Supplementary for “Adaptive demand response with heterogeneous types of thermostatically controlled loads”

Appendix A
Proof of Theorem 1

Proof: Consider the observable canonical state-space form (15) with matrices as in (16). Because $R_p(z)$ is coprime to $Z_{q,1}(z)$ and $Z_{q,2}(z)$, then (A,B) is stabilizable.

Let $\boldsymbol{x}_o(k) \triangleq \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k)$ be the state-observation error. Substracting (25) from (15) and using (24) in (25), we obtain

$$\begin{bmatrix}\boldsymbol{x}_o(k+1)\\ \hat{\boldsymbol{x}}(k+1)\end{bmatrix} = \begin{bmatrix}A-K_oC^\top & \mathbf{0}\\ K_oC^\top & A-BK\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_o(k)\\ \hat{\boldsymbol{x}}(k)\end{bmatrix}. \tag{29}$$

Since the eigenvalues of the state matrix in (29) are the eigenvalues of $A-K_oC^\top$, and of $A-BK$, where $A-K_oC^\top$ is a stable matrix by design, it follows that the origin of (29) is exponentially stable if and only if $A-BK$ is a stable matrix. So, consider

$$\breve{\boldsymbol{x}}(k+1) = (A-BK)\breve{\boldsymbol{x}}(k) \tag{30}$$

and we aim to prove that its origin is exponentially stable. Using (24) and $Q = CC^\top$, let us manipulate the Riccati equation (20) as

$$\begin{aligned}
P &= A^\top PA - (A^\top PB)(B^\top PB+R)^{-1}(B^\top PB+R)\\
&\quad (B^\top PB+R)^{-1}(B^\top PA)+Q\\
&= A^\top PA - K^\top(B^\top PB+R)K + Q\\
&= (A-BK)^\top P(A-BK) + K^\top B^\top PA + A^\top PBK\\
&\quad - K^\top B^\top PBK - K^\top(B^\top PB+R)K + Q\\
&= (A-BK)^\top P(A-BK) + 2K^\top(B^\top PB+R)K\\
&\quad - K^\top B^\top PBK - K^\top(B^\top PB+R)K + Q\\
&= (A-BK)^\top P(A-BK) + K^\top RK + CC^\top.
\end{aligned}$$

Choose the Lyapunov function

$$V(k) = \breve{\boldsymbol{x}}^\top(k)P\breve{\boldsymbol{x}}(k)$$

and compute the change of V along the trajectory of (30)

$$\begin{aligned}
\Delta V(k) &= \breve{\boldsymbol{x}}^\top(k+1)P\breve{\boldsymbol{x}}(k+1) - \breve{\boldsymbol{x}}^\top(k)P\breve{\boldsymbol{x}}(k)\\
&= \breve{\boldsymbol{x}}^\top(k)(A-BK)^\top P(A-BK)\breve{\boldsymbol{x}}(k) - \breve{\boldsymbol{x}}^\top(k)P\breve{\boldsymbol{x}}(k)\\
&= \breve{\boldsymbol{x}}^\top(k)[(A-BK)^\top P(A-BK)-P]\breve{\boldsymbol{x}}(k)\\
&= -\breve{\boldsymbol{x}}^\top(k)C^\top C\breve{\boldsymbol{x}}(k) - \breve{\boldsymbol{x}}^\top(k)K^\top RK\breve{\boldsymbol{x}}(k) \le 0
\end{aligned}$$

which implies that the origin of (30) is stable. Since

$$(A-BK)^\top P(A-BK) - P = -\begin{bmatrix}C^\top\\ R^{\frac{1}{2}}K\end{bmatrix}^\top\begin{bmatrix}C^\top\\ R^{\frac{1}{2}}K\end{bmatrix},$$

according to [34, Thm. A.12.22], if $\left(A-BK, \begin{bmatrix}C^\top\\ R^{\frac{1}{2}}K\end{bmatrix}\right)$ is observable, then the origin of (30) is exponentially stable.

