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Problem 1 (Covering of a topological group)

- (1) Let G be a topological group which is path-connected and locally path-connected, and $p: \tilde{G} \rightarrow G$ be a covering map. Suppose \tilde{G} is path-connected and fix $\tilde{e} \in p^{-1}(e)$. Prove that there exists a unique group structure on \tilde{G} with \tilde{e} its identity element, such that p is a group homomorphism.
- (2) Let \tilde{G} and G be connected topological groups, and $p: \tilde{G} \rightarrow G$ a covering map. Moreover, suppose that p is also a group homomorphism. Prove that G is abelian if and only if \tilde{G} is abelian.

Proof (1) Consider the map

$$\mu: \tilde{G} \times \tilde{G} \rightarrow G, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto p(\tilde{g}_1)p(\tilde{g}_2).$$

We want to lift this map to a map $\tilde{\mu}: \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} & (\tilde{G}, \tilde{e}) & \\ \tilde{\mu} \swarrow & & \downarrow p \\ (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) & \xrightarrow{\mu} & (G, e) \end{array} \quad (\text{P1-1})$$

Observe that

- ◊ Since \tilde{G} is path-connected, $\tilde{G} \times \tilde{G}$ is also path-connected.
- ◊ Since G is locally path-connected, so is $G \times G$. Moreover, because $p \times p$ is a local homeomorphism from $\tilde{G} \times \tilde{G}$ to $G \times G$, $\tilde{G} \times \tilde{G}$ is also locally path-connected.

By Theorem 10.16, the lift in (P1-1) exists if and only if

$$\mu_*\left(\pi_1(\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}))\right) \subset p_*\left(\pi_1(\tilde{G}, \tilde{e})\right). \quad (\text{P1-2})$$

To prove (P1-2), first observe that the image under μ of any loop in $\tilde{G} \times \tilde{G}$ based at (\tilde{e}, \tilde{e}) is of the form $\gamma_1 \cdot \gamma_2$, where γ_1, γ_2 are loops in G based at e , and \cdot denotes the multiplication in G . By Problem 7 (2) ② from last time, $\gamma_1 \cdot \gamma_2$ is path-homotopic to $\gamma_1 * \gamma_2$, thus

$$\begin{aligned} \mu_*\left(\pi_1(\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}))\right) &= p_*\left(\pi_1(\tilde{G}, \tilde{e})\right) * p_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \\ &= p_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \cdot p_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \\ &= p_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \end{aligned}$$

The first equality follows from the functoriality of p_* , and the last equality holds because $p_*\left(\pi_1(\tilde{G}, \tilde{e})\right)$ is a subgroup of $\pi_1(G, e)$. Hence (P1-2) holds, and the lift $\tilde{\mu}$ exists. With $\tilde{\mu}$ as the multiplication map on \tilde{G} , let us verify the group axioms:

Associativity Consider the maps

$$\begin{aligned} \alpha: \tilde{G} \times \tilde{G} \times \tilde{G} &\rightarrow \tilde{G}, \quad (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \mapsto \tilde{\mu}(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3), \\ \beta: \tilde{G} \times \tilde{G} \times \tilde{G} &\rightarrow \tilde{G}, \quad (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \mapsto \tilde{\mu}(\tilde{g}_1, \tilde{\mu}(\tilde{g}_2, \tilde{g}_3)). \end{aligned}$$

Using the commutative diagram (P1-1), we have

$$\begin{aligned}
 p(\alpha(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)) &= p(\tilde{\mu}(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3)) \\
 &= \mu(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2), \tilde{g}_3) \\
 &= p(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2))p(\tilde{g}_3) \\
 &= \mu(\tilde{g}_1, \tilde{g}_2)p(\tilde{g}_3) \\
 &= p(\tilde{g}_1)p(\tilde{g}_2)p(\tilde{g}_3),
 \end{aligned}$$

and similarly

$$p(\beta(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)) = p(\tilde{g}_1)p(\tilde{g}_2)p(\tilde{g}_3).$$

So $p \circ \alpha = p \circ \beta$. As always, $\tilde{G} \times \tilde{G} \times \tilde{G}$ is path-connected, and $p \circ \alpha(\tilde{e}, \tilde{e}, \tilde{e}) = p(\tilde{e})^3 = e^3 = e$.

The diagram

$$\begin{array}{ccc}
 & \nearrow \alpha & \rightarrow (\tilde{G}, \tilde{e}) \\
 (\tilde{G} \times \tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}, \tilde{e})) & \xrightarrow[p \circ \alpha = p \circ \beta]{} & (G, e) \\
 & \searrow \beta & \downarrow p
 \end{array}$$

together with the uniqueness of lifting with base point (Theorem 10.16) implies $\alpha = \beta$.

