) = (0, \operatorname{sgn}(y)\min\{|y|,1\}, z) \in A, \quad \forall (x,y,z) \in X$$

and

$$F(t,0,y,z) = (0,(1-t)y + \operatorname{sgn}(y)t|y|, z) = (0,y,z), \quad \forall (0,y,z) \in A.$$

That is, A is a strong deformation retract of X. So $X \sim A \sim \mathbb{S}^1$. Thus $\pi_1(X) \simeq \mathbb{Z}$.

(6) Consider $\mathbb{R}^3$ minus the z-axis in cylindrical coordinates (ρ,φ,z) where

- ◇ ρ is the Euclidean distance from the z-axis to the point,
- ◇ φ is the angle between the x direction and the line from the origin to the projection of the point on the x-y plane,
- ◇ z is the signed distance from the x-y plane to the point.

Then $X = \mathbb{R}^3 \setminus (\{(1,\varphi,0) : \varphi \in [0,2\pi)\} \cup \{(0,0,z) : z \in \mathbb{R}\})$. Define the map

$$F\colon [0,1] \times X \to X, \quad (\rho,\varphi,z) \mapsto (1-t)(\rho,\varphi,z) + t\left(1 + \frac{\rho-1}{2\sqrt{(\rho-1)^2+z^2}}, \varphi, \frac{z}{2\sqrt{(\rho-1)^2+z^2}}\right).$$

Then we have

$$F(0,\rho,\varphi,z) = (\rho,\varphi,z), \quad F(1,\rho,\varphi,z) \in \mathbb{T}^2, \quad \forall (\rho,\varphi,z) \in X$$

and

$$F(t,\rho,\varphi,z) = (1-t)(\rho,\varphi,z) + t(1+(\rho-1),\varphi,z) = (\rho,\varphi,z), \quad \forall (\rho,\varphi,z) \in \mathbb{T}^2,$$

where the 2-torus is represented by

$$\mathbb{T}^2 = \left\{(\rho,\varphi,z) \in X : (\rho-1)^2 + z^2 = \tfrac{1}{4}\right\}.$$

That is, $\mathbb{T}^2$ is a strong deformation retract of X. Thus $\pi_1(X) \simeq \pi_1(\mathbb{T}^2) \simeq \mathbb{Z} \times \mathbb{Z}$.

(7) Let $A = \{(x,y,0) : x^2+y^2=1\}$.

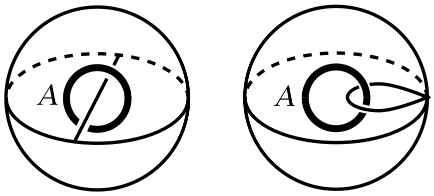


Figure 5: $\mathbb{S}^2 \cup \{\text{diameter}\}$ and $\mathbb{S}^1 \vee \mathbb{S}^2$

As shown in the first picture of Figure 5, $\mathbb{R}^3 \setminus A$ deformation retracts onto the union of $\mathbb{S}^2$ with a diameter, where points outside $\mathbb{S}^2$ deformation retract onto $\mathbb{S}^2$, and points inside $\mathbb{S}^2$ and not in A can be pushed away from A toward $\mathbb{S}^2$ or the diameter. Then we can gradually move the two endpoints of the diameter toward each other along the equator until they coincide, forming $\mathbb{S}^1 \vee \mathbb{S}^2$, as shown in the second picture. Thus by (4), $\pi_1(X) \simeq \pi_1(\mathbb{S}^1 \vee \mathbb{S}^2) \simeq \mathbb{Z}$.

(8) Let

$$U = \left\{(x,y,z) \in \mathbb{R}^3 : x > \tfrac{1}{3}\right\}, \quad V = \left\{(x,y,z) \in \mathbb{R}^3 : x < \tfrac{2}{3}\right\}.$$

Then both U and V are open in X, and $U \cap V$ is path-connected. Let $p = \left\{(\frac{1}{2}, 0, 0)\right\} \in U \cap V$. By Problem 90, any loop in X based at p is path-homotopic to a product of the form $\gamma_1 * \cdots * \gamma_n$, where each γ_i is a loop in X based at p that lies either in U or in V. Note that $\mathbb{S}^1$ is a strong deformation retract of U and $\mathbb{S}^2$ is a strong deformation retract of V, and since $\pi_1(\mathbb{S}^2) \simeq \{e\}$, the loop $\gamma_1 * \cdots * \gamma_n$ is path-homotopic to a loop in U. Hence $\pi_1(X) \simeq \pi_1(U) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$. $\square$

Problem 94 (Maps with trivial induced homomorphism)

(1) Suppose $h\colon \mathbb{S}^1 \to X$ is a continuous map. Prove that the following are equivalent.

① The induced homomorphism $h_*\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(X, h(1))$ is the trivial homomorphism.

② h is null-homotopic.

③ h can be extended to a continuous map $H\colon \mathbb{D}^2 \to X$.

(2) Now suppose $X = \mathbb{S}^1$. Prove: ①-③ are equivalent to

④ h can be lifted to a continuous map $\tilde{h}\colon \mathbb{S}^1 \to \mathbb{R}$ so that $p \circ \tilde{h} = h$.

Proof (1) $\boxed{① \Rightarrow ②}$ Denote $\gamma_1\colon [0,1] \to \mathbb{S}^1$, $t \mapsto \mathrm{e}^{2\pi i t}$. By ①, the loop $h \circ \gamma_1$ is path-homotopic to the constant loop at $h(1)$, so there exists a path-homotopy $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = h \circ \gamma_1(t)$ and $F(1,t) = h(1)$. Then

$$G\colon [0,1] \times \mathbb{S}^1 \to X, \quad \left(s, \mathrm{e}^{2\pi \mathrm{i} t}\right) \mapsto F(s,t)$$

is a homotopy from h to the constant map $h(1)$. It is well-defined since $F(s,0) = F(s,1)$.

$\boxed{② \Rightarrow ③}$ Suppose $F\colon [0,1] \times \mathbb{S}^1 \to X$ is a homotopy from h to a constant map. Then

$$H\colon \mathbb{D}^2 \to X, \quad r\mathrm{e}^{\mathrm{i}\theta} \mapsto F\left(1 - r, \mathrm{e}^{\mathrm{i}\theta}\right)$$

is a continuous extension of h. It is well-defined at $0 \in \mathbb{D}^2$ since $F\left(1, \mathrm{e}^{\mathrm{i}\theta}\right)$ is constant.

$\boxed{③ \Rightarrow ①}$ Denote $\gamma_1 \colon [0,1] \to \mathbb{S}^1$, $t \mapsto \mathrm{e}^{2\pi \mathrm{i} t}$. Then

$$F \colon [0,1] \times [0,1] \to X, \quad (s,t) \mapsto H\big((1-s)\mathrm{e}^{2\pi \mathrm{i} t} + s\big)$$

is a path-homotopy from $h \circ \gamma_1$ to the constant loop at $h(1)$, i.e., $h_*([\gamma_1]_p) = e$. Since $[\gamma_1]_p$ is the generator of $\pi_1(\mathbb{S}^1, 1)$, h_* is the trivial homomorphism.

(2) $\boxed{② \Rightarrow ④}$ Suppose $F \colon [0,1] \times \mathbb{S}^1 \to X = \mathbb{S}^1$ is a homotopy from a constant map to h. Since $F|_{\{0\}\times\mathbb{S}^1}$ is constant, it lifts to a continuous map $\widetilde{F}_0 \colon \{0\} \times \mathbb{S}^1 \to \mathbb{R}$ such that $p \circ \widetilde{F}_0 = F|_{\{0\}\times\mathbb{S}^1}$. By the lifting lemma (Lemma 3.5.8), there is a lifting $\widetilde{F} \colon [0,1] \times \mathbb{S}^1 \to \mathbb{R}$ of F such that $p \circ \widetilde{F} = F$. Then $\tilde{h} \colon \mathbb{S}^1 \to \mathbb{R}$ defined by $\tilde{h}(x) := \widetilde{F}(1,x)$ satisfies $p \circ \tilde{h}(x) = p \circ \widetilde{F}(1,x) = F(1,x) = h(x)$.

$\boxed{④ \Rightarrow ②}$ If $\tilde{h} \colon \mathbb{S}^1 \to \mathbb{R}$ satisfies $p \circ \tilde{h} = h$, then $F(t,x) := p\Big((1-t)\tilde{h}(x)\Big)$ is a homotopy from h to the constant map $p(0) = 1$. □

Problem 95 (Degree of maps between circles) For any continuous map $f \colon \mathbb{S}^1 \to \mathbb{S}^1$, there exists $n \in \mathbb{Z}$ such that $f_*([\gamma_1]_p) = [\gamma_n]_p$. The integer n is called the *degree* of the map f, and is denoted by $\deg(f)$.

(1) Prove: if $f \in \mathcal{C}(\mathbb{S}^1, \mathbb{S}^1)$ is not surjective, then $\deg(f) = 0$.

(2) Prove: if $f, g \in \mathcal{C}(\mathbb{S}^1, \mathbb{S}^1)$, then $\deg(f \circ g) = \deg(f)\deg(g)$.

(3) Prove: f is homotopic to g if and only if $\deg(f) = \deg(g)$.

(4) Read the following paragraph which gives a descriptive definition of the winding number.

Suppose $\gamma \colon \mathbb{S}^1 \to \mathbb{R}^2$ is a continuous map and $p \notin \mathrm{Im}(\gamma)$. The *winding number* $W(\gamma, p)$ of the closed curve γ around the point p is defined to be the integer representing the total number of times that curve travels counterclockwise around the point.

Use the language of mapping degree to give a rigorous definition of winding number $W(\gamma, p)$.

Proof (1) If $f \in \mathcal{C}(\mathbb{S}^1, \mathbb{S}^1)$ is not surjective, then so is $f \circ \gamma_1 \colon [0,1] \to \mathbb{S}^1$. By Problem 83 (1), $f \circ \gamma_1$ is null-homotopic, so $\deg(f) = 0$.

(2) We have

$$\begin{aligned}(f \circ g)_*([\gamma_1]_p) &= f_* \circ g_*([\gamma_1]_p) = f_*\Big(\big[\gamma_{\deg(g)}\big]_p\Big) = f_*(\deg(g)[\gamma_1]_p) = \deg(g) f_*([\gamma_1]_p) \\ &= \deg(g)\big[\gamma_{\deg(f)}\big]_p = \deg(f)\deg(g)[\gamma_1]_p = \big[\gamma_{\deg(f)\deg(g)}\big]_p.\end{aligned}$$

Hence $\deg(f \circ g) = \deg(f)\deg(g)$.

(3) $(\Rightarrow)$ Suppose $F \colon [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ is a homotopy from f to g. Then

$$H \colon [0,1] \times [0,1] \to \mathbb{S}^1, \quad (s,t) \mapsto F(s, \gamma_1(t))$$

is a homotopy from $f \circ \gamma_1$ to $g \circ \gamma_1$. Hence $\deg(f) = \deg(g)$.

$(\Leftarrow)$ If $\deg(f) = \deg(g)$, then $f \circ \gamma_1 \sim g \circ \gamma_1$. By rotation we can assume $f \circ \gamma_1 \underset{p}{\sim} g \circ \gamma_1$. Suppose $F \colon [0,1] \times [0,1] \to \mathbb{S}^1$ is a path-homotopy from $f \circ \gamma_1$ to $g \circ \gamma_1$. Then

$$H \colon [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1, \quad (s, \mathrm{e}^{2\pi \mathrm{i} t}) \mapsto F(s,t)$$

is a homotopy from f to g. It is well-defined since $F(s,0) = F(s,1)$ for all $s \in [0,1]$.

(4) Consider the map

$$r\colon \mathbb{R}^2 \setminus \{p\} \to \partial\mathbb{B}(p,1), \quad x \mapsto p + \frac{x-p}{\|x-p\|}.$$

Then we can define $W(\gamma, p) := \deg(r \circ \gamma)$. $\square$

Problem 96 (Equivalent statements of Borsuk–Ulam) Prove the equivalent statements stated in Remark 3.5.19:

(1) *There exists no antipodal-preserving continuous map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$.*

(2) *For any antipodal-preserving continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$, there exists $x_0 \in \mathbb{S}^n$ such that $f(x_0) = 0$.*

(3) *There exists no continuous map $f\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ such that the restriction to the boundary of f, $f|_{\mathbb{S}^{n-1}}\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, is antipodal-preserving.*

(4) *Let $F_1, \cdots, F_{n+1}$ be a covering of $\mathbb{S}^n$ by closed sets, then there exists $1 \leqslant i \leqslant n+1$ such that $F_i \cap (-F_i) \neq \varnothing$.*

(5) *Let $U_1, \cdots, U_{n+1}$ be a covering of $\mathbb{S}^n$ by open sets, then there exists $1 \leqslant i \leqslant n+1$ such that $U_i \cap (-U_i) \neq \varnothing$.*

Proof We prove by the implications

$$\begin{array}{ccccccc} & & \text{(B–U)} & \Longleftrightarrow & (2) & & \\ & & \Downarrow & & \Updownarrow & & \\ (5) & \Longleftrightarrow & (4) & \Longrightarrow & (1) & \Longleftrightarrow & (3) \end{array}.$$

(B–U) $\Rightarrow$ (2) If $f \in \mathcal{C}(\mathbb{S}^n, \mathbb{R}^n)$ is antipodal-preserving, then by the Borsuk–Ulam theorem, there exists $x_0 \in \mathbb{S}^n$ such that $f(x_0) = f(-x_0) = -f(x_0)$, so $f(x_0) = 0$.

(2) $\Rightarrow$ (B–U) Apply (2) to the antipodal-preserving map given by $g(x) := f(x) - f(-x)$.

(2) $\Rightarrow$ (1) An antipodal-preserving map $\mathbb{S}^n \to \mathbb{S}^{n-1}$ is also a nowhere zero antipodal-preserving map $\mathbb{S}^n \to \mathbb{R}^n$.

(1) $\Rightarrow$ (2) Assume that $f \in \mathcal{C}(\mathbb{S}^n, \mathbb{R}^n)$ is nowhere zero and antipodal-preserving. Then the antipodal-preserving map $g\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ given by $g(x) := \dfrac{f(x)}{\|f(x)\|}$ contradicts (1).

(1) $\Leftrightarrow$ (3) Note that the projection $\pi\colon (x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_n)$ is a homomorphism of the upper closed hemisphere U of $\mathbb{S}^n$ with $\mathbb{D}^n$. An antipodal-preserving map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ as in (1) would yield a map $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ antipodal on $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ by $g(x) = f\big(\pi^{-1}(x)\big)$.