Rewrite R as $R = (R^{\frac{1}{2}})^\top R^{\frac{1}{2}}$. Then $|R^{\frac{1}{2}}| \neq 0$, so that $B = MR^{\frac{1}{2}}$ for some M. Therefore, we have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix}zI-A\\ C^\top\\ R^{\frac{1}{2}}K\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}I & 0 & M\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}zI-A\\ C^\top\\ R^{\frac{1}{2}}K\end{bmatrix}\\
&= \operatorname{rank}\begin{bmatrix}zI-(A-BK)\\ C^\top\\ R^{\frac{1}{2}}K\end{bmatrix}.
\end{aligned}$$

Since $(C^\top, A)$ is observable thanks to the observable canonical form (15), then by the Popov-Belevitch-Hautus rank tests [38, Thm. 2.4-9], $\operatorname{rank}\begin{pmatrix}zI-A\\ C^\top\end{pmatrix} = 3$ for every z, so that $\left(A-BK, \begin{bmatrix}C^\top\\ R^{\frac{1}{2}}K\end{bmatrix}\right)$ is observable. It follows that the origin of (30) is exponentially stable.

Therefore, the origin of (29) is exponentially stable. It follows that $\boldsymbol{x}_o(k), \hat{\boldsymbol{x}}(k) \in \ell_\infty$ and converge to $\mathbf{0}$ as k goes to ∞. From $\boldsymbol{x}_o = \boldsymbol{x} - \hat{\boldsymbol{x}}$, $e(k) = C^\top\boldsymbol{x}(k)$ and $\Delta\boldsymbol{T}_{\mathbf{set}}(k) = -K\hat{\boldsymbol{x}}(k)$, it follows that all closed-loop signal are bounded and $e(k)$ converges to 0 as k goes to ∞.

This completes the proof of Theorem 1. □

Appendix B
Proof of Theorem 2

Proof: The proof is organized according to the following four steps: first, write the tracking error dynamics as a homogeneous term perturbed by estimation error terms; second, prove that the homogeneous term is exponentially stable; third, prove that the tracking error and other closed-loop signals of interest can be bounded using the estimation errors; fourth, using the norm-boundedness properties of the estimation errors, prove that the closed-loop signals are norm-bounded and the tracking error converges to zero.

Step 1. Calculate the closed-loop state error dynamics.

Consider the observable canonical state-space form (15) with matrices as in (16), let $\boldsymbol{x}_o(k) \triangleq \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k)$ be the state observation error. The relation between the tracking error and the state observation error, and between the tracking error and the inputs can be written respectively as

$$e(k) = C^\top\boldsymbol{x}_o(k) + C^\top\hat{\boldsymbol{x}}(k), \tag{31}$$

$$e(k) = \frac{\hat{Z}_{q,1}(k)}{\hat{R}_p(k)}\Delta T_{\text{set}}^{\text{zone}}(k) + \frac{\hat{Z}_{q,2}(k)}{\hat{R}_p(k)}\Delta T_{\text{set}}^{\text{tank}}(k). \tag{32}$$

From (15), (16) and (26)-(27), we have

$$\begin{bmatrix}\boldsymbol{x}_o(k+1)\\ \hat{\boldsymbol{x}}(k+1)\end{bmatrix}=\begin{bmatrix}A_o & \mathbf{0}\\ \hat{K}_o(k)C^\top & A_c(k)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_o(k)\\ \hat{\boldsymbol{x}}(k)\end{bmatrix}+\begin{bmatrix}-\tilde{\boldsymbol{\theta}}_{q,1}(k) & -\tilde{\boldsymbol{\theta}}_{q,2}(k)\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\Delta\boldsymbol{T}_{\mathbf{set}}(k)+\begin{bmatrix}\tilde{\boldsymbol{\theta}}_p(k)\\ \mathbf{0}\end{bmatrix}e(k), \quad (33)$$

where $A_o \triangleq \left[\begin{array}{c|c} -\boldsymbol{p}^* & \begin{array}{c} I_2 \\ \hline 0 \end{array}\end{array}\right]$ is a stable matrix,

$$\begin{aligned}
A_c(k) &\triangleq \hat{A}(k)-\hat{B}(k)\hat{K}(k),\\
\tilde{\boldsymbol{\theta}}_p(k) &\triangleq \boldsymbol{\theta}_p(k)-\boldsymbol{\theta}_p^* = [-\tilde{a}_{1d}\ \ -\tilde{a}_{2d}\ \ 0]^\top,\\
\tilde{\boldsymbol{\theta}}_{q,1}(k) &\triangleq \boldsymbol{\theta}_{q,1}(k)-\boldsymbol{\theta}_{q,1}^* = [\tilde{b}_1\ \tilde{b}_2\ \tilde{b}_3]^\top,\\
\tilde{\boldsymbol{\theta}}_{q,2}(k) &\triangleq \boldsymbol{\theta}_{q,2}(k)-\boldsymbol{\theta}_{q,2}^* = [\tilde{c}_1\ \tilde{c}_2\ \tilde{c}_3]^\top,
\end{aligned}$$

and where we have used the fact that

$$\begin{bmatrix}e & 1 & 0\\ f & 0 & 1\\ g & 0 & 0\end{bmatrix}\begin{bmatrix}q\\ r\\ s\end{bmatrix}=\begin{bmatrix}e\\ f\\ g\end{bmatrix}C^\top\begin{bmatrix}q\\ r\\ s\end{bmatrix}+\begin{bmatrix}r\\ s\\ 0\end{bmatrix}.$$