Identity element Define the maps $f_1, f_2: \tilde{G} \rightarrow \tilde{G}$ by $f_1(\tilde{g}) = \tilde{g}$ and $f_2(\tilde{g}) = \tilde{\mu}(\tilde{e}, \tilde{g})$. Then

$$p(f_2(\tilde{g})) = p(\tilde{\mu}(\tilde{e}, \tilde{g})) = \mu(\tilde{e}, \tilde{g}) = p(\tilde{e})p(\tilde{g}) = ep(\tilde{g}) = p(\tilde{g}) = p(f_1(\tilde{g})).$$

So $p \circ f_1 = p \circ f_2$. Since $p \circ f_1(\tilde{e}) = p(\tilde{e}) = e$ and \tilde{G} is path-connected, the diagram

$$\begin{array}{ccc}
 & \nearrow f_1 & \rightarrow (\tilde{G}, \tilde{e}) \\
 (\tilde{G}, \tilde{e}) & \xrightarrow[p \circ f_1 = p \circ f_2]{} & (G, e) \\
 & \searrow f_2 & \downarrow p
 \end{array}$$

together with the uniqueness of lifting with base point (Theorem 10.16) implies $f_1 = f_2$. Hence $\tilde{\mu}(\tilde{e}, \tilde{g}) = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$. Similarly, $\tilde{\mu}(\tilde{g}, \tilde{e}) = \tilde{g}$ for all $\tilde{g} \in \tilde{G}$.

Inverse element Consider the map

$$i: \tilde{G} \rightarrow G, \quad \tilde{g} \mapsto p(\tilde{g})^{-1}.$$

We want to lift this map to a map $\tilde{i}: \tilde{G} \rightarrow \tilde{G}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & \nearrow \tilde{i} & \rightarrow (\tilde{G}, \tilde{e}) \\
 (\tilde{G}, \tilde{e}) & \xrightarrow[i]{} & (G, e) \\
 & \downarrow p &
 \end{array} \tag{P1-3}$$

Since \tilde{G} is path-connected and locally path-connected, by Theorem 10.16, the lift in (P1-3)

exists if and only if

$$i_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \subset p_*\left(\pi_1(\tilde{G}, \tilde{e})\right). \quad (\text{P1-4})$$

By definition of i , we have

$$i_*\left(\pi_1(\tilde{G}, \tilde{e})\right) \subset p_*\left(\pi_1(\tilde{G}, \tilde{e})\right)^{-1} = p_*\left(\pi_1(\tilde{G}, \tilde{e})\right),$$

where the last equality follows since $p_*\left(\pi_1(\tilde{G}, \tilde{e})\right)$ is a subgroup of $\pi_1(G, e)$. Hence (P1-4) holds, and the lift \tilde{i} exists. By Theorem 10.16, \tilde{i} is unique, so we can define the inverse of any $\tilde{g} \in \tilde{G}$ as $\tilde{i}(\tilde{g})$. Now define the maps $\ell_1, \ell_2: \tilde{G} \rightarrow \tilde{G}$ by $\ell_1(\tilde{g}) = \tilde{\mu}(\tilde{i}(\tilde{g}), \tilde{g})$ and $\ell_2(\tilde{g}) = \tilde{e}$. Then

$$\begin{aligned} p(\ell_1(\tilde{g})) &= p(\tilde{\mu}(\tilde{i}(\tilde{g}), \tilde{g})) = \mu(\tilde{i}(\tilde{g}), \tilde{g}) = p(\tilde{i}(\tilde{g}))p(\tilde{g}) = i(\tilde{g})p(\tilde{g}) = p(\tilde{g})^{-1}p(\tilde{g}) \\ &= e = p(\ell_2(\tilde{g})). \end{aligned}$$

So we obtain $p \circ \ell_1 = p \circ \ell_2$. Since $p \circ \ell_2(\tilde{e}) = e$ and \tilde{G} is path-connected, the diagram

$$\begin{array}{ccc} & & (\tilde{G}, \tilde{e}) \\ & \nearrow \ell_1 & \downarrow p \\ (\tilde{G}, \tilde{e}) & \xrightarrow{\ell_2} & (G, e) \\ & \xrightarrow[p \circ \ell_1 = p \circ \ell_2]{} & \end{array}$$

together with the uniqueness of lifting with base point (Theorem 10.16) implies $\ell_1 = \ell_2$. Hence $\tilde{\mu}(\tilde{i}(\tilde{g}), \tilde{g}) = \tilde{e}$ for all $\tilde{g} \in \tilde{G}$. Similarly, $\tilde{\mu}(\tilde{g}, \tilde{i}(\tilde{g})) = \tilde{e}$ for all $\tilde{g} \in \tilde{G}$.

