BOUNDEDNESS AND MODULI OF TRADITIONAL STABLE MINIMAL MODELS

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ABSTRACT. In [Bir22], Birkar constructs projective coarse moduli spaces of good minimal models with semi-log canonical (slc) singularities polarized by effective divisors. When considering good minimal models with Kawamata log terminal (klt) singularities polarized by possibly non-effective divisors, we prove that they belong to a bounded family after fixing appropriate numerical invariants. As an application, we construct separated coarse moduli spaces of klt good minimal models polarized by line bundles.

CONTENTS

1. INTRODUCTION

Throughout this paper, we work over an algebraically closed field k of characteristic zero unless otherwise stated.

The central problem in birational geometry is the classification of algebraic varieties. According to the standard Minimal Model Conjecture and Abundance Conjecture, any variety is birational to either a Mori fiber space or a good minimal model. Therefore, the next main goal of birational geometry is to construct moduli spaces for these objects.

In this paper, we will focus on good minimal models, that is, varieties with semi-ample canonical bundles. For canonical polarized varieties, i.e., good minimal models with maximal Kodaira dimension, the projective coarse moduli spaces of these varieties have been established by the contributions of many people over the past three decades [KS88, Kol90, Ale94, KM97, AH11, Fuj18, KP17, Kol23]. Notably, advancements in the higher-dimensional minimal model program [BCHM10, Bir12, HX13, HMX13, HMX14, HMX18] have played a crucial role in these developments. For Calabi-Yau varieties, i.e., good minimal models with Kodaira dimension zero, there is no natural choice of polarization. Various techniques, such as GIT, Hodge theory, minimal model program, mirror symmetry and differential geometry, have been applied to study the moduli of Calabi-Yau varieties and have been successful in specific examples; refer to [Ale02, GHK15, Laz16, Zha16, HKY20, Oda22, ADL23, AE23, ABB$^+$23] for a partial list. In [Bir23b], Birkar establishes a boundedness result for slc Calabi-Yau varieties polarized by effective ample Weil divisors and it is crucial for constructing their projective moduli spaces [Bir22].

For good minimal models with arbitrary Kodaira dimension, Birkar has recently proven a boundedness result for slc good minimal models polarized by effective divisors and has constructed their projective coarse moduli spaces [Bir22]. This achievement is based on a series of recent works [BZ16, Bir19, Bir21c, Bir23b, Bir21a, BH22]. The reason for taking effective divisors as polarization is that one can obtain proper moduli spaces. Traditionally, people use line bundles as polarization. For example, Viehweg considers ample line bundles as polarization for treating the moduli of smooth good minimal models [Vie95]. Then Taji constructs moduli of good minimal models with canonical singularities [Taj20]. More recently, Hashizume-Hattori consider klt good minimal models of Kodaira dimension one polarized by line bundles [HH23].

In this paper, we will consider klt good minimal models of arbitrary Kodaira dimension polarized by line bundles. We define our main concepts in a general form and state our main results on the boundedness and moduli of these objects.

Traditional stable minimal models. We define the main object studied in this paper, as introduced in Birkar's survey note [Bir23a].

Definition 1.1. A *traditional stable minimal model* $(X, B), A$ over k consists of a projective connected pair (X, B) and $\mathbb{Q}$-Cartier Weil divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \to Z$, and
- $K_X + B + tA$ is ample for some $t > 0$.

We usually denote the model by $(X, B), A$ or more precisely $(X, B), A \to Z$. We say $(X, B), A$ is *strongly traditional stable* if in addition $K_X + B + A$ is ample.

Note that we do not assume that A is ample globally. Indeed, if $X \to Z$ is the contraction defined by $K_X + B$, then $K_X + B + tA$ being ample for some $t > 0$ is equivalent to the fact that A is ample over Z. Moreover, t is not fixed in our setting.

The term "traditional" originates from our adoption of the traditional approach, wherein we consider using a line bundle or, more generally, a possibly non-effective $\mathbb{Q}$-Cartier Weil divisor as the polarization. When we choose an effective divisor A as the polarization, $(X, B), A$ is referred to as a *stable minimal model* in [Bir22].

Definition 1.2. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $u, v, \lambda \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. We will define various classes of traditional stable minimal models.

(1) A (d, Φ)-traditional stable minimal model is a traditional stable minimal model $(X, B), A$ such that
   - $\dim X = d$, and
   - the coefficients of B are in Φ.

   The set of all the (d, Φ)-traditional stable minimal models is denoted by $\mathcal{TS}_{klt}(d, \Phi)$.

(2) A (d, Φ, u, v)-traditional stable minimal model is a (d, Φ)-traditional stable minimal model $(X, B), A$ such that
   - $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$, and
   - $\mathrm{Ivol}(K_X + B) = v$ (see Definition 2.2).

   Let $\mathcal{TS}_{klt}(d, \Phi, u, v)$ denote the set of all (d, Φ, u, v)-traditional stable minimal models.

(3) A $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model is a (d, Φ)-traditional stable minimal model $(X, B), A$ such that
   - $\mathrm{vol}(A|_F) \in \Gamma$, where F is any general fiber of $f : X \to Z$, and
   - we have

$$\mathrm{vol}(K_X + B + tA) = \sigma(t), \text{ for all } 0 \leq t \ll 1.$$

   Let $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ denote the set of all $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models.

(4) A $(d, \Phi, \Gamma, \sigma, \lambda)$-traditional stable minimal model is a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model $(X, B), A$ such that $K_X + B + \lambda A$ is big (and hence $K_X + B + tA$ is big for any $0 < t \leq \lambda$).

   Let $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda)$ denote the set of all $(d, \Phi, \Gamma, \sigma, \lambda)$-traditional stable minimal models.

Example 1.3. Given a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model $(X, B), A$, if A is a pseudo-effective $\mathbb{Q}$-Cartier Weil divisor, then $(X, B), A$ is a $(d, \Phi, \Gamma, \sigma, 1)$-traditional stable minimal model. Indeed,

$$K_X + B + A = K_X + B + \epsilon A + (1 - \epsilon)A$$

for some $0 < \epsilon \ll 1$, then $K_X + B + \epsilon A$ is ample and $(1 - \epsilon)A$ is pseudo-effective, thus $K_X + B + A$ is big. In particular, a klt stable minimal model $(X, B), A$ as defined in [Bir22] is a $(d, \Phi, \Gamma, \sigma, 1)$-traditional stable minimal model.

Boundedness of traditional stable minimal models. We define the notion of boundedness of traditional stable minimal models and then state one of the main results of this paper.

Definition 1.4 (Boundedness of traditional stable minimal models). A subset $\mathcal{E} \subset \mathcal{TS}_{klt}(d, \Phi)$ is said to be a bounded family if there is a fixed $r \in \mathbb{N}$ such that for any $(X, B), A$ in $\mathcal{E}$ we can find a very ample divisor H on X satisfying

$$H^d \leq r, (K_X + B) \cdot H^{d-1} \leq r \text{ and } H - A \text{ is pseudo-effective.}$$

This in particular bounds $(X, \mathrm{Supp} B)$. If in addition A is effective, then $(X, \mathrm{Supp}(B+A))$ belongs to a bounded family.

Our definition of boundedness slightly differs from the original definition in [Bir22, Section 2.9]. In [Bir22], Birkar assumes A to be an effective divisor, then the notion of boundedness is defined as follows: a subset $\mathcal{E} \subset \mathcal{TS}_{klt}(d, \Phi)$ is said to be a bounded family if there is a fixed $r \in \mathbb{N}$ such that for any $(X, B), A$ in $\mathcal{E}$ we can find a very ample divisor H on X satisfying

$$H^d \leq r \text{ and } (K_X + B + A) \cdot H^{d-1} \leq r.$$

The requirements of our definition imply that of his, and by [Bir22, Lemma 4.6], they are equivalent when A is an effective divisor with coefficients belonging to a DCC set.

Theorem 1.5. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a bounded family.*

Birkar has proved this theorem when $\dim Z = 0$ (see Theorem 3.3). When $\dim Z = 1$, Hashizume-Hattori show that the set of (X, B) appearing in $\mathcal{TS}_{klt}(d, \Phi, u, v)$ belongs to a log bounded family [HH23, Theorem 1.4] without using the polynomial σ. Recently, there have been some other related results on the birational boundedness or boundedness of klt good minimal models, see [FS20, Fil20, FHS21, Jia22, Zhu23].

Conjecturally, we can remove the condition on $\mathrm{vol}(A|_F)$, which is related to the effective b-semiampleness conjecture [PS09, Conjecture 7.13]. For related discussions, refer to [Bir21a, Bir22].

Moduli of traditional stable minimal models. We now define families of traditional minimal models and the corresponding moduli functors. See Section 2.2 for the relevant definitions of families of pairs over reduced bases.

Definition 1.6. Let S be a reduced Noetherian scheme over k.

(1) When $S = \mathrm{Spec}\, K$ for a field K, we define a traditional stable minimal model over K as in Definition 1.1 by replacing k with K and replacing connected with geometrically connected. Similarly we can define $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over K.

(2) A *family of traditional stable minimal models* over S consists of a projective morphism $X \to S$ of schemes, a $\mathbb{Q}$-divisor B and a line bundle A on X such that

- $(X, B) \to S$ is a locally stable family,
- $(X_s, B_s), A_s$ is a traditional stable minimal model over $k(s)$ for every $s \in S$. Here X_s is the fiber of $X \to S$ over s and B_s is the divisorial pullback of B to X_s. $K_{X_s} + B_s$ is semi-ample defining a contraction $X_s \to Z_s$, A_s is a line bundle on X_s which is relatively ample over Z_s. We will denote the family by $(X, B), A \to S$.

