Name	SOLUTION
PID #	
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STAT 598G Spring 2011
Quiz #1
February 3, 2011

You are not allowed to use books or notes. Please read the directions carefully. The quiz is graded out of 3 points. You have 15 minutes to complete it. Please show all your work.

Write down pseudocode for an algorithm that uses a bisection method to compute the cubic root of a positive real number x. The root is to be found with a given precision $\epsilon > 0$. Please make the inputs, outputs, and assumptions made in the algorithm. What is the worst-case running time complexity for your algorithm? **No credit will be given if the algorithm is not using a bisection method.**

Solution: We are interested in y such $y^3 - x = 0$. Instead of finding the exact value of y we want $y > 0$ such that $|y - \sqrt[3]{x}| < \epsilon$. We will keep track of the value y_- such that $y_-^3 - x \leq 0$ and the value y_+ such that $y_+^3 - x \geq 0$. Then there exists $y \in [y_-, y_+]$. (y^3 is an increasing function for $y > 0$.) Initially, y_- can be set to 0, guaranteeing $y_-^3 < x$ as $x > 0$. y_+ can be set to x if $x > 1$, and to 1 otherwise.

There are two possible approaches, non-recursive (Algorithm 1) and recursive (Algorithm 2). Note that the recursion will require four inputs. To start the recursive version, one needs to call CubicRootRecursive($x, 0, \max(1, x), \epsilon$).

Algorithm 1 Cubic root by bisection without recursion

function CubicRoot(x,ϵ)
    Inputs: $x > 0,\ \epsilon > 0$
    Output: $y > 0$ such that $|y - \sqrt[3]{x}| < \epsilon$
    $y_- = 0$ ▷ left end-point
    $y_+ = \max(1, x),\ y = y_+$ ▷ right end-point
    while $y_+ - y_- \geq \epsilon$ **do**
        $y = \frac{1}{2}(y_- + y_+)$ ▷ current value is the midpoint of the interval
        if $y^3 < x$ **then**
            $y_- = y$ ▷ updating left end-point
        else
            $y_+ = y$ ▷ updating right end-point
        end if
    end while
    return y
end function

Algorithm 2 Cubic root by bisection with recursion

function CubicRootRecursive(x, y_-, y_+, ϵ)
    Inputs: $x > 0$, y_-, s.t. $y_-^3 \leq x$, y_+, s.t. $y_+^3 \geq x$, $\epsilon > 0$
    Output: $y > 0$ such that $|y - \sqrt[3]{x}| < \epsilon$
    if $|y_+ - y_-| < \epsilon$ **then**
        $y = y_-$ ▷ interval within precision, pick any value within the interval
    else
        $y = \frac{1}{2}(y_- + y_+)$
        if $y^3 < x$ **then**
            $y =$CubicRootRecursive(x, y, y_+, ϵ) ▷ updating left end-point
        else
            $y =$CubicRootRecursive(x, y_-, y, ϵ) ▷ updating right end-point
        end if
    end if
    return y
end function

Initially, the length of the interval is $\max(x, 1)$. Each iteration of the algorithm cuts the interval's length in half. Assuming n iterations, $2^{-n}\max(x, 1) < \epsilon \Rightarrow n > \log\max(x, 1) - \log\epsilon$, so $n = \lceil \log_2\{\max(x, 1)/\epsilon\} \rceil \in O(\max(-\log\epsilon, \log x - \log\epsilon))$.