Therefore, \tilde{G} admits a group structure with \tilde{e} as the identity element. The fact that p is a group homomorphism follows immediately from the commutativity of (P1-1), i.e.,

$$p(\tilde{\mu}(\tilde{g}_1, \tilde{g}_2)) = \mu(\tilde{g}_1, \tilde{g}_2) = p(\tilde{g}_1)p(\tilde{g}_2).$$

The uniqueness of the group structure follows from the uniqueness of the lift in (P1-1).

(2) (\Rightarrow) Consider the map

$$d: \tilde{G} \times \tilde{G} \rightarrow \tilde{G}, \quad (\tilde{g}_1, \tilde{g}_2) \mapsto \tilde{g}_1 \tilde{g}_2 \tilde{g}_1^{-1} \tilde{g}_2^{-1}.$$

If G is abelian, then $p(d(\tilde{g}_1, \tilde{g}_2)) = e_G$, which means that the image of d is contained in $p^{-1}(e_G)$. Since p is a covering, $p^{-1}(e_G)$ is discrete. Since d is continuous and $\tilde{G} \times \tilde{G}$ is connected, the image of d must be connected. Thus the image of d is the single point $e_{\tilde{G}}$, i.e., \tilde{G} is abelian.

(\Leftarrow) Since G is connected, the map p is a group epimorphism. Hence G is abelian whenever \tilde{G} is abelian. \square

Problem 2 (Abelianization) Let G be a group, and let $[G, G]$ denote the subgroup of G generated by all elements of the form $xyx^{-1}y^{-1}$, where $x, y \in G$. Prove the following statements:

- (1) $[G, G]$ is a normal subgroup of G .
- (2) The group $\text{Ab}(G) := G/[G, G]$ is abelian (called the **abelianization** of G).
- (3) The abelianization defines a functor from **Grp** to **Ab**.

$$(4) \text{ Ab}(\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_n) = \mathbb{Z}^n.$$

$$(5) \text{ Ab}(\langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle) = \mathbb{Z}^{2n}.$$

$$(6) \text{ Ab}(\langle a_1, \dots, a_n \mid a_1^2 \cdots a_n^2 = 1 \rangle) = \mathbb{Z}^{n-1} \times \mathbb{Z}_2.$$

Proof (1) For any $u \in [G, G]$ and $g \in G$, we have $gug^{-1} = u(u^{-1}gug^{-1}) \in [G, G]$. Thus $g[G, G]g^{-1} \subset [G, G]$ and $[G, G]$ is normal.

(2) Since $[g][h][g]^{-1}[h]^{-1} = [ghg^{-1}h^{-1}] = [e]$, we have $[g][h] = [h][g]$ for all $[g], [h] \in \text{Ab}(G)$.

(3) ① By (2), Ab associates each object G in **Grp** to an object $\text{Ab}(G)$ in **Ab**.

② For any group homomorphism $f: G \rightarrow H$, define $\text{Ab}(f): \text{Ab}(G) \rightarrow \text{Ab}(H)$ by $\text{Ab}(f)([g]) = [f(g)]$. If $g_1, g_2 \in G$ satisfy $[g_1] = [g_2]$, then $g_1 g_2^{-1} = \prod_{k=1}^n x_k y_k x_k^{-1} y_k^{-1}$ for some $x_i, y_i \in G$ and

$$f(g_1) f(g_2)^{-1} = \prod_{k=1}^n f(x_k) f(y_k) f(x_k)^{-1} f(y_k)^{-1} \in [H, H].$$

So $[f(g_1)] = [f(g_2)]$, and $\text{Ab}(f): \text{Ab}(G) \rightarrow \text{Ab}(H)$ is a well-defined morphism in **Ab**.

③ $\text{Ab}(\text{Id}_G) = \text{Id}_{\text{Ab}(G)}$ for every $G \in \text{Grp}$, and $\text{Ab}(g \circ f) = \text{Ab}(g) \circ \text{Ab}(f)$ for all morphisms $f: G \rightarrow H$ and $g: H \rightarrow K$ in **Grp**.