(3) Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A *family of $(d, \Phi, \Gamma, \sigma)$-marked traditional stable minimal models* over S is a family of traditional stable minimal models $(X, B), A \to S$ such that

- $B = \sum\limits_{i=1}^{m} a_i D_i$, where $D_i \geq 0$ are relative Mumford divisors, and
- $(X_s, B_s), A_s$ is a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model over $k(s)$ for every $s \in S$, where $B_s = \sum\limits_{i=1}^{m} a_i D_{i,s}$.

(4) We define the moduli functor $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models from the category of reduced Noetherian schemes to the category of groupoids by choosing

- On objects: for a reduced Noetherian scheme S, one take

  $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$

  $= \{$family of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over $S\}$.

  We define an isomorphism $(f' : (X', B'), A' \to S) \to (f : (X, B), A \to S)$ of any two objects in $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$ to be an S-isomorphism $\alpha_X : (X', B') \to (X, B)$ such that $A' \sim_S \alpha_X^* A$.

- On morphisms: $(f_T : (X_T, B_T), A_T \to T) \to (f : (X, B), A \to S)$ consists of morphisms of schemes $\alpha : T \to S$ such that the natural map $g : X_T \to X \times_S T$ is an isomorphism, B_T is the divisorial pullback of B and $A_T \sim_T g^* \alpha_X^* A$. Here $\alpha_X : X \times_S T \to X$ is the base change of α.

Now we can state our main result on moduli.

Theorem 1.7. $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ *is a separated Deligne-Mumford stack of finite type, which admits a coarse moduli space $TS_{klt}(d, \Phi, \Gamma, \sigma)$ as a separated algebraic space.*

2. Preliminaries

We refer readers to [KM98] for the background of singularities of pairs.

Notations and conventions. We collect notations and conventions used in this paper.

(1) A projective morphism $f : X \to Y$ of schemes is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (f is not necessarily birational). In particular, f is surjective with connected fibers.

(2) For any scheme S and positive integer n, l, we denote $\mathbb{P}_k^n \times_{\mathrm{Spec}\,k} S$ and $\mathbb{P}_k^l \times_{\mathrm{Spec}\,k} S$ by $\mathbb{P}_S^n$ and $\mathbb{P}_S^l$, respectively. When $S = \mathrm{Spec}\,k$, we simply write $\mathbb{P}^n$ and $\mathbb{P}^l$ if there is no risk of confusion. Let $\mathbb{P}_S^n \times_S \mathbb{P}_S^l \cong \mathbb{P}^n \times \mathbb{P}^l \times S$ be the natural isomorphism, and
$$\mathbb{P}^n \overset{p_1}{\leftarrow} \mathbb{P}^n \times \mathbb{P}^l \times S \overset{p_2}{\to} \mathbb{P}^l$$
be the projections. Then for any $a, b \in \mathbb{Z}$, we denote $p_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(b)$ by $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S}(a, b)$.

(3) We say that a surjective morphism $f : S' \to S$ between schemes is an *étale covering* if $S' := \sqcup_{i \in I} S_i$, and $f|_{S_i} : S_i \to S$ is étale for any $i \in I$. Given a covering $S' \to S$, we will use the convention that S_{ij} denotes $S_i \times_S S_j$ and S_{ijk} denotes $S_i \times_S S_j \times_S S_k$.

(4) Let $f : X \to S$ be a morphism of schemes. Let L_1 and L_2 be line bundles on X. We say that L_1 and L_2 are *linearly equivalent over S*, denoted by $L_1 \sim_S L_2$, if there is a line bundle M on S such that $L_1 \cong L_2 \otimes f^*M$.

(5) Suppose that X is a normal variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. We say that D_1 and D_2 are $\mathbb{Q}$-linear equivalent, denoted by $D_1 \sim_{\mathbb{Q}} D_2$, if there exists $m \in \mathbb{Z}_{>0}$ such that mD_1 and mD_2 are Cartier and $mD_1 \sim mD_2$. Moreover, fixed $l \in \mathbb{Z}_{>0}$, the notation $D_1 \sim_l D_2$ means that $lD_1 \sim lD_2$.

(6) We say that a line bundle L on a variety X is *strongly ample* if L is very ample and $H^q(X, kL) = 0$ for any $k, q > 0$.

(7) We say that a set $\Phi \subset \mathbb{Q}$ satisfies the *descending chain condition* (DCC, for short) if Φ does not contain any strictly decreasing infinite sequence. Similarly, we say that a set $\Phi \subset \mathbb{Q}$ satisfies the *ascending chain condition* (ACC, for short) if Φ does not contain any strictly increasing infinite sequence.

2.1. **Volume of divisors.** We define various kinds of volumes of divisors.

Definition 2.1 (Volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. The *volume* of D is
$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{d!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^d}.$$

Definition 2.2 (Iitaka volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X such that Iitaka dimension $\kappa(D)$ is non-negative. Then the *Iitaka volume* of D, denoted by $\mathrm{Ivol}(D)$, is
$$\mathrm{Ivol}(D) = \limsup_{m \to \infty} \frac{\kappa(D)!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$

Definition 2.3 (Restricted volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. Let $S \subset X$ be a normal irreducible subvariety of dimension n. Suppose that S is not contained in the augmented base locus

$\mathbf{B}_+(D)$ (see [ELM$^+$06]). Then the *restricted volume of D along S* is

$$\text{vol}_{X|S}(D) = \limsup_{m \to \infty} \frac{n!(\dim \text{Im}(H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \to H^0(S, \mathcal{O}_S(\lfloor mD|_S \rfloor))))}{m^n}.$$

The restricted volume $\text{vol}_{X|S}(D)$ measures asymptotically the number of sections of the restriction $\mathcal{O}_S(\lfloor mD|_S \rfloor)$ that can be lifted to X. If D is ample, then the restriction maps are eventually surjective, and hence

$$\text{vol}_{X|S}(D) = \text{vol}(D|_S).$$

In general, it can happen that $\text{vol}_{X|S}(D) < \text{vol}(D|_S)$.

Theorem 2.4. ([LM09, Corollary 4.27]). *Let X be an irreducible projective variety of dimension d, and let $S \subset X$ be an irreducible (and reduced) Cartier divisor on X. Suppose that D is a big $\mathbb{R}$-divisor such that $S \nsubseteq \mathbf{B}_+(D)$. Then the function $t \mapsto \text{vol}(D + tS)$ is differentiable at $t = 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{vol}(D + tS))\Big|_{t=0} = d \, \text{vol}_{X|S}(D).$$

By [LM09, Remark 4.29], volume function has continuous partials in all directions at any point $D \in \text{Big}(X)$, i.e., the function $\text{vol} : \text{Big}(X) \to \mathbb{R}$ is $\mathcal{C}^1$.

2.2. **Family of pairs.** In this subsection, we define a family of pairs over reduced base scheme. We firstly recall the notion of relative Mumford divisor from [Kol23, Definition 4.68].

Definition 2.5 (Relative Mumford divisor). Let $f : X \to S$ be a flat finite type morphism with S_2 fibers of pure dimension d. A subscheme $D \subset X$ is a *relative Mumford divisor* if there is an open set $U \subset X$ such that

- $\text{codim}_{X_s}(X_s \setminus U_s) \geq 2$ for each $s \in S$,
- $D|_U$ is a relative Cartier divisor,
- D is the closure of $D|_U$, and
- X_s is smooth at the generic points of D_s for every $s \in S$.

By $D|_U$ being relative Cartier we mean that $D|_U$ is a Cartier divisor on U and that its support does not contain any irreducible component of any fiber U_s.

If $D \subset X$ is a relative Mumford divisor for $f : X \to S$ and $T \to S$ is a morphism, then the *divisorial pullback* D_T on $X_T := X \times_S T$ is the relative Mumford divisor defined to be the closure of the pullback of $D|_U$ to U_T. In particular, for each $s \in S$, we define $D_s = D|_{X_s}$ to be the closure of $D|_{U_s}$ which is the divisorial pullback of D to X_s.

Definition 2.6 (Locally stable family). A *locally stable family of projective klt pairs* $(X, B) \to S$ over a reduced Noetherian scheme S is a flat finite type morphism $X \to S$ with S_2 fibers and a $\mathbb{Q}$-divisor B on X satisfying

- each prime component of B is a relative Mumford divisor,
- $K_{X/S} + B$ is $\mathbb{Q}$-Cartier, and
- (X_s, B_s) is a klt pair for any point $s \in S$.

3. BOUNDEDNESS

In this section, we will prove boundedness of traditional stable minimal models with fixed data.

3.1. **Boundedness of generalised klt pairs.** To bound the bases of the traditional stable minimal models, we need the theory of generalised pairs [BZ16, Bir21b]. We recall notions and boundedness results about generalised pairs.

Definition 3.1. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. Let $\mathcal{F}_{gklt}(d, \Phi, v)$ be the set of projective generalised pairs $(X, B + M)$ with nef part M' such that

- $(X, B + M)$ is generalised klt of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M_i'$ where $\mu_i \in \Phi$ and M_i' are nef Cartier, and
- $K_X + B + M$ is ample with volume $\operatorname{vol}(K_X + B + M) = v$.

Theorem 3.2 ([Bir21a])**.** *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exists $l \in \mathbb{N}$ depending only on d, Φ, u, v such that for any traditional stable minimal model*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, u, v),$$

we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

such that the corresponding set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. Moreover, $l(K_Z + B_Z + M_Z)$ is very ample.

Proof. By applying [Bir23b, Corollary 1.4] to $(F, B|_F), A|_F$, we can find $m \in \mathbb{Z}_{>0}$, depending only on d, Φ such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. Thus we have $G \sim mA$ for some Weil divisor G whose horizontal part, which we denote by E, is an effective Weil divisor. By applying [Bir21a, Lemma 7.4] to $f : (X, B), E \to Z$, there exist $p, q \in \mathbb{N}$ depending only on d, Φ, u such that we can write an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where B_Z (resp. M_Z) is the discriminant part (resp. moduli part) of the canonical bundle formula such that $pM_{Z'}$ is Cartier on some higher resolution $Z' \to Z$.