Step 2. Establish the exponential stability of the homogeneous part of (33).

When $\hat{R}_p(z,k)$ is strongly coprime to $\hat{Z}_{q,1}(z,k)$ and $\hat{Z}_{q,2}(z,k)$ at each time k, then $(\hat{A}(k),\hat{B}(k))$ is always stabilizable, and we can establish that $P(k),\hat{K}(k)\in\ell_\infty$. In addition, $\hat{A}(k),\hat{B}(k)\in\ell_\infty$ due to the properties of the adaptive law in Lemma 1. Also, $\Delta\hat{A}(k)\in\ell_2$ is guaranteed by the adaptive law. The following manipulations hold

$$\begin{aligned}
&P(k+2)-P(k+1)-\big(\hat{A}(k+1)-\hat{B}(k+1)\hat{K}(k+1)\big)^\top\\
&\big(P(k+1)-P(k)\big)\big(\hat{A}(k+1)-\hat{B}(k+1)\hat{K}(k+1)\big)\\
=\ &\hat{A}^\top(k+1)P(k)\hat{A}(k+1)-\hat{A}^\top(k)P(k)\hat{A}(k)\\
&+\hat{K}^\top(k)R\hat{K}(k)+\hat{K}^\top(k+1)R\hat{K}(k+1)\\
&-\hat{K}^\top(k+1)\hat{B}^\top(k+1)P(k)\hat{A}(k+1)\\
&-\hat{A}^\top(k+1)P(k)\hat{B}(k+1)\hat{K}(k+1)\\
&+\hat{K}^\top(k)\hat{B}^\top(k)P(k)\hat{B}(k)\hat{K}(k)\\
&+\hat{K}^\top(k+1)\hat{B}^\top(k+1)P(k)\hat{B}(k+1)\hat{K}(k+1),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\hat{K}^\top(k)\big(\hat{B}^\top(k)P(k)\hat{B}(k)+R\big)&=\hat{A}^\top(k)P(k)\hat{B}(k),\\
\hat{K}^\top(k)\big(\hat{B}^\top(k)P(k)\hat{B}(k)+R\big)\hat{K}(k)&=\hat{K}^\top(k)\hat{B}^\top(k)P(k)\hat{A}(k).
\end{aligned}$$

It follows from $P(k),\hat{K}(k),\hat{A}(k),\hat{B}(k)\in\ell_\infty$ that

$$\begin{aligned}
&P(k+2)-P(k+1)-\big(\hat{A}(k+1)-\hat{B}(k+1)\hat{K}(k+1)\big)^\top\\
&\big(P(k+1)-P(k)\big)\big(\hat{A}(k+1)-\hat{B}(k+1)\hat{K}(k+1)\big)\in\ell_\infty.
\end{aligned}$$

Let us decompose $P(k)$ as $P(k)=S^\top(k)S(k)$. From the fact that

$$\begin{aligned}
&\hat{K}^\top(k)\hat{B}^\top(k)P(k)\hat{B}(k)\hat{K}(k)\in\ell_\infty,\\
&\hat{K}^\top(k+1)\hat{B}^\top(k+1)P(k)\hat{B}(k+1)\hat{K}(k+1)\in\ell_\infty,
\end{aligned}$$

we can establish

$$S(k)\hat{B}(k)\hat{K}(k),\ S(k)\hat{B}(k+1)\hat{K}(k+1)\in\ell_\infty\cap\ell_2.$$

From $A_c(k)=\hat{A}(k)-\hat{B}(k)\hat{K}(k)$, we obtain

$$\begin{aligned}
\|\Delta A_c(k)\|_2 =&\|\hat{A}(k+1)-S^{-1}(k)S(k)\hat{B}(k+1)\hat{K}(k+1)\\
&-\hat{A}(k)+S^{-1}(k)S(k)\hat{B}(k)\hat{K}(k)\|_2\\
\le&\|\Delta\hat{A}(k)\|_2+\|S^{-1}(k)\|_2\|S(k)\hat{B}(k)\hat{K}(k)\|_2\\
&+\|S^{-1}(k)\|_2\|S(k)\hat{B}(k+1)\hat{K}(k+1)\|_2. \quad (34)
\end{aligned}$$

It follows from (34) that $\Delta A_c(k)\in\ell_2$. Because $A_c(k)$ is a stable matrix at each time k, we can conclude that $A_c(k)$ is uniformly asymptotically stable. Also, as A_o is a stable matrix, the homogeneous part of (33) is exponentially stable.

