Generic Vanishing In Birational Geometry

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Abstract

In this note, we study morphisms from smooth projective varieties to abelian varieties. We prove that direct images of pluricanonical bundles are GV-sheaves, and we construct an example such that a higher direct image of pluricanonical bundle is not a GV-sheaf. We also discuss applications of generic vanishing theory to birational geometry, such as characterization of abelian varieties and Iitaka conjecture over abelian varieties.

Contents

1 Introduction

Throughout this note, we always assume that all the varieties are defined over the complex number field. Generic vanishing theorem of Green-Lazarsfeld [GL91] [GL87] states that on a smooth projective variety X, the cohomology of a generic line bundle

$L \in \mathrm{Pic}^0(X)$ vanishes in degree below $\dim \mathrm{alb}_X(X)$, where $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ denotes the Albanese map of X.

One can consider the set of line bundles for which the cohomology in a given degree does not vanish. Denoting by

$$V^i(\mathcal{F}) := \{L \in \mathrm{Pic}^0(X) | H^i(X, \mathcal{F} \otimes L) \neq 0\} \subseteq \mathrm{Pic}^0(X)$$

the i-th cohomological support locus of a coherent sheaf $\mathcal{F}$, one has $\mathrm{codim} V^i(\omega_X) \geq i - \mathrm{dim} X + \dim \mathrm{alb}_X(X)$ for all $i \geq 0$ [GL91] [GL87]. This implies the previous generic vanishing theorem via Serre duality. In particular, one has $\mathrm{codim} V^i(\omega_X) \geq i$ for all $i \geq 0$ if X has maximal Albanese dimension. Moreover, Simpson studied the structure of $V^i(\omega_X)$ and proved that every irreducible components of $V^i(\omega_X)$ are torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$ [Sim93]. For irregular varieties, we usually study their properties by their albanese maps to abelian varieties, since abelian varieties have maximal Albanese dimension obviously.

We now concentrate on the morphism $f : X \to A$, where X is smooth projective variety and A is an abelian variety.

Definition 1.1. *Let A be an abelian variety, a coherent sheaf $\mathcal{F}$ on A is called a GV-sheaf if it satisfies*

$$\mathrm{codim} V^i(\mathcal{F}) \geq i, \ \ for\ all\ i \geq 0.$$

When $f : X \to A$ is generically finite, $f_*\omega_X$ is a GV-sheaf [GL91] [GL87]. This result was generalised to the case that the higher direct images $R^j f_*\omega_X$ are GV-sheaves on A for all $j \geq 0$ and for arbitrary map f by Hacon.

Theorem 1.2. *([Hac04]). Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety, then $R^j f_*\omega_X$ is a GV-sheaf for every $j \geq 0$.*

The original proof of generic vanishing theorem by Green-Lazarsfeld used deformation theory and classical Hodge theory, while the one by Hacon was based on derived category and Fourier-Mukai transform for abelian varieties. There is also a more recent proof by Popa-Schnell via mixed Hodge module [PS13].

By an effective freeness result (see Theorem 3.2), Popa-Schnell gave another direction generalization of [GL91] [GL87] to direct images of pluricanonical bundles.

Theorem 1.3. *([PS14]). Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety, then $f_*\omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.*

Therefore, it is natural to ask that whether can we combine the results of Hacon and Popa-Schnell?

Question 1.4. *Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety, then is $R^j f_*\omega_X^{\otimes k}$ a GV-sheaf for any $k \geq 2$ and $j \geq 1$?*

Unfortunately, Shibata gave a negative answer to this question by constructing a counterexample [Shi16] (See Example 3.7).

After discussing the generic vanishing theory, we give some applications in birational geometry, namely characterization of abelian varieties due to Chen-Hacon and Iitaka conjecture over maximal Albanese dimension varieties due to Cao-Păun.

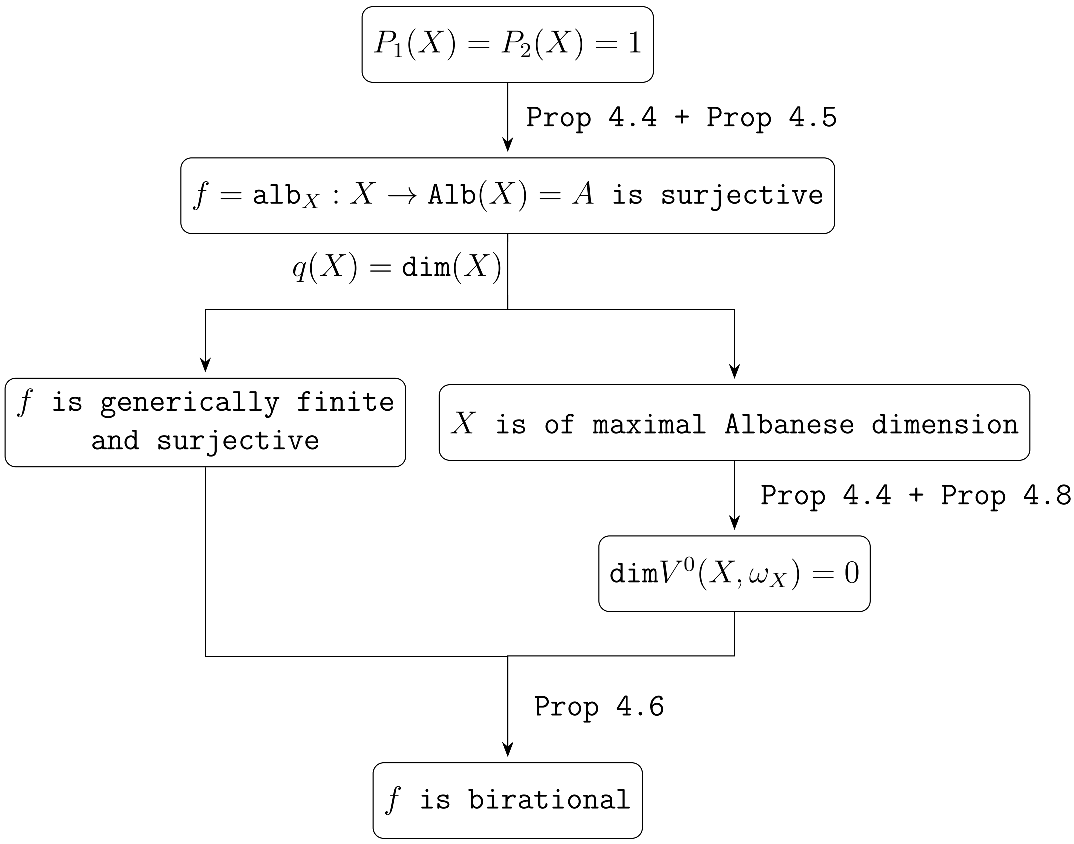
Theorem 1.5. *([CH01]). Let X be a smooth projective variety with $P_1(X) = P_2(X) = 1$ and $q(X) = \dim X$. Then X is birational to an abelian variety.*

Theorem 1.6. *([CP17] [HPS18]). Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.*

Sketch of proof of Theorem 1.5. $P_1(X) = P_2(X) = 1$ implies that the Albanese map $f : X \to \mathrm{Alb}(X) = A$ is surjective by a result of Ein-Lazarsfeld (see Proposition 4.4 and Proposition 4.5). Since $q(X) = \dim X$, we obtain that f is generically finite surjective and that X is of maximal Albanese dimension. We only need to show that deg $f = 1$.

If we further assume that $\dim V^0(X, \omega_X) = 0$, then together with the fact that f is generically finite surjective, we can show that deg $f = 1$, hence f is birational (see Proposition 4.6).