By definition of the discriminant part of the canonical bundle formula and the ACC for lc thresholds [HMX14, Theorem 1.1], we see that the coefficients of B_Z belong to a DCC set of $\mathbb{Q}^{>0}$ depending only on d and Φ, which we denote by Ψ. Moreover, $(Z, B_Z + M_Z)$ is generalised klt pair and

$$\operatorname{Ivol}(K_X + B) = \operatorname{vol}(K_Z + B_Z + M_Z) = v.$$

Adding $\frac{1}{p}$, we can assume $\frac{1}{p} \in \Psi$, we see that

$$(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, \Psi, v).$$

By [Bir21a, Theorem 1.4], the set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. By the argument after [Bir22, Theorem 4.3], there exists $l \in \mathbb{N}$ depending only on $\dim Z, \Psi, v$, hence depending only on d, Φ, u, v such that $l(K_Z + B_Z + M_Z)$ is very ample. Replacing q, l with ql, we conclude the proof. $\qquad \square$

3.2. **Boundedness of traditional stable Calabi-Yau pairs.** The fibers of traditional stable minimal models are so-called *traditional stable Calabi-Yau pairs*, i.e., klt Calabi-Yau pairs polarized by possibly non-effective ample Weil divisors.

Theorem 3.3 ([Bir23b])**.** *Let* $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ *be a DCC set,* $\Gamma \subset \mathbb{Q}^{>0}$ *be a finite set,* $\sigma \in \mathbb{Q}[t]$ *be a polynomial. Consider those*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma)$$

with $\dim Z = 0$. *Then such* $(X, B), A$ *form a bounded family.*

Proof. This is a special case of Theorem 3.7. $\qquad \square$

3.3. **Boundedness of traditional stable minimal models.** In this subsection, we prove Theorem 1.5.

The following crucial result is a key ingredient in the proof of Theorem 1.5. It indicates that the singularities of the pairs (X, B) can be controlled uniformly. Moreover, the coefficients of B belong to a fixed finite set.

Lemma 3.4. *Let* $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ *be a DCC set, and* $u, v \in \mathbb{Q}^{>0}$. *Then the set*

$$\{a(D, X, B) \leq 1 | (X, B), A \in \mathcal{TS}_{klt}(d, \Phi, u, v), D \text{ prime divisor over } X\}$$

is finite.

Proof. By applying [Bir23b, Corollary 1.4] to $(F, B|_F), A|_F$, we can find $m \in \mathbb{Z}_{>0}$, depending only on d, Φ such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. Thus we have $G \sim mA$ for some Weil divisor G whose horizontal part, which we denote by E, is an effective Weil divisor. By applying [Bir21a, Lemma 8.2] to $f : (X, B), E \to Z$ (the proof uses E only in the relative sense over Z, the condition of $K_X + B + E$ being ample is not required), the log discrepancies of (X, B) in the interval $[0, 1]$ belong to a fixed finite set depending only on d, Φ, u, v. $\qquad \square$

We can recover some invariants of a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model from the given data $(d, \Phi, \Gamma, \sigma)$ by the following lemma.

Lemma 3.5. *Let* $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ *be a DCC set,* $\Gamma \subset \mathbb{Q}^{>0}$ *be a finite set,* $\sigma \in \mathbb{Q}[t]$ *be a polynomial. Then for any*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma),$$

- *we have*

$$\sigma(t) = \sum_{i=0}^{d} \binom{d}{i} (K_X + B)^{d-i} \cdot A^i t^i,$$

hence for $0 \leq i \leq d$, the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ are uniquely determined by d and σ.

- $d - \dim Z$ *is the smallest degree of t that appears in σ.*
- *For a general fiber F of $X \to Z$, we have*

$$(K_X + B)^{\dim Z} \cdot A^{d-\dim Z} = \mathrm{Ivol}(K_X + B) \, \mathrm{vol}(A|_F).$$

Proof. This is [Bir22, Lemma 4.12], the condition that A is effective is not used in his proof. $\qquad\square$

First, we consider a special case of Theorem 1.5, where we add the condition that $K_X + B + A$ is ample. We recall the notation of a set of strongly traditional stable minimal models with fixed data, and then we prove that it belongs to a bounded family.

Definition 3.6. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u, v \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Recall that a strongly traditional stable minimal model is a traditional stable minimal model$(X, B), A$ such that $K_X + B + A$ is ample. A (d, Φ, u, v, σ)- strongly traditional stable minimal model is a strongly traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ,
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$,
- $\mathrm{Ivol}(K_X + B) = v$, and
- $\mathrm{vol}(K_X + B + tA) = \sigma(t)$, for all $t \in [0, 1]$.

Let $\mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$ denote the set of all (d, Φ, u, v, σ)-strongly traditional stable minimal models.

Theorem 3.7. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u, v \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then $\mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$ is a bounded family.*

Proof. Pick $(X, B), A \in \mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$, let $f : X \to Z$ be the contraction defined by the semi-ample divisor $K_X + B$. By Lemma 3.4, the log discrepancies of (X, B) in the interval $[0, 1]$ belong to a fixed finite set depending only on d, Φ, u, v. Therefore, (X, B) is ϵ-lc and $l(K_X + B)$ is integral for some $\epsilon > 0$ and $l \in \mathbb{N}$ depending only on d, Φ, u, v. Let $N := l(K_X + B + A)$ be an ample $\mathbb{Q}$-Cartier Weil divisor, then

$$\mathrm{vol}(K_X + B + N) = (l+1)^d \mathrm{vol}(K_X + B + \frac{l}{l+1}A) = (l+1)^d \sigma(\frac{l}{l+1})$$

is fixed. Thus by [Bir23b, Theorem 1.5], the set of (X, B) forms a log bounded family.

Since $\mathcal{P} = \{(X, B)\}$ is a set of ϵ-lc pairs that belongs to a log bounded family, by [HLQ23, Theorem 1.10], for any $\mathbb{Q}$-Cartier Weil divisor D on X, $I_0 D$ is Cartier for some $I_0 \in \mathbb{N}$ depending only on ϵ and $\mathcal{P}$ and hence depending only on d, Φ, u, v, σ. Let

$$L := 2l(K_X + B) + lA$$

be an ample $\mathbb{Q}$-Cartier Weil divisor, then $I_0 L$ is an ample Cartier divisor. We see that

$$I_0 L - (K_X + B) = (I_0 l - 1)(K_X + B) + I_0 l(K_X + B + A)$$

is nef and big, then by effective base point freeness theorem [Kol93, Theorem 1.1], there exists a positive integer I_1 depending only on d, I_0 such that $I_1 L$ is base point free. Since $I_1 L - (K_X + B)$ is nef and big, by very ampleness lemma [Fuj17, Lemma 7.1], there exists a positive integer I depending only on d, I_1 and hence depending only on d, Φ, u, v, σ such that IL is very ample. Note that

$$\mathrm{vol}(IL) = (2Il)^d \, \mathrm{vol}(K_X + B + \frac{1}{2}A) = (2Il)^d \sigma(\frac{1}{2})$$

is fixed, and $(K_X + B) \cdot (IL)^{d-1}$ is determined by $\sigma(t)$ by Lemma 3.5. Also we see that

$$IL - A = (Il + 1)(K_X + B) + (Il - 1)(K_X + B + A)$$

is pseudo-effective. Let H be a general element of $|IL|$, then $(X, B), A$ belongs to a bounded family in the sense of Definition 1.4. $\qquad\square$

Under the condition that $K_X + B + \lambda A$ is big for some positive rational number λ (and hence $K_X + B + tA$ is big for any $0 < t \leq \lambda$), we can reduce Theorem 1.5 to Theorem 3.7 by the following theorem. We follow the argument of [Bir22, Proof of Theorem 4.1] with some modifications. In Step 6 of the proof of the following theorem, since we are dealing with a possibly non-effective big Weil divisor, after replacing it with an effective $\mathbb{Q}$-divisor in the $\mathbb{Q}$-linear system, we may lose control of its coefficients. Therefore, we require a stronger boundedness result on singularities, as discussed in [Bir21c, Theorem 1.8], compared to the one presented in [Bir22, Lemma 4.7]. This motivates the modification of the definition of boundedness for traditional stable minimal models in our context (Definition 1.4) from the original definition in [Bir22, Section 2.9].

Theorem 3.8. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial, $\lambda \in \mathbb{Q}^{>0}$. Then there is a positive rational number $\tau \in (0, 1]$ depending only on $d, \Phi, \Gamma, \sigma, \lambda$ such that $K_X + B + \tau A$ is ample for all*

$$(X, B), A \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda).$$

Moreover, $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda)$ is a bounded family.

Proof. We prove theorem by induction on dimension of X.

Step 1. Choose

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda).$$

By Lemma 3.5, for a general fiber F of $X \to Z$, $\mathrm{Ivol}(K_X + B) \, \mathrm{vol}(A|_F)$ is a fixed number depending only on d and σ. Since $\mathrm{vol}(A|_F)$ belongs to a finite set Γ, there are finitely many possibilities for $\mathrm{Ivol}(K_X + B)$. Therefore, we may fix both $u = \mathrm{vol}(A|_F)$ and $v = \mathrm{Ivol}(K_X + B)$.

Again by Lemma 3.5,

$$\sigma(t) = \sum_{i=0}^{d} \binom{d}{i} (K_X + B)^{d-i} \cdot A^i t^i,$$

and the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ are determined by d and σ for any $0 \le i \le d$.

