BOUNDEDNESS AND MODULI OF TRADITIONAL STABLE MINIMAL MODELS

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Abstract. For good minimal models with semi-log canonical (slc) singularities, polarized by effective divisors that are relatively ample over the bases of Iitaka fibration, Birkar proves that they belong to a bounded family after fixing appropriate numerical invariants recently. Subsequently, he constructs their projective coarse moduli spaces. In this paper, we consider good minimal models with only Kawamata log terminal (klt) singularities but polarized by possibly non-effective divisors. We prove that they still belong to a bounded family after fixing the same invariants. As an application, we construct separated coarse moduli spaces for klt good minimal models polarized by line bundles.

Contents

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic zero unless otherwise stated.

The central problem in birational geometry is the classification of algebraic varieties. According to the standard Minimal Model Conjecture and Abundance Conjecture, any variety is birational to either a Mori fiber space or a good minimal model. Therefore, the next main goal of birational geometry is to construct moduli spaces for these objects. For this purpose, one big problem is whether or not there are only finitely many families for these objects after fixing certain numerical invariants, i.e., they belong to a bounded family. In this paper, we will focus on good minimal models, that is, varieties with semi-ample canonical divisors.

For canonical polarized varieties, i.e., ample models of good minimal models with maximal Kodaira dimension, the projective coarse moduli spaces of these varieties have

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been established in any dimension by the contributions of many people over the past three decades, see Kollár's book [Kol23] and the references therein for details. Notably, advancements in the higher-dimensional minimal model program [BCHM10, Bir12, HX13, HMX13, HMX14, HMX18] have played a crucial role in these developments, particularly in the boundedness and properness of the relevant moduli functors.

For Calabi-Yau varieties, i.e., good minimal models with Kodaira dimension zero, there is no natural choice of polarization. In general, they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces and abelian varieties of any fixed dimension are not bounded. Nevertheless, there have been some recent progresses toward (birational) boundedness of elliptic Calabi-Yau varieties and rationally connected Calabi-Yau varieties, see [BDCS20, FHS21, Bir23c].

When studying moduli of Calabi-Yau varieties, a polarization is typically fixed despite the non-uniqueness of the choice. Various techniques, such as GIT, Hodge theory, minimal model program and mirror symmetry, have been applied to study the moduli of polarized Calabi-Yau varieties and have been successful in specific examples. Recently, Birkar establishes a boundedness result for slc (resp. klt) Calabi-Yau varieties polarized by effective (resp. possibly non-effective) ample Weil divisors with fixed volume in any dimension, and it is crucial for constructing their projective (resp. separated) coarse moduli spaces [Bir23b].

For good minimal models with arbitrary non-negative Kodaira dimension, Birkar has recently proven a boundedness result for slc good minimal models polarized by effective Weil divisors that are relatively ample over the bases of Iitaka fibration, and he has constructed their projective coarse moduli spaces [Bir22]. When restricted to good minimal models with maximal Kodaira dimension, he also obtains a new moduli functor for varieties whose canonical divisors are nef and big but not ample, a similar construction in this case has appeared in the work of Filipazzi-Inchiostro [FI21].

The reason for taking effective divisors as polarization in [Bir22] is that one can obtain proper or even projective moduli spaces. However, this often leads to larger moduli spaces than desired, as it also parametrizes all the additional effective divisors. Traditionally, people use line bundles as polarization, although it is difficult to meaningfully compactify these moduli spaces. For example, Viehweg considers ample line bundles as polarization for treating the moduli of smooth good minimal models [Vie95]. Then Taji constructs moduli of good minimal models with canonical singularities [Taj20]. More recently, Hashizume-Hattori consider klt good minimal models of Kodaira dimension one, polarized by line bundles that are relatively ample over the bases of Iitaka fibration [HH23].

One naturally wonders if there exists a boundedness result for klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of the Iitaka fibration. This paper addresses this question in a more general context and proceeds to construct their coarse moduli spaces.

1.1. **Traditional stable minimal models.** We define the main object studied in this paper, as introduced in Birkar's survey note [Bir23a].

Definition 1.1. A *traditional stable minimal model* $(X, B), A$ over k consists of a projective connected pair (X, B) and $\mathbb{Q}$-Cartier Weil divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \to Z$, and
- $K_X + B + tA$ is ample for some $t > 0$.

We usually denote the model by $(X, B), A$ or more precisely $(X, B), A \to Z$. We say $(X, B), A$ is *strongly traditional stable* if in addition $K_X + B + A$ is ample.

Note that we do not assume that A is ample globally. Indeed, if $X \to Z$ is the contraction defined by $K_X + B$, then $K_X + B + tA$ being ample for some $t > 0$ is equivalent to the fact that A is ample over Z. Moreover, t is not fixed in our setting.

The terminology "traditional" originates from our adoption of the traditional approach, wherein we consider using a line bundle or, more generally, a possibly non-effective $\mathbb{Q}$-Cartier Weil divisor as the polarization. When (X, B) is slc, and we choose an effective divisor A as the polarization such that $(X, B + tA)$ is slc for some $t > 0$, then $(X, B), A$ is referred to as a *stable minimal model* in [Bir22].

Definition 1.2. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $u, v, \lambda \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. We will define various classes of traditional stable minimal models.

(1) A (d, Φ)-traditional stable minimal model is a traditional stable minimal model $(X, B), A$ such that
- $\dim X = d$, and
- the coefficients of B are in Φ.

The set of all the (d, Φ)-traditional stable minimal models is denoted by $\mathcal{TS}_{klt}(d, \Phi)$.

(2) A (d, Φ, u, v)-traditional stable minimal model is a (d, Φ)-traditional stable minimal model $(X, B), A$ such that
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$, and
- $\mathrm{Ivol}(K_X + B) = v$ (see Definition 2.2).

Let $\mathcal{TS}_{klt}(d, \Phi, u, v)$ denote the set of all (d, Φ, u, v)-traditional stable minimal models.

(3) A $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model is a (d, Φ)-traditional stable minimal model $(X, B), A$ such that
- $\mathrm{vol}(A|_F) \in \Gamma$, where F is any general fiber of $f : X \to Z$, and
- we have

$$\mathrm{vol}(K_X + B + tA) = \sigma(t), \text{ for all } 0 \leq t \ll 1.$$

Let $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ denote the set of all $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models.