Conversely, for $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ as in (3) we can define $f(x) = g(\pi(x))$ and $f(-x) = -g(\pi(x))$ for $x \in U$. This specifies f on the whole of $\mathbb{S}^n$; it is consistent because g is antipodal on the equator of $\mathbb{S}^n$. The resulting f is continuous by Problem 16 (1).

(B–U) $\Rightarrow$ (4) For a closed cover $F_1, \cdots, F_{n+1}$ of $\mathbb{S}^n$, we define $f \in \mathcal{C}(\mathbb{S}^n, \mathbb{R}^n)$ by

$$f(x) := (\operatorname{dist}(x, F_1), \cdots, \operatorname{dist}(x, F_n)).$$

By the Borsuk–Ulam theorem, there exists $x \in \mathbb{S}^n$ with $f(x) = f(-x) =: y$. If the i-th coordinate of y is 0, then by Problem 7 (2), both x and $-x$ are in F_i. If all coordinates of y are nonzero, then both x and $-x$ lie in F_{n+1}.

$\boxed{(4) \Rightarrow (1)}$ Consider an n-simplex in $\mathbb{R}^n$ containing 0 in its interior, and we project the facets centrally from 0 on $\mathbb{S}^{n-1}$. Then we obtain a covering of $\mathbb{S}^{n-1}$ by closed sets $F_1, \cdots, F_{n+1}$ such that no F_i contains a pair of antipodal points. Then if a continuous antipodal-preserving map $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ exists, the closed cover $f^{-1}(F_1), \cdots, f^{-1}(F_{n+1})$ of $\mathbb{S}^n$ would contradict (4).

$\boxed{(4) \Rightarrow (5)}$ Given any open cover $U_1, \cdots, U_{n+1}$ of $\mathbb{S}^n$, since $\mathbb{S}^n$ is a topological manifold, by Lemma 2.10.12, there exists a closed cover $F_1, \cdots, F_{n+1}$ of $\mathbb{S}^n$ such that $F_i \subset U_i$. Then the existence of i such that $F_i \cap (-F_i) \neq \varnothing$ implies $U_i \cap (-U_i) \neq \varnothing$.

$\boxed{(5) \Rightarrow (4)}$ Suppose $F_1, \cdots, F_{n+1}$ form a closed cover of $\mathbb{S}^n$ with $F_i \cap (-F_i) = \varnothing$ for all i. Since each F_i is compact and every two points of it have distance strictly smaller than 2, there exists $\varepsilon > 0$ such that all the F_i have diameter at most $2 - \varepsilon$. Then the open cover $F_1^{\frac{\varepsilon}{2}}, \cdots, F_{n+1}^{\frac{\varepsilon}{2}}$, where $F_i^{\frac{\varepsilon}{2}} := \left\{x \in \mathbb{S}^n : \mathrm{dist}(x, F_i) < \frac{\varepsilon}{2}\right\}$, contradicts (5). □

PSet 13, Part 1

Problem 97 (Smallest normal subgroup) Let G be a group and $S \subset G$ be a subset.

(1) Show that the smallest normal subgroup of G containing S is

$$N_S = \bigcap_{H \text{ is a normal subgroup of } G \text{ and } S \subset H} H.$$

(2) Prove: N_S is generated by all conjugates of elements of S in G, i.e.,

$$N_S = \left\{c_1 \cdots c_n : n \geqslant 0, c_i = g_i s_i g_i^{-1} \text{ for some } g_i \in G,\ s_i \in S \cup S^{-1}\right\}.$$

Proof (1) It suffices to show that the intersection of normal subgroups is normal. First, N_S is a subgroup of G since it is an intersection of subgroups. For any $g \in G$ and any normal subgroup H of G, $gN_Sg^{-1} \subset gHg^{-1} = H$. Hence $gN_Sg^{-1} \subset N_S$ and $gN_S \subset N_Sg$. Similarly, from $g^{-1}N_Sg \subset N_S$ we deduce $N_Sg \subset gN_S$. Therefore $gN_S = N_Sg$, and N_S is normal.

(2) By definition, RHS $\subset N_S$. Since it is clear that RHS is a subgroup of G and $g(\mathrm{RHS})g^{-1} \subset$ RHS for all $g \in G$, we have RHS $\supset N_S$. Thus the equality holds. □

Problem 98 (Abelianization) Let G be a group.

(1) Let $[G, G]$ be the subgroup of G that is generated by all elements of the form $xyx^{-1}y^{-1}$ for all $x, y \in G$. Prove: $[G, G]$ is a normal subgroup of G.

(2) Prove: the group $\mathrm{Ab}(G) := G/[G, G]$ is abelian (called the *abelianization* of G).

(3) Prove: the abelianization defines a functor from **Grp** to **Ab**.

(4) What is the abelianization of $\mathbb{Z} * \cdots * \mathbb{Z}$?

(5) Prove: $\mathrm{Ab}\big(\big\langle a_1, b_1, \cdots, a_n, b_n \,\big|\, a_1b_1a_1^{-1}b_1^{-1} \cdots a_nb_na_n^{-1}b_n^{-1} = 1 \big\rangle\big) = \mathbb{Z}^{2n}$.

(6) Prove: $\mathrm{Ab}\big(\big\langle a_1, \cdots, a_n \,\big|\, a_1^2 \cdots a_n^2 = 1 \big\rangle\big) = \mathbb{Z}^{n-1} \times \mathbb{Z}_2$.

Proof (1) For any $u \in [G,G]$ and $g \in G$, we have $gug^{-1} = u\big(u^{-1}gug^{-1}\big) \in [G,G]$. Thus $g[G,G]g^{-1} \subset [G,G]$ and $[G,G]$ is normal.

(2) Since $[g][h][g]^{-1}[h]^{-1} = \big[ghg^{-1}h^{-1}\big] = [e]$, we have $[g][h] = [h][g]$ for all $[g],[h] \in \mathrm{Ab}(G)$.

(3) ① By (2), Ab associates each object G in **Grp** to an object $\mathrm{Ab}(G)$ in **Ab**.

② For any group homomorphism $f\colon G \to H$, define $\mathrm{Ab}(f)\colon \mathrm{Ab}(G) \to \mathrm{Ab}(H)$ by $\mathrm{Ab}(f)([g]) = [f(g)]$. If $g_1, g_2 \in G$ satisfy $[g_1] = [g_2]$, then $g_1 g_2^{-1} = \displaystyle\prod_{k=1}^{n} x_k y_k x_k^{-1} y_k^{-1}$ for some $x_i, y_i \in G$ and

$$f(g_1)f(g_2)^{-1} = \prod_{k=1}^{n} f(x_k)f(y_k)f(x_k)^{-1}f(y_k)^{-1} \in [H,H].$$

So $[f(g_1)] = [f(g_2)]$, and $\mathrm{Ab}(f)\colon \mathrm{Ab}(G) \to \mathrm{Ab}(H)$ is a well-defined morphism in **Ab**.

③ $\mathrm{Ab}(\mathrm{Id}_G) = \mathrm{Id}_{\mathrm{Ab}(G)}$ for every $G \in$ **Grp**, and $\mathrm{Ab}(g \circ f) = \mathrm{Ab}(g) \circ \mathrm{Ab}(f)$ for all morphisms $f\colon G \to H$ and $g\colon H \to K$ in **Grp**.

(4) $\mathrm{Ab}(\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}) = \langle a_1, \cdots, a_n \,|\, a_i a_j = a_j a_i,\ 1 \leqslant i < j \leqslant n \rangle = \mathbb{Z}^n$.

(5) $\mathrm{Ab}\big(\big\langle a_1, b_1, \cdots, a_n, b_n \,\big|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \big\rangle\big) = \mathrm{Ab}(\langle a_1, b_1, \cdots, a_n, b_n \rangle) = \mathbb{Z}^{2n}$.

(6) $\mathrm{Ab}\big(\big\langle a_1, \cdots, a_n \,\big|\, a_1^2 \cdots a_n^2 = 1 \big\rangle\big) = \mathbb{Z}^n / \langle (2, \cdots, 2) \rangle$. Since the Smith normal form of the matrix $\begin{pmatrix} 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 0 & \cdots & 0 \end{pmatrix}$ is $\mathrm{diag}(2, \underbrace{0, \cdots, 0}_{n-1})$, we have

$$\mathbb{Z}^n / \langle (2, \cdots, 2) \rangle \simeq \mathbb{Z}^n / (2\mathbb{Z} \times \underbrace{0\mathbb{Z} \times \cdots \times 0\mathbb{Z}}_{n-1}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times \underbrace{(\mathbb{Z}/\{0\}) \times \cdots \times (\mathbb{Z}/\{0\})}_{n-1} \simeq \mathbb{Z}_2 \times \mathbb{Z}^{n-1}.$$

□

Problem 99 (Wedge sum of circles)

(1) **(Finite wedge sum and applications)**

① Prove: $\pi_1\big(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \cdots \mathbb{S}^1\big) \simeq \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$.

② What is the fundamental group of $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$?

③ What is the fundamental group of $\mathbb{R}^2 \setminus \mathbb{Z}^2$?

④ Find the fundamental group of $\mathbb{R}^3 \setminus \{\text{finitely many lines passing through } 0\}$.

⑤ A group is called *finitely presented* if it has a presentation $G = \langle S \,|\, R \rangle$ where both S and R are finite sets. Prove: any finitely presented group is the fundamental group of some compact Hausdorff space.

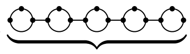
(2) **(Infinite wedge sum)**

① Let $X = \displaystyle\bigcup_{n \geqslant 1} C_n$, where C_n is the circle in $\mathbb{R}^2$ of radius n centered at $(n, 0)$. Compute $\pi_1(X)$.

② Let $Y = \{(x, 0) : x \in \mathbb{R}\} \cup \displaystyle\bigcup_{n \geqslant 1} \widetilde{C}_n$, where $\widetilde{C}_n$ is the circle in $\mathbb{R}^2$ of radius $\frac{1}{3}$ centered at $\big(n, \frac{1}{3}\big)$. Compute $\pi_1(Y)$. Are X and Y homeomorphic?

Proof (1) ① View $\bigvee_{n=1}^{\infty} \mathbb{S}^1$ as a connected graph with $n+1$ vertex and $2n$ edges. By Example 3.6.13,

$$\pi_1\left(\bigvee_{n=1}^{\infty} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-(n+1)+1=n}.$$

② $\mathbb{R}^2 \setminus \{n \text{ points}\}$ is homotopy equivalent to the connected graph  n with $3n$ vertices and $4n-1$ edges. By Example 3.6.13, its fundamental group is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(4n-1)-3n+1=n}$.

③ $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is homotopy equivalent to an "infinite grid" (the union of all lines $x = m$ and $y = n$, $m, n \in \mathbb{Z}$), which admits a maximal subtree consisting of all lines $x = m$ ($m \in \mathbb{Z}$) together with $y = 0$. The entire grid has countably many edges and the maximal subtree misses infinitely many edges. By Example 3.6.13, its fundamental group is $*_{n\in\mathbb{N}}\mathbb{Z}$.

④ $\mathbb{R}^3 \setminus \{n \text{ lines passing through } 0\}$ is homotopy equivalent to $\mathbb{S}^2 \setminus \{2n \text{ points}\}$, which is homeomorphic to $\mathbb{R}^2 \setminus \{(2n-1) \text{ points}\}$. By ②, its fundamental group is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1}$.

⑤ Let $G = \langle S \mid R\rangle$ be a finitely presented group, where $S = \{s_1, \ldots, s_m\}$ and R is a finite set of relations. Begin with the wedge sum of m circles, whose fundamental group is the free group $\langle S\rangle$ by ①. Each circle C_i represents the generator s_i. Write each relation $r \in R$ as a word

$$r = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1.$$

For each such r, attach a disk to the wedge sum by identifying its boundary with the loop that successively traverses the circles $C_{i_1}, \ldots, C_{i_k}$, using the reverse orientation when $\varepsilon_j = -1$. Let X be the resulting space. Since X is obtained from finitely many circles and disks by gluing, it is compact and Hausdorff. Attaching a disk along a loop makes that loop null-homotopic, so by van Kampen's theorem the fundamental group of X is obtained from $\langle S\rangle$ by imposing exactly the relations in R, i.e., $\pi_1(X) \simeq G$.

(2) ① Let U be an open ball centered at $(0,0)$ with radius less than 1. Then $V := X \cap U$ is contractible. Let $A_n = V \cup (C_n \setminus \{0\})$. Then A_n is open in X since it is the union of two open sets, and $A_m \cap A_n = V$ for distinct m, n. Now $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a path-connected open set, and $A_k \cap A_m \cap A_n$ is path-connected for all k, m, n. By van Kampen's theorem, $\pi_1(X) \simeq *_{n\in\mathbb{N}}\mathbb{Z}$.

② Y is a strong deformation retract of $\mathbb{R}^2 \setminus \left\{\left(n, \frac{1}{3}\right) : n \geqslant 1\right\}$, and the latter is homotopy equivalent to the graph $\{(x,0) : x \geqslant 0\} \cup \{(x,1) : x \geqslant 0\} \cup \{(m,y) : m \in \mathbb{N} \cup \{0\}, 0 \leqslant y \leqslant 1\}$, which admits a maximal subtree consisting of all the two horizontal rays together with $\{0\} \times [0,1]$. The entire graph has countably many edges and the maximal subtree misses infinitely many edges. By Example 3.6.13, its fundamental group is $*_{n\in\mathbb{N}}\mathbb{Z}$. However, $X \setminus \{(0,0)\}$ has infinitely many path components, while Y minus a point has at most 3 path components. Therefore X and Y are not homeomorphic. $\square$

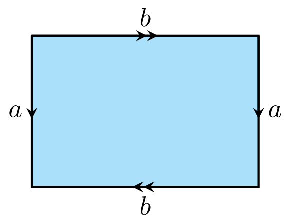
Problem 100 (Application of van Kampen) Use van Kampen's theorem to compute the fundamental groups of the following spaces.

(1) $\mathbb{RP}^2$.

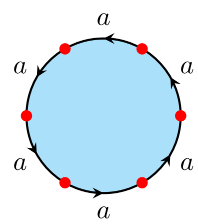
(2) The Klein bottle.

(3) $\Sigma_g = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$.

(4) The n-fold dunce cap. [Split the boundary circle of a closed disk into n parts (by n red dots), and identify the boundary segments according to the picture below (but keep the interior of the disk unchanged).]