Step 2. By Lemma 3.4, we may choose an $\alpha \in \mathbb{N}$ depending only on d, Φ, u, v, λ such that $\alpha(K_X + B + \lambda A)$ is a big Weil divisor. Moreover, for $0 \le t \ll 1$, $K_X + B + t\alpha(K_X + B + \lambda A)$ is ample and

$$\mathrm{vol}(K_X + B + t\alpha(K_X + B + \lambda A))$$
$$= \mathrm{vol}((1 + t\alpha)(K_X + B) + t\alpha\lambda A)$$
$$= ((1 + t\alpha)(K_X + B) + t\alpha\lambda A)^d$$

is a polynomial γ in t whose coefficients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$, α and λ. Therefore, γ is determined by $d, \Phi, \Gamma, \sigma, \lambda$.

Replacing A, u, σ with $\alpha(K_X + B + \lambda A), (\alpha\lambda)^{\dim F} u, \gamma$, we may assume that A is a big Weil divisor, and

$$(X, B), A \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, 1).$$

Step 3. If $K_X + B + A$ is ample, i.e., $(X, B), A$ is a strongly traditional stable minimal model, then for any $t \in [0, 1]$,

$$\mathrm{vol}(K_X + B + tA) = (K_X + B + tA)^d$$

is a polynomial in t in the interval $[0, 1]$. Thus for every $t \in [0, 1]$,

$$\mathrm{vol}(K_X + B + tA) = \sigma(t)$$

since both sides of the equality are polynomials which agree for t sufficiently small. Thus by Theorem 3.7, $(X, B), A$ is bounded in the sense of Definition 1.4. In particular, when $\dim X = 1$, $K_X + B + A$ is ample, the theorem holds.

Step 4. From now on we assume that $\dim X \ge 2$. Our goal is to find a positive rational number $\tau \in (0, 1]$ depending only on d, Φ, Γ, σ such that $K_X + B + \tau A$ is ample. Once this condition is guaranteed, then by Lemma 3.4, we may choose $\beta \in \mathbb{N}$ depending only on d, Φ, u, v, τ such that $\beta(K_X + B + \tau A)$ is an ample Weil Divisor. By the same argument as Step 2, after replacing A, u, σ with $\beta(K_X + B + \tau A), (\beta\tau)^{\dim F} u, \zeta$, where ζ is a polynomial in t determined by d, σ, τ, and β, we may assume that A is an ample Weil divisor, and $(X, B), A$ is strongly traditional stable belonging to

$$\mathcal{STS}_{klt}(d, \Phi, u, v, \sigma).$$

Hence by Step 3, $(X, B), A$ belongs to a bounded family in the sense Definition 1.4.

We claim that it is enough to find $\tau \in (0, 1]$ depending only on d, Φ, Γ, σ such that $K_X + B + \tau A$ is nef. Indeed, we can show that $K_X + B + tA$ is ample for any $t \in (0, \tau)$: we may pick $0 < t' \ll t$ such that $K_X + B + t'A$ is ample by Definition 1.1, then $K_X + B + tA$ is a positive linear combination of $K_X + B + t'A$ and $K_X + B + \tau A$, hence it is ample. So

replacing τ with $\frac{\tau}{2}$ we can assume that $K_X + B + \tau A$ is ample. We aim to find such τ in the subsequent steps.

Step 5. By Definition 1.1, $K_X + B$ is semi-ample defining the contraction $f : X \to Z$. If $\dim Z = 0$, then $(X, B), A$ is a traditional stable Calabi-Yau pair belonging to a bounded family by Theorem 3.3. We can assume that $\dim Z > 0$. By Theorem 3.2, there exists $l \in \mathbb{N}$ depending only on d, Φ, u, v such that we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

and the generalised klt pair $(Z, B_Z + M_Z)$ belongs to a bounded family. Moreover,

$$L := l(K_Z + B_Z + M_Z)$$

is very ample.

Let T be a general member of $|L|$ and let S be its pullback to X. Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$. Then $(S, B_S), A_S \to T$ is a traditional stable minimal model: indeed, we may choose a general $T \in |L|$ such that $(X, B + S)$ is plt, hence (S, B_S) is a projective klt pair. Moreover, $K_S + B_S$ is semi-ample defining the contraction $g : S \to T$, and $K_S + B_S + tA_S$ is ample for every small $t > 0$ since $K_X + B + S + tA$ is ample for every small $t > 0$.

If G is a general fiber of $S \to T$, then

$$\mathrm{vol}(A_S|_G) = \mathrm{vol}(A|_G) = u,$$

since G is among the general fibers of $X \to Z$. Moreover, for $0 \le t \ll 1$, we have

$$
\begin{aligned}
\psi(t) &= \mathrm{vol}(K_S + B_S + tA_S) \\
&= \mathrm{vol}((K_X + B + S + tA)|_S) \\
&= (K_X + B + S + tA)^{d-1} \cdot S \\
&= ((l+1)(K_X + B) + tA)^{d-1} \cdot S \\
&= ((l+1)(K_X + B) + tA)^{d-1} \cdot l(K_X + B),
\end{aligned}
$$

where $\psi(t)$ is a polynomial in t whose coefficients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ and by l, and hence by d, σ, l. Furthermore, since S is the pullback of a general element of a very ample linear system, $A_S = A|_S$ is still a big Weil divisor. Thus $(S, B_S), A_S$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d - 1, \Phi, \Gamma, \psi, 1).$$

Therefore, $(S, B_S), A_S$ belongs to a bounded family in the sense of Definition 1.4 by induction.

Step 6. By Lemma 3.5, $(S, B_S), A_S$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d - 1, \Phi, u, v')$$

for some fixed $v' = \mathrm{Ivol}(K_S + B_S)$ that depends on d, σ, u. Then (S, B_S) is ϵ-lc for some $\epsilon \in \mathbb{Q}^{>0}$ depending only on $d - 1, \Phi, u, v'$ by Lemma 3.4. By Step 5, $(S, B_S), A_S$ belongs to a bounded family in the sense of Definition 1.4, hence there is a fixed $r \in \mathbb{N}$ such that for any $(S, B_S), A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1} \le r, (K_S + B_S) \cdot H_S^{d-2} \le r, \text{ and } H_S - A_S \text{ is pseudo-effective.}$$

By [Bir22, Lemma 4.6], we may assume that $H_S - B_S$ is pseudo-effective. Since A is big, there is an effective $\mathbb{Q}$-divisor E such that $A \sim_{\mathbb{Q}} E$. Since S is a pullback of a general element of a very ample linear system, $E_S := E|_S$ is an effective $\mathbb{Q}$-divisor and $A_S \sim_{\mathbb{Q}} E_S$. Moreover, $H_S - E_S \sim_{\mathbb{Q}} H_S - A_S$ is also pseudo-effective.

Thus by [Bir21c, Theorem 1.8], there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $d - 1, \epsilon, r$ such that

$$\mathrm{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt. Then by inversion of adjunction [KM98, Theorem 5.50], $(X, B + S + \tau E)$ is plt near S. Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z: otherwise, the non-lc locus of $(X, B + \tau E)$ maps onto a closed subset of Z positive dimension which intersects T, hence S intersects the non-lc locus of $(X, B + \tau E)$, a contradiction.

Step 7. In this step, we assume that $K_X + B + \tau E$ is not nef. Otherwise, $K_X + B + \tau A \sim_{\mathbb{Q}} K_X + B + \tau E$ is nef, and we are done by Step 4.

Let R be a $(K_X + B + \tau E)$-negative extremal ray, since $K_X + B + \tau E$ is ample over Z, R is not contained in fibers of $X \to Z$. Since $K_X + B$ is the pullback of ample divisor on Z, $(K_X + B) \cdot R > 0$. By Step 6, the non-lc locus of $(X, B + \tau E)$ maps to finitely many points of Z, so R is not contained in the image

$$\mathrm{Im}(\overline{\mathrm{NE}}(\Pi) \to \overline{\mathrm{NE}}(X)),$$

where Π is the non-lc locus of $(X, B + \tau E)$.

Then by the the length of extremal ray [Fuj11, Theorem 1.1], R is generated by a curve C with

$$(K_X + B + \tau E) \cdot C \ge -2d.$$

Since L is very ample, $f^*L \cdot C = L \cdot f_*C \ge 1$, we see that

$$(K_X + B + 2df^*L + \tau E) \cdot C \ge 0.$$

It follows that

$$K_X + B + 2df^*L + \tau E$$

is nef. Since $f^*L \sim l(K_X + B)$, we see that

$$K_X + B + \frac{\tau}{1 + 2dl}E \sim_{\mathbb{Q}} \frac{1}{1 + 2dl}(K_X + B + 2df^*L + \tau E)$$

is nef. Hence after replacing τ with $\frac{\tau}{1+2dl}$, we can assume that $K_X + B + \tau E$ is nef. $\quad\square$

To find a positive rational number λ such that $K_X + B + \lambda A$ is big for any $(X, B), A \to Z \in \mathcal{S}_{klt}(d, \Phi, \Gamma, \sigma)$, one way is to focus on the volume function $\theta(t) = \mathrm{vol}(K_X + B + tA)$, as its graph provide information about the big thresholds of $K_X + B$ with respect to A. However, in our setting, we only fix the top self-intersection number function $\sigma(t) = (K_X + B + tA)^d$. The idea is to compare $\sigma(t)$ with $\theta(t)$. Since $\sigma(t) = \theta(t)$ for $0 < t \ll 1$, we can compare them by investigating their derivatives. For the purpose of simplifying the computation, we will consider the variations of the volume function $\vartheta(t)$ and the top self-intersection function $\varsigma(t)$ instead.

