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Given a family of topological spaces $(X_\alpha, \mathcal{T}_\alpha)$, we can define the **box topology** \mathcal{T}_{box} on $\prod_\alpha X_\alpha$ using the base

$$\mathcal{B} = \left\{ \prod_\alpha U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\}.$$

Then

$$\mathcal{T}_{\text{box}} = \left\{ U \subset \prod_\alpha X_\alpha : \text{for any } (x_\alpha) \in U \text{ there exists } U_\alpha \in \mathcal{T}_\alpha \text{ such that } (x_\alpha) \in \prod_\alpha U_\alpha \subset U \right\}.$$

Also recall that the **product topology** $\mathcal{T}_{\text{product}}$ on $\prod_\alpha X_\alpha$ is defined using the subbase

$$\mathcal{S} = \bigcup_\beta \left\{ \pi_\beta^{-1}(V_\beta) : V_\beta \in \mathcal{T}_\beta \right\},$$

so that the open sets are the unions of subsets $\prod_\alpha U_\alpha$, where $U_\alpha \in \mathcal{T}_\alpha$ with the additional condition that $U_\alpha = X_\alpha$ for all but finitely many indices α . The reason for this choice of open sets is that these are the *least* needed to make the projection onto the α -th factor π_α continuous for all indices α .

Any map $f: Y \rightarrow \prod_\alpha X_\alpha$ is continuous with respect to the product topology if and only if each of its component maps $\pi_\alpha \circ f: Y \rightarrow X_\alpha$ is continuous. However, this is not true for the box topology when there are infinitely many factors. For example, the map

$$f: \mathbb{R} \rightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\text{box}}), \quad t \mapsto (t, t, t, \dots),$$

is not continuous, since

$$(-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2} \right) \times \left(-\frac{1}{3}, \frac{1}{3} \right) \times \dots \in \mathcal{T}_{\text{box}}$$

but

$$f^{-1}\left((-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2} \right) \times \left(-\frac{1}{3}, \frac{1}{3} \right) \times \dots \right) = \{0\},$$

which is not open in \mathbb{R} .

Problem 1 Let X_α be topological spaces, and $A_\alpha \subset X_\alpha$. Consider the box topology \mathcal{T}_{box} and the product topology $\mathcal{T}_{\text{product}}$ on $\prod_\alpha X_\alpha$.

(1) Show that in both topologies, we have $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$.

(2) Prove that for \mathcal{T}_{box} , $\text{Int}\left(\prod_\alpha A_\alpha\right) = \prod_\alpha \text{Int}(A_\alpha)$, and give an example showing that this equality may fail for $\mathcal{T}_{\text{product}}$.

Solution (1) Let us show the inclusion in both directions for either topology.

□ Let (x_α) be a point of $\prod_\alpha \overline{A_\alpha}$ and $U = \prod_\alpha U_\alpha$ be a basis element for either \mathcal{T}_{box} or $\mathcal{T}_{\text{product}}$ that contains (x_α) . Since $x_\alpha \in \overline{A_\alpha}$ and U_α is an open neighborhood of x_α , we can choose a point

$y_\alpha \in U_\alpha \cap (A_\alpha \setminus \{x_\alpha\})$ for each α . Then $(y_\alpha) \in U \cap \left(\prod_{\alpha} A_\alpha \setminus \{(x_\alpha)\} \right)$. Since U is arbitrary, it follows that $(x_\alpha) \in \overline{\prod_{\alpha} A_\alpha}$.

□ Suppose $(x_\alpha) \in \overline{\prod_{\alpha} A_\alpha}$ in either topology. For each given index β , let V_β be an arbitrary open set of X_β containing x_β . Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_{\alpha} X_\alpha$ in either topology, if $x_\beta \notin A_\beta$, then $\pi_\beta^{-1}(V_\beta)$ contains a point $(y_\alpha) \in \prod_{\alpha} A_\alpha$. Then $y_\beta \in V_\beta \cap (A_\beta \setminus \{x_\beta\})$. Since V_β is arbitrary, it follows that $x_\beta \in \overline{A_\beta}$.

(2) We first show the inclusion in both directions for \mathcal{T}_{box} .

□ Since each $\text{Int}(A_\alpha)$ is open in X_α , the inclusion holds.