(4) A $(d,\Phi,\Gamma,\sigma,\lambda)$-traditional stable minimal model is a (d,Φ,Γ,σ)-traditional stable minimal model $(X,B),A$ such that $K_X+B+\lambda A$ is big (and hence K_X+B+tA is big for any $0<t\le\lambda$).

Let $\mathcal{TS}_{klt}(d,\Phi,\Gamma,\sigma,\lambda)$ denote the set of all $(d,\Phi,\Gamma,\sigma,\lambda)$-traditional stable minimal models.

Example 1.3. Given a (d,Φ,Γ,σ)-traditional stable minimal model $(X,B),A$, if A is a pseudo-effective $\mathbb{Q}$-Cartier Weil divisor, then $(X,B),A$ is a $(d,\Phi,\Gamma,\sigma,1)$-traditional stable minimal model. Indeed,

$$K_X+B+A=K_X+B+\epsilon A+(1-\epsilon)A$$

for some $0<\epsilon\ll 1$, then $K_X+B+\epsilon A$ is ample and $(1-\epsilon)A$ is pseudo-effective, thus K_X+B+A is big. In particular, a klt stable minimal model $(X,B),A$ as defined in [Bir22] is a $(d,\Phi,\Gamma,\sigma,1)$-traditional stable minimal model.

1.2. **Boundedness of traditional stable minimal models.** We define the notion of boundedness of traditional stable minimal models and then state one of the main results of this paper.

Definition 1.4 (Boundedness of traditional stable minimal models)**.** A subset $\mathcal{E}\subset\mathcal{TS}_{klt}(d,\Phi)$ is said to be a bounded family if there is a fixed $r\in\mathbb{N}$ such that for any $(X,B),A$ in $\mathcal{E}$ we can find a very ample divisor H on X satisfying

$$H^d\le r,(K_X+B)\cdot H^{d-1}\le r \text{ and } H-A \text{ is pseudo-effective.}$$

This in particular bounds $(X,\mathrm{Supp}B)$. If in addition A is effective, then $(X,\mathrm{Supp}(B+A))$ belongs to a bounded family.

Our definition of boundedness slightly differs from the original definition in [Bir22, Section 2.9]. In [Bir22], Birkar assumes A to be an effective divisor, then the notion of boundedness is defined as follows: a subset $\mathcal{E}\subset\mathcal{TS}_{klt}(d,\Phi)$ is said to be a bounded family if there is a fixed $r\in\mathbb{N}$ such that for any $(X,B),A$ in $\mathcal{E}$ we can find a very ample divisor H on X satisfying

$$H^d\le r \text{ and } (K_X+B+A)\cdot H^{d-1}\le r.$$

The requirements of our definition imply that of his, and by [Bir22, Lemma 4.6], they are equivalent when A is an effective divisor with coefficients belonging to a DCC set.

Theorem 1.5. *Let $d\in\mathbb{N}$, $\Phi\subset\mathbb{Q}^{\ge 0}$ be a DCC set, $\Gamma\subset\mathbb{Q}^{>0}$ be a finite set, $\sigma\in\mathbb{Q}[t]$ be a polynomial. Then $\mathcal{TS}_{klt}(d,\Phi,\Gamma,\sigma)$ is a bounded family.*

Birkar has proved this theorem when $\dim Z=0$ (see Theorem 3.3). When $\dim Z=1$, Hashizume-Hattori show that the set of (X,B) appearing in $\mathcal{TS}_{klt}(d,\Phi,u,v)$ belongs to a log bounded family [HH23, Theorem 1.4] without using the polynomial σ. Recently, there have been some other related results on the (birational) boundedness of klt good minimal models, see [FS20, Fil20, FHS21, Jia22, Zhu23].

Conjecturally, we can remove the condition on $\mathrm{vol}(A|_F)$, which is related to the effective b-semiampleness conjecture [PS09, Conjecture 7.13]. For related discussions, refer to [Bir21a, Bir22].

Let us briefly explain some key ideas behind the proof of Theorem 1.5. When A is effective, this theorem is a special case of [Bir22, Theorem 4.1]. Given a lc stable minimal model $(X, B), A \in \mathcal{S}_{lc}(d, \Phi, \Gamma, \sigma)$ ([Bir22, Definition 1.8]), the main idea in his proof is to find a fixed $\tau \in \mathbb{Q}^{>0}$ such that $(X, B + \tau A)$ is lc and $K_X + B + \tau A$ is ample. Then by [HMX18, Theorem 1.1], $(X, B), A$ belongs to a bounded family. In our situation, (X, B) is klt, by Lemma 3.4, we may assume that (X, B) is ϵ-lc for some fixed $\epsilon \in \mathbb{Q}^{>0}$. However, A may not be an effective divisor, we cannot apply the length of extremal ray as in Step 7 of [Bir22, Proof of Theorem 4.1] to find $\tau \in \mathbb{Q}^{>0}$ such that $K_X + B + \tau A$ is ample directly. So, we first assume that $(X, B), A$ belongs to $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda)$ for some fixed $\lambda \in \mathbb{Q}^{>0}$, i.e., $K_X + B + \lambda A$ is big. Then, we can replace $K_X + B + \lambda A$ with an effective $\mathbb{Q}$-divisor E, but we lose control of the coefficients of E. A key observation is that if we define boundedness for $(X, B), A$ as in Definition 1.4, then a similar argument as [Bir22, Theorem 4.1] works, see Theorem 3.8. Since $K_X + B + \tau A$ is ample for some fixed τ, we can apply [Bir23b, Theorem 1.5] to conclude that (X, B) belongs to a log bounded family. Additionally, by applying [HLQ23, Theorem 1.10], effective base point free theorem and very ampleness lemma, we can establish that $(X, B), A$ belongs to a bounded family in the sense of Definition 1.4. It suffices to find a positive rational number λ such that $K_X + B + \lambda A$ is big for any $(X, B), A \in \mathcal{S}_{klt}(d, \Phi, \Gamma, \sigma)$. Note that the graph of the volume function $\theta(t) = \operatorname{vol}(K_X + B + tA)$ provides information about the big thresholds of $K_X + B$ with respect to A. Since we have already fixed the top self-intersection number function $\sigma(t) = (K_X + B + tA)^d$, and $\sigma(t) = \theta(t)$ for $0 < t \ll 1$, we aim to compare $\sigma(t)$ with $\theta(t)$ by investigating their derivatives. The notion of restricted volumes (Definition 2.3) will play a crucial role in these computations, see Proposition 3.9 for details.