Finally, we show that $\dim V^0(X, \omega_X) = 0$ by combining Proposition 4.4 with Proposition 4.8.



Sketch of proof of Theorem 1.6. Firstly, we can reduce the problem to the case that $\kappa(X) = 0$ by Iitaka fibration, and that $Y = A$ is an abelian variety by the structure of maximal Albanese dimension varieties (see Lemma 4.10).

Then, we study the GV-sheaf $\mathcal{F}_m = f_*\omega_X^{\otimes m}$ which encodes the information of plurigenera of fiber F. We fixed an m such that $h^0(A, \mathcal{F}_m) = h^0(X, \omega_X^{\otimes m}) = 1$. To obtain the conclusion, it is enough to show $\mathcal{F}_m$ has rank 1 generically. With the help of generic vanishing theory, we show that $\mathcal{F}_m$ is a successive extension of trivial bundles (see Lemma 4.11).

Finally, by an analytic result (see Theorem 4.12), $\mathcal{F}_m \simeq \mathcal{O}_A^{\oplus r}$, hence the rank r of $\mathcal{F}_m$ is equal to $h^0(A, \mathcal{F}_m) = 1$.

Acknowledgments. content...

2 Preliminaries

2.1 GV-sheaves

In this subsection, we recall the definition of GV-sheaves. And we proved that higher direct images of canonical bundles of morphisms to abelian varieties are GV-sheaves. Let us first recall Fourier-Mikai transform on abelian varieties.

Let A be an abelian variety of dimension g, and $\hat{A}$ be the dual abelian variety $\mathrm{Pic}^0(A)$, P be the normalized Poincare bundle on $A \times \hat{A}$. It induces Fourier-Mukai transforms

$$\mathbf{R}\Phi_P : D^b(A) \to D^b(\hat{A}),\ \mathbf{R}\Phi_P\mathcal{F} = \mathbf{R}p_{2_*}(p_1^*\mathcal{F} \otimes P),$$

and

$$\mathbf{R}\Psi_P : D^b(\hat{A}) \to D^b(A),\ \mathbf{R}\Psi_P\mathcal{G} = \mathbf{R}p_{1_*}(p_2^*\mathcal{G} \otimes P).$$

These functors are known to be equivalences of derived categories, moreover,

$$\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A) * [-g] \ \ and\ \ \mathbf{R} \circ \Phi_P \mathbf{R}\Psi_P = (-1_{\hat{A}}) * [-g],$$

where $[-g]$ denotes shifting g places to the right.

By the work of Hacon, we have the following equivalent characterization of GV-sheaf.

Theorem 2.1. *([Hac04]). Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. Then the following four conditions are equivalent to each other.*

(1) $\mathcal{F}$ is a GV-sheaf, i.e.

$$\mathrm{codim} V^i(\mathcal{F}) \geq i \ \ for\ \ all\ \ i \geq 0, \ \ where\ \ V^i(\mathcal{F}) := \{\alpha \in \hat{A} | H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0\}.$$

(2)The Fourier-Mukai transform $\boldsymbol{R}\Phi_P\mathcal{F}$ satisfies

$$\mathrm{codim}\,\mathrm{Supp} R^i\Phi_P(\mathcal{F}) \geq i \ \ for\ all\ \ i \geq 0.$$

(3) For every finite étale morphism $\phi : B \to A$ of abelian varieties, and every ample line bundle L on B, one has

$$H^i(B, L \otimes \phi^*\mathcal{F}) = 0 \quad for\ \ i > 0.$$

(4) There is a coherent sheaf $\mathcal{G}$ with the property that $\boldsymbol{R}\Phi_P(\mathcal{F}) = \boldsymbol{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_{\hat{A}})$.

We give a sketch of proof the equivalence of part (1) and (2): note that the restriction of $p_1^*\mathcal{F} \otimes P$ to a fiber $A \times \alpha$ of p_2 is isomorphic to the sheaf $\mathcal{F} \otimes P_\alpha$ on A, and so fiberwise we are looking at the cohomology groups $H^i(A, \mathcal{F} \otimes P_\alpha)$. A simple application of the theorem on cohomology and base change then shows for every $m \geq 0$ that

$$\bigcup_{i \geq m} \mathrm{Supp} R^i \Phi_P \mathcal{F} = \bigcup_{i \geq m} V^i(\mathcal{F}).$$

This implies the result by decending induction on i.

Proof of Theorem 1.2 . Let $\phi : B \to A$ be a finite *é*tale morphism of abelian varieties, then by Theorem 2.1(3), it is enough to show $H^i(B, L \otimes \phi^* R^j f_* \omega_X) = 0$ for every ample line bundle L on B and every $i > 0$.

If we let $Y = B \times_A X$ be the fiber product, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ B \arrow[r, "\phi"] & A \end{tikzcd}$$

in which ψ is also finite *é*tale. By flat base change,

$$\phi^* R^j f_* \omega_X \simeq R^j g_* \psi^* \omega_X \simeq R^j g_* \omega_X,$$

and so the assertion follow from Koll*á*r Vanishing (see Theorem 2.3(2)), applied to the morphism g. $\square$

Moreover, after twisting by torsion line bundles, the higher direct images of canonical bundles are still GV-sheaves.

Corollary 2.2. *Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. Suppose that $L \in \mathrm{Pic}^0(X)$ is a line bundle with $L^d \simeq \mathcal{O}_X$. Then the sheaf $R^j f_*(\omega_X \otimes L)$ is a GV-sheaf for every $j \geq 0$.*

Proof. Since $L^d \simeq \mathcal{O}_X$, L determines a finite *é*tale covering $p : Y \to X$ of degree d such that

$$p_* \mathcal{O}_Y \simeq \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(d-1)}.$$

Because Y is *é*tale over X, we get

$$p_* \omega_Y \simeq p_* p^* \omega_X \simeq \omega_X \otimes p_* \mathcal{O}_Y.$$

In particular, $\omega_X \otimes L$ is a direct summand of $p_* \omega_Y$. This means that $R^j f_*(\omega_X \otimes L)$ is a direct summand of $R^j (f \circ p)_* \omega_Y$, and therefore a GV-sheaf. $\square$

We need Koll*á*r's important results about higher direct images of dualizing sheaves.

Theorem 2.3. *(Kollár Vanishing [Kol86a]). Let $f : X \to Y$ be a morphism of projective varieties, with X smooth. Then*

(1) $R^j f_* \omega_X$ are torsion free sheaves on Y for all $j \geq 0$.

(2) if L is an ample line bundle on Y, then

$$H^i(Y, L \otimes R^j f_* \omega_X) = 0$$

for all $i > 0$ and all $j \geq 0$.

Theorem 2.4. *(Kollár [Kol86b]). Let $f : X \to Y$ be a morphism of projective varieties, with X smooth. then one has a non-canonical isomorphism*

$$\mathbf{R} f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i]$$

in the derived category $D^b(X)$.

2.2 Properties of GV-sheaves

In this subsection, we discuss some properties of GV-sheaves.