Step 3. Use the properties of $\ell_{2\delta}$ norm and the discrete-time Bellman-Gronwall lemma to establish boundedness.

From (31), (32) and (33), we obtain the following bounds

$$\|\hat{\boldsymbol{x}}_k\|_{2\delta}\le c\|C^\top\boldsymbol{x}_{o_k}\|_{2\delta}, \quad (35)$$

$$\|e_k\|_{2\delta}\le c\|C^\top\boldsymbol{x}_{o_k}\|_{2\delta}+c\|\hat{\boldsymbol{x}}_k\|_{2\delta}\le c\|C^\top\boldsymbol{x}_{o_k}\|_{2\delta}, \quad (36)$$

$$\|\Delta T^{\mathrm{zone}}_{\mathrm{set}_k}\|_{2\delta},\ \|\Delta T^{\mathrm{tank}}_{\mathrm{set}_k}\|_{2\delta}\le c\|e_k\|_{2\delta}\le c\|C^\top\boldsymbol{x}_{o_k}\|_{2\delta}, \quad (37)$$

for some constant $c>0$. We relate the term $C^\top\boldsymbol{x}_o(k)$ with the estimation error by using (33) to express $C^\top\boldsymbol{x}_o(k)$ as

$$\begin{aligned}
C^\top\boldsymbol{x}_o(k)=\ &C^\top\big(zI-A_o\big)^{-1}\big(\tilde{\boldsymbol{\theta}}_p(k)e(k)-\tilde{\boldsymbol{\theta}}_{q,1}(k)\Delta T^{\mathrm{zone}}_{\mathrm{set}}(k)\\
&-\tilde{\boldsymbol{\theta}}_{q,2}(k)\Delta T^{\mathrm{tank}}_{\mathrm{set}}(k)\big).
\end{aligned}$$

Noting that (C,A_o) is in the observer canonical form, i.e. $C^\top\big(zI-A_o\big)^{-1}=\dfrac{\bar{\boldsymbol{\alpha}}^\top(z)}{A_o^*(z)}$, with $A_o^*(z)=\det\big(zI-A_o\big)$, we have

$$\begin{aligned}
C^\top\boldsymbol{x}_o(k)=&-\sum_{v=1}^{2}\frac{z^{3-v}}{A_o^*(z)}\tilde{a}_{vd}(k)e(k)\\
&-\sum_{v=1}^{3}\frac{z^{3-v}}{A_o^*(z)}\Big(\tilde{b}_v(k)\Delta T^{\mathrm{zone}}_{\mathrm{set}}(k)+\tilde{c}_v(k)\Delta T^{\mathrm{tank}}_{\mathrm{set}}(k)\Big).
\end{aligned}$$

Let us denote $\Lambda_p(z)=z^3+\boldsymbol{\lambda}_p^\top\bar{\boldsymbol{\alpha}}(z)$, with $\boldsymbol{\lambda}_p=[\lambda_2,\lambda_1,\lambda_0]^\top$ a Hurwitz polynomial. Applying the discrete-time swapping lemma [34, Lem. A.12.35] with $W(z)=\dfrac{z^{3-v}}{\Lambda_p(z)}$ to each term under the summation, we have

$$\begin{aligned}
\frac{z^{3-v}}{A_o^*(z)}\tilde{a}_{vd}(k)e(k)=\ &\frac{\Lambda_p(z)}{A_o^*(z)}\Bigg(\tilde{a}_{vd}(k)\frac{z^{3-v}}{\Lambda_p(z)}e(k)\\
&+W_{c_v}(z)\big(W_{b_v}(z)e(k)\big)\Delta\tilde{a}_{vd}(k)\Bigg),
\end{aligned}$$