1.3. **Moduli of traditional stable minimal models.** We will use the boundedness result from the previous subsection to construct moduli spaces for traditional stable minimal models. We firstly define families of traditional minimal models and the corresponding moduli functors. See Section 2.3 for the relevant definitions of families of pairs over reduced bases.

Definition 1.6. Let S be a reduced Noetherian scheme over k.

(1) When $S = \operatorname{Spec} K$ for a field K, we define a traditional stable minimal model over K as in Definition 1.1 by replacing k with K and replacing connected with geometrically connected. Similarly we can define $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over K.

(2) A *family of traditional stable minimal models* over S consists of a projective morphism $X \to S$ of schemes, a $\mathbb{Q}$-divisor B and a line bundle A on X such that
- $(X, B) \to S$ is a locally stable family,
- $(X_s, B_s), A_s$ is a traditional stable minimal model over $k(s)$ for every $s \in S$.

Here X_s is the fiber of $X \to S$ over s and B_s is the divisorial pullback of B to X_s. Moreover, $K_{X_s} + B_s$ is semi-ample which defines a contration $X_s \to Z_s$, and

A_s is a line bundle on X_s which is ample over Z_s. We will denote this family by $(X, B), A \to S$.

(3) Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A *family of $(d, \Phi, \Gamma, \sigma)$-marked traditional stable minimal models* over S is a family of traditional stable minimal models $(X, B), A \to S$ such that
- $B = \sum_{i=1}^{m} a_i D_i$, where $D_i \geq 0$ are relative Mumford divisors, and
- $(X_s, B_s), A_s$ is a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model over $k(s)$ for every $s \in S$, where $B_s = \sum_{i=1}^{m} a_i D_{i,s}$.

(4) We define the moduli functor $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)$ of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models from the category of reduced Noetherian schemes to the category of groupoids by choosing:
- On objects: for a reduced Noetherian scheme S, one take

$$\begin{aligned} &\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(S) \\ =&\{\text{family of } (d, \Phi, \Gamma, \sigma)\text{-traditional stable minimal models over } S\}. \end{aligned}$$

We define an isomorphism $(f' : (X', B'), A' \to S) \to (f : (X, B), A \to S)$ of any two objects in $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(S)$ to be an S-isomorphism $\alpha_X : (X', B') \to (X, B)$ such that $A' \sim_S \alpha_X^* A$.
- On morphisms: $(f_T : (X_T, B_T), A_T \to T) \to (f : (X, B), A \to S)$ consists of morphisms of schemes $\alpha : T \to S$ such that the natural map $g : X_T \to X \times_S T$ is an isomorphism, B_T is the divisorial pullback of B and $A_T \sim_T g^* \alpha_X^* A$. Here $\alpha_X : X \times_S T \to X$ is the base change of α.

Now we can state our main result on moduli.

Theorem 1.7. *$\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)$ is a separated Deligne-Mumford stack of finite type, which admits a coarse moduli space $TS_{klt}(d, \Phi, \Gamma, \sigma)$ as a separated algebraic space.*

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2. Preliminaries

2.1. **Notations and conventions.** We collect some notations and conventions used in this paper.

(1) For any scheme S and positive integer n, l, we denote $\mathbb{P}^n_k \times_{\operatorname{Spec} k} S$ and $\mathbb{P}^l_k \times_{\operatorname{Spec} k} S$ by $\mathbb{P}^n_S$ and $\mathbb{P}^l_S$, respectively. When $S = \operatorname{Spec} k$, we simply write $\mathbb{P}^n$ and $\mathbb{P}^l$ if there

is no risk of confusion. Let $\mathbb{P}^n_S \times_S \mathbb{P}^l_S \cong \mathbb{P}^n \times \mathbb{P}^l \times S$ be the natural isomorphism, and

$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times S \xrightarrow{p_2} \mathbb{P}^l$$

be the projections. Then for any $a, b \in \mathbb{Z}$, we denote $p_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(b)$ by $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^l\times S}(a, b)$.

(2) A projective morphism $f : X \to Y$ of schemes is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (f is not necessarily birational). In particular, f is surjective with connected fibers.

(3) Let $f : X \to S$ be a morphism of schemes. Let L_1 and L_2 be line bundles on X. We say that L_1 and L_2 are *linearly equivalent over* S, denoted by $L_1 \sim_S L_2$, if there is a line bundle M on S such that $L_1 \cong L_2 \otimes f^*M$.

(4) Suppose that X is a normal variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. We say that D_1 and D_2 are $\mathbb{Q}$-linear equivalent, denoted by $D_1 \sim_{\mathbb{Q}} D_2$, if there exists $m \in \mathbb{Z}_{>0}$ such that mD_1 and mD_2 are Cartier and $mD_1 \sim mD_2$. Moreover, fixed $l \in \mathbb{Z}_{>0}$, the notation $D_1 \sim_l D_2$ means that $lD_1 \sim lD_2$.

(5) We say that a line bundle L on a variety X is *strongly ample* if L is very ample and $H^q(X, kL) = 0$ for any $k, q > 0$.

(6) We say that a set $\Phi \subset \mathbb{Q}$ satisfies the *descending chain condition* (DCC, for short) if Φ does not contain any strictly decreasing infinite sequence. Similarly, we say that a set $\Phi \subset \mathbb{Q}$ satisfies the *ascending chain condition* (ACC, for short) if Φ does not contain any strictly increasing infinite sequence.

(7) A *pair* (X, B) consists of a normal variety X and a boundary divisor B with rational coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Fixed $\epsilon \in \mathbb{Q}^{>0}$, singularities of (X, B) are defined by taking a log resolution $\phi : W \to X$ and writing

$$K_W + B_W = \phi^*(K_X + B).$$

We say (X, B) is *lc* (resp. ϵ-*lc*) (resp. *klt*) if every coefficient of B_W is ≤ 1 (resp. $\leq 1 - \epsilon$) (resp. < 1).