Corollary 2.5. *If $\mathcal{F}$ is a GV-sheaf on A, then*

$$V^g(\mathcal{F}) \subseteq \cdots \subseteq V^1(\mathcal{F}) \subseteq V^0(\mathcal{F}) \subseteq \hat{A}.$$

Proof. The assertion is that $H^i(A, \mathcal{F} \otimes \alpha) = 0$ implies that $H^{i+1}(A, \mathcal{F} \otimes \alpha) = 0$. Tis turn out to be a formal consequence of the fact that $\mathbf{R}\Phi_P(\mathcal{F}) = \mathbf{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_{\hat{A}})$ for a coherent sheaf $\mathcal{G}$ on $\hat{A}$. By the base change,

$$H^i(A, \mathcal{F} \otimes \alpha) \simeq R^i\Gamma(\hat{A}, \mathbf{R}\Phi_P(\mathcal{F}) \otimes \mathcal{O}_\alpha) \simeq R^i\Gamma(\hat{A}, \mathbf{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_{\hat{A}}) \otimes \mathcal{O}_\alpha).$$

This may be rewritten in the form

$$R^i\Gamma(\hat{A}, \mathbf{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_\alpha)) \simeq H^0(\hat{A}, \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_\alpha))$$

because the support of $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_\alpha)$ is a point. This reduced the problem to show that $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_\alpha) = 0$ implies $\mathcal{E}xt^{i+1}(\mathcal{G}, \mathcal{O}_\alpha) = 0$.

This follows from the following result about commutative algebra: Let (A, m) be a local ring with residue field $k = A/m$, then

$$Ext^i_A(M, k) = 0 \quad \text{implies} \quad Ext^{i+1}_A(M, k) = 0$$

for every finitely generated A-module M.

□

Corollary 2.6. *If $\mathcal{F}$ is a GV-sheaf on A, then $\mathcal{F} = 0$ if and only if $V^0(\mathcal{F}) = \emptyset$.*

Proof. By Corollary 2.5, we see that $V^0(\mathcal{F}) = \emptyset$ is equivalent to $V^k(\mathcal{F}) = \emptyset$ for all $k \geq 0$, which is equivalent to $\mathbf{R}\Phi_P\mathcal{F} = 0$ by base change and cohomology. This is equivalent to $\mathcal{F} = 0$ by Mukai' s equivalence of derived categories.

□

The following two properties of GV-sheaf are used in the proof of Chen-Hacon's characterization of abelian varieties (Theorem 1.5).

Proposition 2.7. *If $\mathcal{F}$ is a GV-sheaf on A, one has $R^i\Phi_P\mathcal{F} = 0$ for $i < \mathrm{codim}V^0(\mathcal{F})$.*

Proof. By Theorem 2.1(4), there is a coherent sheaf $\mathcal{G}$ with the property that $\mathbf{R}\Phi_P(\mathcal{F}) = \mathbf{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_{\hat{A}})$. Now

$$\text{Supp } \mathcal{G} = \text{Supp}\mathbf{R}\Phi_P(\mathcal{F}) = \bigcup_{i\geq 0} \text{Supp}R^i\Phi_P(\mathcal{F}) = \bigcup_{i\geq 0} V^i(\mathcal{F}) = V^0(\mathcal{F})$$

by base change and Corollary 2.5. Therefore the assertion is that

$$R^i\Phi_P(\mathcal{F}) \simeq \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\hat{A}}) = 0$$

for every $i < \text{codim Supp}\mathcal{G}$.

This follows from the following result about commutative algebra: Let (A, m) be a regular local ring, and let M be a finitely generated A-module. Then

$$\min\{Ext^i_A(M, A) \neq 0\} = \text{codim Supp}M = \dim A/\mathrm{Ann}(M).$$

□

Proposition 2.8. *If $\mathcal{F}$ is a GV-sheaf on A, suppose that $Z \subseteq \mathrm{codim}V^0(\mathcal{F})$ is an irreducible component of codimension k in $\hat{A}$. Then Z is actually an irreducible component of $V^k(\mathcal{F})$. In particular, we must have* $\dim \text{Supp}\mathcal{F} \geq k$.

Proof. We have already seen that

$$\text{Supp}\mathcal{G} = \text{Supp}\mathbf{R}\Phi_P(\mathcal{F}) = V^0(\mathcal{F}),$$

we know that Z is also an irreducible component of $\text{Supp}\mathcal{G}$. By applying the same argument as before to the local ring at the generic point of Z, we deduced that the sheaves

$$R^i\Phi_P(\mathcal{F}) \simeq \mathcal{E}xt(\mathcal{G}, \mathcal{O}_{\hat{A}})$$

for $i < k$ have to be zero in a Zariski open neighborhood of the generic point of Z. By base change, we have

$$\text{Supp}\mathcal{G} = \text{Supp}\mathbf{R}\Phi_P(\mathcal{F}) = \bigcup_{i\geq k} \text{Supp}R^i\Phi_P\mathcal{F} = \bigcup_{i\geq k} V^i(\mathcal{F})$$

in a Zariski open neighborhood of the generic point of Z. Therefore Z is actually an irreducible component of $V^k(\mathcal{F})$ by Corollary 2.5. Because

$$V^k(\mathcal{F}) := \{\alpha \in \hat{A} | H^k(A, \mathcal{F} \otimes P_\alpha) \neq 0\},$$

this can only happen if $\dim \text{Supp}\mathcal{F} \geq k$.

□

2.3 Unipotent vector bundle

A vector bundle U on an abelian variety A is called *unipotent* if it has a filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = U$$

such that $U_i/U_{i+1} \simeq \mathcal{O}_A$ for all $i = 1, ..., n$. This notation is used in the proof of Iitaka conjecture (Theorem 1.6).

Proposition 2.9. *Let $\mathcal{F}$ be a GV-sheaf on an abelian variety A, if $V^0(\mathcal{F}) = \{0\}$, then $\mathcal{F}$ is a unipotent vector bundle.*

Proof. By [Muk81, Example 2.9], if $\dim A = g$, then $\mathcal{F}$ is unipotent if and only if $R^i\Phi_P\mathcal{F} = 0$ for all $i \neq g$, and $R^g\Phi_P\mathcal{F} = \mathcal{G}$, where $\mathcal{G}$ is a coherent sheaf supported at the origin $0 \in \hat{A}$.

We now check the two conditions are satisfied. By Corollary 2.5, $V^i(\mathcal{F}) \subseteq \{0\}$ for all $i \geq 0$. By base change and cohomology we obtain that $R^i\Phi_P\mathcal{F}$ is supported at most $0 \in \hat{A}$ for $0 \leq i \leq g$. It remains to show that $R^i\Phi_P\mathcal{F} = 0$ for all $i \neq g$. Note that

$$H^j(\hat{A}, R^i\Phi_P\mathcal{F} \otimes \alpha) = 0$$

for all $j > 0$, $0 \leq i \leq g$, and $\alpha \in \mathrm{Pic}^0(\hat{A})$. So by base change we have

$$R^j\Psi_P(R^i\Phi_P\mathcal{F}) = 0$$

for all $j > 0$, $0 \leq i \leq g$. By an easy Grothendieck spectral sequence argument, it follows that $R^0\Psi_P(R^i\Phi_P\mathcal{F}) = \mathcal{H}^i((-1_A)^*\mathcal{F}[-g])$ since $\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^*[-g]$. So we have

$$R^0\Psi_P(R^i\Phi_P\mathcal{F}) = 0$$

for $i < g$. But then $\mathbf{R}\Psi_P(R^i\Phi_P\mathcal{F}) = 0$ for $i < g$, hence $R^i\Phi_P\mathcal{F} = 0$ for $i < g$ by equivalence of derived categories.

□

2.4 M-regular

In this subsection, we recall the M-regular sheaves and their properties.