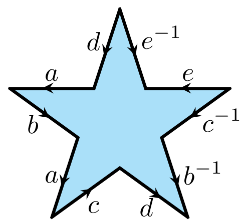


The Klein bottle



The n-fold dunce cap

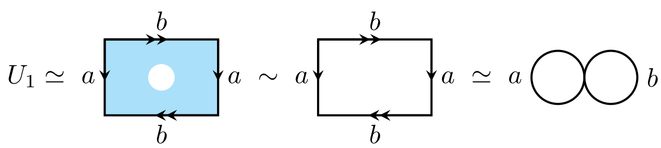
(5) The surface X obtained by gluing the sides of a star as shown below (a "letter edge" is glued counterclockwise, and an "inverse letter edge" is glued clockwise).



(6) $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$.

Solution (1) If we form $\mathbb{RP}^2$ by identifying antipodal points of $\mathbb{S}^2$, and obtain a hemisphere with antipodal points on the equator identified, then it reduces to the case in (4) where $n = 2$. So $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$.

(2) We first write the Klein bottle K as the union of two open sets $U_1 = K \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disc and $\widetilde{D}$ is a small disc containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1(\mathbb{S}^1 \wedge \mathbb{S}^1) \simeq \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_*\colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

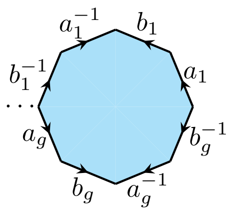
The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $baba^{-1}$. In other words,

$$\iota_*(1) = baba^{-1}.$$

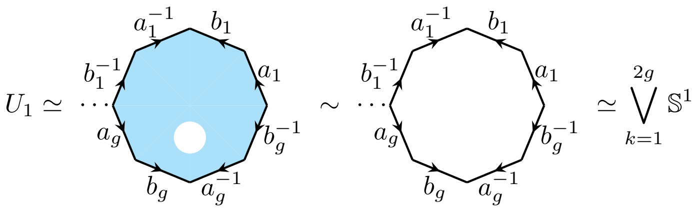
Hence by van Kampen's theorem,

$$\pi_1(K) \simeq (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \big\langle a, b \,\big|\, baba^{-1} = 1 \big\rangle.$$

(3) Consider the polygonal presentation of Σ_g:



We first write Σ_g as the union of two open sets $U_1 = \Sigma_g \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disc and $\widetilde{D}$ is a small disc containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\bigvee_{k=1}^{2g} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g} = \langle a_1, b_1, \cdots, a_g, b_g \rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_*\colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. In other words,

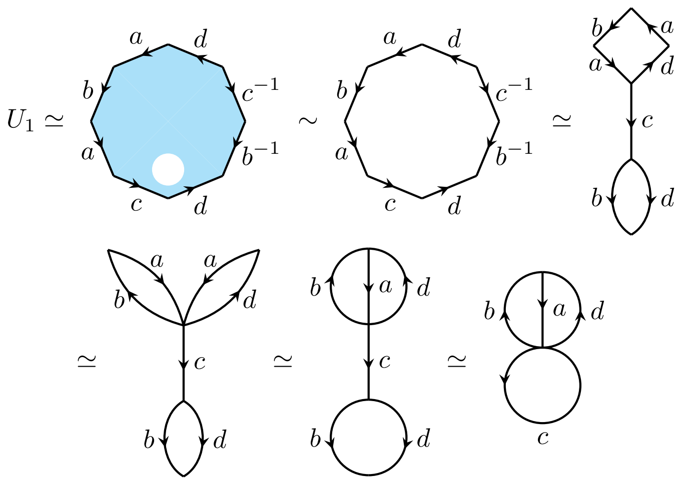
$$\iota_*(1) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Hence by van Kampen's theorem,

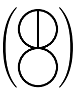
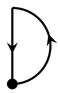
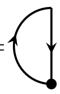
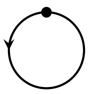
$$\pi_1(\Sigma_g) \simeq (\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g}) *_{\mathbb{Z}} \{e\} = \big\langle a_1, b_1, \cdots, a_g, b_g \,\big|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \big\rangle.$$

(4) This is exactly the way we used to construct a presentation complex in Problem 99 (1) ⑤, so the fundamental group of the n-fold dunce cap is $\langle a \mid a^n = 1 \rangle \simeq \mathbb{Z}_n$.

(5) We first write X as the union of two open sets $U_1 = X \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disc and $\widetilde{D}$ is a small disc containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\right.$$  $$\left.\right) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \left\langle \alpha := \right.$$  $$, \beta :=$$  $$, \gamma :=$$  $$\left.\right\rangle$$

by Example 3.6.13, for the last graph has 2 vertices and 4 edges.

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_* \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

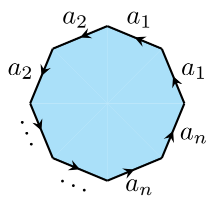
The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $dabacdb^{-1}c^{-1}$, which is represented by the loop $\alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1}$ in the last graph. In other words,

$$\iota_*(1) = \alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1}.$$

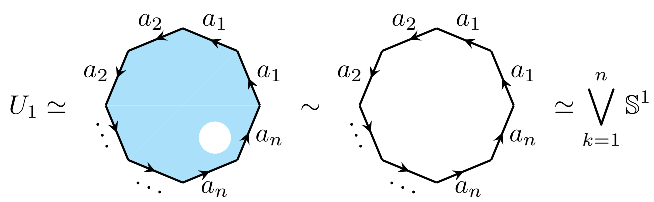
Hence by van Kampen's theorem,

$$\pi_1(X) \simeq (\mathbb{Z} * \mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} \{e\} = \left\langle \alpha, \beta, \gamma \,\middle|\, \alpha\beta\gamma\alpha\beta^{-1}\gamma^{-1} = 1 \right\rangle.$$

(6) Consider the polygonal presentation of $\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$:



We first write $\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}$ as the union of two open sets $U_1 = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n} \setminus \overline{D}$ and $U_2 = \widetilde{D}$, where D is a small disc and $\widetilde{D}$ is a small disc containing $\overline{D}$. Since



we have

$$\pi_1(U_1) \simeq \pi_1\left(\bigvee_{k=1}^{n} \mathbb{S}^1\right) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n} = \langle a_1, \cdots, a_n \rangle.$$

Since U_2 is contractible, and $U_1 \cap U_2$ is an annulus, which is homotopy equivalent to $\mathbb{S}^1$, we have

$$\pi_1(U_2) \simeq \{e\} \quad \text{and} \quad \pi_1(U_1 \cap U_2) \simeq \pi_1\big(\mathbb{S}^1\big) \simeq \mathbb{Z}.$$

Consider the inclusion-induced group homomorphism

$$\iota_* \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1).$$

The generator of $\pi_1(U_1 \cap U_2)$, that is, the circle, can be deformed inside U_1 to the boundary loop $a_1^2 a_2^2 \cdots a_n^2$. In other words,

$$\iota_*(1) = a_1^2 a_2^2 \cdots a_n^2.$$

Hence by van Kampen's theorem,

$$\pi_1\big(\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n}\big) \simeq (\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}) *_{\mathbb{Z}} \{e\} = \big\langle a_1, \cdots, a_n \,\big|\, a_1^2 a_2^2 \cdots a_n^2 = 1 \big\rangle.$$

□

PSet 13, Part 2

Problem 101 (Products of coverings)

(1) Prove: if X is connected, $\widetilde{X} \neq \varnothing$, then p is surjective, and the cardinality of $p^{-1}(x)$ is independent of x.

(2) Prove: if $p \colon \widetilde{X} \to X$ and $p' \colon \widetilde{X'} \to X'$ are covering maps, so is their product $p \times p' \colon \widetilde{X} \times \widetilde{X'} \to X \times X'$.

(3) Construct a covering map $p \colon \mathbb{H} = \{x + y\mathrm{i} : y > 0\} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by identifying $\mathbb{C}^*$ with $\mathbb{S}^1 \times \mathbb{R}_+$ (via polar coordinates).

(4) Let $p \colon \mathbb{R} \to \mathbb{S}^1$ be the standard covering map. Prove: the infinite product

$$P = \prod_{n \in \mathbb{N}} p \colon \prod_{n \in \mathbb{N}} \mathbb{R} \to \prod_{n \in \mathbb{N}} \mathbb{S}^1$$

is not a covering map.

Proof (1) Any $x \in X$ has an open neighborhood U_x homeomorphic to an open neighborhood V_α in $\widetilde{X}$ with $p(V_\alpha) = U_x$. Hence p is surjective. Note that for any $y \in U_x$, we have $\left|p^{-1}(y)\right| = \left|p^{-1}(x)\right|$, since $p^{-1}(y) \cap V_\alpha$ contains exactly one point. Now fix $x_0 \in X$ and let $A = \{x \in X : |p^{-1}(x)| = |p^{-1}(x_0)|\}$. The above remark shows that both A and A^c are open. Since X is connected and $A \neq \varnothing$, we must have $A = X$, i.e., the cardinality of $p^{-1}(x)$ is constant.

(2) For any $(x, x') \in X \times X'$, let U be an open neighborhood of x in X such that $p^{-1}(U)$ is a disjoint union of open sets V_α in $\widetilde{X}$ and $p|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism for each α. Similarly, let U' be an open neighborhood of x' in X' such that $p'^{-1}(U')$ is a disjoint union of open sets V'_β in $\widetilde{X'}$ and $p'|_{V'_\beta} \colon V'_\beta \to U'$ is a homeomorphism for each β. Then $U \times U'$ is an open neighborhood of (x, x') in $X \times X'$, and $(p \times p')^{-1}(U \times U') = p^{-1}(U) \times p'^{-1}(U')$ is a disjoint union of open sets $V_\alpha \times V'_\beta$ in $\widetilde{X} \times \widetilde{X'}$. Also, $(p \times p')|_{V_\alpha \times V'_\beta} = (p|_{V_\alpha}) \times (p'|_{V'_\beta})$ is a homeomorphism for all α, β.

(3) Consider

$$p \colon \mathbb{H} \to \mathbb{C}^*, \quad x + y\mathrm{i} \mapsto y\mathrm{e}^{\mathrm{i}x}.$$

For any $z = r_0 \mathrm{e}^{\mathrm{i}\theta_0} \in \mathbb{C}^*$, consider the open neighborhood

$$U_z = \left\{ r\mathrm{e}^{\mathrm{i}\theta} : \tfrac{r_0}{2} < r < 2r_0,\ \theta_0 - 1 < \theta < \theta_0 + 1 \right\}.$$

Then $p^{-1}(U_z)$ is a disjoint union of open sets V_k ($k \in \mathbb{Z}$) in $\mathbb{H}$ where

$$V_k = (\theta_0 - 1 + 2k\pi, \theta_0 + 1 + 2k\pi) \times \left(\tfrac{r_0}{2}, 2r_0\right),$$

and $p|_{V_k} \colon V_k \to U_z$ is a homeomorphism for all $k \in \mathbb{Z}$.

(4) If it were a covering map, then some open neighborhood U of $(0, 0, \cdots)$ in $\prod_{n \in \mathbb{N}} \mathbb{R}$ would be mapped homeomorphically to some neighborhood V of $(1, 1, \cdots)$ in $\prod_{n \in \mathbb{N}} \mathbb{S}^1$. By shrinking we can assume $U = (-a_1, a_1) \times \cdots \times (-a_k, a_k) \times \mathbb{R} \times \mathbb{R} \times \cdots$, so that U is contractible, and $P(U)$ has infinitely many $\mathbb{S}^1$ factors. By Problem 86 (2), $\pi_1(U) \simeq \pi_1(P(U))$ is nontrivial, a contradiction. $\square$

Problem 102 (Covering over subspace)

(1) Let $p \colon \widetilde{X} \to X$ be a covering map, and $A \subset X$ a subset. Denote $\widetilde{A} = p^{-1}(A)$. Show that $p_A = p|_{\widetilde{A}} \colon \widetilde{A} \to A$ is a covering map.

(2) Recall that $\mathbb{S}^1 \vee \mathbb{S}^1$ (the figure 8) can be realized as the subspace $(\mathbb{S}^1 \times \{a_0\}) \cup (\{a_0\} \times \mathbb{S}^1)$ in $\mathbb{T}^2$. What is the restricted covering of "the standard covering $p \colon \mathbb{R}^2 \to \mathbb{T}^2$" to $\mathbb{S}^1 \vee \mathbb{S}^1$?

Proof (1) By Problem 13 (4) and (5), the map $p_A = p|_{\widetilde{A}} \colon \widetilde{A} \to A$ is continuous. For any $a \in A$, let U be an open neighborhood of a in X such that $p^{-1}(U)$ is a disjoint union of open sets V_α in $\widetilde{X}$ and $p|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism for each α. Then $U \cap A$ is an open neighborhood of a in A, and $p_A^{-1}(U \cap A) = p^{-1}(U) \cap \widetilde{A}$ is a disjoint union of open sets $V_\alpha \cap \widetilde{A}$ in $\widetilde{A}$, and $p_A|_{V_\alpha \cap \widetilde{A}} = p_{V_\alpha \cap \widetilde{A}}$ is a homeomorphism from $V_\alpha \cap \widetilde{A}$ to $U \cap A$ for each α.

(2) For the standard covering $p \colon \mathbb{R}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (\mathrm{e}^{2\pi\mathrm{i}x}, \mathrm{e}^{2\pi\mathrm{i}y})$, the preimage of $\mathbb{S}^1 \vee \mathbb{S}^1$ is given by

$$\begin{aligned} \widetilde{A} &= p^{-1}\big((\mathbb{S}^1 \times \{a_0\}) \cup (\{a_0\} \times \mathbb{S}^1)\big) = p^{-1}(\mathbb{S}^1 \times \{a_0\}) \cup p^{-1}(\{a_0\} \times \mathbb{S}^1) \\ &= \left(\mathbb{R} \times \left\{\tfrac{1}{2\pi} \arg a_0 + k : k \in \mathbb{Z}\right\}\right) \cup \left(\left\{\tfrac{1}{2\pi} \arg a_0 + k : k \in \mathbb{Z}\right\} \times \mathbb{R}\right). \end{aligned}$$

So $p|_{\widetilde{A}} \colon \widetilde{A} \to \mathbb{S}^1 \wedge \mathbb{S}^1 \subset \mathbb{T}^2$ is the restricted covering of p. $\square$

Problem 103 (Fundamental groups of covering spaces) Suppose $X, \widetilde{X}$ are path-connected, $p\colon \widetilde{X} \to X$ is a covering map, and $p(\tilde{x}_0) = x_0$.

(1) Prove: the index of the subgroup $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$ in $\pi_1(X, x_0)$ is the cardinality of $p^{-1}(x_0)$.