□ Let $(x_\alpha) \in \text{Int}\left(\prod_{\alpha} A_\alpha\right)$, then for each α there is an open set $U_\alpha \subset X_\alpha$ containing x_α with

$$\prod_{\alpha} U_\alpha \subset \text{Int}\left(\prod_{\alpha} A_\alpha\right) \subset \prod_{\alpha} A_\alpha.$$

This implies $U_\alpha \subset A_\alpha$ for each α , hence $U_\alpha \subset \text{Int}(A_\alpha)$ and $(x_\alpha) \in \prod_{\alpha} \text{Int}(A_\alpha)$.

This equality fails for $\mathcal{T}_{\text{product}}$. Take $X_n = \mathbb{R}$ and $A_n = (0, 1)$ for each $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty} \text{Int}(A_n) = \prod_{n=1}^{\infty} (0, 1)$. However, by the definition of product topology, this cannot be an open set in $\prod_{n=1}^{\infty} X_n$, and hence $\text{Int}\left(\prod_{n=1}^{\infty} A_n\right) \neq \prod_{n=1}^{\infty} \text{Int}(A_n)$. □

Problem 2 Let (X_n, d_n) be compact metric spaces. Define a metric on $X = \prod_{n=1}^{\infty} X_n$ by

$$d((x_n), (y_n)) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{[1 + \text{diam}(X_n)] \cdot 2^n}.$$

Prove that on X the metric topology \mathcal{T}_d coincides with the product topology $\mathcal{T}_{\text{product}}$.

Proof \$\mathcal{T}_d \subset \mathcal{T}_{\text{product}} Suppose $U \in \mathcal{T}_d$. Then for any $(a_n) \in U$ there exists $r > 0$ such that $B((a_n), r) \subset U$. Consider

$$V = B_1(a_1, \frac{r}{2}) \times \cdots \times B_N(a_N, \frac{r}{2}) \times X_{N+1} \times X_{N+2} \times \cdots \in \mathcal{T}_{\text{product}}.$$

Take $N \in \mathbb{N}$ such that $2^{-N} < \frac{r}{2}$, then for all $(b_n) \in V$ we have

$$\begin{aligned} d((a_n), (b_n)) &= \sum_{n=1}^{\infty} \frac{d_n(a_n, b_n)}{[1 + \text{diam}(X_n)] \cdot 2^n} \leq \sum_{n=1}^N \frac{\frac{r}{2}}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &= \frac{r}{2} \left(1 - \frac{1}{2^N}\right) + \frac{1}{2^N} < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Hence $(a_n) \in V \subset \mathbb{B}((a_n), r) \subset U$, which implies $U \in \mathcal{T}_{\text{product}}$.

$\boxed{\mathcal{T}_{\text{product}} \subset \mathcal{T}_d}$ Suppose $V \in \mathcal{T}_{\text{product}}$, which has the form

$$V = V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \cdots,$$

where each V_i is open in X_i . For any $(x_n) \in V$, there exists $r_1, \dots, r_k > 0$ such that $\mathbb{B}(x_i, r_i) \subset V_i$ for $1 \leq i \leq k$. Let

$$r_0 = \min \left\{ \frac{r_1}{[1 + \text{diam}(X_1)] \cdot 2^1}, \dots, \frac{r_k}{[1 + \text{diam}(X_k)] \cdot 2^k} \right\} > 0,$$

then $(x_n) \in \mathbb{B}((x_n), r_0) \subset V$. Hence $V \in \mathcal{T}_d$. \square

Problem 3 Show that \mathbb{S}^n is path-connected for all $n \geq 1$.

Proof 1 Given distinct points $x, y \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$:

◇ If $y \neq -x$, then the path

$$\gamma: [0, 1] \rightarrow \mathbb{S}^n, \quad t \mapsto \frac{(1-t)x + ty}{\|(1-t)x + ty\|}$$

is continuous and connects x to y .