where $W_{c_v}(z)$, $W_{b_v}(z)$ are strictly proper transfer functions, having the same poles as $\dfrac{1}{\Lambda_p(z)}$, and similar for the terms with $\Delta T^{\mathrm{zone}}_{\mathrm{set}}(k)$ and $\Delta T^{\mathrm{tank}}_{\mathrm{set}}(k)$. Therefore, $C^\top\boldsymbol{x}_o(k)$ can be expressed as

$$\begin{aligned}
C^\top\boldsymbol{x}_o(k)=\ &r_1(k)+\frac{\Lambda_p(z)}{A_o^*(z)}\Bigg[-\sum_{v=1}^{2}\tilde{a}_{vd}(k)\frac{z^{3-v}}{\Lambda_p(z)}e(k)\\
&-\sum_{v=1}^{3}\Bigg(\tilde{b}_v(k)\frac{z^{3-v}}{\Lambda_p(z)}\Delta T^{\mathrm{zone}}_{\mathrm{set}}(k)+\tilde{c}_v(k)\frac{z^{3-v}}{\Lambda_p(z)}\Delta T^{\mathrm{tank}}_{\mathrm{set}}(k)\Bigg)\Bigg]\\
=\ &r_1(k)+\frac{\Lambda_p(z)}{A_o^*(z)}\Bigg(\tilde{\boldsymbol{\theta}}_p^\top(k)\frac{\bar{\boldsymbol{\alpha}}(z)}{\Lambda_p(z)}e(k)\\
&-\tilde{\boldsymbol{\theta}}_{q,1}^\top(k)\frac{\bar{\boldsymbol{\alpha}}(z)}{\Lambda_p(z)}\Delta T^{\mathrm{zone}}_{\mathrm{set}}(k)-\tilde{\boldsymbol{\theta}}_{q,2}^\top(k)\frac{\bar{\boldsymbol{\alpha}}(z)}{\Lambda_p(z)}\Delta T^{\mathrm{tank}}_{\mathrm{set}}(k)\Bigg),
\end{aligned} \quad (38)$$

where

$$r_1(k) \triangleq \frac{\Lambda_p(z)}{A_o^*(z)}\Bigg[-\sum_{v=1}^{2} W_{c_v}(z)\big(W_{b_v}(z)e(k)\big)\Delta\tilde{a}_{vd}(k) - \sum_{v=1}^{3} W_{c_v}(z)W_{b_v}(z)\Bigg(\Delta T_{\text{set}}^{\text{zone}}(k)\Delta\tilde{b}_v(k) + \Delta T_{\text{set}}^{\text{tank}}(k)\Delta\tilde{c}_v(k)\Bigg)\Bigg].$$

We now note that (32) can be rewritten as

$$\hat{\zeta}(k) = \boldsymbol{\theta}^\top(k)\boldsymbol{\Phi}(k),$$

where

$$\begin{aligned}\hat{\zeta}(k) &= \frac{z^3}{\Lambda_p(z)}e(k),\ \boldsymbol{\theta}(k) = [\boldsymbol{\theta}_{q,1}^\top(k)\ \boldsymbol{\theta}_{q,2}^\top(k)\ \boldsymbol{\theta}_p^\top(k)]^\top,\\ \boldsymbol{\Phi}(k) &= \frac{\bar{\boldsymbol{\alpha}}^\top(z)}{\Lambda_p(z)}\Big[\Delta T_{\text{set}}^{\text{zone}}(k)\quad \Delta T_{\text{set}}^{\text{tank}}(k)\quad -e(k)\Big]^\top\\ &= [\boldsymbol{\phi}_1^\top(k)\quad \boldsymbol{\phi}_2^\top(k)\quad \boldsymbol{\phi}_3^\top(k)]^\top. \end{aligned} \tag{39}$$

Using (39) and the discrete-time swapping lemma [34, Lem. A.12.35] in (38), we obtain

$$C^\top \boldsymbol{x}_o(k) = -\frac{\Lambda_p(z)}{A_o^*(z)}\tilde{\boldsymbol{\theta}}^\top(k)\boldsymbol{\Phi}(k) + r_1(k), \tag{40}$$

where $\tilde{\boldsymbol{\theta}}(k)$ is the estimation error of $\boldsymbol{\theta}(k)$ in (39).