$$(4) \text{ Ab}(\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_n) = \langle a_1, \dots, a_n \mid a_i a_j = a_j a_i, 1 \leq i < j \leq n \rangle = \mathbb{Z}^n.$$

$$(5) \text{ Ab}(\langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle) = \text{Ab}(\langle a_1, b_1, \dots, a_n, b_n \rangle) = \mathbb{Z}^{2n}.$$

$$(6) \text{ Ab}(\langle a_1, \dots, a_n \mid a_1^2 \cdots a_n^2 = 1 \rangle) = \mathbb{Z}^n / \langle (2, \dots, 2) \rangle. \text{ Since the Smith normal form of the matrix } \begin{pmatrix} 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 0 & \cdots & 0 \end{pmatrix} \text{ is } \text{diag}(2, \underbrace{0, \dots, 0}_{n-1}), \text{ we have}$$

$$\mathbb{Z}^n / \langle (2, \dots, 2) \rangle \simeq \mathbb{Z}^n / (2\mathbb{Z} \times \underbrace{0\mathbb{Z} \times \cdots \times 0\mathbb{Z}}_{n-1}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times \underbrace{(\mathbb{Z}/\{0\}) \times \cdots \times (\mathbb{Z}/\{0\})}_{n-1} \simeq \mathbb{Z}_2 \times \mathbb{Z}^{n-1}.$$

□

Problem 3 Describe all the covering spaces of the torus, projective plane, Klein bottle, Möbius strip, and cylinder.

Solution Recall the following statement from page 232 of the textbook:

Suppose X has a universal covering space \tilde{X} . Then the **covering transformations** form a group isomorphic to $\pi_1(X)$. Given any subgroup H of $\pi_1(X)$, it acts on \tilde{X} and the associated orbit space \tilde{X}/H is a covering space of X whose fundamental group is isomorphic to H .

Torus \mathbb{T}^2 The universal covering space of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is \mathbb{R}^2 , with $\pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$ acting by translations.

Every subgroup $H \leq \mathbb{Z}^2$ is free abelian of rank 0, 1, or 2.

◇ If rank $H = 2$, then \mathbb{R}^2/H is a torus.

- ◊ If $\text{rank } H = 1$, then \mathbb{R}^2/H is a cylinder $\mathbb{S}^1 \times \mathbb{R}$.
- ◊ If $\text{rank } H = 0$, then \mathbb{R}^2/H is \mathbb{R}^2 itself.

Projective plane \mathbb{RP}^2 The universal covering space of \mathbb{RP}^2 is \mathbb{S}^2 , with $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ acting by the antipodal map. The only subgroups of \mathbb{Z}_2 are the trivial group and \mathbb{Z}_2 itself.

- ◊ If $H = \mathbb{Z}_2$, then \mathbb{S}^2/H is \mathbb{RP}^2 itself.
- ◊ If $H = \{0\}$, then \mathbb{S}^2/H is \mathbb{S}^2 .

Klein bottle K The universal covering space of the Klein bottle K is \mathbb{R}^2 . Its fundamental group has presentation

$$\pi_1(K) = \langle a, b \mid aba = b \rangle,$$

and acts on \mathbb{R}^2 by

$$\begin{aligned} a: (x, y) &\mapsto (x + 1, y), \\ b: (x, y) &\mapsto (-x + 1, y + 1). \end{aligned}$$

By $ab = ba^{-1}$ and $ba = a^{-1}b$, any element in $\pi_1(K)$ can be expressed as $a^m b^n$ for some $m, n \in \mathbb{Z}$.

- ◊ Suppose $H = \langle a^k \rangle$ for some $k \geq 0$. Then \mathbb{R}^2/H is a cylinder $\mathbb{S}^1 \times \mathbb{R}$ when $k > 0$ and \mathbb{R}^2 when $k = 0$.
- ◊ Now suppose H is not generated solely by powers of a .
 - Suppose every power of b appearing in elements of H is even. Since a commutes with b^2 , H is a subgroup of $\langle a, b^2 \mid ab^2 = b^2a \rangle = \text{Ab}(\mathbb{Z} * \mathbb{Z}) \simeq \mathbb{Z}^2$ by Problem 2 (4). Note that

$$\begin{aligned} a: (x, y) &\mapsto (x + 1, y), \\ b^2: (x, y) &\mapsto (x, y + 2). \end{aligned}$$

Hence we reduce to the torus case, so \mathbb{R}^2/H is either a torus or a cylinder $\mathbb{S}^1 \times \mathbb{R}$.

- Suppose there exists an element in H with an odd power of b . Then

$$T := \{t \in \mathbb{Z} : a^n b^t \in H \text{ for some } n \in \mathbb{Z}\}$$

is a nonempty additive subgroup of \mathbb{Z} , so $T = k\mathbb{Z}$ for some $k \geq 1$. Thus

$$H = \{a^n b^{kt} : n, t \in \mathbb{Z}\}. \quad (\text{P3-1})$$

Moreover, **k is odd** by assumption. Fix an element $a^{n_1} b^k \in H$. Since $\{a^n \in H : n \in \mathbb{Z}\}$ is a subgroup of \mathbb{Z} , it is of the form $\langle a^{n_0} \rangle$ for some $n_0 \geq 0$.