Definition 2.10. *Let A be an abelian variety, a coherent sheaf $\mathcal{F}$ on A is called M-regular if it satisfies* $\mathrm{codim} V^i(\mathcal{F}) > i$ *for all $i > 0$.*

Lemma 2.11. *([LPS20, Lemma 2.2]) If $\mathcal{F} \neq 0$ is an M-regular sheaf on an abelian variety A. then the Euler characteristic $\chi(A, \mathcal{F}) > 0$, and in particular $V^0(\mathcal{F}) = \mathrm{Pic}^0(A)$.*

Definition 2.12. *We say that a coherent sheaf $\mathcal{F}$ on an abelian variety A has the Chen-Jiang decomposition property if $\mathcal{F}$ admits a finite direct sum decompoition*

$$\mathcal{F} \simeq \bigoplus_{i \in I} (\alpha_i \otimes q_i^*\mathcal{F}_i),$$

into pullacks of M-regular cohenrent sheaves $\mathcal{F}_i$ from quotients $q_i : A \to A_i$ of abelian variety, tensored by torsion line bundles $\alpha_i \in \hat{A}$.

Lemma 2.13. *If $\mathcal{F}$ admits a Chen-Jiang decomposition, then*

$$V^0(\mathcal{F}) = \bigcup_{i \in I} \alpha_i^{-1} \otimes p_i^* \mathrm{Pic}^0(A_i).$$

Proof. This is an easy computation, using projection formula and the fact that $V^0(\mathcal{F}_i) = \mathrm{Pic}^0(A_i)$ by Lemma 2.11. □

3 Direct images of pluricanonical bundles

3.1 Direct images of pluricanonical bundles

In this subsection, we explain the proof of Theorem 1.3. First we discuss the effective freeness due to Popa-Schnell.

The famous Fujita Conjecture predicts that if X is a smooth projective variety of dimension n, and L is an ample line bundle on X, then $\omega_X \otimes L^{\otimes l}$ is globally generated for $l \geq n+1$. It is well known that $\omega_X \otimes L^{\otimes l}$ is nef by length of extremal ray. If we further assume L is globally generated, Fujita Conjecture holds by Kodaira Vanishing and Castelnuovo-Mumford regularity. Moreever, this can be extended to the relative setting via Kollár Vanishing (see Theorem 2.3) and Castelnuovo-Mumford regularity.

Theorem 3.1. *(Kollár). Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. If L is an ample and globally generated line bundle on Y, then*

$$R^j f_* \omega_X \otimes L^{\otimes l}$$

is globally generated for $l \geq n+1$.

When $j = 0$, we can generalize the previous theorem to pluricanonical bundles.

Theorem 3.2. *(Effective freeness). Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n, and k be a positive integer. If L is an ample and globally generated line bundle on Y, then we have*

$$H^i(Y, f_* \omega_X^{\otimes k} \otimes L^{\otimes l}) = 0,$$

for every $i > 0$ and every $l \geq k(n+1) - n$. By Castelnuovo-Mumford, regularity

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes l} \textit{ is globally generated}, \ for\ every\ l \geq k(n+1).$$

Proof. Let us consider

$$M = \mathrm{Im}(f^* f_* \omega_X^{\otimes k} \to \omega_X^{\otimes k}).$$

By taking blow ups, we may assume M is invertible sheaf such that $\omega_X^{\otimes k} = M \otimes \mathcal{O}_X(E)$ for some effective divisor E on X, we may further assume that $\mathrm{Supp} E$ is a simple normal crossing divisor. Since L is ample, we can take the smallest integer $m \geq 0$ such that $f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$ is globally generated. Since $\omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) = M$, $f_* M = f_* \omega_X^{\otimes k}$, and

$f^*f_*M \to M$ is surjective, we have that $\omega_X^{\otimes k} \otimes \mathcal{O}_X(-E) \otimes f^*L^{\otimes m}$ is also globally generated. Therefore, we can take general smooth effective divisor D such that $\mathrm{Supp}(D+E)$ is simple normal crossing on X and that

$$kK_X + mf^*L \sim D + E.$$

Then we have

$$(k-1)K_X \sim_{\mathbb{Q}} \frac{k-1}{k}D + \frac{k-1}{k}E - \frac{k-1}{k}mf^*L.$$

So we obtain

$$kK_X - \lfloor \frac{k-1}{k}E \rfloor + lf^*L$$

$$\sim_{\mathbb{Q}} K_X + \frac{k-1}{k}D + \{\frac{k-1}{k}E\} + (l - \frac{k-1}{k}m)f^*L.$$

Note that

$$f_*\mathcal{O}_X(kK_X - \lfloor \frac{k-1}{k}E \rfloor) = f_*\omega_X^{\otimes k}$$

by the definition of E, so if $l - \frac{k-1}{k}m > 0$, we have

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}) = 0$$

for every $i > 0$ by Ambro-Fujino Vanishing (Theorem 3.3) . Therefore, if $l - \frac{k-1}{k}m > n$, $f_*\omega_X^{\otimes k} \otimes L^{\otimes l}$ is globally generated by Castelnuovo-Mumford regularity. By the choice of m, we hace

$$m \leq \frac{k-1}{k}m + n + 1.$$

This implies $m \leq k(n+1)$, Therefore we obtain that if

$$l > \frac{k-1}{k}k(n+1) = kn + k - n - 1$$

then

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}) = 0$$

for every $i > 0$. □

We need a generalization of Kollár vanishing (Theorem 2.3) to pairs case.

Theorem 3.3. *(Ambro-Fujino Vanishing [Amb03] [Fuj11]). Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n. Let (X, Δ) be a log canonical log smooth pair. Consider a line bundle B on X such that $B \sim_{\mathbb{R}} K_X + \Delta + f^*L$, where L is an ample $\mathbb{R}$-Cartier divisor on Y. Then*

$$H^i(Y, R^jf_*B) = 0$$

for all $i > 0$ and all $j \geq 0$.

Before proving Theorem 1.3, we recall that an ample line bundle M on $\hat{A}$ induces an isogeny

$$\phi_M : \hat{A} \to A, \quad \alpha \to t_\alpha^* M^{-1} \otimes M,$$

where t_α denotes translation by $\alpha \in \hat{A}$.

Proof of Theorem 1.3. Let $M = L^{\otimes l}$, where L is an ample and globally generated line bundle on $\hat{A}$, and l is an integer that can be chosen arbitrarily large. Let $\phi_M : \hat{A} \to A$ be the isogeny induced by M. According to Theorem 2.1(3), it is enough to show that

$$H^i(\hat{A}, \phi_M^* f_* \omega_X^{\otimes k} \otimes M) = 0$$

for all $i > 0$. Equivalently, we need to show that

$$H^i(\hat{A}, h_* \omega_Y^{\otimes k} \otimes L^{\otimes l}) = 0$$

for all $i > 0$, where $h : Y \to \hat{A}$ is the base change of $f : X \to A$ via ϕ_M.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "h"] & X \arrow[d, "f"] \\ \hat{A} \arrow[r, "\phi_M"] & A \end{tikzcd}$$

We conclude immediately since by Theorem 3.2 we know that there exists a bound $l = l(g, k)$ depending only on $g = \dim A$ and k, such that the vanishing in question holds for any morphism h. (Note that we cannot apply Serre Vanishing here, as construction depends on the original choice of M.) $\square$

Moreover, direct images of pluricanonical bundles under morphisms to abelian varieties have Chen-Jiang decomposition property.

Theorem 3.4. *([LPS20, Theorem A]). Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety, then we have*

(1) $f_* \omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.

(2) The cohomology support loci $V^i(f_* \omega_X^{\otimes k})$ are finite unions of torsion subvarieties (= abelian subvarieties translated by torsion points) of the dual abelian variety $\hat{A}$ for every $i \geq 0$.