Proposition 3.9. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then for any*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma),$$

there is a positive rational number λ depending only on d, Φ, Γ, σ such that $K_X + B + tA$ is big for all $0 < t < \lambda$.

Proof. Step 1. Let $\varsigma \in \mathbb{Q}[t]$ be a polynomial function such that $\varsigma(t) = \mathrm{vol}(A + t(K_X + B))$ for $t \gg 1$. We have

$$\varsigma(t) = (A + t(K_X + B))^d = \sum_{i=0}^{d} \tbinom{d}{i} A^{d-i} \cdot (K_X + B)^i t^i$$

for all t, since both left hand and right hand side of the equality are polynomials in t that agree for all large values of t. By Lemma 3.5, ς is determined by σ. Again by Lemma 3.5, we may fix both $u = \mathrm{vol}(A|_F)$ and $v = \mathrm{Ivol}(K_X + B)$. It is enough to show that there exists a positive rational number τ depending only on d, Φ, Γ, σ such that $A + t(K_X + B)$ is big for all $t > \tau$. Let $\vartheta(t) = \mathrm{vol}(A + t(K_X + B))$ be a function on t, then $\vartheta(t)$ is a non-negative non-decreasing real function on t, which agrees with $\varsigma(t)$ for $t \gg 0$. It suffices to show that $\vartheta(t) > 0$ for all $t > \tau$.

Step 2. We prove this proposition by induction on the dimension of Z. By Lemma 3.5, $\dim Z$ is determined by σ, we may assume that $\dim Z = m$ is fixed. If $\dim Z = 0$, then $(X, B), A$ is a stable Calabi-Yau pair, proposition holds trivially. We may assume that $\dim Z > 0$.

If $\dim Z = 1$, then

$$\varsigma(t) = (A + t(K_X + B))^d = A^d + dA^{d-1} \cdot (K_X + B)t = A^d + duvt.$$

Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, then $\varsigma'(t) = duv$. Let F be a general fiber of $X \to Z$, then we see that

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z) \sim_{\mathbb{Q}} vF,$$

and we have

$$\vartheta(t) = \mathrm{vol}(A + t(K_X + B)) = v^d \, \mathrm{vol}(\tfrac{1}{v}A + tF).$$

For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose a sufficiently general fiber F_t of $X \to Z$ such that $F_t \not\subseteq \mathbf{B}_+(\frac{1}{v}A + tF_t)$. Then by Theorem 2.4, the function $s \mapsto \mathrm{vol}(\frac{1}{v}A + tF_t + sF_t)$ is differentiable at $s = 0$.
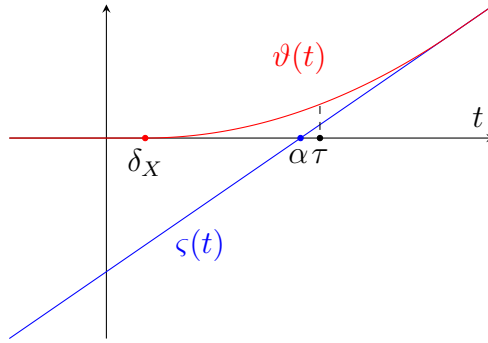
Let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t, we have

$$\frac{1}{v^d}\vartheta'(t) = \frac{1}{v^d}\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{vol}(\frac{1}{v}A + tF_t + sF_t))\Big|_{s=0} = d\,\mathrm{vol}_{X|F_t}(\frac{1}{v}A + tF_t),$$

hence

$$\vartheta'(t) = dv^d\,\mathrm{vol}_{X|F_t}(\frac{1}{v}A + tF_t) \le dv^d\,\mathrm{vol}((\frac{1}{v}A + tF_t)|_{F_t}) = dv^d\frac{1}{v^{d-1}}u = duv = \varsigma'(t).$$

FIGURE 1. The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z = 1$

Let α be the root of $\varsigma(t)$ and $\tau = \max\{\lceil\alpha\rceil + 1, 1\}$, then τ is a positive rational number. Let δ_X be the real number such that $\vartheta(t) = 0$ for all $t \le \delta_X$ and $\vartheta(t) > 0$ for all $t > \delta_X$, where δ_X may depend on X. We claim that $\vartheta(\tau) > 0$. Otherwise, $\vartheta(\tau) = 0$, and so $\tau \le \delta_X$. Since $\vartheta(t)$ agrees with $\varsigma(t)$ for all $t \gg 0$, there exists a real number $\delta'_X \gg 0$ such that $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, where δ'_X may depend on X for the moment. By [Laz04, Theorem 2.2.44], $\vartheta(t)$ is a continuous function. Since both $\varsigma(t)$ and $\vartheta(t)$ are differentiable in (δ_X, δ'_X), by [Rud76, Theorem 5.9], there is a point $\gamma_X \in (\delta_X, \delta'_X)$ at which

$$[\vartheta(\delta'_X) - \vartheta(\delta_X)]\varsigma'(\gamma_X) = [\varsigma(\delta'_X) - \varsigma(\delta_X)]\vartheta'(\gamma_X).$$

Since $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, $\vartheta(\delta_X) = 0$ and $\varsigma(\delta_X) \ge \varsigma(\tau) > 0$, it follows that $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$, but by the previous paragraph, $\vartheta'(t) \le \varsigma'(t)$ on $(\delta_X, +\infty)$, which is a contradiction. Therefore, $\vartheta(\tau) > 0$, and we can conclude that $\vartheta(t) \ge \vartheta(\tau) > 0$ for all $t > \tau$.

Step 3. From now on we assume that $\dim Z = m > 1$. Recall that in the Step 5 of the proof of Theorem 3.8, we pick a a general element T in the very ample linear system $|l(K_Z + B_Z + M_Z)|$ and let S be its pullback to X, then

$$S \sim_{\mathbb{Q}} l(K_X + B).$$

Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$, then

$$K_S + B_S \sim_{\mathbb{Q}} (\frac{1}{l} + 1)S|_S.$$

Moreover, $(S, B_S), A_S \to T$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d - 1, \Phi, \Gamma, \psi)$$

for some fixed polynomial $\psi(t) \in \mathbb{Q}[t]$ that depends on d, Φ, Γ, σ, and $\dim T = m - 1$. By induction, there is a positive rational number λ' depending only on d, Φ, Γ, σ such that $K_S + B_S + tA_S$ is big for all $0 < t < \lambda'$. Hence by Theorem 3.8, there is a positive rational number β depending only on d, Φ, Γ, σ such that $A_S + t(K_S + B_S)$ is ample for all $t > \beta$.

Step 4. We have

$$\varsigma(t) = (A + t(K_X + B))^d$$
$$= A^d + \binom{d}{1} A^{d-1} \cdot (K_X + B)t + \cdots + \binom{d}{m} A^{d-m} \cdot (K_X + B)^m t^m.$$

Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, if $t > \beta(l+1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$ is ample by Step 3, we have

$$\varsigma'(t) = \binom{d}{1} A^{d-1} \cdot (K_X + B) + \cdots + m\binom{d}{m} A^{d-m} \cdot (K_X + B)^m t^{m-1}$$
$$= dA^{d-1} \cdot (K_X + B) + \cdots + d\binom{d-1}{m-1} A^{d-m} \cdot (K_X + B)^{m-1} \cdot (K_X + B)t^{m-1}$$
$$= d(A + t(K_X + B))^{d-1} \cdot (K_X + B)$$
$$= \frac{d}{l}(A + t(K_X + B))^{d-1} \cdot S$$
$$= \frac{d}{l}(A_S + \frac{t}{l+1}(K_S + B_S))^{d-1}$$
$$> 0.$$

Hence $\varsigma(t)$ is an increasing function on $(\beta(l + 1), +\infty)$.

If $\varsigma(t)$ has no roots (which occurs only when $\dim X$ is even), set $\tau = \beta(l + 1) + 1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$, set $\tau = \max\{\beta(l + 1), \lceil \alpha \rceil\} + 1$. Note that τ is a positive rational number. Moreover, on $[\tau, +\infty)$, $\varsigma(t)$ is a positive, increasing real function, and $\vartheta(t)$ is a non-negative, non-decreasing real function.

Step 5. In this step, we conclude the proof. We see that

$$\vartheta(t) = \text{vol}(A + t(K_X + B)) = \frac{1}{l^d} \text{vol}(lA + tS),$$

for any $S \sim_{\mathbb{Q}} l(K_X + B)$. For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose S_t as the pullback of a sufficiently general element $T_t \in |l(K_Z + B_Z + M_Z)|$ such that $S_t \not\subseteq \mathbf{B}_+(lA + tS_t)$. Then by Theorem 2.4, the function $s \mapsto \text{vol}(lA + tS_t + sS_t)$ is differentiable at $s = 0$, let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t, we have

$$l^d \vartheta'(t) = l^d \frac{\text{d}}{\text{d}s} \vartheta(t + s)\Big|_{s=0} = \frac{\text{d}}{\text{d}s}(\text{vol}(lA + tS_t + sS_t))\Big|_{s=0} = d\,\text{vol}_{X|S_t}(lA + tS_t).$$

Moreover, if $t \geq \tau$ and if $A + t(K_X + B)$ is big, we have

$$
\begin{aligned}
\vartheta'(t) =& \frac{d}{l^d} \operatorname{vol}_{X|S_t}(lA + tS_t) \\
\leq& \frac{d}{l} \operatorname{vol}((A + \frac{t}{l}S_t)|_{S_t}) \\
=& \frac{d}{l} \operatorname{vol}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})) \\
=& \frac{d}{l}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t}))^{d-1} \\
=& \varsigma'(t),
\end{aligned}
$$

where the third equality follows from the fact that $A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})$ is ample on $[\tau, +\infty)$.