◇ If $y = -x$, then we can choose some $z \in \mathbb{S}^n$ such that $z \neq \pm x$. By the previous case, there exist paths $\gamma_1, \gamma_2 \in C([0, 1], \mathbb{S}^n)$ such that γ_1 connects x to z and γ_2 connects z to y . Then the concatenation of γ_1 and γ_2 ,

$$\gamma_1 * \gamma_2: [0, 1] \rightarrow \mathbb{S}^n, \quad t \mapsto \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2, \\ \gamma_2(2t-1), & 1/2 \leq t \leq 1, \end{cases}$$

is a path connecting x to y . \square

Proof 2 Consider the spherical coordinates representation of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$:

$$\begin{aligned} x_1 &= \cos(\theta_1), \\ x_2 &= \sin(\theta_1) \cos(\theta_2), \\ x_3 &= \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \\ &\vdots \\ x_n &= \sin(\theta_1) \cdots \sin(\theta_{n-1}) \cos(\theta_n), \\ x_{n+1} &= \sin(\theta_1) \cdots \sin(\theta_{n-1}) \sin(\theta_n). \end{aligned}$$

For any two points

$$\boldsymbol{\theta}^{(i)} = (\theta_1^{(i)}, \dots, \theta_n^{(i)}) \in \mathbb{S}^n, \quad i = 1, 2,$$

we can define a path $\gamma \in C([0, 1], \mathbb{S}^n)$ from $\boldsymbol{\theta}^{(1)}$ to $\boldsymbol{\theta}^{(2)}$ by

$$\gamma(t) = (1-t)\boldsymbol{\theta}^{(1)} + t\boldsymbol{\theta}^{(2)}.$$

\square

Proof 3 Choose the north pole $N = (0, 0, \dots, 0, 1) \in \mathbb{S}^n$, and define the stereographic projection

$$\sigma: \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$$

by projecting points from N onto the hyperplane $x_{n+1} = 0$.

Given distinct points $x, y \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$:

◇ If $x = -y = N$, then the path

$$\gamma: [0, 1] \rightarrow \mathbb{S}^n, \quad t \mapsto (0, \dots, 0, \sin(\pi t), \cos(\pi t))$$

connects x to y .

- ◇ If $x, y \neq N$, then we can find a path $\gamma \in C([0, 1], \mathbb{R}^n)$ connecting $\sigma(x)$ to $\sigma(y)$ in \mathbb{R}^n . Then the composition $\sigma^{-1} \circ \gamma$ is a path in \mathbb{S}^n connecting x to y .
- ◇ If $x = N$ and $y \neq -N$, then we can first reflect both points about the hyperplane $x_{n+1} = 0$ to get x' and y' , then find a path from x' to y' as in the previous case, and finally reflect the path back to get a path from x to y . \square

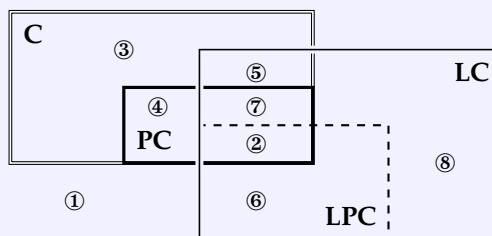
Problem 4 A topological space X is **locally (path-)connected** if for each $x \in X$, and each open neighborhood U of x , there is a (path-)connected open neighborhood V of x with $V \subset U$.

(1) Is $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ connected/locally connected/path-connected/locally path-connected?

(2) For simplicity, let us denote

$$\begin{aligned} \mathbf{C} &= \text{connected}, & \mathbf{LC} &= \text{locally connected}, \\ \mathbf{PC} &= \text{path-connected}, & \mathbf{LPC} &= \text{locally path-connected}. \end{aligned}$$

Give examples for regions ① to ⑧ in the following diagram:



- (3) Prove that if X is compact and locally connected, then X has finitely many connected components. Can we remove the local connectedness condition?
- (4) Prove that X is locally connected if and only if for any open set U in X , any connected component of U is open. In particular, any connected component of a locally connected space is open.
- (5) Suppose X is locally connected, $f: X \rightarrow Y$ is continuous. Prove that if f is either open or closed, then $f(X)$ is locally connected. Can we remove the assumption " f is either open or closed"?

Proof (1) ① Since \mathbb{R} is uncountable, there are no disjoint non-empty open subsets in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$, so it is connected.