2.2. **Volume of divisors.** We recall the definition of various types of volumes for divisors. In this paper, we mainly consider $\mathbb{Q}$-divisors. However, for the proof of Proposition 3.9, we need to deal with $\mathbb{R}$-divisors.

Definition 2.1 (Volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. The *volume* of D is

$$\mathrm{vol}(X, D) = \limsup_{m\to\infty} \frac{d!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^d}.$$

Definition 2.2 (Iitaka volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X such that Iitaka dimension $\kappa(D)$ is non-negative. Then the *Iitaka volume* of D, denoted by $\mathrm{Ivol}(D)$, is

$$\mathrm{Ivol}(D) = \limsup_{m\to\infty} \frac{\kappa(D)!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$

Definition 2.3 (Restricted volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. Let $S \subset X$ be a normal irreducible subvariety of dimension n. Suppose that S is not contained in the augmented base locus $\mathbf{B}_+(D)$. Then the *restricted volume of D along S* is

$$\mathrm{vol}_{X|S}(D) = \limsup_{m\to\infty} \frac{n!(\dim \mathrm{Im}(H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \to H^0(S, \mathcal{O}_S(\lfloor mD|_S \rfloor))))}{m^n}.$$

For the precise definition of the augmented base locus $\mathbf{B}_+(D)$, see [ELM+06]. In this paper, we only use the fact that $\mathbf{B}_+(D)$ is a Zariski-closed subset of X such that $\mathbf{B}_+(D) \subsetneq X$ if and only if D is big. The restricted volume $\mathrm{vol}_{X|S}(D)$ measures asymptotically the number of sections of the restriction $\mathcal{O}_S(\lfloor mD|_S \rfloor)$ that can be lifted to X. If D is ample, then the restriction maps are eventually surjective, and hence

$$\mathrm{vol}_{X|S}(D) = \mathrm{vol}(D|_S).$$

In general, it can happen that $\mathrm{vol}_{X|S}(D) < \mathrm{vol}(D|_S)$.

Theorem 2.4 ([LM09, Corollary 4.27])**.** *Let X be an irreducible projective variety of dimension d, and let $S \subset X$ be an irreducible (and reduced) Cartier divisor on X. Suppose that D is a big $\mathbb{R}$-divisor such that $S \not\subseteq \mathbf{B}_+(D)$. Then the function $t \mapsto \mathrm{vol}(D + tS)$ is differentiable at $t = 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{vol}(D + tS))\bigg|_{t=0} = d\,\mathrm{vol}_{X|S}(D).$$

By [LM09, Remark 4.29], volume function has continuous partials in all directions at any point $D \in \mathrm{Big}(X)$, i.e., the function $\mathrm{vol} : \mathrm{Big}(X) \to \mathbb{R}$ is $\mathcal{C}^1$.

2.3. **Family of pairs.** In this subsection, we define a family of pairs over reduced base scheme. We firstly recall the notion of relative Mumford divisor from [Kol23, Definition 4.68].

Definition 2.5 (Relative Mumford divisor)**.** Let $f : X \to S$ be a flat finite type morphism with S_2 fibers of pure dimension d. A subscheme $D \subset X$ is a *relative Mumford divisor* if there is an open set $U \subset X$ such that

- $\mathrm{codim}_{X_s}(X_s \setminus U_s) \geq 2$ for each $s \in S$,
- $D|_U$ is a relative Cartier divisor,
- D is the closure of $D|_U$, and
- X_s is smooth at the generic points of D_s for every $s \in S$.

By $D|_U$ being relative Cartier we mean that $D|_U$ is a Cartier divisor on U and that its support does not contain any irreducible component of any fiber U_s.

If $D \subset X$ is a relative Mumford divisor for $f : X \to S$ and $T \to S$ is a morphism, then the *divisorial pullback* D_T on $X_T := X \times_S T$ is the relative Mumford divisor defined to be the closure of the pullback of $D|_U$ to U_T. In particular, for each $s \in S$, we define $D_s = D|_{X_s}$ to be the closure of $D|_{U_s}$ which is the divisorial pullback of D to X_s.

Definition 2.6 (Locally stable family)**.** A *locally stable family of projective klt pairs* $(X, B) \to S$ over a reduced Noetherian scheme S is a flat finite type morphism $X \to S$ with S_2 fibers and a $\mathbb{Q}$-divisor B on X satisfying

- each prime component of B is a relative Mumford divisor,
- $K_{X/S} + B$ is $\mathbb{Q}$-Cartier, and
- (X_s, B_s) is a klt pair for any point $s \in S$.

3. Boundedness

In this section, we will prove Theorem 1.5, that is, boundedness of traditional stable minimal models with fixed certain numerical invariants.

3.1. **Boundedness of generalised klt pairs.** The bases of traditional stable minimal models are so called generalised klt pairs, we refer readers to [BZ16, Bir21b] for the background of the theory of generalised pairs and their singularities. We recall notions and boundedness results about generalised klt pairs.

Definition 3.1. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. Let $\mathcal{F}_{gklt}(d, \Phi, v)$ be the set of projective generalised pairs $(X, B + M)$ with nef part M' such that

- $(X, B + M)$ is generalised klt of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M_i'$ where $\mu_i \in \Phi$ and M_i' are nef Cartier, and
- $K_X + B + M$ is ample with volume $\mathrm{vol}(K_X + B + M) = v$.

The following theorem, although not explicitly stated in this form, is due to [Bir21a].

Theorem 3.2. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exists $l \in \mathbb{N}$ depending only on d, Φ, u, v such that for any traditional stable minimal model*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, u, v),$$

we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

such that the corresponding set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. Moreover, $l(K_Z + B_Z + M_Z)$ is very ample.

Proof. By applying [Bir23b, Corollary 1.4] to $(F, B|_F), A|_F$, we can find $m \in \mathbb{Z}_{>0}$, depending only on d, Φ such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. Thus we have $G \sim mA$ for some Weil divisor G whose horizontal part, which we denote by E, is an effective Weil divisor. By applying [Bir21a, Lemma 7.4] to $f : (X, B), E \to Z$, there exist $p, q \in \mathbb{N}$ depending only on d, Φ, u such that we can write an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where B_Z (resp. M_Z) is the discriminant part (resp. moduli part) of the canonical bundle formula such that $pM_{Z'}$ is Cartier on some higher resolution $Z' \to Z$.