(2) Prove: if the base space X is simply connected, then p is a homeomorphism.

(3) Suppose $\tilde{x}_1 \in p^{-1}(x_0)$. Prove: as subgroups of $\pi_1(X, x_0)$, the two groups $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$ and $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_1\Big)\Big)$ are conjugate to each other.

Proof (1) Let $p^{-1}(x_0) = \{x_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, choose a path $\tilde{\gamma}_\alpha$ in $\widetilde{X}$ from $\tilde{x}_0$ to x_α. We shall show that $\{[p(\tilde{\gamma}_\alpha)]_p : \alpha \in \Lambda\}$ is a set of representatives for the right coset of $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$ in $\pi_1(X, x_0)$.

- ◇ If $[p(\tilde{\gamma}_\alpha)]_p$ and $[p(\tilde{\gamma}_\beta)]_p$ are in the same right coset, then there exists $\gamma_0 \in \Omega(X, x_0)$ such that $\gamma_0 * p(\tilde{\gamma}_\alpha)$ is path-homotopic to $p(\tilde{\gamma}_\beta)$. By homotopy lifting property (Corollary 3.7.10), $p^{-1}(\gamma_0 * p(\tilde{\gamma}_\alpha))$ and $p^{-1}(p(\tilde{\gamma}_\beta))$ are path-homotopic in $\widetilde{X}$, which requires $\alpha = \beta$.
- ◇ For any $\gamma \in \Omega(X, x_0)$, suppose its lifting $\tilde{\gamma}$ ends at x_α. Then $\tilde{\gamma} * \overline{\tilde{\gamma}_\alpha} \in \Omega\Big(\widetilde{X}, \tilde{x}_0\Big)$. Hence $[\tilde{\gamma}]_p * [\tilde{\gamma}_\alpha]_p^{-1} \in \pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)$ and then $[\gamma]_p * [p(\tilde{\gamma}_\alpha)]_p^{-1} \in p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$, i.e., $[\gamma]_p$ is in the right coset with representative $[p(\tilde{\gamma}_\alpha)]_p$.

Therefore, the index of $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$ in $\pi_1(X, x_0)$ is $|\Lambda|$, the cardinality of $p^{-1}(x_0)$.

(2) Since $\pi_1(X, x_0) = \{e\}$, by (1), $\left|p^{-1}(x_0)\right| = 1$. And since $x_0 \in X$ is arbitrary, p is bijective. By the definition of a covering map, p is a local homeomorphism, in particular, p^{-1} is continuous at each point. Hence p is a homeomorphism.

(3) Let $\tilde{\gamma}_1$ be a path in $\widetilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. The same argument as in (1) shows that $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big) * [p(\tilde{\gamma}_1)]_p$ is the class of all loops in X based at x_0 whose liftings are from $\tilde{x}_0$ to $\tilde{x}_1$, and it is the same for $[p(\tilde{\gamma}_1)]_p * p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_1\Big)\Big)$. Thus these two cosets are the same, and it follows that

$$p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big) = [p(\tilde{\gamma}_1)]_p * p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_1\Big)\Big) * [p(\tilde{\gamma}_1)]_p^{-1}.$$

Therefore, the two subgroups $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_0\Big)\Big)$ and $p_*\Big(\pi_1\Big(\widetilde{X}, \tilde{x}_1\Big)\Big)$ are conjugate. □

Problem 104 (Covering of topological groups)

(1) Let G be a topological group which is path-connected and locally path-connected, and $p\colon \widetilde{G} \to G$ be a covering map. Suppose $\widetilde{G}$ is path-connected and fix $\tilde{e} \in p^{-1}(e)$. Prove: there exists a unique group structure on $\widetilde{G}$ with $\tilde{e}$ its identity element, such that p is a group homomorphism.

(2) Suppose $\widetilde{G}$ and G are connected topological groups, and suppose $p\colon \widetilde{G} \to G$ is a covering map. Moreover, suppose p is also a group homomorphism. Prove: G is abelian if and only if $\widetilde{G}$ is abelian.

Proof (1) Consider the map

$$\mu\colon \widetilde{G} \times \widetilde{G} \to G, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto p(\tilde{g}_1)p(\tilde{g}_2).$$

We want to lift this map to a map $\tilde{\mu}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that the following diagram commutes:

$$\begin{tikzcd} & \left(\widetilde{G}, \tilde{e}\right) \arrow[d, "p"] \\ \left(\widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e})\right) \arrow[r, "\mu"] \arrow[ur, dashed, "\tilde{\mu}"] & (G, e) \end{tikzcd} \tag{104–1}$$

Observe that

- ◇ Since $\widetilde{G}$ is path-connected, so is $\widetilde{G} \times \widetilde{G}$.
- ◇ Since G is locally path-connected, so is $G \times G$. And since $p \times p$ is a local homeomorphism from $\widetilde{G} \times \widetilde{G}$ to $G \times G$, $\widetilde{G} \times \widetilde{G}$ is also locally path-connected.

By Theorem 3.7.14, the lift in (104–1) exists if and only if

$$\mu_*\left(\pi_1\left(\widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e})\right)\right) \subset p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right). \tag{104–2}$$

To prove (104–2), first note that images of paths in $\Omega\left(\widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e})\right)$ under μ are of the form $\gamma_1 \cdot \gamma_2$, where $\gamma_1, \gamma_2 \in \Omega(G, e)$ and $\cdot$ denotes the multiplication in G. By Problem 88 (2) ②, $\gamma_1 \cdot \gamma_2$ is path-homotopic to $\gamma_1 * \gamma_2$, thus

$$\mu_*\left(\pi_1\left(\widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e})\right)\right) = p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right) \cdot p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right) = p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right),$$

where the last equality follows since $p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right)$ is a subgroup of $\pi_1(G, e)$. Hence (104–2) holds, and the lift $\tilde{\mu}$ exists. With $\tilde{\mu}$ as the multiplication map on $\widetilde{G}$, let us verify the group axioms:

(Associativity) Consider the maps

$$\begin{aligned} \alpha\colon \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad & (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \mapsto \tilde{\mu}(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3), \\ \beta\colon \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad & (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \mapsto \tilde{\mu}(\tilde{g}_1, \tilde{\mu}(\tilde{g}_2, \tilde{g}_3)). \end{aligned}$$

Using the commutativity in (104–1), we have

$$\begin{aligned} p(\alpha(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)) &= p(\tilde{\mu}(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3)) \\ &= \mu(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3) \\ &= p(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2))p(\tilde{g}_3) \\ &= \mu(\tilde{g}_1, \tilde{g}_2)p(\tilde{g}_3) \\ &= p(\tilde{g}_1)p(\tilde{g}_2)p(\tilde{g}_3), \end{aligned}$$

and similarly

$$p(\beta(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)) = p(\tilde{g}_1)p(\tilde{g}_2)p(\tilde{g}_3).$$

So we obtain $p \circ \alpha = p \circ \beta$. As before, $\widetilde{G} \times \widetilde{G} \times \widetilde{G}$ is path-connected, and $p \circ \alpha(\tilde{e}, \tilde{e}, \tilde{e}) = p(\tilde{e})^3 =$

$e^3 = e$. The diagram

$$\begin{tikzcd} & \left(\widetilde{G}, \tilde{e}\right) \arrow[d, "p"] \\ \left(\widetilde{G} \times \widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e}, \tilde{e})\right) \arrow[ur, bend left, "\alpha"] \arrow[ur, bend right, "\beta"'] \arrow[r, "p\circ\alpha = p\circ\beta"] & (G, e) \end{tikzcd}$$

together with the uniqueness of lifting with base point (Proposition 3.7.13) implies $\alpha = \beta$.

(Identity element) Define the maps $f_1, f_2 \colon \widetilde{G} \to \widetilde{G}$ by $f_1(\tilde{g}) = \tilde{g}$ and $f_2(\tilde{g}) = \tilde{\mu}(\tilde{e}, \tilde{g})$. Then

$$p(f_2(\tilde{g})) = p(\tilde{\mu}(\tilde{e}, \tilde{g})) = \mu(\tilde{e}, \tilde{g}) = p(\tilde{e})p(\tilde{g}) = ep(\tilde{g}) = p(\tilde{g}) = p(f_1(\tilde{g})).$$

So we obtain $p \circ f_1 = p \circ f_2$. Since $p \circ f_1(\tilde{e}) = p(\tilde{e}) = e$ and $\widetilde{G}$ is path-connected, the diagram

$$\begin{tikzcd} & \left(\widetilde{G}, \tilde{e}\right) \arrow[d, "p"] \\ \left(\widetilde{G}, \tilde{e}\right) \arrow[ur, bend left, "f_1"] \arrow[ur, bend right, "f_2"'] \arrow[r, "p\circ f_1 = p\circ f_2"] & (G, e) \end{tikzcd}$$

together with the uniqueness of lifting with base point (Proposition 3.7.13) implies $f_1 = f_2$. Hence $\tilde{e}\tilde{g} = \tilde{g}$ for all $\tilde{g} \in \widetilde{G}$. Similarly, $\tilde{g}\tilde{e} = \tilde{g}$ for all $\tilde{g} \in \widetilde{G}$.

(Inverse element) Consider the map

$$i \colon \widetilde{G} \to G, \quad \tilde{g} \mapsto p(\tilde{g})^{-1}.$$

We want to lift this map to a map $\tilde{i} \colon \widetilde{G} \to \widetilde{G}$ such that the following diagram commutes:

$$\begin{tikzcd} & \left(\widetilde{G}, \tilde{e}\right) \arrow[d, "p"] \\ \left(\widetilde{G}, \tilde{e}\right) \arrow[ur, dashed, "\tilde{i}"] \arrow[r, "i"] & (G, e) \end{tikzcd} \tag{104–3}$$

Since $\widetilde{G}$ is path-connected and locally path-connected, by Theorem 3.7.14, the lift in (104–3) exists if and only if

$$i_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right) \subset p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right). \tag{104–4}$$

By the definition of i, we have

$$i_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right) \subset p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right)^{-1} = p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right),$$

where the last equality follows since $p_*\left(\pi_1\left(\widetilde{G}, \tilde{e}\right)\right)$ is a subgroup of $\pi_1(G, e)$. Hence (104–4) holds, and the lift $\tilde{i}$ exists. By Proposition 3.7.13, $\tilde{i}$ is unique, so we can define the inverse of any $\tilde{g} \in \widetilde{G}$ as $\tilde{i}(\tilde{g})$. Now define the maps $\ell_1, \ell_2 \colon \widetilde{G} \to \widetilde{G}$ by $\ell_1(\tilde{g}) = \tilde{\mu}\left(\tilde{i}(\tilde{g}), \tilde{g}\right)$ and $\ell_2(\tilde{g}) = \tilde{e}$.

Then

$$p(\ell_1(\tilde{g})) = p\big(\tilde{\mu}\big(\tilde{i}(\tilde{g}), \tilde{g}\big)\big) = \mu\big(\tilde{i}(\tilde{g}), \tilde{g}\big) = p\big(\tilde{i}(\tilde{g})\big)p(\tilde{g}) = i(\tilde{g})p(\tilde{g}) = p(\tilde{g})^{-1}p(\tilde{g})$$
$$= e = p(\ell_2(\tilde{g})).$$

So we obtain $p \circ \ell_1 = p \circ \ell_2$. Since $p \circ \ell_2(\tilde{e}) = e$ and $\widetilde{G}$ is path-connected, the diagram

$$\begin{tikzcd} & \big(\widetilde{G}, \tilde{e}\big) \arrow[d, "p"] \\ \big(\widetilde{G}, \tilde{e}\big) \arrow[ur, bend left, "\ell_1"] \arrow[ur, "\ell_2"'] \arrow[r, "p\circ\ell_1 = p\circ\ell_2"] & (G, e) \end{tikzcd}$$

together with the uniqueness of lifting with base point (Proposition 3.7.13) implies $\ell_1 = \ell_2$. Hence $\tilde{\mu}\big(\tilde{i}(\tilde{g}), \tilde{g}\big) = \tilde{e}$ for all $\tilde{g} \in \widetilde{G}$. Similarly, $\tilde{\mu}\big(\tilde{g}, \tilde{i}(\tilde{g})\big) = \tilde{e}$ for all $\tilde{g} \in \widetilde{G}$.

Therefore, $\widetilde{G}$ admits a group structure with $\tilde{e}$ as the identity element. The fact that p is a group homomorphism is just the commutativity in (104–1), i.e., $p(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2)) = \mu(\tilde{g}_1, \tilde{g}_2) = p(\tilde{g}_1)p(\tilde{g}_2)$. The uniqueness of the group structure follows from the uniqueness of the lift in (104–1).

(2) ($\Rightarrow$) Consider the map

$$d\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto \tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1}\tilde{g}_2^{-1}.$$

If G is abelian, then $p(d(\tilde{g}_1, \tilde{g}_2)) = e_G$, which means that the image of d is contained in $p^{-1}(e_G)$. Since p is a covering, $p^{-1}(e_G)$ is discrete. Since d is continuous and $\widetilde{G} \times \widetilde{G}$ is connected, the image of d must be connected. Thus $d\Big(\widetilde{G} \times \widetilde{G}\Big) = d\big(e_{\widetilde{G}}, e_{\widetilde{G}}\big) = e_{\widetilde{G}}$, i.e., $\widetilde{G}$ is abelian.

($\Leftarrow$) Since G is connected, by Problem 101 (1), p is a surjective group homomorphism. Hence G is abelian whenever $\widetilde{G}$ is abelian. □

PSet 14, Part 1

Problem 105 (Properly discontinuous action)

(1) Let $G = \big\langle a, b \,\big|\, a^{-1}bab = 1 \big\rangle$. Consider the action of G on $\mathbb{R}^2$ generated by

$$a \cdot (x, y) \coloneqq (-x, y-1), \quad b \cdot (x, y) = (x+1, y).$$

① Show that this action is properly discontinuous, and the quotient space is the Klein bottle. What is the fundamental group of the Klein bottle?

② Also check that the quotient space in Example 3.7.6 is the Klein bottle, and thus $\mathbb{T}^2$ is a double covering of the Klein bottle.

(2) Suppose group G acts on $\widetilde{X}$. We say the action is *free* if

for any $g \neq e$ and any $x \in \widetilde{X}$, $g \cdot x \neq x$.

Prove: if $\widetilde{X}$ is Hausdorff, G is a finite group, and the G-action on $\widetilde{X}$ is free, then the action is properly discontinuous.