We claim that $\vartheta(\tau) > 0$. Otherwise, $\vartheta(\tau) = 0$, employing the same argument as in the last paragraph of Step 2, we can get a contradiction. We conclude that $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t > \tau$. $\square$

Remark 3.10. In the case when $\dim X = 2$, by Zariski decomposition [Laz04, Theorem 2.3.19] [Eno20, Theorem 3.1], the volume of a big divisor should be greater than or equal to its self-intersection. Thus when $\dim X = 2$, Proposition 3.9 quickly follows from this fact. However, this property does not necessarily hold for higher-dimensional varieties. For example, let Y be a smooth 3-fold with K_Y ample, and $\pi : X = \operatorname{Bl}_P Y \to X$ be the blowing up of Y at a closed point P. Then $K_X = \pi^* K_Y + 2E$, where E is the exceptional divisor over $P \in Y$, and K_X is big. Therefore, $\operatorname{vol}(K_X) = \operatorname{vol}(K_Y) = K_Y^3$, but $K_X^3 = K_Y^3 + 8E^3 = K_Y^3 + 8 > \operatorname{vol}(K_X)$.

Proof of Theorem *1.5.* This directly follows from Theorem 3.8 and Proposition 3.9.
                                                                        $\square$

4. Moduli spaces

In this section, we prove Theorem 1.7, that is, we show that there is a separated Deligne-Mumford stack of finite type parametrizing families of traditional stable minimal models and it admits a coarse moduli space. We refer readers to [Alp23] for the notions of algebraic spaces, algebraic stacks and Deligne-Mumford stacks.

Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. In this section, we will fix these data.

4.1. **Moduli stacks of traditional stable minimal models.** The general strategy for constructing moduli stacks of varieties is to embed the varieties into a single projective space and then employ Hilbert scheme arguments. Moreover, the theory of relative Mumford divisors developed in [Kol23] also works for varieties polarized by effective divisors. However, for varieties polarized by non-canonical line bundles, to obtain a universal object for these line bundles, we consider embedding the varieties into the product of two projective spaces, following the approach in [Vie95, Section 1.7].

For any $(X, B), A \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(k)$, we will use the following lemma to produce two different very ample line bundles $\mathcal{O}_X(L_1), \mathcal{O}_X(L_2)$ on X such that $\mathcal{O}_X(L_1) \sim \mathcal{O}_X(L_2) \otimes A$.

Lemma 4.1. *Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Let K be a field of characteristic zero. Then there exist natural number τ and I depending only on d, Φ, Γ, σ such that $\tau \Phi \subset \mathbb{N}$ and they satisfy the following. For any $(X, B), A \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(K)$ and nef Cartier divisor M on X, we have*

- *$\tau(K_X + B)$ is a base point free divisor, $A + \tau(K_X + B)$ is an ample Cartier divisor,*
- *Let $L_M := I(A + \tau(K_X + B)) + M$, then L_M is strongly ample, i.e. L_M is very ample and $H^q(X, kL_M) = 0$ for any $k, q > 0$,*

Proof. By the same argument as [Bir22, Proof of Lemma 10.2], it is enough to find τ and I when $K = \operatorname{Spec} \mathbb{C}$. Note that A is a line bundle in our setting, hence by the proof of Theorem 1.5, there exist τ, I_0 such that $\tau(K_X + B)$ is base point free, $A + \tau(K_X + B)$ is ample Cartier, and $L_0 := I_0(A + \tau(K_X + B))$ is very ample.

After replacing I_0 with a bounded multiple, we may assume that $L_0 - (K_X + B)$ is nef and big. Let $I = (d + 2)I_0$ and $\mathcal{F} := L_M - I_0(A + \tau(K_X + B))$, then

$$H^i(X, \mathcal{F} \otimes L_0^{\otimes(-i)}) = 0$$

for all $i > 0$ by Kawamata-Viehweg vanishing theorem. Thus $\mathcal{F}$ is 0-regular with respect to L_0 ([Laz04, Definition 1.8.4]), and hence $\mathcal{F}$ is base point free by [Laz04, Theorem 1.8.5]. Therefore,

$$L_M = L_0 + \mathcal{F}$$

is very ample by [Har77, Exercise II 7.5(d)]. Again we have $L_M - (K_X + B)$ is nef and big, hence $H^q(X, kL_M) = 0$ for any $k, q > 0$. $\qquad\square$

We study the behavior of some invariants of traditional stable minimal models in families.

Lemma 4.2. *Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Let I and τ be the positive natural number as Lemma 4.1. Let $(X, B = \sum_{i=1}^{m} a_i D_i), A \to S$ be a family of $(d, \Phi, \Gamma, \sigma)$-marked traditional stable minimal models over reduced Noetherian scheme S. Let $L_1 := I(A + \tau(K_{X/S} + B)) + I(A + \tau(K_{X/S} + B))$, $L_2 := I(A + \tau(K_{X/S} + B)) + (I - 1)(A + \tau(K_{X/S} + B)) + \tau(K_{X/S} + B)$, and $L_3 := L_1 + L_2$, then for every $k \in \mathbb{Z}_{>0}$, the functions $S \to \mathbb{Z}$ sending*

(1) $s \mapsto h^0(X_s, kL_{j,s})$ for $j = 1, 2, 3$ and

(2) $s \mapsto \deg_{L_{3,s}}(D_{i,s})$ for $i = 1, 2, \dots, m$

are locally constant on S, where $\deg_{L_{3,s}}(D_{i,s}) := D_{i,s} \cdot L_{3,s}^{d-1}$.

Proof. (1). For $j = 1, 2, 3$, L_j are strongly ample over S by Lemma 4.1, it is enough to show that L_i are flat over S. Since $X \to S$ is flat, it suffices to show that $\mathcal{O}_X(L_i)$ are line bundles by [Har77, Proposition III 9.2(c)(e)].

Since $(X, B) \to S$ is locally stable family and $\mathcal{O}_{X_s}(\tau(K_{X_s}+B_s))$ is base point free for any $s \in S$, we see that $\tau(K_{X/S} + B)$ is $\mathbb{Q}$-Cartier and $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a mostly flat family ([Kol23, Definition 3.26]) of line bundles. By [Kol23, Corollary 4.34 and Proposition 5.29], $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a line bundle. Moreover, since A is a line bundle on X, so $\mathcal{O}_X(L_j)$ are line bundles for $j = 1, 2, 3$.

(2). It follows from [Kol23, Theorem 4.3.5]. $\square$

We recall the definition of stacks [Alp23, Definition 2.4.1 and Exercise 2.4.4]. Let **Sets** be the category of sets and **Sch**$_S$ the category of schemes over S. If $S = \operatorname{Spec} k$, we denote **Sch**.

Definition 4.3 (Stack). Let $p : \mathfrak{X} \to \mathbf{Sch}$ be a category fibered in groupoids. $\mathfrak{X}$ is called a stack if the following two conditions hold.

(1) (morphisms glue) For any scheme $S \in \mathbf{Sch}$ and any two objects $a, b \in \mathfrak{X}(S) := p^{-1}(S)$, the presheaf $\mathfrak{Isom}_S(a, b) : \mathbf{Sch}_S \to \mathbf{Sets}$ defined by

$$\mathfrak{Isom}_S(a, b)(f : T \to S) = \operatorname{Isom}_T(f^*a, f^*b)$$

is an étale sheaf.

(2) (objects glue) For any étale covering $S' \to S$ of an object $S \in \mathbf{Sch}$, any descent data with respect to this covering is effective: for objects a_i over S_i and isomorphisms $\alpha_{ij} : a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, which satisfy the cocycle condition

$$\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$$

on S_{ijk}, then there exists an object a over S and isomorphisms $\Phi_i : a|_{S_i} \to a_i$ over id_{S_i} such that $\Phi_i|_{S_{ij}} = \Phi_j|_{S_{ij}} \circ \alpha_{ij}$ on S_{ij}.

Lemma 4.4. $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a stack.

Proof. Take two objects $f : (X, B), A \to S$ and $f' : (X', B'), A' \to S$ in $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$. Then $\mathfrak{Isom}_S(f, f') \hookrightarrow \mathfrak{Isom}_S((X, B), (X', B'))$ is a closed immersion by separatedness of Pircard scheme [FGI$^+$05, Theorem 9.4.8]. For $i = 1, 2$, let $\mathcal{O}_X(L_i)$ and $\mathcal{O}_{X'}(L_i')$ be the relatively strongly ample line bundles as Lemma 4.2. Let $\mathbb{P} = \mathbb{P}(f_*\mathcal{O}_X(L_1)) \times_S \mathbb{P}(f_*\mathcal{O}_X(L_2))$ and $\mathbb{P}' = \mathbb{P}(f'_*\mathcal{O}_{X'}(L_1')) \times_S \mathbb{P}(f'_*\mathcal{O}_{X'}(L_2'))$. Then

$$\mathfrak{Isom}_S((X, B), (X', B')) \hookrightarrow \mathfrak{Isom}_S(\mathbb{P}, \mathbb{P}')$$

is a closed immersion, where the presheaf $\mathfrak{Isom}_S(\mathbb{P}, \mathbb{P}') : \mathbf{Sch}_S \to \mathbf{Sets}$ is defined by

$$\mathfrak{Isom}_S(\mathbb{P}, \mathbb{P}')(T) = \operatorname{Isom}_T(\mathbb{P} \times_S T, \mathbb{P}' \times_S T).$$

Moreover, $\mathfrak{Isom}_S(\mathbb{P}, \mathbb{P}')$ is represented by a locally closed subscheme of $\operatorname{Hilb}_S(\mathbb{P} \times_S \mathbb{P}')$ [FGI$^+$05, Theorem 5.23]. Hence $\mathfrak{Isom}_S(f, f')$ is represented by a scheme, in particular, it is an étale sheaf.