By definition of the discriminant part of the canonical bundle formula and the ACC for lc thresholds [HMX14, Theorem 1.1], we see that the coefficients of B_Z belong to a DCC subset of $\mathbb{Q}^{>0}$ depending only on d and Φ, which we denote by Ψ. Moreover, (Z, B_Z+M_Z) is generalised klt pair and

$$\mathrm{Ivol}(K_X+B)=\mathrm{vol}(K_Z+B_Z+M_Z)=v.$$

Adding $\frac{1}{p}$, we can assume $\frac{1}{p}\in\Psi$, we see that

$$(Z, B_Z+M_Z)\in\mathcal{F}_{gklt}(\dim Z,\Psi,v).$$

By [Bir21a, Theorem 1.4], the set of generalized pairs (Z, B_Z+M_Z) forms a bounded family. By the argument after [Bir22, Theorem 4.3], there exists $l\in\mathbb{N}$ depending only on $\dim Z,\Psi,v$, hence depending only on d,Φ,u,v such that $l(K_Z+B_Z+M_Z)$ is very ample. Replacing q,l with ql, we conclude the proof. $\square$

3.2. **Boundedness of traditional stable Calabi-Yau pairs.** The fibers of traditional stable minimal models are so-called *traditional stable Calabi-Yau pairs*, i.e., klt Calabi-Yau pairs polarized by possibly non-effective ample Weil divisors.

Theorem 3.3 ([Bir23b, Corollary 1.6]). *Let $d\in\mathbb{N}$, $\Phi\subset\mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma\subset\mathbb{Q}^{>0}$ be a finite set, $\sigma\in\mathbb{Q}[t]$ be a polynomial. Consider those*

$$(X,B),A\to Z\in\mathcal{TS}_{klt}(d,\Phi,\Gamma,\sigma)$$

with $\dim Z=0$. *Then such $(X,B),A$ form a bounded family.*

Proof. This is a special case of Theorem 3.7. $\square$

3.3. **Boundedness of traditional stable minimal models.** In this subsection, we prove Theorem 1.5.

The following crucial result is a key ingredient in the proof of Theorem 1.5. It indicates that the singularities of the pairs (X,B) can be controlled uniformly. Moreover, the coefficients of B belong to a fixed finite set.

Lemma 3.4. *Let $d\in\mathbb{N}$, $\Phi\subset\mathbb{Q}^{\geq 0}$ be a DCC set, and $u,v\in\mathbb{Q}^{>0}$. Then the set*

$$\{a(D,X,B)\leq 1|(X,B),A\in\mathcal{TS}_{klt}(d,\Phi,u,v), D \textit{ prime divisor over } X\}$$

is finite.

Proof. By applying [Bir23b, Corollary 1.4] to $(F,B|_F),A|_F$, we can find $m\in\mathbb{Z}_{>0}$, depending only on d,Φ such that $H^0(F,\mathcal{O}_X(mA|_F))\neq 0$. Thus we have $G\sim mA$ for some Weil divisor G whose horizontal part, which we denote by E, is an effective Weil divisor. By applying [Bir21a, Lemma 8.2] to $f:(X,B),E\to Z$ (the proof uses E only in the relative sense over Z, the condition of K_X+B+E being ample is not required), the log discrepancies of (X,B) in the interval $[0,1]$ belong to a fixed finite set depending only on d,Φ,u,v. $\square$

We can recover some invariants of a (d,Φ,Γ,σ)-traditional stable minimal model from the given data (d,Φ,Γ,σ) by the following lemma.

Lemma 3.5. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then for any*

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma),$$

- *we have*
$$\sigma(t) = \sum_{i=0}^{d} \binom{d}{i} (K_X + B)^{d-i} \cdot A^i t^i,$$
hence for $0 \leq i \leq d$, the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ are uniquely determined by d and σ.
- *$d - \dim Z$ is the smallest degree of t that appears in σ.*
- *For a general fiber F of $X \to Z$, we have*
$$(K_X + B)^{\dim Z} \cdot A^{d-\dim Z} = \mathrm{Ivol}(K_X + B) \operatorname{vol}(A|_F).$$

Proof. This is [Bir22, Lemma 4.12], the condition that A is effective is not used in his proof. $\square$

First, we consider a special case of Theorem 1.5, where we add the condition that $K_X + B + A$ is ample. We recall the notation of a set of strongly traditional stable minimal models with fixed data, and then we prove that it belongs to a bounded family.

Definition 3.6. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u, v \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Recall that a strongly traditional stable minimal model is a traditional stable minimal model$(X, B), A$ such that $K_X + B + A$ is ample. A (d, Φ, u, v, σ)- strongly traditional stable minimal model is a strongly traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ,
- $\operatorname{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$,
- $\mathrm{Ivol}(K_X + B) = v$, and
- $\operatorname{vol}(K_X + B + tA) = \sigma(t)$, for all $t \in [0, 1]$.

Let $\mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$ denote the set of all (d, Φ, u, v, σ)-strongly traditional stable minimal models.

Theorem 3.7. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u, v \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then $\mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$ is a bounded family.*

Proof. Pick $(X, B), A \in \mathcal{STS}_{klt}(d, \Phi, u, v, \sigma)$, let $f : X \to Z$ be the contraction defined by the semi-ample divisor $K_X + B$. By Lemma 3.4, the log discrepancies of (X, B) in the interval $[0, 1]$ belong to a fixed finite set depending only on d, Φ, u, v. Therefore, (X, B) is ϵ-lc and $l(K_X + B)$ is integral for some $\epsilon > 0$ and $l \in \mathbb{N}$ depending only on d, Φ, u, v. Let $N := l(K_X + B + A)$ be an ample $\mathbb{Q}$-Cartier Weil divisor, then

$$\operatorname{vol}(K_X + B + N) = (l + 1)^d \operatorname{vol}(K_X + B + \frac{l}{l + 1} A) = (l + 1)^d \sigma(\frac{l}{l + 1})$$

is fixed. Thus by [Bir23b, Theorem 1.5], the set of (X, B) forms a log bounded family.