(3) More generally, let $\widetilde{X}$ be a LCH space. Suppose the G-action on $\widetilde{X}$ is free, and satisfies the following condition (known as *proper action*):

$$\text{for any compact subset } C \subset \widetilde{X}, \text{ the set } \{g \in G : g \cdot C \cap C \neq \varnothing\} \text{ is finite.}$$

Prove: the G-action is properly discontinuous, and $\widetilde{X}/G$ is a LCH space.

Proof (1) ① For any $(x, y) \in \mathbb{R}^2$, choose $U = \mathbb{B}_\infty\big((x, y), \frac{1}{3}\big)$, then $g \cdot U \cap U = \varnothing$ for any $g \in G \setminus \{e\}$. Thus the action is properly discontinuous. For any $(x, y) \in \mathbb{R}^2$, first apply a or a^{-1} to make $y \in [0, 1)$, then apply b or b^{-1} to make $x \in [0, 1)$. Thus $[0, 1) \times [0, 1)$ is a fundamental domain. The identifications on its boundary are $(0, y) \sim (1, y)$ for $y \in [0, 1)$ and $(x, 1) \sim (-x, 0) \sim (1-x, 0)$ for $x \in [0, 1)$. Thus the quotient space is the Klein bottle. Since $\mathbb{R}^2$ is simply connected, by Proposition 3.7.20 (3), the fundamental group of the Klein bottle is G.

② Regard $\mathbb{T}^2$ as $[0, 2\pi] \times [0, 2\pi]$ with identifications $(0, y) \sim (2\pi, y)$ and $(x, 0) \sim (x, 2\pi)$. The G-action further identifies (x, y) with $(2\pi - x, y + \pi)$, which gives the Klein bottle.

(2) Suppose $G = \{e, g_1, \cdots, g_n\}$. For any $\tilde{x} \in \widetilde{X}$, $g_i\tilde{x} \neq \tilde{x}$. Since $\widetilde{X}$ is Hausdorff, there exist open neighborhoods U_i of $\tilde{x}$ and V_i of $g_i\tilde{x}$ such that $U_i \cap V_i = \varnothing$. Then $U = \bigcap_{i=1}^{n} U_i \cap g_i^{-1}(V_i)$ is an open neighborhood of $\tilde{x}$ such that $g_i \cdot U \cap U = \varnothing$ for $1 \leqslant i \leqslant n$. Thus the action is properly discontinuous.

(3) $\boxed{G\text{-action is properly discontinuous}}$ For any $\tilde{x} \in \widetilde{X}$, since $\widetilde{X}$ is LCH, there exists a precompact open neighborhood U of $\tilde{x}$. Since the action is proper, the set $\big\{g \in G : g \cdot \overline{U} \cap \overline{U} \neq \varnothing\big\}$ is finite, and thus the set $\{g \in G : g \cdot U \cap U \neq \varnothing\}$ is finite. Let $g_1, \cdots, g_n$ be the elements in this set that are not identity (if this set contains more than one element). Since the action is free, the same argument as in (2) shows that the action is properly discontinuous.

$\boxed{\widetilde{X}/G \text{ is locally compact}}$ Let $p\colon \widetilde{X} \to \widetilde{X}/G$ be the covering map. For any $x \in \widetilde{X}/G$, choose an open neighborhood U of x such that $p^{-1}(U) = \bigcup_\alpha V_\alpha$ is a disjoint union of open sets V_α and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism. Fix any $\tilde{x} \in p^{-1}(x)$ and suppose $\tilde{x} \in V_\beta$. Since $\widetilde{X}$ is LCH, by Proposition 2.4.16, we can find a compact neighborhood K of $\tilde{x}$ that is contained in V_β. Then $p(K)$ is a compact neighborhood of x.

$\boxed{\widetilde{X}/G \text{ is Hausdorff}}$ For any $x, y \in \widetilde{X}/G$ with $x \neq y$, fix $\tilde{x} \in p^{-1}(x)$ and $\tilde{y} \in p^{-1}(y)$. Since $\widetilde{X}$ is LCH, there exist precompact open neighborhoods U_x of $\tilde{x}$ and U_y of $\tilde{y}$. Since $\overline{U_x} \cup \overline{U_y}$ is compact, from the proper action condition, there are only finitely many $g \in G$ such that $g \cdot \big(\overline{U_x} \cup \overline{U_y}\big) \cap \big(\overline{U_x} \cup \overline{U_y}\big) \neq \varnothing$. It follows that there are only finitely many $g \in G$ such that

$$g \cdot \overline{U_y} \cap \overline{U_x} \neq \varnothing.$$

Call these exceptional elements $g_1, \cdots, g_n$. For each $j = 1, \cdots, n$, since $\widetilde{X}$ is Hausdorff and $x \neq y$ (which means $\tilde{x} \neq g_j\tilde{y}$), there exist disjoint open neighborhoods V_j' of $\tilde{x}$ and W_j' of $g_j\tilde{y}$. Let $V_j = V_j' \cap U_x$ and $W_j = g_j^{-1} \cdot W_j' \cap U_y$. Then for all $1 \leqslant j \leqslant n$, the sets V_j and W_j are open, $x \in V_j \subset U_x$, $y \in W_j \subset U_y$, and $V_j \cap g_j \cdot W_j = \varnothing$. Define

$$V = \bigcap_{j=1}^{n} V_j, \quad W = \bigcap_{j=1}^{n} W_j.$$

Now $x \in V$ and $y \in W$. By construction, for any $g \in G$, we have $V \cap g \cdot W = \varnothing$ and then

$$V \cap \bigcup_{g \in G} g \cdot W = \varnothing.$$

Thus for any $h \in G$ we have

$$\varnothing = h \cdot \left(V \cap \bigcup_{g \in G} g \cdot W \right) = (h \cdot V) \cap \left(h \cdot \bigcup_{g \in G} g \cdot W \right) = (h \cdot V) \cap \bigcup_{g \in G} g \cdot W.$$

Hence

$$\left(\bigcup_{h \in G} h \cdot V \right) \cap \left(\bigcup_{g \in G} g \cdot W \right) = \varnothing.$$

Let $\widetilde{V} = \bigcup_{h \in G} h \cdot V$ and $\widetilde{W} = \bigcup_{g \in G} g \cdot W$. Then $\widetilde{V}$ and $\widetilde{W}$ are disjoint open neighborhoods of x and y, and

$$p^{-1}\Big(p(\widetilde{V})\Big) = \widetilde{V}, \quad p^{-1}\Big(p(\widetilde{W})\Big) = \widetilde{W}.$$

Hence $p(\widetilde{V})$ and $p(\widetilde{W})$ are disjoint open neighborhoods of x and y, and $\widetilde{X}/G$ is Hausdorff. $\square$

Problem 106 (SU(2) and SO(3)) Let SU(2) be the special unitary group, i.e., the group of 2×2 unitary matrices with determinant 1, and SO(3) the special orthogonal group, i.e., the group of 3×3 orthogonal matrices with determinant 1.

(1) Prove: SU(2) is homeomorphic to $\mathbb{S}^3$ (and thus simply connected).

(2) Prove: SU(2) is a double covering of SO(3) (and thus SO$(3) \simeq \mathbb{RP}^3$).

(3) Find the fundamental group of SO(3). (Try to find a video on Dirac's belt trick from internet and try to understand it.)

Proof (1) Elements in SU(2) are of the form $\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$, where $|z|^2 + |w|^2 = 1$. Thus the map

$$f\colon \mathrm{SU}(2) \to \mathbb{S}^3 \subset \mathbb{C}^2, \quad \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mapsto (z, w)$$

is well-defined and is clearly a homeomorphism.

(2) There is a group isomorphism $f\colon$ SU$(2) \to \{$unit quaternions$\}$, generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1, \quad \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \mapsto \mathrm{i}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \mathrm{j}, \quad \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \mapsto \mathrm{k}.$$

Now identify $\mathbb{R}^3$ with the space of pure quaternions $\{b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : b, c, d \in \mathbb{R}\}$. For any unit quaternion q, observe that

$$\overline{q(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\bar{q}} = q\overline{b\mathrm{i} + c\mathrm{j} + d\mathrm{k}}\bar{q} = -q(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\bar{q},$$

so $q(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\bar{q}$ is also a pure quaternion. Thus we can consider the group action of SU(2) on

$\mathbb{R}^3 \simeq \mathbb{R}\mathrm{i} \oplus \mathbb{R}\mathrm{j} \oplus \mathbb{R}\mathrm{k}$ defined by

$$A \cdot (b\mathrm{i} + c\mathrm{j} + d\mathrm{k}) = f(A)(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\overline{f(A)}.$$

If $f(A) = \mathrm{e}^{\frac{\theta}{2}(u_x\mathrm{i}+u_y\mathrm{j}+u_z\mathrm{k})} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(u_x\mathrm{i} + u_y\mathrm{j} + u_z\mathrm{k})$, then the action of A on $\mathbb{R}^3$ is in fact a rotation of angle θ around the axis defined by the unit vector $u_x\mathrm{i} + u_y\mathrm{j} + u_z\mathrm{k}$, which can be represented by an element in SO(3). Thus we obtain a covering map SU(2) $\to$ SO(3), and since the actions of $f(A)$ and $f(-A) = -f(A)$ on $\mathbb{R}^3$ are the same, this is a double covering.

(3) By (1), $\pi_1(\mathrm{SU}(2)) = \{e\}$. By (2) and Problem 103 (1), the index of the subgroup $p_*(\pi_1(\mathrm{SU}(2), \tilde{x}_0)) = \{e\}$ in $\pi_1(\mathrm{SO}(3), x_0)$ is 2. Thus $\pi_1(\mathrm{SO}(3))$ has order 2, which implies $\pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}_2$. $\square$

Problem 107 (Covering of covering spaces) Let X, Y, Z be path-connected and locally path-connected spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ be continuous maps.

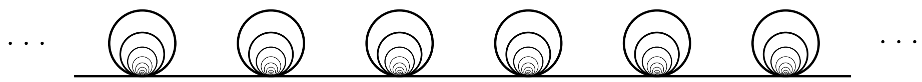
(1) Suppose both g and $g \circ f$ are covering maps. Prove: f is a covering map.

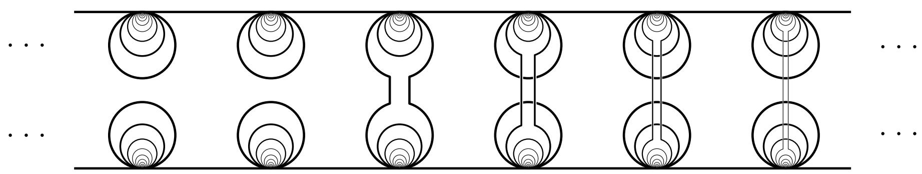
(2) Suppose both f and $g \circ f$ are covering maps. Prove: g is a covering map.

(3) Suppose f is a covering, and g is a finite covering. Prove: $g \circ f$ is a covering.

(4) Suppose f and g are coverings, and suppose Z is semi-locally simply connected. Prove: $g \circ f$ is a covering.

(5) Let X be the second space below, Y be the first space below, and Z be the Hawaiian earring. Construct a natural covering map $g\colon Y \to Z$, and a natural double covering map $f\colon X \to Y$ (as a double covering), so that the composition $g \circ f$ is not a covering map. (So in general the composition of covering maps may fail to be a covering map.)





Proof (1) Fix $x_0 \in X$ and set $y_0 = f(x_0)$, $z_0 = g(y_0)$. We first show that f is surjective. Given $y \in Y$, choose a path $\tilde{\alpha}$ in Y from y_0 to y. Then $\alpha = g \circ \tilde{\alpha}$ is a path in Z beginning at z_0. Let $\tilde{\tilde{\alpha}}$ be a lifting of α to a path in X beginning at x_0. Then $f \circ \tilde{\tilde{\alpha}}$ is a lifting of α to Y that begins at y_0. By uniqueness of path liftings, $\tilde{\alpha} = f \circ \tilde{\tilde{\alpha}}$. Then f maps the end point of $\tilde{\tilde{\alpha}}$ to the end point y of $\tilde{\alpha}$. Thus f is surjective. Given $y \in Y$, we find an open neighborhood of y that is evenly covered by f. Let $z = g(y)$. Since $g \circ f$ and g are covering maps, and Z is locally path-connected, we can find a path-connected open neighborhood U of z that is evenly covered by both $g \circ f$ and g. Let V be the slice of $g^{-1}(U)$ that contains the point y; we show V is evenly covered by f. Let $\{U_\alpha\}$ be the collection of slices of $(g \circ f)^{-1}(U)$. Now f maps each set U_α into the set $g^{-1}(U)$; because U_α is connected, it must be mapped by f into a single one of the slices of $g^{-1}(U)$. Therefore, $f^{-1}(V)$ equals the union of those slices U_α that are mapped by f into V. Let $f_0 = f|_{U_\alpha}$, $g_0 = g|_V$, and $h = (g \circ f)|_{U_\alpha}$. Since h_0 and g_0 are homeomorphisms, so is $f_0 = g_0^{-1} \circ h_0$.

(2) Since $g \circ f$ is surjective, g is also surjective. Given $z \in Z$, let U be a path-connected open neighborhood of z that is evenly covered by $g \circ f$. We show that U is also evenly covered by g. Let $\{V_\beta\}$ be the collection of path components of $g^{-1}(U)$; these sets are disjoint and open in Y (see remarks on page 181). We show that for each β, the map g carries V_β homeomorphically onto U. Let $\{U_\alpha\}$ be the collection of slices of $(g \circ f)^{-1}(U)$; they are disjoint, open, and path-connected, so they are the path-components of $(g \circ f)^{-1}(U)$. Now f maps each U_α into the set $g^{-1}(U)$; because U_α is connected, it must be mapped by f into one of the sets V_β. Therefore $f^{-1}(V_\beta)$ equals the union of a subcollection of the collection $\{U_\alpha\}$. By Remark 3.7.2, for each α, since U_α is a path component of $q^{-1}(V_\beta)$, the map $f_0 \colon U_\alpha \to V_\beta$ obtained by restricting f is a covering map. In particular, f_0 is surjective. Hence f_0 is a homeomorphism, being continuous, open, and injective as well. Let $g_0 = g|_{V_\beta}$ and $h_0 = h|_{U_\alpha}$. Then both h_0 and f_0 are homeomorphisms, so is $g_0 = h_0 \circ f_0^{-1}$.

(3) Suppose $g \colon Y \to Z$ is an n-sheeted covering map. Given $z \in Z$, let W be an open neighborhood of z that is evenly covered by g and let $\{V_1, \cdots, V_n\}$ be the slices of $g^{-1}(W)$. For each $1 \leqslant k \leqslant n$, find $y_k \in V_k$ such that $g(y_k) = z$. Let $V_k' \subset V_k$ be an open neighborhood of y_k that is evenly covered by f. Then $W' := \bigcap_{k=1}^{n} g(V_k')$ is an open neighborhood of z that is evenly covered by $g \circ f$.