We claim that $H = \langle a^{n_0}, a^{n_1} b^k \rangle$. Indeed, using (P3-1), from

$$(a^n b^{kt}) (a^{n_1} b^k)^{-t} = a^{(n-n_1)t} \in \langle a^{n_0} \rangle$$

we see the claim holds.

When $n_0 = 0$, we have $H = \langle a^{n_1} b^k \rangle$. Note that

$$b^k: (x, y) \mapsto (-x + 1, y + k),$$

$$a^{n_1}b^k : (x, y) \mapsto (-x + 1 + n_1, y + k).$$

In this case, \mathbb{R}^2/H is a [Möbius strip].
When $n_0 > 0$, we have

$$\begin{aligned} a^{n_0} : (x, y) &\mapsto (x + n_0, y), \\ a^{n_1}b^k : (x, y) &\mapsto (-x + 1 + n_1, y + k). \end{aligned}$$

In this case, \mathbb{R}^2/H is a [Klein bottle].

Möbius strip M The universal covering space of the (infinite) Möbius strip M is \mathbb{R}^2 , with $\pi_1(M) \simeq \mathbb{Z}$ acting by a glide reflection:

$$n \cdot (x, y) = (x + n, (-1)^n y).$$

Subgroups of \mathbb{Z} are of the form $n\mathbb{Z}$ for some $n \geq 0$.

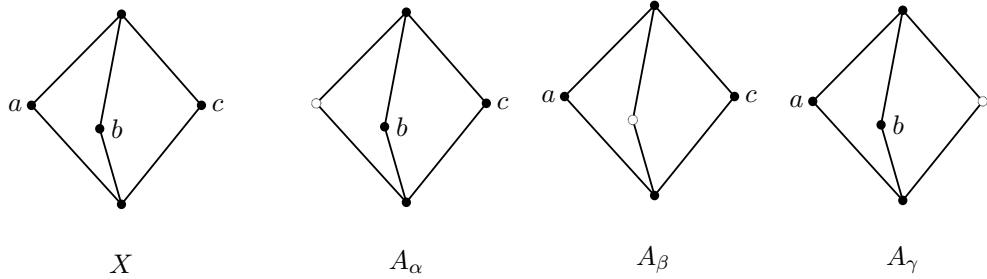
- ◊ If n is even, then $\mathbb{R}^2/n\mathbb{Z}$ is a [cylinder $\mathbb{S}^1 \times \mathbb{R}$] when $n > 0$ and [\mathbb{R}^2] when $n = 0$.
- ◊ If n is odd, then $\mathbb{R}^2/n\mathbb{Z}$ is a [Möbius strip].

Cylinder C The universal covering space of the (infinite) cylinder $C = \mathbb{S}^1 \times \mathbb{R}$ is \mathbb{R}^2 , with $\pi_1(C) \simeq \mathbb{Z}$ acting by translations in one direction. Any nontrivial subgroup of \mathbb{Z} yields a quotient homeomorphic to a [cylinder $\mathbb{S}^1 \times \mathbb{R}$], while the trivial subgroup yields [\mathbb{R}^2]. \square

Problem 4 (van Kampen's theorem) If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi: *_\alpha \pi_1(A_\alpha) \rightarrow \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \simeq *_\alpha \pi_1(A_\alpha)/N$.

Remark 1 (1) The proof of this theorem can be found in [Allen Hatcher's Algebraic Topology](#), p. 44.

(2) For an example showing that triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ need to be path-connected, let X be the suspension¹ of three points a, b, c , and let A_α, A_β , and A_γ be the complements of these three points.



The theorem does apply to the covering $\{A_\alpha, A_\beta\}$, so there are isomorphisms

$$\pi_1(X) \simeq \pi_1(A_\alpha) * \pi_1(A_\beta) \simeq \mathbb{Z} * \mathbb{Z}$$

¹The **suspension** of a topological space Y is the quotient space of the cylinder space $Y \times [-1, 1]$ by the relation which identifies all the points at either end.

since $A_\alpha \cap A_\beta$ is contractible. If we tried to use the covering $\{A_\alpha, A_\beta, A_\gamma\}$, which has each of the twofold intersections path-connected but not the triple intersection, then we would get

$$\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z},$$

but this is not isomorphic to $\mathbb{Z} * \mathbb{Z}$ by Problem 2 (4).