Since $\mathcal{P} = \{(X, B)\}$ is a set of ϵ-lc pairs that belongs to a log bounded family, by [HLQ23, Theorem 1.10], for any $\mathbb{Q}$-Cartier Weil divisor D on X, $I_0 D$ is Cartier for some $I_0 \in \mathbb{N}$ depending only on ϵ and $\mathcal{P}$ and hence depending only on d, Φ, u, v, σ. Let

$$L := 2l(K_X + B) + lA$$

be an ample $\mathbb{Q}$-Cartier Weil divisor, then $I_0 L$ is an ample Cartier divisor. We see that

$$I_0 L - (K_X + B) = (I_0 l - 1)(K_X + B) + I_0 l(K_X + B + A)$$

is nef and big, then by effective base point freeness theorem [Kol93, Theorem 1.1], there exists a positive integer I_1 depending only on d, I_0 such that $I_1 L$ is base point free. Since $I_1 L - (K_X + B)$ is nef and big, by very ampleness lemma [Fuj17, Lemma 7.1], there exists a positive integer I depending only on d, I_1 and hence depending only on d, Φ, u, v, σ such that IL is very ample. Note that

$$\operatorname{vol}(IL) = (2Il)^d \operatorname{vol}(K_X + B + \frac{1}{2}A) = (2Il)^d \sigma(\frac{1}{2})$$

is fixed, and $(K_X + B) \cdot (IL)^{d-1}$ is determined by $\sigma(t)$ by Lemma 3.5. Also we see that

$$IL - A = (Il + 1)(K_X + B) + (Il - 1)(K_X + B + A)$$

is pseudo-effective. Let H be a general element of $|IL|$, then $(X, B), A$ belongs to a bounded family in the sense of Definition 1.4. □

Under the condition that $K_X + B + \lambda A$ is big for some positive rational number λ (and hence $K_X + B + tA$ is big for any $0 < t \le \lambda$), we can find a fixed positive rational number $\tau \in (0, 1]$ such that $K_X + B + \tau A$ is ample by the following theorem. We follow the argument of [Bir22, Proof of Theorem 4.1] with some modifications. In Step 6 of the proof of the following theorem, since we are dealing with a possibly non-effective big Weil divisor, after replacing it with an effective $\mathbb{Q}$-divisor in the $\mathbb{Q}$-linear system, we may lose control of its coefficients. Therefore, we require a stronger boundedness result on singularities, as discussed in [Bir21c, Theorem 1.8], compared to the one presented in [Bir22, Lemma 4.7]. This motivates the modification of the definition of boundedness for traditional stable minimal models in our context (Definition 1.4) from the original definition in [Bir22, Section 2.9].

Theorem 3.8. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\ge 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial, $\lambda \in \mathbb{Q}^{>0}$. Then there is a positive rational number $\tau \in (0, 1]$ depending only on $d, \Phi, \Gamma, \sigma, \lambda$ such that $K_X + B + \tau A$ is ample for all*

$$(X, B), A \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda).$$

Moreover, $\mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda)$ is a bounded family.

Proof. We will prove this theorem by induction on the dimension of X.

Step 1. Choose

$$(X, B), A \to Z \in \mathcal{TS}_{klt}(d, \Phi, \Gamma, \sigma, \lambda).$$

By Lemma 3.5, for a general fiber F of $X \to Z$, $\mathrm{Ivol}(K_X+B)\,\mathrm{vol}(A|_F)$ is a fixed number depending only on d and σ. Since $\mathrm{vol}(A|_F)$ belongs to a finite set Γ, there are finitely many possibilities for $\mathrm{Ivol}(K_X+B)$. Therefore, we may fix both $u=\mathrm{vol}(A|_F)$ and $v=\mathrm{Ivol}(K_X+B)$.

Again by Lemma 3.5,

$$\sigma(t)=\sum_{i=0}^{d}\binom{d}{i}(K_X+B)^{d-i}\cdot A^it^i,$$

and the intersection numbers $(K_X+B)^{d-i}\cdot A^i$ are determined by d and σ for any $0\le i\le d$.

Step 2. By Lemma 3.4, we may choose an $\alpha\in\mathbb{N}$ depending only on d,Φ,u,v,λ such that $\alpha(K_X+B+\lambda A)$ is a big Weil divisor. Moreover, for $0\le t\ll 1$, $K_X+B+t\alpha(K_X+B+\lambda A)$ is ample and

$$\begin{aligned}&\mathrm{vol}(K_X+B+t\alpha(K_X+B+\lambda A))\\ =&\,\mathrm{vol}((1+t\alpha)(K_X+B)+t\alpha\lambda A)\\ =&((1+t\alpha)(K_X+B)+t\alpha\lambda A)^d\end{aligned}$$

is a polynomial γ in t whose coefficients are uniquely determined by the intersection numbers $(K_X+B)^{d-i}\cdot A^i$, α and λ. Therefore, γ is determined by $d,\Phi,\Gamma,\sigma,\lambda$.

Replacing A,u,σ with $\alpha(K_X+B+\lambda A),(\alpha\lambda)^{\dim F}u,\gamma$, we may assume that A is a big Weil divisor, and

$$(X,B),A\in\mathcal{TS}_{klt}(d,\Phi,\Gamma,\sigma,1).$$

Step 3. If K_X+B+A is ample, i.e., $(X,B),A$ is a strongly traditional stable minimal model, then for any $t\in[0,1]$,

$$\mathrm{vol}(K_X+B+tA)=(K_X+B+tA)^d$$

is a polynomial in t in the interval $[0,1]$. Thus for every $t\in[0,1]$,

$$\mathrm{vol}(K_X+B+tA)=\sigma(t)$$

since both sides of the equality are polynomials which agree for t sufficiently small. Thus by Theorem 3.7, $(X,B),A$ is bounded in the sense of Definition 1.4. In particular, when $\dim X=1$, K_X+B+A is ample, the theorem holds.