(4) Since Z is path-connected, locally path-connected and semi-locally simply connected, it admits a universal covering space $\widehat{Z}$. Suppose $p_Z \colon \widehat{Z} \to Z$ is the covering map. By Theorem 3.8.9, there exists a covering map $p_Y \colon \widehat{Z} \to Y$ such that $p_Z = g \circ p_Y$. It follows that there exists a covering map $p_X \colon \widehat{Z} \to X$ such that $p_Y = f \circ p_X$. Now $p_Z = g \circ f \circ p_X$ and p_X are both covering maps, by (2), $g \circ f$ is a covering map.

(5) ① Wrapping the horizontal line around the outermost circle of the Hawaiian earring gives a natural covering map $g \colon Y \to Z$.

② By gluing the two horizontal lines of X in the same direction, we obtain a double of Y (note that those curves connecting the two horizontal lines will become cicrles).

③ Any open neighborhood of the intersection point of the Hawaiian earring contains some small circle, whose preimage under $g \circ f$ is a curve connecting the two horizontal lines. Thus there is a slice of the preimage that is not homeomorphic to the given open neighborhood. □

Problem 108 (Covering of topological manifolds) Let M be a connected topological manifold.

(1) Prove: any topological manifold admits a universal covering.

(2) Prove: the fundamental group of any topological manifold is countable.

(3) Prove: any covering space of a topological manifold is still a topological manifold.

Proof (1) Since M is locally Euclidean, it is also locally path-connected and semi-locally simply connected. By Theorem 3.8.3, M admits a universal covering space.

(2) By Proposition 2.7.14 / Problem 58 (1), any second countable space is Lindelöf. So we can take a countable cover $\mathcal{U}$ of M by coordinate balls. For each $U, U' \in \mathcal{U}$, the intersection $U \cap U'$ has at most countably many path components, since M is separable. Choose a point in each such path component and let X denote the (countable) set consisting of all the chosen points as U, U' range over all the sets in $\mathcal{U}$. For each $U \in \mathcal{U}$ and $x, x' \in X$ such that $x, x' \in U$, choose a definite path $\gamma_{x,x'}^U$ from x to x' in U.

Now choose any point $p \in X$ as base point. Let us say that a loop based at p is *special* if it is a finite product of paths of the form $\gamma^U_{x,x'}$. Because both $\mathcal{U}$ and X are countable sets, there are only countably many special loops. Each special loop determines an element of $\pi_1(M,p)$. If we can show that every element of $\pi_1(M,p)$ is obtained in this way, we are done, because we will have exibited a surjective map from a countable set onto $\pi_1(M,p)$.

So suppose γ is any loop based at p. By the Lebesgue number lemma there is an integer n such that γ maps each subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ into one of the balls in $\mathcal{U}$; call this ball U_k. Let $\gamma_k = \gamma|_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}$ reparametrized on the unit interval, so that $[\gamma]_p = [\gamma_1]_p * \cdots * [\gamma_n]_p$.

For each $k = 1, \cdots, n-1$, the point $\gamma\left(\frac{k}{n}\right)$ lies in $U_k \cap U_{k+1}$. Therefore, there is some $x_k \in X$ that lies in the same path component of $U_k \cap U_{k+1}$ as $\gamma\left(\frac{k}{n}\right)$. Choose a path δ_k in $U_k \cap U_{k+1}$ from x_k to $\gamma\left(\frac{k}{n}\right)$, and set $\tilde{\gamma}_k = \delta_{k-1} * \gamma_k * \overline{\delta_k}$ (taking $x_k = p$ and $\delta_k = \gamma_p$ when $k = 0$ or n). It is immediate that $[\gamma]_p = [\tilde{\gamma}_1]_p * \cdots * [\tilde{\gamma}_n]_p$, because all the δ_k's cancel out. But for each k, $\tilde{\gamma}_k$ is a path in U_k from x_{k-1} to x_k, and since U_k is simply connected, $\tilde{\gamma}_k$ is path-homotopic to $\gamma^{U_k}_{x_{k-1},x_k}$. This shows that γ is path-homotopic to a special loop and completes the proof.

(3) Let $\widetilde{M}$ be a path-connected covering space of M and let $p\colon \widetilde{M} \to M$ be the covering map.

$\boxed{\text{Locally Euclidean}}$ Since every point $\tilde{x} \in M$ has an open neighborhood homeomorphic to an open neighborhood of $p(\tilde{x})$, and $p(\tilde{x})$ has an open neighborhood homeomorphic to an open subset in $\mathbb{R}^n$ (for some n), it follows by restriction that $\tilde{x}$ has an open neighborhood homeomorphic to an open subset in $\mathbb{R}^n$.

$\boxed{\text{(T2)}}$ For any distinct $\tilde{x}, \tilde{y} \in \widetilde{M}$, if $p(\tilde{x}) = p(\tilde{y})$, then choose an open neighborhood U of $p(\tilde{x})$ that is evenly covered by p. Then $p^{-1}(U)$ is a disjoint union of open sets, and two of them contain $\tilde{x}$ and $\tilde{y}$, respectively. If $p(\tilde{x}) \neq p(\tilde{y})$, then we can find disjoint open neighborhoods U and V of $p(\tilde{x})$ and $p(\tilde{y})$, respectively. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open neighborhoods of $\tilde{x}$ and $\tilde{y}$, respectively.

$\boxed{\text{(A2)}}$ For each $x \in M$, choose an open neighborhood U_x of x that is evenly covered by p. Since M is (A2), it is Lindelöf, so we can take a countable subcover $\{U_n\}_{n=1}^{\infty}$ of $\{U_x : x \in M\}$. Since $\widetilde{M}$ is path-connected, by Proposition 3.7.16, the degree of the covering is not greater than the cardinality of $\pi_1(M)$, which is countable by (2). Thus $\widetilde{M}$ is a countable union of open sets V_k, each of which is homeomorphic to U_n for some n. Since M is (A2) and by Problem 59 (1) (A2) is hereditary, V_k is (A2) as well. For each k, let $\mathcal{B}_k$ be a countable basis for V_k. Then $\displaystyle\bigcup_{k=1}^{\infty} \mathcal{B}_k$ is a countable basis for $\widetilde{M}$ since each V_k is open in $\widetilde{M}$ and $\widetilde{M} = \displaystyle\bigcup_{k=1}^{\infty} V_k$. $\square$

PSet 14, Part 2

Problem 109 (Applications of Brouwer's fixed point theorem)

(1) **(A special case of Poincaré–Hopf theorem, proved by Hadamard)** Let $f\colon \mathbb{D}^n \to \mathbb{R}^n$ be a continuous map (i.e., f is a vector field on $\mathbb{D}^n$) such that $x \cdot f(x) > 0$ for all $x \in \mathbb{S}^{n-1} = \partial\mathbb{D}^n$. Prove: there exists $x \in \mathbb{B}^n$ such that $f(x) = 0$.

(2) **(Poincaré–Bohl)** Let $f\colon \mathbb{D}^n \to \mathbb{R}^n$ be a continuous map such that $f(x) \notin \{\alpha x : \alpha > 0\}$ for any $x \in \mathbb{S}^{n-1}$. Prove: there exists $x \in \mathbb{D}^n$ such that $f(x) = 0$.

(3) **(Perron–Frobenius)** Any $n \times n$ real matrix with positive entries has a positive eigenvalue, and the corresponding eigenvector can be chosen to have strictly positive entries.

(4) **(Kuratowski–Steinhaus)** Let $f\colon \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map such that $f\left(\mathbb{S}^{n-1}\right) \subset \mathbb{S}^{n-1}$, and suppose for any $x \in \mathbb{S}^{n-1}$, $f(x) \neq x$. Prove: $f(\mathbb{D}^n) = \mathbb{D}^n$.

Proof (1) If $f(x) \neq 0$ for all $x \in \mathbb{B}^n$, then let $g(x) = -\dfrac{f(x)}{\|f(x)\|} \in \mathcal{C}\left(\mathbb{D}^n, \mathbb{S}^{n-1}\right)$. By Brouwer's fixed point theorem, there exists $x_0 \in \mathbb{S}^{n-1}$ such that $g(x_0) = x_0$, i.e., $f(x_0) = -\|f(x_0)\|x_0$. Then $x_0 \cdot f(x_0) = -\|f(x_0)\| < 0$, a contradiction.

(2) If $f(x) \neq 0$ for all $x \in \mathbb{D}^n$, then let $g(x) = \dfrac{f(x)}{\|f(x)\|} \in \mathcal{C}\left(\mathbb{D}^n, \mathbb{S}^{n-1}\right)$. By Brouwer's fixed point theorem, there exists $x_0 \in \mathbb{S}^{n-1}$ such that $g(x_0) = x_0$, i.e., $f(x_0) = \|f(x_0)\|x_0$, a contradiction.

(3) Let $K = \{x \in \mathbb{R}^n : \|x\| = 1,\, x_i \geqslant 0,\, \forall i\}$. Then by Problem 110 (1), $K \simeq \mathbb{D}^{n-1}$ has the fixed point property. Since all entries of A are positive, the map $f(x) = \dfrac{Ax}{\|Ax\|}$ is a well-defined continuous map from K to K, so it has a fixed point x_0, i.e., $Ax_0 = \|Ax_0\|x_0$. To see that $x_0 \notin \partial K$, just notice that all components of Ax are strictly positive for any $x \in K$.

(4) For any $x \in \mathbb{B}^n$, we show that there is a homeomorphism $\varphi_x\colon \mathbb{D}^n \to \mathbb{D}^n$ with $\varphi_x(x) = 0$ and $\varphi_x|_{\mathbb{S}^{n-1}} = \mathrm{Id}_{\mathbb{S}^{n-1}}$. First, consider the homeomorphism

$$\psi\colon \mathbb{B}^n \to \mathbb{R}^n, \quad x \mapsto x \cdot \arctan\frac{\pi\|x\|}{2}.$$

Then define

$$\varphi_x\colon \mathbb{D}^n \to \mathbb{D}^n, \quad a \mapsto \begin{cases} a, & a \in \mathbb{S}^{n-1}, \\ \psi^{-1} \circ T_{\psi(x),0} \circ \psi(a), & a \in \mathbb{B}^n, \end{cases}$$

where $T_{\psi(x),0}\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes the translation from $\psi(x)$ to 0. It is easy to check that φ_x is a homeomorphism (via sequential continuity), as desired. Now apply (2) to $\varphi_x \circ f$, we see that there exists $x_0 \in \mathbb{D}^n$ such that $\varphi_x(f(x_0)) = 0$, and since φ_x is a homeomorphism, this implies $f(x_0) = x$. Since $x \in \mathbb{B}^n$ is arbitrary, we obtain $\mathbb{B}^n \subset f(\mathbb{D}^n)$. Finally, since $f(\mathbb{D}^n)$ is compact, we must have $f(\mathbb{D}^n) = \mathbb{D}^n$. □

Problem 110 (Fixed point property) We say a topological space X has the *fixed point property* if for every continuous map $f\colon X \to X$, there exists $p \in X$ such that $f(p) = p$.

(1) Show that the fixed point property is a topological property.

(2) Prove: if X is disconnected, then X cannot have the fixed point property.

(3) Suppose X, Y have the fixed point property. Prove: $X \vee Y$ has the fixed point property.

(4) Prove: if $X \times Y$ has the fixed point property, then X and Y have the fixed point property. (There exist complicated examples: X, Y have the fixed point property, while $X \times Y$ does not.)

(5) Prove: if X has the fixed point property, and $A \subset X$ is a retract of X, then A has the fixed point property.

(6) Let X be the subset of $\mathbb{R}^2$ that consists of the union of the line segment from $(0, 0)$ to $(1, 0)$ and all line segments from $\left(\frac{1}{n}, 0\right)$ to $\left(\frac{1}{n}, 1\right)$. Show that X has the fixed point property. (As a result, X has the fixed point property does not imply X is compact.)

Proof (1) Suppose X has the fixed point property, and $h: X \to Y$ is a homeomorphism. Then for any continuous map $f: Y \to Y$, the composition $h^{-1} \circ f \circ h: X \to X$ has a fixed point $p \in X$, i.e., $f(h(p)) = h(p)$, so $h(p) \in Y$ is a fixed point of f. Therefore, Y has the fixed point property.

(2) If X is disconnected, then X can be written as the disjoint union of two nonempty open sets U and V. Fix $u \in U$ and $v \in V$, then the map

$$f: X \to X, \quad x \mapsto \begin{cases} v, & x \in U, \\ u, & x \in V \end{cases}$$

is continuous and has no fixed point.

(3) Suppose $f: X \vee Y \to X \vee Y$ is continuous and let p denote the intersection. Without loss of generality, assume $f(p) \in X$. Let g be the natural map $X \to X \vee Y$, and consider the map

$$h: X \sqcup Y \to X, \quad z \mapsto \begin{cases} z, & z \in X, \\ p_X, & z \in Y. \end{cases}$$

Here p_X is the base point for X. The map h descends to a continuous map $\tilde{h}: X \vee Y \to X$, thus we get a continuous map by the composition

$$X \xrightarrow{g} X \vee Y \xrightarrow{f} X \vee Y \xrightarrow{\tilde{h}} Y, \quad x \mapsto \begin{cases} f(x), & \text{if } f(x) \in X, \\ p, & \text{if } f(x) \notin X. \end{cases}$$

By assumption, there exists $x_0 \in X$ such that $f(x_0) = x_0$, which implies $f(x_0) = x_0 \in X$ or $f(x_0) \notin X$ and $x_0 = p$. But the latter is impossible since we assumed $f(p) \in X$. Hence x_0 is a fixed point of f.

(4) If $f: X \to X$ is continuous, then the map

$$g: X \times Y \to X \times Y, \quad (x, y) \mapsto (f(x), y)$$

is continuous. By assumption, there exists $(x_0, y_0) \in X \times Y$ such that $g(x_0, y_0) = (x_0, y_0)$, which implies $f(x_0) = x_0$. Similarly, we can show that Y has the fixed point property.

(5) By Problem 67 (2), any continuous map $f: A \to A$ has an extension $\tilde{f}: X \to A$. By assumption, there exists $p \in X$ such that $\tilde{f}(p) = p$, which implies $p \in A$ and then $f(p) = p$.