(3) One has a canonical decomposition (Chen-Jiang decomposition)

$$f_* \omega_X^{\otimes k} \simeq \bigoplus_{i \in I} (\alpha_i \otimes q_i^* \mathcal{F}_i),$$

into pullacks of M-regular cohenrent sheaves $\mathcal{F}_i$ from quotients $q_i : A \to A_i$ of abelian variety, tensored by torsion line bundles $\alpha_i \in \hat{A}$.

3.2 Higher direct images of pluricanonical bundles

In this subsection, we discuss higher direct images of pluricanonical bundles to abelian varieties.

Lemma 3.5. *There exists an irregular smooth projective variety with big anti-canonical bundle.*

Proof. Let A br an abelian variety. We take an ample line bundle L on A and define a vector bundle $E = L^{-1} \oplus \mathcal{O}_A$. Let $\pi : X = \mathbb{P}_A(E) \to A$ be the projective bundle. Clearly, the irregularity $q(X)\, 0$. The canonical bundle ω_X of X is isomorphic to $\pi^*(\omega_A \otimes \det E) \otimes \mathcal{O}_X(-\mathrm{rank} E) = \pi^* L^{-1} \otimes \mathcal{O}_X(-2)$.

We will see that ω_X^{-1} is big. Let ξ and l be the numerical classes of $\mathcal{O}_X(1)$ and L. Note that ξ is an effective class since $H^0(X, \mathcal{O}_X(1)) = H^0(A, E) \neq 0$. The numerical class of ω_X^{-1} is equal to

$$2\xi + \pi^* l = \frac{N-1}{N} 2\xi + \frac{1}{N} 2\xi + \pi^* l,$$

where N is sufficiently large integer such that $\frac{1}{N}2\xi + \pi^* l$ is ample. Therefore, ω_X^{-1} is big. $\square$

Lemma 3.6. *Let X be a smooth projective variety of dimension n , and D be a big Cartier divisor on X. Then $V^0(mD) = \mathrm{Pic}^0(X)$ for any sufficiently large and divsible m.*

Proof. Since D is big, there exist a positive integer m_0, a very ample Cartier divisor H, and an effective Cartier divisor E such that $m_0 D \sim H + E$. For any positive integer m, we have

$$V^0(mm_0 D) = V^0(mH + mE) \supseteq V^0(mH).$$

We can take a positive integer m_1 satisfying

$$H^i(X, \mathcal{O}_X(mH + \alpha)) = 0$$

for every $\alpha \in \mathrm{Pic}^0(X)$, $m \geq m_1$, and $i > 0$ (take m_1 such that $m_1 H - K_X$ is ample). By Castelnuovo-Mumford regularity, $mH + \alpha$ is globally generated for every $\alpha \in \mathrm{Pic}^0(X)$ and $m \geq m_1 + n$. Hence we have $V^0(mH) = \mathrm{Pic}^0(X)$ for every $m \geq m_1 + n$. Therefore $V^0(mm_0 D) = \mathrm{Pic}^0(X)$ for every $m \geq m_1 + n$. $\square$

Now we can give an example of a higher direct image of pluricanonical bundle that is not a GV-sheaf.

Example 3.7. *Let X be a irregular smooth projective variety of dimension $n \geq 2$ with big anticanonical bundle, and let $f = \mathrm{alb}_X : X \to A$ be the Albanese mprphism of X. Then $R^j f_* \omega_X^{\otimes k}$ is not a GV-sheaf for some $k \geq 2$ and $j \geq 1$.*

Proof. Let P be the poincaré bundle on $A \times \hat{A}$, and $Q = (f \times id_{\hat{A}})^* P$ on $X \times \hat{A}$. Then we have the Fourier-Mukai transforms $\Phi_P : \mathrm{Coh}(A) \to \mathrm{Coh}(\hat{A})$ and $\Phi_Q : \mathrm{Coh}(X) \to \mathrm{Coh}(\hat{A})$ with $\Phi_Q \simeq \Phi_P \circ f_*$.

Let $k \geq 2$ be an integer, then $\omega_X^{\otimes(1-k)}$ is big. By the previous lemma, we can pick k such that $V^n(\omega_X^{\otimes k}) = -V^0(\omega_X^{\otimes(1-k)}) = \mathrm{Pic}^0(X)$. Then it follows that

$$\mathrm{Supp} R^n\Phi_Q(\omega_X^{\otimes k}) = \mathrm{Pic}^0(X).$$

Consider the Grothendieck spectral sequence

$$E_2^{i,j} = R^i\Phi_P R^j f_*(\omega_X^{\otimes k}) \Rightarrow R^{i+j}\Phi_Q(\omega_X^{\otimes k}).$$

Then there exists an integer i such that $\mathrm{Supp} R^{n-j}\Phi_P R^j f_*(\omega_X^{\otimes k}) = \mathrm{Pic}^0(X)$. Note that $\dim f(X) > 0$ since X is irregular. Therefore, $R^n f_*(\omega_X^{\otimes k}) = 0$, and so $n - j > 0$. By Theorem 2.1(2), $R^j f_*(\omega_X^{\otimes k})$ is not a GV-sheaf. aNote that j must be positive since $f_*(\omega_X^{\otimes k})$ is a GV-sheaf by Theorem 1.3. $\square$

Remark 3.8. *Let $f : X \to Y$ be a morphism of projective varieties, with X smooth and Y of dimension n, and $k > 0$ and $j \geq 0$ be integers, L be an ample and globally generated line bundle on Y. We cannot expect that there exists a positive integer $N = N(n, k, j)$ depending only on n, k, j such that*

$$H^i(Y, R^j f_*\omega_X^{\otimes k} \otimes L^{\otimes l}) = 0$$

for every $i > 0$ and every $l \geq N$.

Indeed, if there exists such N, by the same argument as the proof of Theorem 1.3 via Theorem 3.2, it follows that $R^j f_*\omega_X^{\otimes k}$ is a GV-sheaf for any $k \geq 1$ and $j \geq 0$, which contradicts Example 3.7.

4 Applications to birational geometry

4.1 Characterization of abelian varieties

In this subsection, we discuss characterization of abelian varieties due to Chen-Hacon. The problem was motivated by the following conjecture.

Conjecture 4.1. *(Ueno's Conjecture K). Let X be a smooth projective variety such that $\kappa(X) = 0$, and let $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ be the Albanese map. Then*

(1) alb_X is surjective and has connected fibers; that is, a is an algebraic fiber space.

(2) If F is a general fiber of alb_X, $\kappa(F) = 0$.

(3) There is an étale covering $B \to A$ such that $X \times_{\mathrm{Alb}(X)} B$ id birationally equivalent to $F \times B$ over B.

Now the conjecture is solved except the third statement.

Theorem 4.2. *(1) Conjecture K(1) is true [Kaw81, Theorem 1].*

(2) Conjecture K(2) is true.

(3) If F has a good minimal model, then Conjecture K(3) is true [Kaw85].

Conjecture K(2) follows from Theorem 1.6 since Iitaka Conjecture over abelian varieties holds. Under the assumption of Conjecture K, $\kappa(X) = \kappa(Y) = 0$, hence $\kappa(F) = 0$. Note that $\kappa(F)$ can not be $-\infty$ by easy addition formula [Mor87, Corollary 2.3].

As a consequence of [Kaw81, Theorem 1], one see the following corollary.

Corollary 4.3. *(Kawamata). If $\kappa(X) = 0$, then $q(X) \leq \dim(X)$. Moreover, if $q(X) = \dim(X)$, then $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a birational morphism.*

Chen-Hacon improved the previous result by weakening the hypothesis $\kappa(X) = 0$ to $P_1(X) = P_2(X) = 0$. Recall that, by the work of Ein-Lazarsfeld, $P_1(X) = P_2(X) = 0$ guarantees that the Albanese map of X is surjective.