- ② For any open neighborhood U of x and any two open sets $A, B \subset \mathbb{R}$ such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, we have $U \cap A, U \cap B \in \mathcal{T}_{\text{cocountable}}$. Hence $A \cap B \cap U = (U \cap A) \cap (U \cap B)$ is the intersection of two nonempty open sets, which is nonempty for the same reason as in ①. Therefore any open neighborhood of x is connected, so $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is locally connected.
- ③ For any $f \in C([0, 1], (\mathbb{R}, \mathcal{T}_{\text{cocountable}}))$, the image $f([0, 1])$ is compact. However, compact sets in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ must be finite (any infinite subset A contains some countable subset $\{x_n\}_{n=1}^{\infty}$, and the open cover $\left\{\left(\bigcup_{k \neq n} \{x_k\}\right)^c\right\}_{n=1}^{\infty}$ of A has no finite subcover). Since the topology on any finite set in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is discrete, $f([0, 1])$ would be totally disconnected (*i.e., a topological space that has only singletons as connected subsets*) if it has more than one point. But now it is the continuous image of $[0, 1]$, so $f([0, 1])$ must be a singleton. Therefore f must be constant, which means $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is not path-connected.
- ④ Since $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is connected but not path-connected, it is not locally path-connected by the implication

$$\mathbf{C} + \mathbf{LPC} \implies \mathbf{PC}.$$

Hint For any fixed $x \in X$, the set of points path-connected to x is a clopen subset of X .

- (2) ① \mathbb{Q} is neither **C** nor **LC**.
- ② \mathbb{R} is both **PC** and **LPC**.
- ③ The topologist's sine curve is **C** but neither **PC** nor **LC**.
- ④ The topologist's sine curve with an additional path from $(0, 0)$ to $(1, 0)$ is **PC** but not **LC**.
- ⑤ $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is **C** and **LC** but not **PC**, as shown in (1).
- ⑥ $(0, 1) \cup (1, 2)$ is **LPC** but not **PC**.
- ⑦ Let L^* be the space $\Omega \times [0, 1] \cup \{(\omega_1, 0)\}$ with the lexicographic order topology, where $\Omega = [0, \omega_1)$ is the minimal uncountable well-ordered set. Denote by C the quotient of L^* identifying its initial point $(0, 0)$ and final point $(\omega_1, 0)$. Then C is **PC**, **LC**, but not **LPC**. (It is not locally path-connected at $[(\omega_1, 0)]$.)
- ⑧ The disjoint union of two copies of ⑤-type space is **LC** but neither **C** nor **LPC**.
- (3) Since X is compact and locally connected, it can be covered by finitely many connected open sets. Therefore X has finitely many connected components. The local connectedness condition is necessary. For example, the Cantor set is compact, but it has uncountably many connected components.
- (4) (\Rightarrow) Let C be a connected component of U . Since any point in C has a connected open neighborhood, which must lie in C , C is open.
- (\Leftarrow) For any $x \in X$ and any neighborhood V of x , V contains an open neighborhood U of x . Then the connected component of U containing x is the desired connected open neighborhood of x which is contained in V .
- (5) We shall use the characterization of local connectedness in (4). By our assumptions, $f: X \rightarrow f(X)$ is an identification map. Let V be an open set in $f(X)$ and C be a connected component of V . We want to show that C is open, i.e., $f^{-1}(C)$ is open in X . For any $x \in f^{-1}(C)$, we have $x \in f^{-1}(V)$, which is open in X . Since X is locally connected, there exists a connected open neighborhood U of x such that $U \subset f^{-1}(V)$. Then $f(U)$ is also connected and $f(U) \cap C \ni f(x)$, so $f(U) \cup C$ is connected and contained in V . Since C is a connected component of V , we must have $f(U) \cup C = C$, i.e.,

$f(U) \subset C$. Therefore $x \in U \subset f^{-1}(C)$, which implies $f^{-1}(C)$ is open. It follows that C is connected and by (4) we see that $f(X)$ is locally connected.

The assumption “ f is either open or closed” is necessary. Let (X, \mathcal{T}_X) be any non-locally connected space. Then $(X, \mathcal{T}_{\text{discrete}})$ is locally connected and the identity map $\text{Id}_X : (X, \mathcal{T}_{\text{discrete}}) \rightarrow (X, \mathcal{T}_X)$ is continuous. However, the image is not locally connected. \square