(6) Let $f\colon X \to X$ be an arbitrary continuous map. Let $I = [0,1] \times \{0\}$ be the bottom interval and let $\pi\colon X \to I$ be the projection $\pi(x,y) = (x,0)$. The map $\pi \circ f$ maps I to I, so it has a fixed point in I. Hence, there exist x_0 and y_0 such that $f(x_0,0) = (x_0,y_0)$. If $y_0 = 0$ then we are done. Otherwise, $0 < y_0 \leqslant 1$, then by the continuity of f, there exists $\varepsilon > 0$ such that $\{x_0\} \times [0,\varepsilon]$ is mapped into $\{x_0\} \times [0,1]$. Thus, we define

$$y_1 := \max\{y \in [0,1] : f(\{x_0\} \times [0,y]) \subset \{x_0\} \times [0,1]\} \geqslant \varepsilon > 0.$$

If $y_1 = 1$, then f maps $\{x_0\} \times [0,1]$ to itself, so it has a fixed point. Now assume $0 < y_0 < 1$ and consider the retraction

$$r\colon \{x_0\} \times [0,1] \to \{x_0\} \times [0,y_1], \quad (x_0,y) \mapsto (x_0, \min\{y,y_1\}).$$

Since $r \circ f$ maps $\{x_0\} \times [0,y_1]$ to itself, it has a fixed point (x_0,y_*) in $\{x_0\} \times [0,y_1]$. If $y_* < y_1$, then we are done. Otherwise, $y_* = y_1$, and $r(f(x_0,y_1)) = (x_0,y_1)$ implies $f(x_0,y_1) \in \{x_0\} \times [y_1,1]$. But then, by the continuity of f, there exists $\delta > 0$ such that $\{x_0\} \times [y_1-\delta, y_1+\delta]$ is mapped into $\{x_0\} \times [0,1]$. This contradicts the maximality of y_1 and completes the proof. □

Problem 111 (Brouwer's fixed point theorem, the second version) Let $K \subset \mathbb{R}^n$ be any nonempty compact convex set.

(1) Suppose K has nonempty interior. Prove: K is homeomorphic to $\mathbb{D}^n$.

(2) Prove: K has nonempty interior if and only if K is not contained in a proper hyperplane (i.e., a set of the form $x_0 + V$, where $V \subset \mathbb{R}^n$ is a linear subspace).

(3) Prove Theorem 4.1.8:

Let $K \subset \mathbb{R}^n$ be nonempty, compact and convex. Then any continuous map $f\colon K \to K$ has a fixed point.

Proof (1) By translation we may assume $0 \in \operatorname{Int} K$. Since $K \subset \mathbb{R}^n$ is compact, we can define

$$p\colon \mathbb{R}^n \to [0,+\infty), \quad x \mapsto \inf\Big\{\lambda > 0 : \frac{x}{\lambda} \in K\Big\}.$$

It is easy to check that

- ◇ $p(x+y) \leqslant p(x) + p(y), \forall x,y \in \mathbb{R}^n$.
- ◇ $p(\lambda x) = \lambda p(x), \forall \lambda > 0, \forall x \in \mathbb{R}^n$.
- ◇ $p(x) = 0$ if and only if $x = 0$.

Therefore, there exist $C_1, C_2 > 0$ such that

$$C_1\|x\| \leqslant p(x) \leqslant C_2\|x\|, \quad \forall x \in \mathbb{R}^n,$$

where $\|\cdot\|$ denotes the Euclidean norm. Now define

$$\varphi\colon K \to \mathbb{D}^n, \quad x \mapsto \begin{cases} \dfrac{p(x)x}{\|x\|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Since $0 \in \operatorname{Int} K$, there exists $r > 0$ such that $\mathbb{B}(0,r) \subset K$. Then

$$\frac{rx}{2\|x\|} \in K, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

It follows that $p(x) \leqslant \dfrac{2\|x\|}{r}$ for all $x \in \mathbb{R}^n$. Hence

$$|p(x) - p(y)| \leqslant \max\{p(x-y), p(y-x)\} \leqslant \frac{2}{r}\|x-y\|, \quad \forall x, y \in \mathbb{R}^n,$$

which implies that φ is uniformly continuous. Thus φ is a continuous bijection from the compact set K to the Hausdorff space $\mathbb{D}^n$, so it is a homeomorphism.

(2) The "only if" part is trivial. For the "if" part, first assume $0 \in K$. By assumption, K contains a basis $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$. Now the point $p_0 := \dfrac{1}{2n}\displaystyle\sum_{k=1}^{n} e_k$ must lie in $\operatorname{Int} K$, since $p_0 \in K$ and

$$\left\{p_0 + \sum_{k=1}^{n} \lambda_k e_k : |\lambda_k| \leqslant \frac{1}{2n}\right\} \subset K.$$

(3) By lowering the dimension, we may assume $K \subset \mathbb{R}^m$ ($m \leqslant n$) and K is not contained in a proper hyperplane of $\mathbb{R}^m$. By (2), K has nonempty interior in $\mathbb{R}^m$. Then by (1), K is homeomorphic to $\mathbb{D}^m$. By Problem 110 (1), K has the fixed point property. $\square$

Problem 112 (A proof or not?) Here is a proof of two-dimensional Brouwer's fixed point theorem:

Proof We first observe (by the intermediate value theorem) that

Lemma *Any continuous map $h\colon [0,1] \to [0,1]$ has a fixed point.*

Now write $F = (f, g)$, where both f and g are continuous functions from $[0,1]^2$ to $[0,1]$. For each $y \in [0,1]$, we define a function $\tilde{f}_y\colon [0,1] \to [0,1]$ by $\tilde{f}_y(x) = f(x,y)$. According to the lemma above, there exists $a(y) \in [0,1]$ such that $\tilde{f}_y(a(y)) = a(y)$, i.e., $f(a(y), y) = a(y)$ for any $y \in [0,1]$. Using the function $a(y)$ we can define another function $\tilde{g}\colon [0,1] \to [0,1]$ by $\tilde{g}(y) = g(a(y), y)$. Again according to the lemma above, we can find $b \in [0,1]$ such that $\tilde{g}(b) = b$, i.e., $g(a(b), b) = b$. It follows that $F(a(b), b) = (a(b), b)$. In other words, the point $(a(b), b)$ is a fixed point of F.

Question Is this proof correct or wrong? If you think this is correct, then generalize this proof to give a proof of the Brouwer's fixed point theorem for n-dimensional cubes; if you think this is wrong, then point out the mistake in the proof and also provide a counterexample for which the proof fails.

Solution This proof is wrong since the continuity of the function $a(y)$ is not guaranteed. In general, $a(y)$ cannot be chosen to be continuous. A counterexample is given by the function

$$f\colon [0,1]^2 \to [0,1], \quad (x,y) \mapsto \begin{cases} 2xy, & \text{if } y \leqslant \frac{1}{2}, \\ 1 - 2(1-x)(1-y), & \text{if } y \geqslant \frac{1}{2}. \end{cases}$$

Then following the "proof", we find

$$a(y) = \begin{cases} 0, & \text{if } y < \frac{1}{2}, \\ 1, & \text{if } y > \frac{1}{2}, \end{cases}$$

and when $y = \frac{1}{2}$, all $x \in [0, 1]$ are fixed points of $\tilde{f}_y$. In this case, $a(y)$ cannot be chosen to be continuous.□

PSet 15, Part 1

Problem 113 (Continuity and injectivity does not imply homeomorphism)

(1) Suppose $U \subset \mathbb{R}^n$ is open, and $f\colon U \to \mathbb{R}^n$ is continuous and injective. Prove: $f\colon U \to f(U)$ is a homeomorphism.

(2) Construct a continuous and injective map $f\colon \mathbb{R} \to \mathbb{R}^2$ such that $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

Proof (1) By Brouwer's invariance of domain theorem, f is an open map. So $f\colon U \to f(U)$ is a continuous open bijection, hence a homeomorphism.

(2) Consider the curve $\beta\colon (-\pi, \pi) \to \mathbb{R}^2$ defined by $\beta(t) = (\sin 2t, \sin t)$. Its image is a set that looks like a figure-eight in the plane (Figure 6). It is easy to see that β is continuous and injective.

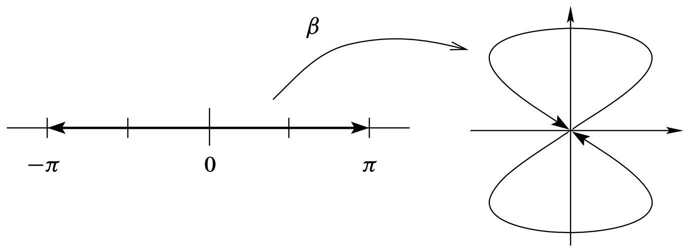


Figure 6: The figure-eight curve

Consider the homeomorphism $\alpha\colon \mathbb{R} \to (-\pi, \pi)$ given by $\alpha(x) = 2 \arctan x$. Since $(-\pi, \pi)$ is non-compact, but the image $\beta((-\pi, \pi))$ is compact, the map β is not a homeomorphism. It follows that $f := \beta \circ \alpha\colon \mathbb{R} \to \mathbb{R}^2$ is continuous and injective, but $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism. □

Problem 114 (Applications of Brouwer's invariance of domain theorem)

(1) Suppose M and N are n-dimensional topological manifolds, and $f\colon M \to N$ is continuous and locally injective (what is a reasonable definition of "locally injective"?). Prove: f is an open map.

(2) Prove: there is no injective continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$.

(3) Show that there is no proper subset of $\mathbb{S}^n$ that is homeomorphic to $\mathbb{S}^n$ itself.

Proof (1) A function $M \to N$ is said to be locally injective at $p \in M$ if there exists an open neighborhood U of p in M such that $f|_U$ is injective, and f is said to be locally injective if it is locally injective at every point in M. Suppose $f\colon M \to N$ is continuous and locally injective. For any $p \in M$, we can find a local chart (U, φ) around p and a local chart (V, ψ) around $f(p)$. By restricting U if necessary, we can assume that $f|_U$ is injective and f maps U into V. Then the composition $\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is an injective continuous map between open subsets of $\mathbb{R}^n$. Since $\varphi(U)$ is open, by Brouwer's invariance of domain theorem, $\psi \circ f \circ \varphi^{-1}$ is an open map. It follows that f is a locally open map and hence an open map.

(2) If there exists an injective continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$, then by (1), f is an open map. In particular, $f(\mathbb{S}^n)$ is open in $\mathbb{R}^n$. But $f(\mathbb{S}^n)$ is compact in $\mathbb{R}^n$, which means $f(\mathbb{S}^n)$ is bounded and closed in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is connected, there is no nonempty subset of $\mathbb{R}^n$ that is bounded and clopen. This is a contradiction.

(3) If A is a proper subset of $\mathbb{S}^n$, then A misses at least one point of $\mathbb{S}^n$, which means A is homeomorphic to a subset of $\mathbb{R}^n$. By (2), there is no injective continuous map from $\mathbb{S}^n$ to A, so A cannot be homeomorphic to $\mathbb{S}^n$. $\square$

Problem 115 (Manifolds with boundary)

(1) Show that the concept of the boundary point is well-defined in the definition of "topological manifold with boundary".

(2) Prove: if X is an n-dimensional topological manifold with boundary, then its boundary ∂X is an $(n-1)$-dimensional topological manifold (without boundary).

(3) Let $f\colon M \to N$ be a homeomorphism, where M, N are topological manifolds of dimension n. Show that $f\colon \partial M \to \partial N$ is a homeomorphism.

(4) Show that $[0,1]\times\mathbb{R}$ is not homeomorphic to $[0,\infty)\times\mathbb{R}$.

Proof (1) Let M be an n-manifold with boundary and denote $\mathbb{H}^n = \{(x_1,\cdots,x_n)\in\mathbb{R}^n : x_n \geqslant 0\}$. If $p \in M$ is a boundary point via a chart (U,φ) and also an interior point via a chart (V,ψ), then consider an open neighborhood $W \subset U\cap V$ of p such that $\varphi(W) = \mathbb{B}(\varphi(p), r)\cap\mathbb{H}^n$ for some $r>0$ and $\psi(W)$ is an open subset of $\mathbb{R}^n$. Now $\varphi\circ\psi^{-1}\colon \psi(W)\to\varphi(W)$ is a homeomorphism. Let $\iota\colon \varphi(W)\hookrightarrow\mathbb{R}^n$ be the inclusion map. Then $\iota\circ\varphi\circ\psi^{-1}\colon \psi(W)\to\mathbb{R}^n$ is a continuous injection, which is impossible by Brouwer's invariance of domain theorem, for $\iota\circ\varphi(W)$ is not open in $\mathbb{R}^n$.

(2) By Problem 59, both (A2) and (T2) are hereditary properties. If $p\in\partial X$ and (U,φ) is a lcoal chart around p in X, then $\varphi|_{U\cap\partial M}$ is a homeomorphism from $U\cap\partial M$ to an open subset of $\mathbb{R}^{n-1}\times\{0\}\simeq\mathbb{R}^{n-1}$. Hence ∂X is an $(n-1)$-manifold without boundary.

(3) By (1), $f(\partial M)=\partial N$. So by restricting f to ∂M, we get a homeomorphism $f\colon\partial M\to\partial N$.

(4) If $[0,1]\times\mathbb{R}$ is homeomorphic to $[0,\infty)\times\mathbb{R}$ (as 2-manifolds), then by (3), their boundaries $(\{0\}\times\mathbb{R})\cup(\{1\}\times\mathbb{R})$ and $\{0\}\times\mathbb{R}$ are homeomorphic. However, the former is disconnected while the latter is connected, a contradiction. $\square$

Problem 116 (Invariance of domain via Borsuk–Ulam) Assume the following version of Borsuk–Ulam theorem:

> **(Borsuk–Ulam)** *There does not exist any continuous map $f\colon\mathbb{D}^n\to\mathbb{S}^{n-1}$ such that $f|_{\mathbb{S}^{n-1}}$ preserves antipodal points (i.e., $f(-x)=-f(x)$).*

Prove the invariance of domain theorem.