Proposition 4.4. *Let X be a smooth projective variety such that $P_1(X) = P_2(X) = 1$. Then there is no positive-dimensional subvariety Z of $\mathrm{Pic}^0(X)$ such that both Z and $-Z$ are contained in $V^0(X, \omega_X)$. In particular, the origin is an isolated point of $V^0(X, \omega_X)$.*

Proof. Assume that there is a positive-dimensional subvariety Z of $\mathrm{Pic}^0(X)$ as in the statement. Then the images of the multiplication map of global section

$$H^0(X, \omega_X \otimes L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X^{\otimes 2})$$

is nonzero for all $L \in Z$. Since a given divisor in the linear system $|\omega_X^{\otimes 2}|$ has only finitely many irreducible components, it follows that as L varies over the positive-dimensional subvariety Z, the image of the multiplication map must vary as well. Therefor $P_2(X) > 1$, a contradiction.

Since $P_1(X) = 1$, $H^0(X, \omega_X) \neq 0$, so $\mathcal{O}_X \in V^0(X, \omega_X)$. Suppose that it is not an isolated point, then by a result of Simpson [Sim93], a positive-dimensional component Z of $V^0(X, \omega_X)$ containing $\mathcal{O}_X$ would be a subtorus, hence $-Z$ is also contained in $V^0(X, \omega_X)$, a contradiction.

$\square$

Proposition 4.5. *If the origin is the isolated point of $V^0(X, \omega_X)$, then the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is surjective.*

Proof. Denote $f = \mathrm{alb}_X$ and $A = \mathrm{Alb}(X)$ for simplicity. We always have $V^0(X, \omega_X) = V^0(A, f_*\omega_X)$, and $f_*\omega_X$ is a GV-sheaf. By Proposition 2.8, the origin is a point of $V^g(A, f_*\omega_X)$, where $g = \dim A$. In particular, we must have $\dim f(X) \geq g$, which means that f is surjective.

$\square$

If we assume in addition that $q(X) = \dim X$, then we can conclude from $P_1(X) = P_2(X) = 0$ that the Albanese mao of X is generically finite and surjective. In particular, X has maximal Albanese dimension. To prove Theorem 1.5, we just need to show that the degree of Albanese map equal to 1.

Proposition 4.6. *Auume that the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is surjective and generically finite, and that $\dim V^0(X, \omega_X) = 0$. Then alb_X is birational.*

Proof. Denote $f = \mathrm{alb}_X$ and $A = \mathrm{Alb}(X)$ for simplicity. f is surjective and generically finite, so $g = \dim A = \dim X = n$. We have to prove $\deg f$=1, in fact, we shall show that $f_*\omega_X \simeq \mathcal{O}_A$.

To begin with, $f_*\omega_X$ is a GV-sheaf on A. Since f is generically finite, $R^i f_*\omega_X = 0$ for every $i > 0$ by Theorem 2.3(1). So we have

$$V^i(X, \omega_X) = V^i(A, f_*\omega_X)$$

for every $i \geq 0$. By assumption, every point of $V^0(A, f_*\omega_X)$ is an isolated point, and therefore contained in $V^g(A, f_*\omega_X)$ by Proposition 2.8. But $V^g(A, f_*\omega_X) = V^n(X, \omega_X)$ consists of just the origin. We deduced that $V^i(A, f_*\omega_X) = \{\mathcal{O}_A\}$ for every $0 \leq i \leq g$.

By Proposition 2.7, the Fourier-Mukai transform $\mathbf{R}\Phi_P(f_*\omega_X)$ is supported at the origin in $\hat{A}$, and that $R^i\Phi_P(f_*\omega_X) = 0$ for $i < g$. A calculation gives $R^g\Phi_P(f_*\omega_X) = \mathcal{O}_0$. So we have

$$\mathbf{R}\Phi_P(f_*\omega_X) \simeq R^g\Phi_P(f_*\omega_X)[-g] \simeq= \mathcal{O}_0[-g] \simeq \mathbf{R}\Phi_P(\mathcal{O}_A).$$

By Mukai's equivalence of derived category, we have $f_*\omega_X \simeq \mathcal{O}_A$, hence f is birational. $\square$

Remark 4.7. *By using Chen-Jiang decomposition, we can replace the last paragraph of proof of Proposition 4.6 with the following argument.*

By Theorem 3.4, we know $f_*\omega_X$ admits Chen-Jiang decomposition

$$f_*\omega_X \simeq \bigoplus_{i=1}^{m}(\alpha_i \otimes q_i^*\mathcal{F}_i).$$

By Lemma 2.13, we can write

$$\{\mathcal{O}_A\} = V^0(\mathcal{F}) = \bigcup_{i=1}^{m} \alpha_i^{-1} \otimes p_i^*\mathrm{Pic}^0(A_i),$$

which implies $\dim A_i = 0$ and $\alpha_i = \mathcal{O}_A$ for all i. So $f_*\omega_X$ is a trivial bundle of rank m, but then

$$m = h^g(A, f_*\omega_X) = h^n(X, \omega_X) = 1,$$

and so $f_*\omega_X \simeq \mathcal{O}_A$.

To finish the proof Theorem 1.5, it sufficiently to show that $\dim V^0(X, \omega_X) = 0$, the following result describes what happens if $\dim V^0(X, \omega_X) \neq 0$.

Proposition 4.8. *Let X be a smooth projective variety of maximal Albanese dimension. If $\dim V^0(X, \omega_X) \neq 0$, then the intersection $V^0(X, \omega_X) \cap \iota^* V^0(X, \omega_X)$ also has positive dimension.*

We give the summary of the proof. We firstly try to construct a common subset of $V^0(X, \omega_X)$ and $\iota^* V^0(X, \omega_X)$, i.e. W constructed in step 3. Then, we want to show W is nonempty, this is established in step 1 and step 2, which related to a property of GV-sheaf (Corollary 2.6). Finally, in step 4, we show W has positive dimension by contradiction.

Proof. Let $Z \subseteq V^0(X, \omega_X)$ be an irreducible component of positive dimension, and let k denote its codimension. By construction, $k < g = \dim \mathrm{Alb}(X)$. Then we know from Theorem 3.4(2) that Z is a translate of an abelian subvariety $\hat{A}$ by torsion point $L \in \mathrm{Pic}^0(X)$. If Z contains the origin, then we are done: for the remainder of the argument, we may therefore assume that Z does not contain the origin. Let A be the dual abelian variety of $\hat{A}$, we obtain a morphism

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}_X"] \arrow[dr, "f"'] & \mathrm{Alb}(X) \arrow[d, "p"] \\ & A \end{tikzcd}$$

such that $\dim A = g - k$ and $\dim f(X) = \dim \mathrm{alb}_X(X) - k = n - k$ since X is of maximal Albanese dimension.