Given any étale covering $S' \to S$, where $(f' : (X', B'), A' \to S') \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S')$. To prove that f' descends to $(f : (X, B), A \to S) \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$, we use the strongly f'-ample line bundles $\mathcal{O}'_X(L_1')$ and $\mathcal{O}'_X(L_2')$ as Lemma 4.2 instead of $\omega_{\mathcal{C}'/S'}^{\otimes 3}$ in

[Alp23, Proposition 2.4.13]. The same argument as in *loc.cit.* implies that $(X', B') \to S'$ descends to $(X, B) \to S$. Moreover, by applying [Alp23, Proposition B.1.4(2) and Proposition B.1.11] to the étale covering $X' \to X$, A' descends to a line bundle A on X. Every geometric fiber of $f : (X, B), A \to S$ is identified with a geometric fiber of $f' : (X', B'), A' \to S'$, thus $(f : (X, B), A \to S) \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$. $\square$

Now we are ready to prove the main result of this subsection.

Theorem 4.5. $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ *is an algebraic stack of finite type.*

Proof. The strategy for the construction of moduli stacks is standard, we outline the main points of the proof here. In Steps 1 to 3, we fix certain invariants and employ a Hilbert scheme argument along with the theory of relative Mumford divisors to obtain a scheme that parametrizes the total spaces and the relevant boundary divisors. Moving on to Steps 4 to 7, we will shrink the parametrizing space to cut the locus of interest for our moduli problem, ensuring that these loci are locally closed. In the final step, we study the group action of the locus under consideration and conclude the proof using stack theory.

Step 1. Let $f : (X, B), A \to S$ be a family of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models, where S is a reduced scheme. For $j = 1, 2, 3$, let L_j be the Cartier divisor defined in Lemma 4.2, then L_j are strongly ample over S by Lemma 4.1.

Fixed $n, l \in \mathbb{Z}$, $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$ and $h \in \mathbb{Q}[k]$, consider the subfunctor $\mathfrak{TS}_{h,n,l,\mathbf{c}} \subset \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ parametrizing families of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models satisfying

- the Hilbert polynomial of X_s with respect to $L_{3,s}$ is h,
- $h^0(X_s, L_{1,s}) - 1 = n$,
- $h^0(X_s, L_{2,s}) - 1 = l$, and
- $(\deg_{L_{3,s}}(D_{1,s}), \deg_{L_{3,s}}(D_{2,s}), \ldots, \deg_{L_{3,s}}(D_{m,s})) = \mathbf{c}$

for every $s \in S$.

By Theorem 1.5, there are finitely many possibilities for the Hilbert functions

$$h_s(k) = \chi(X_s, kL_{3,s}) = h^0(X_s, kL_{3,s})$$

of X_s with respect to $L_{3,s}$, and for $1 \leq i \leq m$, the numbers

$$n_s = h^0(X_s, L_{1,s}) - 1, \quad l_s = h^0(X_s, L_{2,s}) - 1 \quad \text{and} \quad c_i = \deg_{L_{3,s}}(D_{i,s})$$

belong to a finite set (the finiteness can be reduced to the case when $s = \operatorname{Spec} \mathbb{C}$ by the same argument as [Bir22, Proof of Lemma 10.2]). Furthermore, by Lemma 4.2, for $1 \leq i \leq m$, n_s, l_s, c_i and h_s are locally constant on S. Therefore, we can write

$$\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma) = \bigsqcup_{h,n,l} \mathfrak{TS}_{h,n,l,\mathbf{c}}$$

as disjoint union, and each $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is a union of connected components of $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$. Moreover, there are only finitely many $n, l \in \mathbb{Z}_{>0}$, and $h \in \mathbb{Q}[k]$ such that $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is not empty.

Step 2. Fixing $n, l \in \mathbb{Z}$, $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$ and $h \in \mathbb{Q}[k]$, for any $(f : (X, B), A \rightarrow S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$, we see that $\mathcal{O}_X(L_1)$ and $\mathcal{O}_X(L_2)$ are strongly f-ample line bundles on X, we get an embedding

$$X \hookrightarrow \mathbb{P}(f_* \mathcal{O}_X(L_1)) \times_S \mathbb{P}(f_* \mathcal{O}_X(L_2)).$$

We proceed to parametrize such embedding.

Let $H = \text{Hilb}_h(\mathbb{P}^n \times \mathbb{P}^l)$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n \times \mathbb{P}^l$ with Hilbert polynomial h. Let $X_H = \text{Univ}_h(\mathbb{P}^n \times \mathbb{P}^l) \overset{i}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^l \times H$ be the universal family over H, and

$$\mathbb{P}^n \overset{p_1}{\leftarrow} \mathbb{P}^n \times \mathbb{P}^l \times H \overset{p_2}{\rightarrow} \mathbb{P}^l.$$

be the natural projections. Note that the $\text{PGL}_{n+1} \times \text{PGL}_{l+1}$ action on $\mathbb{P}^n \times \mathbb{P}^l$ induces a $\text{PGL}_{n+1} \times \text{PGL}_{l+1}$ action on H. Let $L_{3,H} := i^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, 1)$ and $A_H := i^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, -1)$ be the universal line bundles on X_H.

Step 3. By [Gro66, Theorem 12.2.1 and Theorem 12.2.4], the locus $s \in H$ such that X_s is geometrically connected and reduced, equidimensional, and geometrically normal is an open subscheme H_1 of H. Let L_{3,H_1} be the pull back of universal line bundle $L_{3,H}$ over H to H_1.

Next, we parametrize the boundary divisors in the moduli problem. Since $f_1 : X_{H_1} \rightarrow H_1$ is equidimensional, and over reduced bases relative Mumford divisors are the same as K-flat divisors [Kol23, Definition 7.1 and comment 7.4.2], there is a separated H_1-scheme $\text{MDiv}_c(X_{H_1}/H_1)$ of finite type which parametrizes relative Mumford divisors of degree c with respect to L_{H_1} by [Kol23, Theorem 7.3]. Fixing $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$, let

$$H_2 := \text{MDiv}_{c_1}(X_{H_1}/H_1) \times_{H_1} \text{MDiv}_{c_2}(X_{H_1}/H_1) \times_{H_1} \cdots \times_{H_1} \text{MDiv}_{c_m}(X_{H_1}/H_1)$$

be the m-fold fiber product, we denote the universal family by

$$(X_{H_2}, B_{H_2} = \sum_{i=1}^m a_i D_{i,H_2}; L_{3,H_2}), A_{H_2} \rightarrow H_2,$$

where D_{i,H_2} are the universal families of relative Mumford divisors on X_{H_2} of degree c_i for $1 \leq i \leq m$. Again the $\text{PGL}_{n+1} \times \text{PGL}_{l+1}$ action on $\mathbb{P}^n \times \mathbb{P}^l$ induces a $\text{PGL}_{n+1} \times \text{PGL}_{l+1}$ action on H_2.

Step 4. By [Kol23, Theorem 4.8], there is a locally closed partial decomposition $H_3 \rightarrow H_2$ satisfying the following: for any reduced scheme W and morphism $q : W \rightarrow H_2$, then the family obtained by base change $f_W : (X_W, B_W) \rightarrow W$ is locally stable iff q factors as $q : W \rightarrow H_3 \rightarrow H_2$.

Since $f_3 : (X_{H_3}, B_{H_3}) \rightarrow H_3$ is locally stable, By [Kol23, Theorem 4.28], there is a locally closed partial decomposition $H_4 \rightarrow H_3$ satisfying the following: for any reduced scheme W and morphism $q : W \rightarrow H_3$, the divisorial pullback of $\tau(K_{X_{H_3}/H_3} + B_{H_3})$ to $W \times_{H_3} X_{H_3}$ is Cartier iff q factors as $q : W \rightarrow H_4 \rightarrow H_3$.

Step 5. Since the fibers X_s of $f_4 : X_{H_4} \to H_4$ are reduced and connected by Step 3, we have $h^0(X_s, \mathcal{O}_{X_s}) = 1$. Since $\tau(K_{X_{H_4}/H_4} + B_{H_4})$ is Cartier, by [Vie95, Lemma 1.19], there is a locally closed subscheme $H_5 \subset H_4$ with the following property: for any scheme W and morphism $q : W \to H_4$,

$$\mathcal{O}_{X_W}(1,0) \sim_W A_W^{2I} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W), \text{ and}$$

$$\mathcal{O}_{X_W}(0,1) \sim_W A_W^{2I-1} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W)$$

iff q factors as $q : W \to H_5 \to H_4$, where $\mathcal{O}_{X_W}(1,0)$ and $\mathcal{O}_{X_W}(0,1)$ are the pullbacks of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(1,0)$ and $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(0,1)$ to X_W, respectively.

Step 6. By [Bir22, Lemma 8.5], there is a locally closed subscheme $H_6 \subset H_5$ such that for any $s \in H_6$, $K_{X_s} + B_s$ is semi-ample defining a contraction $X_s \to Z_s$.

Since ampleness and klt are open conditions, there is an open subscheme $H_7 \subset H_6$ such that $A_s + \tau(K_{X_s} + B_s)$ is ample and (X_s, B_s) is klt for any $s \in H_7$.