Proof In Problem 96, we have shown that the Borsuk–Ulam theorem is equivalent to the following statement:

> *There exists no continuous map $f\colon\mathbb{D}^n\to\mathbb{S}^{n-1}$ such that the restriction to the boundary of f, $f|_{\mathbb{S}^{n-1}}\colon\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$, is antipodal-preserving.*

And recall from Problem 83 (3) that

> *$f \in \mathcal{C}(X, Y)$ is null-homotopic if and only if f has a continuous extension $F \in \mathcal{C}(C(X), Y)$, where $C(X)$ is the cone over X.*

Note that the cone over $\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{D}^n$. Thus the above statements together imply that

> *There exists no continuous map $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ that is null-homotopic and antipodal-preserving.*

To prove the invariance of domain theorem, it suffices to prove its local version (Theorem 4.1.14):

> *Let $f : \mathbb{D}^n \to \mathbb{R}^n$ be continuous and injective. Then $f(0) \in \operatorname{Int} f(\mathbb{D}^n)$.*

Suppose to the contrary that $f(0) \notin \operatorname{Int} f(\mathbb{D}^n)$. Since f is injective, $f(0) \notin f\big(\mathbb{S}^{n-1}\big)$, so there exists $\varepsilon > 0$ scuh that $\mathbb{B}(f(0), \varepsilon) \subset \mathbb{R}^n \setminus f\big(\mathbb{S}^{n-1}\big)$. And since $f(0) \notin \operatorname{Int} f(\mathbb{D}^n)$, we can find $c \in \mathbb{B}(f(0), \varepsilon) \cap (\mathbb{R}^n \setminus f(\mathbb{D}^n))$. Now consider the map

$$g : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad s \mapsto \frac{f(s) - c}{|f(s) - c|}.$$

and the constant map

$$g_0 : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad s \mapsto \frac{f(0) - c}{|f(0) - c|}.$$

Also, define

$$H_1 : [0, 1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad (t, s) \mapsto \frac{f(ts) - c}{|f(ts) - c|}.$$

Since $c \notin f(\mathbb{D}^n)$, these maps are all well-defined, and H_1 is a homotopy from g_0 to g. On the other hand, consider the antipodal-preserving map

$$h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad s \mapsto \frac{f(s) - f(-s)}{|f(s) - f(-s)|},$$

which is well-defined and continuous since f is injective. Define the map

$$h_0 : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad s \mapsto \frac{f(s) - f(0)}{|f(s) - f(0)|}$$

and choose a path $\lambda(t)$ from c to $f(0)$ in $\mathbb{B}(f(0), \varepsilon)$. Then the map

$$H_2 : [0, 1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad (t, s) \mapsto \frac{f(s) - \lambda(t)}{|f(s) - \lambda(t)|}$$

is a homotopy from g to h_0. This map is well-defined since $\mathbb{B}(f(0), \varepsilon) \cap f\big(\mathbb{S}^{n-1}\big) = \varnothing$. At the same time, the map

$$H_3 : [0, 1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad (t, s) \mapsto \frac{f(s) - f(-st)}{|f(s) - f(-st)|}$$

is a homotopy from h_0 to h. This map is well-defined since f is injective. Thus $h \sim h_0 \sim g \sim g_0$, which means $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is null-homotopic and antipodal-preserving, a contradiction. $\square$

PSet 15, Part 2

Problem 117 (Poincaré–Miranda theorem) The following theorem was first announced by H. Poincaré in 1883, which can be viewed as a higher dimensional generalization of the intermediate value theorem. Miranda showed in 1940 that the theorem was equivalent to Brouwer's fixed point theorem.

> **(Poincaré–Miranda)** *Let $f = (f_1, \cdots, f_n)\colon [0,1]^n \to \mathbb{R}^n$ be continuous. Suppose for any $1 \leqslant i \leqslant n$, we have*
>
> $$f_i \leqslant 0 \quad \textit{on} \quad \{x \in [0,1]^n : x_i = 0\},$$
> $$f_i \geqslant 0 \quad \textit{on} \quad \{x \in [0,1]^n : x_i = 1\}.$$
>
> *Then there exists $p \in [0,1]^n$ such that $f(p) = 0$.*

(1) Prove the Poincaré–Miranda theorem via Brouwer's fixed point theorem.

(2) Prove Brouwer's fixed point theorem via the Poincaré–Miranda theorem.

Proof (1) Let $r_0\colon \mathbb{R} \to [0,1]$ be the retraction with $r_0((-\infty,0)) = \{0\}$, $r_0((1,+\infty)) = \{1\}$, and define

$$r\colon \mathbb{R}^n \to [0,1]^n, \quad (x_1, \cdots, x_n) \mapsto (r_0(x_1), \cdots, r_0(x_n)).$$

Consider the map $h(x) = r(x) - f(r(x))$. Since f is continuous and $[0,1]^n$ is compact, $f([0,1]^n)$ is bounded, and thus the image of h is also bounded. Since r_0 is a retraction, the image of h is the same as that of $h|_{[0,1]^n}$. So we can choose $R > 0$ such that $h(\mathbb{R}^n) \cup [0,1]^n \subset \overline{\mathbb{B}(0,R)}$, and this implies $h\left(\overline{\mathbb{B}(0,R)}\right) \subset \overline{\mathbb{B}(0,R)}$. By Brouwer's fixed point theorem, there exists $x = (x_1, \cdots, x_n) \in \overline{\mathbb{B}(0,R)}$ such that $h(x) = x$.

- ◇ If $x_i < 0$, then $r_0(x_i) = 0$ and the i-th component of $h(x)$ is $r_0(x_i) - f_i(r(x)) = -f_i(r(x)) \geqslant 0$, which contradicts $h(x) = x$.
- ◇ If $x_i > 1$, then $r_0(x_i) = 1$ and the i-th component of $h(x)$ is $r_0(x_i) - f_i(r(x)) = 1 - f_i(r(x)) \leqslant 1$, which contradicts $h(x) = x$.

Therefore $x_i \in [0,1]$ for all $1 \leqslant i \leqslant n$ and $x \in [0,1]^n$. So $x = r(x) - f(r(x)) = x - f(x)$, i.e., $f(x) = 0$.

(2) Since $[0,1]^n$ is homeomorphic to $\mathbb{D}^n$, it suffices to show that $[0,1]^n$ has the fixed point property. Let $g\colon [0,1]^n \to [0,1]^n$ be a continuous map and let $f(x) = x - g(x)$. Then for any $1 \leqslant i \leqslant n$,

$$f_i(x) = x_i - g_i(x) = -g_i(x) \leqslant 0, \quad \forall x \in \{x \in [0,1]^n : x_i = 0\},$$
$$f_i(x) = x_i - g_i(x) = 1 - g_i(x) \geqslant 0, \quad \forall x \in \{x \in [0,1]^n : x_i = 1\}.$$

By the Poincaré–Miranda theorem, there exists $p \in [0,1]^n$ such that $f(p) = 0$, i.e., $g(p) = p$. □

Problem 118 (Connectedness of the complement of a Jordan curve) Let M be a surface, and J a Jordan curve in M. Can we conclude that $M \setminus J$ is disconnected? If yes, prove it; if no, give a counterexample.

(1) $M = \mathbb{S}^1 \times \mathbb{R}$.

(2) $M = \Sigma_g$ $(g \geqslant 1)$.

(3) $M =$ Möbius strip.

(4) $M = \mathbb{RP}^2$.

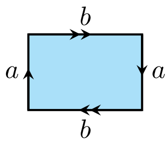
Solution (1) $\boxed{\checkmark}$ The map

$$f\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}_{>0}, \quad (s,t) \mapsto \left(s, \mathrm{e}^t\right)$$

is a homeomorphism. Also note that $\mathbb{S}^1 \times \mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ via polar coordinates. Therefore $\mathbb{S}^1 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, and by the Jordan curve theorem for $\mathbb{R}^2$ we see that $M \setminus J$ is disconnected.

(2) $\boxed{\times}$ The complement of a meridian is connected.

(3) $\boxed{\times}$ Cutting a Möbius strip along the middle line gives a single loop, which is still connected.

(4) $\boxed{\times}$ If we remove the Jordan curve ab from 

, we end up with only one connected component homeomorphic to an open ball. $\square$

Problem 119 (Brouwer's invariance of domain theorem revisited)

(1) **(Higher dimensional analogue of "arc non-separation" theorem)** Prove: if $K \subset \mathbb{R}^n$ is compact and is a retract of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus K$ is connected.

(2) Use the Jordan curve theorem to prove: if $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ is continuous and injective, then $f(\mathbb{B}(0,1))$ is the interior of the Jordan curve $f\left(\mathbb{S}^1\right)$ (i.e., the bounded component).

(3) Assume the Jordan–Brouwer separation theorem holds. State a higher dimensional analogue of (2) and prove it.

Proof (1) Suppose to the contrary that $\mathbb{R}^n \setminus K$ is disconnected. Since K is compact, $\mathbb{R}^n \setminus K$ has at least one bounded connected component A. Take any $x_0 \in A$ and choose $R > 0$ such that $K \subset \mathbb{B}(x_0, R)$. Since K is a retract of $\mathbb{R}^n$, there exists a retraction $r\colon \overline{\mathbb{B}(x_0,R)} \to K$. Define the map

$$h\colon \overline{\mathbb{B}(x_0,R)} \to \overline{\mathbb{B}(x_0,R)}, \quad x \mapsto \begin{cases} r(x), & x \in \overline{A}, \\ x, & x \in A^{\mathrm{c}} \cap \overline{\mathbb{B}(x_0,R)}. \end{cases}$$

Since K is closed, if $x \notin K$, then there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon)$ is contained in a connected component of $\mathbb{R}^n \setminus K$. Then $x \notin \partial A$ and we get $\partial A \subset K$. Since $x_0 \in A$, we have $A \subset \mathbb{B}(x_0, R)$, thus

$$\overline{A} \cap \left(A^{\mathrm{c}} \cap \overline{\mathbb{B}(x_0,R)}\right) = \partial A \cap \overline{\mathbb{B}(x_0,R)} = \partial A \subset K,$$

on which $r(x) = x$. Then by Problem 16 (1), h is continuous on $\overline{\mathbb{B}(x_0,R)}$ (note that A is open by the remarks on page 181). Moreover, since $x_0 \notin K$ and $x_0 \in A$, we have

$$x_0 \notin h\left(\overline{A}\right) \quad \text{and} \quad x_0 \notin h\left(A^{\mathrm{c}} \cap \overline{\mathbb{B}(x_0,R)}\right) \subset A^{\mathrm{c}}.$$

So h is in fact a continuous map into $\overline{\mathbb{B}(x_0,R)} \setminus \{x_0\}$. Composing h with the retraction

$$h_1 \colon \overline{\mathbb{B}(x_0,R)} \setminus \{x_0\} \to \partial\mathbb{B}(x_0,R), \quad x \mapsto x_0 + R\frac{x-x_0}{|x-x_0|},$$

we obtain a continuous map

$$\tilde{h} \coloneqq h_1 \circ h \colon \overline{\mathbb{B}(x_0,R)} \to \partial\mathbb{B}(x_0,R)$$

which is a retraction since for $x \in \partial\mathbb{B}(x_0,R)$ we have $h(x) = x$ and thus $\tilde{h}(x) = x$. This is a contradiction since there is no retraction from $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$.

(2) The map $f\colon \mathbb{D}^2 \to f(\mathbb{D}^2)$ is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. By Remark 2.9.7, we can extend $f^{-1}\colon f(\mathbb{D}^2) \to \mathbb{D}^2$ to a continuous map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$. Let r be the retraction from $\mathbb{R}^2$ to $\mathbb{D}^2$. Then $f \circ r \circ \varphi$ is a retraction from $\mathbb{R}^2$ to $f(\mathbb{D}^2)$. By the Jordan curve theorem, $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ consists of exactly two connected components. Since $f(\mathbb{B}(0,1))$ is connected, it suffices to show that the bounded component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ is contained in $f(\mathbb{B}(0,1))$. Suppose to the contrary that there exists x_0 in the bounded component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ but not in $f(\mathbb{B}(0,1))$. Then there is a retraction from $f(\mathbb{D}^2)$ to $f(\mathbb{S}^1)$. It follows that there is a retraction from $\mathbb{R}^2$ to $f(\mathbb{S}^1)$, which is a contradiction.

(3) *If $f\colon \mathbb{D}^n \to \mathbb{R}^n$ is continuous and injective, then $f(\mathbb{B}(0,1))$ is the bounded component of $\mathbb{R}^n \setminus f(\mathbb{S}^{n-1})$.* Similar to (2), we can show that $f(\mathbb{D}^n)$ is a retract of $\mathbb{R}^n$. By the Jordan–Brouwer separation theorem, if $f(\mathbb{B}(0,1))$ is not the bounded component of $\mathbb{R}^n \setminus f(\mathbb{S}^{n-1})$, then there exists a retraction from $f(\mathbb{D}^n)$ to $f(\mathbb{S}^{n-1})$, and thus one gets a retraction from $\mathbb{R}^n$ to $f(\mathbb{S}^{n-1})$, which is a contradiction. □

Problem 120 (Application to the square peg problem) Let $J \subset \mathbb{R}^2$ be a Jordan curve that is symmetric about the origin (i.e., $P \in J$ if and only if $-P \in J$). Moreover, assume the origin O lies in the bounded connected component of $\mathbb{R}^2 \setminus J$. Prove: J has an inscribed square, i.e., there exist four points on J that are the vertices of a square.

Proof Rotate the curve J by $\frac{\pi}{2}$ and denote the rotated curve by J_0. If we can show that $J \cap J_0$ is nonempty, then by symmetry there exist four points $(\pm x, \pm y)$ on J that are the vertices of a square. For this, we need the following lemma.

Lemma *If J_1, J_2 are two Jordan curves in $\mathbb{R}^2$ such that $J_1 \cap J_2 = \varnothing$ and J_1 is contained in the bounded connected component of $\mathbb{R}^2 \setminus J_2$ (denoted by A_2), then the bounded connected component of $\mathbb{R}^2 \setminus J_1$ (denoted by A_1) is contained in A_2.*

Proof Suppose to the contrary that there exists $x_0 \in A_1 \cap A_2^\complement$. Since $A_2^\complement$ is the closure of the unbounded connected component of $\mathbb{R}^2 \setminus J_2$, by Proposition 3.1.12, $A_2^\complement$ is connected, unbounded, and does not intersect J_1. Thus the connected component of $\mathbb{R}^2 \setminus J_1$ in which x_0 lies must contain $A_2^\complement$, a contradiction to the assumption that A_1 is bounded.

Now we show that $J \cap J_0$ is nonempty. If they are disjoint, then by the Jordan curve theorem, J_0 must be entirely contained in the interior or exterior of J. Without loss of generality, assume J_0 is contained in the interior of J (otherwise J is contained in the interior of J_0). Then rotate J_0 by $\frac{\pi}{2}$ again to get J_1. By symmetry, J_1 is contained in the interior of J_0, and the lemma above implies that J_1 is contained in the bounded connected component of $\mathbb{R}^2 \setminus J$. However, since J is symmetric about the origin, we have $J_1 = J$, which is a contradiction. Therefore $J \cap J_0$ is nonempty, and the proof is complete. □