Step 1. We show that $R^k f_*(\omega_X \otimes L) \neq 0$. Recall that Z has codimension k and is an irreducible component of

$$V^0(X, \omega_X) = V^0(\mathrm{Alb}(X), (\mathrm{alb}_X)_* \omega_X).$$

Since $(alb_X)_* \omega_X$ is a GV-sheaf, Z is also contained in

$$V^k(X, \omega_X) = V^k(\mathrm{Alb}(X), (\mathrm{alb}_X)_* \omega_X),$$

where the equality follows from the fact that X is of maximal Albanese dimension, alb_X is generically finite over its image, and so $R^i(\mathrm{alb}_X)_* \omega_X = 0$ for $i > 0$. By Theorem 2.4, we have a decomposition of cohomology

$$H^k(X, \omega_X \otimes L \otimes f^* \alpha) \simeq \bigoplus_{j=0}^{k} H^{k-j}(A, R^j f_*(\omega_X \otimes L) \otimes \alpha).$$

The left-hand side is nonzero for every $\alpha \in \hat{A}$ because $Z \subseteq V^k(X, \omega_X)$. In terms of cohomology suppport loci, this says that

$$\hat{A} = \bigcup_{j=0}^{k} V^{k-j}(A, R^j f_*(\omega_X \otimes L)).$$

But the sheaves $R^j f_*(\omega_X \otimes L)$ are GV-sheaves on A by Corollary 2.2, and so the cohomology suppport loci with $k - j \geq 1$ are proper subvarieties of $\hat{A}$, hence we have

$$\hat{A} = V^0(A, R^k f_*(\omega_X \otimes L)).$$

In particular, the sheaf $R^k f_*(\omega_X \otimes L)$ has to be nonzero.

Step 2. We show that $R^k f_*(\omega_X \otimes L^{-1}) \neq 0$. Consider the Stein factorization of f:

$$X \xrightarrow{g} Y \xrightarrow{q} A.$$

Let F denote the generalfiber of g, then F is a smooth projective variety of dimension k. We know that $R^k g_*(\omega_X \otimes L)$ is a nontrivial torsion free sheaf on $g(X)$ by Theorem 2.3(1). By base change, it follows that

$$H^k(F, \omega_F \otimes L|_F) \simeq H^k(F, (\omega_X \otimes L)|_F) \neq 0.$$

Since $\dim F = k$, by Serre duality $H^0(F, -L|_F) \neq 0$, hence $L|_F$ is trivial. So

$$L \in \ker(\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(F)).$$

But the kernel is a group, and so it also contains L^{-1}. By running the same argument backwards, we conclude that $R^k f_*(\omega_X \otimes L^{-1}) \neq 0$.

Step 3. We produce a subset of $V^0(X, \omega_X) \cap \iota^* V^0(X, \omega_X)$. By decomposition of cohomology

$$H^k(X, \omega_X \otimes L^{-1} \otimes f^*\alpha) \simeq \bigoplus_{j=0}^{k} H^{k-j}(A, R^j f_*(\omega_X \otimes L^{-1}) \otimes \alpha),$$

if $\alpha \in V^0(A, R^k f_*(\omega_X \otimes L^{-1}))$, then $H^0(A, R^k f_*(\omega_X \otimes L^{-1}) \otimes \alpha) \neq 0$, hence $H^k(X, \omega_X \otimes L^{-1} \otimes f^*\alpha) \neq 0$, which means that $L^{-1} \otimes f^*\alpha \in V^k(X, \omega_X)$. So we have

$$W := L^{-1} \otimes V^0(A, R^k f_*(\omega_X \otimes L^{-1})) \subseteq V^k(X, \omega_X).$$

Since X is of maximal Albanese dimension,

$$V^k(X, \omega_X) = V^k(\mathrm{Alb}(X), (\mathrm{alb}_X)_*\omega_X) \subseteq V^0(\mathrm{Alb}(X), (\mathrm{alb}_X)_*\omega_X) = V^0(X, \omega_X)$$

by Theorem 2.3 and Corollary 2.5. Moreover, we also have

$$W := L^{-1} \otimes V^0(A, R^k f_*(\omega_X \otimes L^{-1})) \subseteq L^{-1} \otimes \hat{A} = L^{-1} \otimes (-\hat{A}) = \iota^* Z \subseteq \iota^* V^0(X, \omega_X),$$

hence $W \subseteq V^0(X, \omega_X) \cap \iota^* V^0(X, \omega_X)$. To finish the proof, we need to show

$$\dim V^0(A, R^k f_*(\omega_X \otimes L^{-1})) \geq 1.$$

Step 4. We prove that $V^0(A, R^k f_*(\omega_X \otimes L^{-1}))$ contains no isolated points. Since $R^k f_*(\omega_X \otimes L^{-1})$ is a nonzero GV-sheaf, $V^0(A, R^k f_*(\omega_X \otimes L^{-1})) \neq \emptyset$ by Corollary 2.6. Suppose that $\alpha \in V^0(A, R^k f_*(\omega_X \otimes L^{-1}))$ was an isolated point. Since $\dim A = g - k$, $\alpha \in V^{g-k}(A, R^k f_*(\omega_X \otimes L^{-1}))$ by Proposition 2.8. Because $\dim f(X) = n - k \leq g - k$, it follows that $f(X) = A$ and $n = g$, and hence that

$$0 \neq H^{n-k}(A, R^k f_*(\omega_X \otimes L^{-1}) \otimes \alpha) \subseteq H^n(X, \omega_X \otimes L^{-1} \otimes f^*\alpha)$$

by Theorem 2.3(1). But then $L^{-1} \otimes f^*\alpha$ would be trivial line bundle, which would mean that Z contains the origin. Since we assume that this is not the case, $V^0(A, R^k f_*(\omega_X \otimes L^{-1}))$ contains no isolated points. Hence we finish the proof.

□

Proof of Theorem 1.5. Let X be a smooth projective variety such that $P_1(X) = P_2(X) = 1$ and that $q(X) = \dim X$. By Proposition 4.4 combined with Proposition 4.5, the Albanese map of X is surjective and generically finite. By Proposition 4.4 together with Proposition 4.8, it follows that $\dim V^0(X, \omega_X) = 0$. Therefore, by Proposition 4.6, X is birational to an abelian variety.

□

4.2 Iitaka conjecture over abelian varieties

In this subsection, we discuss Iitaka conjecture in the case that base of fibration is of maximal Albanese dimension. The famous conjecture by Iitaka predicts the behavior of Kodaira dimension in families.

Conjecture 4.9. *Let $f : X \to Y$ be an algebraic fiber space with general fiber F, then*

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

This conjecture is of fundamental importance in the classification of algebraic varieties. It is known to hold in many special case:

- F has a good minimal model [Kaw85],
- $\dim X \leq 6$ [Bir09],
- Y is of general type [Kaw81],
- Y is of maximal Albanese dimension [HPS18] [CP17],
- $\dim Y = 1$ [Kaw82].
- $\dim Y = 2$ [Cao18].

Using analytic techniques, Cao-Păun [CP17] proved Theorem 1.6. Then Hacon-Popa-Schnell reproved it by combining algebraic tools and analytic tools. We discuss the method in [HPS18] here. Firstly, we can reduce the problem to the case that $\kappa(X) = 0$ and Y is an abelian variety.

Lemma 4.10. *To prove Theorem 1.6, it is enough to assume that $\kappa(X) = 0$ and Y is an abelian variety.*

Proof. We begin by showing that if $\kappa(X) = -\infty$, then $\kappa(F) = -\infty$. If this were not the case, then we could pick some $m > 0$ such that $P_m(F) > 0$ and hence $f_*\omega_X^{\otimes m} \neq 0$. Let $Y \to A$ be the Albanese map of Y, and $g : X \to A$ be the induced morphism. Since F is an irreducible component of the general fiber of $X \to g(X) \subseteq A$, it follows that $g_*\omega_X^{\otimes m} \neq 0$. By Theorem 1.3, $g_*\omega_X^{\otimes m} \neq 0$ is a GV-sheaf, hence $V^0(g_*\omega_X^{\otimes m})$ is nonempty by Corollary 2.6. Now by Theorem 3.4(2), $V^0(g_*\omega_X^{\otimes m})$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(A)$, so it contains a torsion point $P \in \mathrm{Pic}^0(A)$, i.e. there is an integer $k > 0$ such that $P^{\otimes k} \simeq \mathcal{O}_A$. But then

$$h^0(X, \omega_X^{\otimes m} \otimes g^*P) = h^0(A, g_*\omega_X^{\otimes m} \otimes P) \neq 0,$$

and so

$$h^0(X, \omega_X^{\otimes km}) = h^0(X, (\omega_X^{\otimes m} \otimes g^*P)^{\otimes k}) \neq 0.$$

This contradicts the assumption $\kappa(X) = -\infty$.