Step 4. From now on we assume that $\dim X\ge 2$. Our goal is to find a positive rational number $\tau\in(0,1]$ depending only on d,Φ,Γ,σ such that $K_X+B+\tau A$ is ample. Once this condition is guaranteed, then by Lemma 3.4, we may choose $\beta\in\mathbb{N}$ depending only on d,Φ,u,v,τ such that $\beta(K_X+B+\tau A)$ is an ample Weil Divisor. By the same argument as Step 2, after replacing A,u,σ with $\beta(K_X+B+\tau A),(\beta\tau)^{\dim F}u,\zeta$, where ζ is a polynomial in t determined by d,σ,τ, and β, we may assume that A is an ample Weil divisor, and $(X,B),A$ is strongly traditional stable belonging to

$$\mathcal{STS}_{klt}(d,\Phi,u,v,\sigma).$$

Hence by Step 3, $(X,B),A$ belongs to a bounded family in the sense Definition 1.4.

We claim that it is enough to find $\tau\in(0,1]$ depending only on d,Φ,Γ,σ such that $K_X+B+\tau A$ is nef. Indeed, we can show that K_X+B+tA is ample for any $t\in(0,\tau)$: we may pick $0 < t' \ll t$ such that $K_X+B+t'A$ is ample by Definition 1.1, then K_X+B+tA is a positive linear combination of $K_X+B+t'A$ and $K_X+B+\tau A$, hence it is ample. So replacing τ with $\frac{\tau}{2}$ we can assume that $K_X+B+\tau A$ is ample. We aim to find such τ in the subsequent steps.

Step 5. By Definition 1.1, K_X+B is semi-ample defining the contraction $f : X \to Z$. If $\dim Z = 0$, then $(X, B), A$ is a traditional stable Calabi-Yau pair belonging to a bounded family by Theorem 3.3. We can assume that $\dim Z > 0$. By Theorem 3.2, there exists $l \in \mathbb{N}$ depending only on d, Φ, u, v such that we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

and the generalised klt pair $(Z, B_Z + M_Z)$ belongs to a bounded family. Moreover,

$$L := l(K_Z + B_Z + M_Z)$$

is very ample.

Let T be a general member of $|L|$ and let S be its pullback to X. Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$. Then $(S, B_S), A_S \to T$ is a traditional stable minimal model: indeed, we may choose a general $T \in |L|$ such that $(X, B + S)$ is plt, hence (S, B_S) is a projective klt pair. Moreover, $K_S + B_S$ is semi-ample defining the contraction $g : S \to T$, and $K_S + B_S + tA_S$ is ample for every small $t > 0$ since $K_X + B + S + tA$ is ample for every small $t > 0$.

If G is a general fiber of $S \to T$, then

$$\mathrm{vol}(A_S|_G) = \mathrm{vol}(A|_G) = u,$$

since G is among the general fibers of $X \to Z$. Moreover, for $0 \le t \ll 1$, we have

$$\begin{aligned}
\psi(t) =& \mathrm{vol}(K_S + B_S + tA_S) \\
=& \mathrm{vol}((K_X + B + S + tA)|_S) \\
=& (K_X + B + S + tA)^{d-1} \cdot S \\
=& ((l+1)(K_X + B) + tA)^{d-1} \cdot S \\
=& ((l+1)(K_X + B) + tA)^{d-1} \cdot l(K_X + B),
\end{aligned}$$

where $\psi(t)$ is a polynomial in t whose coefficients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ and by l, and hence by d, σ, l. Furthermore, since S is the pullback of a general element of a very ample linear system, $A_S = A|_S$ is still a big Weil divisor. Thus $(S, B_S), A_S$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d-1, \Phi, \Gamma, \psi, 1).$$

Therefore, $(S, B_S), A_S$ belongs to a bounded family in the sense of Definition 1.4 by induction.

Step 6. By Lemma 3.5, $(S, B_S), A_S$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d-1, \Phi, u, v')$$

for some fixed $v' = \mathrm{Ivol}(K_S + B_S)$ that depends on d, σ, u. Then (S, B_S) is ϵ-lc for some $\epsilon \in \mathbb{Q}^{>0}$ depending only on $d-1, \Phi, u, v'$ by Lemma 3.4. By Step 5, $(S, B_S), A_S$ belongs to a bounded family in the sense of Definition 1.4, hence there is a fixed $r \in \mathbb{N}$ such that for any $(S, B_S), A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1} \leq r, (K_S + B_S) \cdot H_S^{d-2} \leq r, \text{ and } H_S - A_S \text{ is pseudo-effective.}$$

By [Bir22, Lemma 4.6], we may assume that $H_S - B_S$ is pseudo-effective. Since A is big, there is an effective $\mathbb{Q}$-divisor E such that $A \sim_{\mathbb{Q}} E$. Since S is a pullback of a general element of a very ample linear system, $E_S := E|_S$ is an effective $\mathbb{Q}$-divisor and $A_S \sim_{\mathbb{Q}} E_S$. Moreover, $H_S - E_S \sim_{\mathbb{Q}} H_S - A_S$ is also pseudo-effective.

Thus by [Bir21c, Theorem 1.8], there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $d-1, \epsilon, r$ such that

$$\mathrm{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt. Then by inversion of adjunction [KM98, Theorem 5.50], $(X, B + S + \tau E)$ is plt near S. Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z: otherwise, the non-lc locus of $(X, B + \tau E)$ maps onto a closed subset of Z positive dimension which intersects T, hence S intersects the non-lc locus of $(X, B + \tau E)$, a contradiction.

Step 7. In this step, we assume that $K_X + B + \tau E$ is not nef. Otherwise, $K_X + B + \tau A \sim_{\mathbb{Q}} K_X + B + \tau E$ is nef, and we are done by Step 4.

Let R be a $(K_X + B + \tau E)$-negative extremal ray, since $K_X + B + \tau E$ is ample over Z, R is not contained in fibers of $X \to Z$. Since $K_X + B$ is the pullback of ample divisor on Z, $(K_X + B) \cdot R > 0$. By Step 6, the non-lc locus of $(X, B + \tau E)$ maps to finitely many points of Z, so R is not contained in the image

$$\mathrm{Im}(\overline{\mathrm{NE}}(\Pi) \to \overline{\mathrm{NE}}(X)),$$

where Π is the non-lc locus of $(X, B + \tau E)$.