By [Har77, Theorem III 12.8], there is an open subscheme $H_8 \subset H_7$ such that for any $s \in H_8$,

$$H^q(X_s, 2IA_s + 2I\tau(K_{X_s} + B_s)) = H^q(X_s, (2I-1)A_s + 2I\tau(K_{X_s} + B_s)) = 0$$

for any $q > 0$. Thus by [Har77, Theorem III 12.11], for any $s \in H_8$,

$$h^0(X_s, 2IA_s + 2I\tau(K_{X_s} + B_s)) = n + 1, \text{ and}$$

$$h^0(X_s, (2I-1)A_s + 2I\tau(K_{X_s} + B_s)) = l + 1.$$

Therefore, $f_{8*}(A_{H_8}^{2I} \otimes \omega_{X_{H_8}/H_8}^{[2I\tau]}(2I\tau B_{H_8}))$ and $f_{8*}(A_{H_8}^{2I-1} \otimes \omega_{X_{H_8}/H_8}^{[2I\tau]}(2I\tau B_{H_8}))$ are locally free sheaves of rank $n + 1$ and $l + 1$, respectively. Shrinking H_8, we may assume that they are free sheaves, and hence

$$\mathbb{P}(f_{8*}(A_{H_8}^{2I} \otimes \omega_{X_{H_8}/H_8}^{[2I\tau]}(2I\tau B_{H_8}))) \cong \mathbb{P}_{H_8}^n, \text{ and}$$

$$\mathbb{P}(f_{8*}(A_{H_8}^{2I-1} \otimes \omega_{X_{H_8}/H_8}^{[2I\tau]}(2I\tau B_{H_8}))) \cong \mathbb{P}_{H_8}^l.$$

Step 7. By [Bir22, Lemma 8.7] (the condition of A_s being effective is not required in the proof), there is a a locally closed subscheme $H_9 \subset H_8$ such that for any $s \in H_9$, $\mathrm{vol}(A_s|_F) \in \Gamma$ for the general fibres F of $X_s \to Z_s$.

For each $s \in H_9$, since $K_{X_s} + B_s$ is nef and $A_s + \tau(K_{X_s} + B_s)$ is ample, $K_{X_s} + B_s + tA_s$ is ample for each $t \in (0, \frac{1}{\tau}]$, then

$$\theta_s(t) = \mathrm{vol}(K_{X_s} + B_s + tA_s) = (K_{X_s} + B_s + tA_s)^d$$

is a polynomial in t of degree $\leq d$ on the interval $(0, \frac{1}{\tau}]$. By Step 3(iv) of [Bir22, Proof of Proposition 9.5], there is an open and closed subscheme $H_{10} \subset H_9$ such that $\theta_s(t) = \sigma(t)$ on the interval $(0, \frac{1}{\tau}]$.

Step 8. By our construction, H_{10} represents the category fibered over **Sch** whose objects consist of a family $f : (X \subset \mathbb{P}^n \times \mathbb{P}^l \times S, B), A \to S$ with isomorphisms $\mathbb{P}(f_*(A_S^{2I} \otimes$

$\omega_{X_S/S}^{[2I\tau]}(2I\tau B_S))) \cong \mathbb{P}_S^n$ and $\mathbb{P}(f_*(A_S^{2I-1} \otimes \omega_{X_S/S}^{[2I\tau]}(2I\tau B_S))) \cong \mathbb{P}_S^l$. We claim that

$$\mathfrak{TS}_{h,n,l,\mathbf{c}} \cong [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}].$$

We follow the arguments of [Alp23, Theorem 3.1.17] and [ABB+23, Proposition 3.9]. Indeed, note that the universal family

$$(X_{H_{10}} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_{10}, B_{H_{10}} = \sum_{i=1}^m a_i D_{i,H_{10}}), A_{H_{10}} \to H_{10}$$

is an object in $\mathfrak{TS}_{h,n,l,\mathbf{c}}(H_{10})$, which induces a morphism $H_{10} \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$, where this morphism just forgets the projective embeddings. Moreover, this morphism is $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$-invariant, hence descends to a morphism $\Psi^{\mathrm{pre}} : [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]^{\mathrm{pre}} \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$ of prestacks. Since $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is a stack by Lemma 4.4, the universal property of stackification [Alp23, Theorem 2.4.17] yields a morphism $\Psi : [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}] \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$.

To show Ψ is fully faithful, by [Alp23, Exercise 2.4.16], it suffices to check that Ψ^{pre} is fully faithful. To see this, let $S \to H_{10}$ be a map corresponding to the family $f : (X \subset \mathbb{P}^n \times \mathbb{P}^l \times S, B), A \to S$. We see that an automorphism of $f : (X, B), A \to S$ induces an automorphism of $\mathcal{O}_X(L_1)$ and $\mathcal{O}_X(L_2)$, which in turn induces an automorphism of

$$\mathbb{P}(f_*\mathcal{O}_X(L_1)) \times_S \mathbb{P}(f_*\mathcal{O}_X(L_2)) \cong \mathbb{P}_S^n \times_S \mathbb{P}_S^l \cong \mathbb{P}^n \times \mathbb{P}^l \times S$$

preserving $(X, B), A \to S$.

It remains to check that Ψ is essentially surjective. Consider $(f : (X, B), A \to S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$, since $f_*\mathcal{O}_X(L_1)$ and $f_*\mathcal{O}_X(L_2)$ are locally free, there exists an open cover $S = \cup_i S_i$ over which their restrictions are free. Choosing trivializations induce embeddings $g_i : (X_{S_i}, B_{S_i}) \hookrightarrow \mathbb{P}^n \times \mathbb{P}^l \times S_i$. Moreover, for any $s \in S_i$, since $A_s \sim g_{i,s}^* \mathcal{O}_{\mathbb{P}_{k(s)}^n \times \mathbb{P}_{k(s)}^l}(1,-1)$, where $g_{i,s} = g_i|_{X_s} : (X_s, B_s) \hookrightarrow \mathbb{P}_{k(s)}^n \times \mathbb{P}_{k(s)}^l$, we have $A_{S_i} \sim_{S_i} g_i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S_i}(1,-1)$ by [Har77, Exercise III 12.4]. Hence by our construction of H_{10}, we have morphisms $\Phi_i : S_i \to H_{10}$. Over the intersections $S_i \cap S_j$, the trivializations differ by a section $s_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1})$. Therefore the Φ_i glue to a morphism $\Phi : S \to [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, which induces a morphism $\mathfrak{TS}_{h,n,l,\mathbf{c}} \to [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, that is the inverse of Ψ.

Note that H_{10} is a finite type scheme, thus $\mathfrak{TS}_{h,n,l,\mathbf{c}} \cong [H_{10}/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$ is a finite type algebraic stack. Taking the disjoint union over all $h, n, l, \mathbf{c}$ yields $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$. $\qquad\square$

4.2. Moduli spaces of traditional stable minimal models.

In this subsection, we prove Theorem 1.7. We need the following separatedness result to construct moduli spaces of traditional stable minimal models. Before stating the result, we review some notations.

Let C be an affine curve. Let $f : (X, B), A \to C$ and $f' : (X', B'), A' \to C$ be two families of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over C. We put

$$((X, B), A) \times_C C^o = (X \times_C C^o, B \times_C C^o), A|_{X \times_C C^o}.$$

We define $g : (X, B), A \to (X', B'), A'$ to be a C-isomorphism $g : X \to X'$ such that $f' \circ g = f$, $g_*B = B'$ and $g^*A' \sim_C A$. We define C^o-isomorphisms between $((X, B), A) \times_C C^o$ and $((X', B'), A') \times_C C^o$ similarly.

Theorem 4.6. ([Bou14, Theorem 2.1], [HH23, Proposition 4.4]). *Let C be an affine curve. Let $f : (X, B), A \to C$ and $f' : (X', B'), A' \to C$ be two families of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over C. Let $0 \in C$ be a closed point and $C^o = C \setminus \{0\}$ the punctured curve. Suppose that there exists a C^o-isomorphism*

$$g^o : ((X, B), A) \times_C C^o \to ((X', B'), A') \times_C C^o,$$

then g^o can be extended to a C-isomorphism $g : (X, B), A \to (X', B'), A'$.

Proof. Consider $L := A' + \tau(K_{X/C} + B)$ and $L' := A' + \tau(K_{X'/C} + B')$, where τ is a positive natural number as Lemma 4.1. By the proof of Lemma 4.2, L is a f-ample Cartier divisor on X (resp. L' is a f'-ample Cartier divisor on X'). Let $g : X \dashrightarrow X'$ be the birational map induced by g^o, then by the same argument as in [HH23, Proof of Proposition 4.4], g and g^{-1} do not contract any divisor, and g is a C-isomorphism. $\square$

The following corollary follows from Theorem 4.6 and the argument of [BX19, Proof of Corollary 3.5]. We include the proof for convenience.

Corollary 4.7. *Let $(X, B), A$ be a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model. Then $\mathrm{Aut}((X, B), A)$ is finite.*

Proof. Since $\mathrm{Aut}((X, B), A)$ is a linear algebraic group, it is affine. To conclude that $\mathrm{Aut}((X, B), A)$ is finite, it suffices to show that it is proper. To see the properness, consider a map $g^o : C^o \to \mathrm{Aut}((X, B), A)$, where $0 \in C$ is a smooth pointed curve and $C^o = C \setminus \{0\}$. The map g^o induces an isomorphism

$$((X \times C, B \times C), A \times C) \times_C C^o \to ((X \times C, B \times C), A \times C) \times_C C^o$$

over C^o. By applying Theorem 4.6 to the above isomorphism, we see g^o extends to a map $g : C \to \mathrm{Aut}((X, B), A)$. Hence, $\mathrm{Aut}((X, B), A)$ is proper, and the proof is complete. $\square$

Proof of Theorem *1.7.* By Theorem 4.5, $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is an algebraic stack of finite type. By Corollary 4.7 and [Alp23, Theorem 3.6.4], $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a Deligne-Mumford stack. Moreover, Theorem 4.6 and [Alp23, Theorem 3.8.2(3)] imply that $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a separated Deligne-Mumford stack. Therefore, we may apply Keel–Mori Theorem [KM97][Alp23, Theorem 4.3.12] to see that $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ has a coarse moduli space $TS_{klt}(d, \Phi, \Gamma, \sigma)$, which is a separated algebraic space. $\square$

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