Now assume that $\kappa(X) \geq 0$. We will first prove the statement in the case that $\kappa(Y) = 0$. By Kawamata's thenorem [Kaw81, Theorem 1], since Y is of maximal Albanese dimension, it is in fact birational to its Albanese variety and so we may assume Y is an abelian variety. Let $h : X \to Z$ the Iitaka fibration of X, we can assume that Z is smooth. We denote by G its general fiber, we have $\kappa(G) = 0$. By [Kaw81, Theorem 1], the $G \to \mathrm{Alb}(G)$ is surjective, so we deduced that $B = f(G) \subseteq Y$ is an abelian subvariety. If $G \to B^{'} \to B$ is the Stein factorization, then $B^{'} \to B$ is an *é*tale map of abelian varieties. We thus have the induced fiber space $G \to B^{'}$ over abelian variety with $\kappa(G) = 0$, and whose general fiber is $H = F \cap G$. Assume that Theorem 1.6 holds for algebraic fiber spaces of Kodaira dimension zero over abelian varieties, we obtain $\kappa(H) \leq \kappa(G) = 0$. Note that H is also an irreducible component of the general fiber of $F \to h(F)$. Consider the Stein factorization of this morphism, the easy addition formula [Mor87, Corollary 2.3] implies that

$$\kappa(F) \leq \kappa(H) + \mathrm{dim} h(F) \leq \mathrm{dim} h(F).$$

Since $\mathrm{dim} h(F) \leq \mathrm{dim} Z = \kappa(X)$, we obtain the required inequality $\kappa(F) \leq \kappa(X)$.

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[d, hook] & F \arrow[r, "h|_F"] \arrow[d, hook] & h(F) \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d, "f|_G"] & X \arrow[r, "h"] \arrow[d, "f"] & Z \\
B \arrow[r, hook] & Y &
\end{tikzcd}$$

Finally we prove the general case. since Y has maximal Albanese dimension, by [Kaw81, Theorem 13] we may assume $Y = Z \times K$ where Z is of general type and K is an abelian variety. In particular, $\kappa(Y) = \mathrm{Z} = \kappa(Z)$. If E is the general fiber of the induced morphism $X \to Z$, then the induced morphism $E \to K$ has general fiber isomorphic to F. By what we have proved above, we deduced that $\kappa(E) \geq \kappa(F)$. We then have the required inequality

$$\kappa(X) = \kappa(Z) + \kappa(E) \geq \kappa(Y) + \kappa(F),$$

where the first equality follows from [Kaw81, Theorem 3], since Z is of general type.

$\square$

Recall that a vector bundle U on an abelian variety A is called *unipotent* if it has a filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = U$$

such that $U_i/U_{i+1} \simeq \mathcal{O}_A$ for all $i = 1, ..., n$. Note in particular that $\mathrm{det} U \simeq \mathcal{O}_A$.

Lemma 4.11. *Let X be a smooth projective variety with $\kappa(X) = 0$, and let $f : X \to A$ be an algebraic fiber space over an abelian variety. If $H^0(X, \omega_X^{\otimes m}) \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathcal{F}_m = f_* \omega_X^{\otimes m}$ is an indecomposable unipotent vector bundle.*

Proof. We fixed an m such that $H^0(A, \mathcal{F}_m) = H^0(X, \omega_X^{\otimes m}) \neq 0$. Since $\kappa(X) = 0$, we have $h^0(A, \mathcal{F}_m) = 1$, and in particular, $0 \in V^0(\mathcal{F}_m)$. We claim that $V^0(\mathcal{F}_m) = \{0\}$, which implies that $\mathcal{F}_m$ is unipotent by Proposition 2.9.

To see this, first note that by Theorem 3.4(2), $V^0(\mathcal{F}_m)$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(A)$. Then, if there were two distinct points $P, Q \in V^0(\mathcal{F}_m)$ we could assume that they are both torsion of the same order k. Since f is a fiber space, the mapping

$$f^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$$

is injective, and so f^*P and f^*Q are distinct as well. Now if $P \in V^0(\mathcal{F}_m)$, then

$$H^0(A, \mathcal{F}_m \otimes P) = H^0(X, \omega_X^{\otimes m} \otimes f^*P) \neq 0,$$

and similarly for Q. Let $D \in |mK_X + f^*P|$ and $G \in |mK_X + f^*Q|$, so that $kD, kG \in |mkK_X|$. Since $h^0(X, \omega_X^{\otimes mk}) = 1$, it follows that $kD = kG$, and hence $f^*P = f^*Q$, which is a contradiction.

Finally, since $h^0(A, \mathcal{F}_m) = 1$, it is clear that $\mathcal{F}_m$ is indecomposable.

□

Proof of Theorem 1.6. By Lemma 4.10, we can assume that $\kappa(X) = 0$ and that $Y = A$ is an abelian variety. Our goal is to prove that $\kappa(F) = 0$, which will follow from the claim: $P_m(F) = 1$ whenever $P_m(X) = 1$. Consider the coherent sheaf $\mathcal{F}_m = f_*\omega_X^{\otimes m}$ on A, whose rank at the generic point of A is equal to $P_m(F)$. We fixed an m such that $h^0(A, \mathcal{F}_m) = h^0(X, \omega_X^{\otimes m}) = 1$. To obtain the conclusion, it is enough to show $\mathcal{F}_m$ has rank 1 generically. We will in fact prove the stronger statement that $\mathcal{F}_m \simeq \mathcal{O}_A$.

By Lemma 4.11, $\mathcal{F}_m$ is an indecomposable unipotent vector bundle on A. In particular, $\det\mathcal{F}_m \simeq \mathcal{O}_A$. Theorem 4.12 implies that $\mathcal{F}_m$ has a smooth hermitian metric that is flat. Thus $\mathcal{F}_m$ is a successive extension of trivial bundles $\mathcal{O}_A$ that can be split off as directed summands with the help of flat metric. It follows that in fact $\mathcal{F}_m \simeq \mathcal{O}_A^{\oplus r}$, the trivial bundle of rank $r \geq 1$. But since $\mathcal{F}_m$ is indecomposable, which implies $r = 1$. We finish the proof of Theorem 1.6.

□

We need the following analytic results to argue that the unipotent vector bundle $\mathcal{F}_m$ is actually trivial bundle $\mathcal{O}_A$. For the moment it seems hopeless to prove it by algebraic method. We recommend the readers to see [HPS18, Section D and E] for details.

Theorem 4.12. *Let $f : X \to Y$ be an algebraic fiber space.*

(1) For any $m \in \mathbb{N}$, the torsion-free sheaf $f_*\omega_{X/Y}^{\otimes m}$ has a canonical singular hermitian metric with semi-positive curvature.

(2) If $c_1(\det f_*\omega_{X/Y}^{\otimes m}) = 0$ in $H^2(Y, \mathbb{R})$, then $f_*\omega_{X/Y}^{\otimes m}$ is locally free, and the singular hermitian metric on it is smooth and flat.

(3) Every nonzero morphism $f_*\omega_{X/Y}^{\otimes m} \to \mathcal{O}_Y$ is split surjective.

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