The normalized estimation error satisfies the equation

$$\begin{aligned}-\tilde{\boldsymbol{\theta}}^\top(k)\boldsymbol{\Phi}(k) &= -\big(\tilde{\boldsymbol{\theta}}^\top(k-1) + \varepsilon(k)\boldsymbol{\Phi}^\top(k)P_1(k)\big)\boldsymbol{\Phi}(k)\\ &= \varepsilon(k)m^2(k)\left(1 - \frac{\boldsymbol{\Phi}^\top(k)P_1(k)\boldsymbol{\Phi}(k)}{m^2(k)}\right),\end{aligned}$$

which can be used in (40) to obtain

$$C^\top \boldsymbol{x}_o(k) = \frac{\Lambda_p(z)}{A_o^*(z)}\varepsilon(k)m^2(k)\left(1-\frac{\boldsymbol{\Phi}^\top(k)P_1(k)\boldsymbol{\Phi}(k)}{m^2(k)}\right)+r_1(k). \tag{41}$$

Defining an auxiliary normalizing signal

$$m_f^2(k) \triangleq 1 + \|\Delta T_{\text{set}_k}^{\text{zone}}\|_{2\delta}^2 + \|\Delta T_{\text{set}_k}^{\text{tank}}\|_{2\delta}^2 + \|e_k\|_{2\delta}^2,$$

we have $\boldsymbol{\Phi}(k)/m_f(k), m(k)/m_f(k) \in \ell_\infty$ for some $\delta > 0$. From the definition of $r_1(k)$ and $\left|1 - \frac{\boldsymbol{\Phi}^\top(k)P_1(k)\boldsymbol{\Phi}(k)}{m^2(k)}\right| < 1$, we obtain

$$\begin{aligned}\|C^\top \boldsymbol{x}_{o_k}\|_{2\delta} &\le c\|\varepsilon_k m_k^2\|_{2\delta} + c\left\|\sum_{v=1}^{2}\Delta\tilde{a}_{vd_k}e_k\right\|_{2\delta}\\ &+ c\left\|\sum_{v=1}^{3}\Delta\tilde{b}_{v_k}\Delta T_{\text{set}_k}^{\text{zone}}\right\|_{2\delta} + c\left\|\sum_{v=1}^{3}\Delta\tilde{c}_{v_k}\Delta T_{\text{set}_k}^{\text{tank}}\right\|_{2\delta}.\end{aligned} \tag{42}$$

Using (42) in (36)-(37) and the properties of $m_f(k)$, we have the following inequality:

$$m_f^2(k) \le c\|g_k m_{f_k}\|_{2\delta}^2 + c,$$

where

$$g^2(k) \triangleq \varepsilon^2(k)m^2(k) + \left|\sum_{v=1}^{2}\Delta\tilde{a}_{vd}(k)\right|^2 + \left|\sum_{v=1}^{3}\Delta\tilde{b}_v(k)\right|^2 + \left|\sum_{v=1}^{3}\Delta\tilde{c}_v(k)\right|^2$$

and $g \in \ell_2$ due to the properties of the adaptive law in Lemma 1. If we now apply the Bellman-Gronwall lemma [34, Lem. A.6.3] to the above inequality we can show that $m_f(k) \in \ell_\infty$. From $m_f(k) \in \ell_\infty$ and the properties of the $\ell_{2\delta}$ norm we can establish boundness for the rest of the closed-loop signals.

Step 4. Convergence of the error to zero.

Since all closed-loop signals are bounded, we can establish that $\Delta(\varepsilon(k)m^2(k)) \in \ell_\infty$, which, together with $\varepsilon(k)m^2(k) \in \ell_\infty \cap \ell_2$, implies that $\varepsilon(k)m^2(k)$ converges to 0, and therefore, $\Delta\boldsymbol{\theta}(k)$ converges to $\mathbf{0}$ as k goes to ∞. From the expression of $r_1(k)$ we can conclude that $r_1(k) \in \ell_2$ and converges to 0 as k goes to ∞. Using $\varepsilon(k)m^2(k), r_1(k) \in \ell_2$ and convergence of $r_1(k)$ in (41), we have that $C^\top \boldsymbol{x}_o(k) \in \ell_2$ and converges to 0 as k goes to ∞.

Consider (33), since $A_c(k)$ is uniformly asymptotically stable, and $C^\top \boldsymbol{x}_o(k) \in \ell_2$ and convergent we get that $\hat{\boldsymbol{x}}(k)$ converges to 0 as k goes to ∞. From $e(k) = C^\top \boldsymbol{x}_o(k) + C^\top \hat{\boldsymbol{x}}(k)$, it follows that $e(k)$ converges to 0 as k goes to ∞.

This completes the proof of Theorem 2. □