Then by the the length of extremal ray [Fuj11, Theorem 1.1], R is generated by a curve C with

$$(K_X + B + \tau E) \cdot C \geq -2d.$$

Since L is very ample, $f^*L \cdot C = L \cdot f_*C \geq 1$, we see that

$$(K_X + B + 2df^*L + \tau E) \cdot C \geq 0.$$

It follows that

$$K_X + B + 2df^*L + \tau E$$

is nef. Since $f^*L \sim l(K_X + B)$, we see that

$$K_X + B + \frac{\tau}{1 + 2dl} E \sim_{\mathbb{Q}} \frac{1}{1 + 2dl}(K_X + B + 2df^*L + \tau E)$$

is nef. Hence after replacing τ with $\frac{\tau}{1+2dl}$, we can assume that $K_X + B + \tau E$ is nef. $\square$

To find a positive rational number λ such that $K_X + B + \lambda A$ is big for any $(X, B), A \to Z \in \mathcal{S}_{klt}(d, \Phi, \Gamma, \sigma)$, one way is to focus on the volume function $\theta(t) = \mathrm{vol}(K_X + B + tA)$, as its graph provide information about the big thresholds of $K_X + B$ with respect to

A. However, in our setting, we only fix the top self-intersection number function $\sigma(t) = (K_X+B+tA)^d$. The idea is to compare $\sigma(t)$ with $\theta(t)$. Since $\sigma(t)=\theta(t)$ for $0<t\ll 1$, we can compare them by investigating their derivatives. For the purpose of simplifying the computation, we will consider the variations of the volume function $\vartheta(t)$ and the top self-intersection function $\varsigma(t)$ instead.

Proposition 3.9. *Let $d\in\mathbb{N}$, $\Phi\subset\mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma\subset\mathbb{Q}^{>0}$ be a finite set, $\sigma\in\mathbb{Q}[t]$ be a polynomial. Then for any*

$$(X,B),A\to Z\in\mathcal{TS}_{klt}(d,\Phi,\Gamma,\sigma),$$

there is a positive rational number λ depending only on d,Φ,Γ,σ such that K_X+B+tA is big for all $0<t<\lambda$.

Proof. Step 1. Let $\varsigma\in\mathbb{Q}[t]$ be a polynomial function such that $\varsigma(t)=\operatorname{vol}(A+t(K_X+B))$ for $t\gg 1$. We have

$$\varsigma(t)=(A+t(K_X+B))^d=\sum_{i=0}^{d}\tbinom{d}{i}A^{d-i}\cdot(K_X+B)^i t^i$$

for all t, since both left hand and right hand side of the equality are polynomials in t that agree for all large values of t. By Lemma 3.5, ς is determined by σ. Again by Lemma 3.5, we may fix both $u=\operatorname{vol}(A|_F)$ and $v=\operatorname{Ivol}(K_X+B)$. It is enough to show that there exists a positive rational number τ depending only on d,Φ,Γ,σ such that $A+t(K_X+B)$ is big for all $t>\tau$. Let $\vartheta(t)=\operatorname{vol}(A+t(K_X+B))$ be a function on t, then $\vartheta(t)$ is a non-negative non-decreasing real function on t, which agrees with $\varsigma(t)$ for $t\gg 0$. It suffices to show that $\vartheta(t)>0$ for all $t>\tau$.

Step 2. We prove this proposition by induction on the dimension of Z. By Lemma 3.5, $\dim Z$ is determined by σ, we may assume that $\dim Z=m$ is fixed. If $\dim Z=0$, then $(X,B),A$ is a stable Calabi-Yau pair, proposition holds trivially. We may assume that $\dim Z>0$.

If $\dim Z=1$, then

$$\varsigma(t)=(A+t(K_X+B))^d=A^d+dA^{d-1}\cdot(K_X+B)t=A^d+duvt.$$

Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, then $\varsigma'(t)=duv$. Let F be a general fiber of $X\to Z$, then we see that

$$K_X+B\sim_{\mathbb{Q}}f^*(K_Z+B_Z+M_Z)\sim_{\mathbb{Q}}vF,$$

and we have

$$\vartheta(t)=\operatorname{vol}(A+t(K_X+B))=v^d\operatorname{vol}(\frac{1}{v}A+tF).$$

For each t such that $A+t(K_X+B)$ is big, i.e., $\vartheta(t)>0$, we may choose a sufficiently general fiber F_t of $X\to Z$ such that $F_t\not\subseteq\mathbf{B}_+(\frac{1}{v}A+tF_t)$. Then by Theorem 2.4, the function $s\mapsto\operatorname{vol}(\frac{1}{v}A+tF_t+sF_t)$ is differentiable at $s=0$.

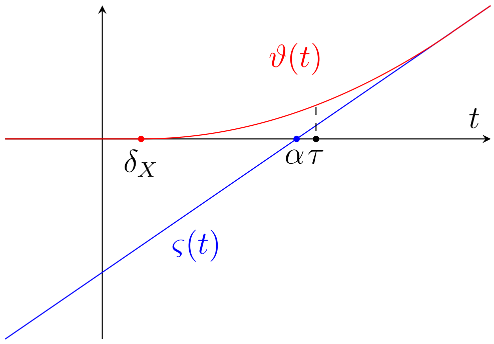
Let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t, we have

$$\frac{1}{v^d}\vartheta'(t)=\frac{1}{v^d}\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}(\operatorname{vol}(\frac{1}{v}A+tF_t+sF_t))\Big|_{s=0}=d\operatorname{vol}_{X|F_t}(\frac{1}{v}A+tF_t),$$

hence

$$\vartheta'(t) = dv^d \operatorname{vol}_{X|F_t}(\frac{1}{v}A + tF_t) \leq dv^d \operatorname{vol}((\frac{1}{v}A + tF_t)|_{F_t}) = dv^d \frac{1}{v^{d-1}} u = duv = \varsigma'(t).$$

FIGURE 1. The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z = 1$



Let α be the root of $\varsigma(t)$ and $\tau = \max\{\lceil \alpha \rceil + 1, 1\}$, then τ is a positive rational number. Let δ_X be the real number such that $\vartheta(t) = 0$ for all $t \leq \delta_X$ and $\vartheta(t) > 0$ for all $t > \delta_X$, where δ_X may depend on X. We claim that $\vartheta(\tau) > 0$. Otherwise, $\vartheta(\tau) = 0$, and so $\tau \leq \delta_X$. Since $\vartheta(t)$ agrees with $\varsigma(t)$ for all $t \gg 0$, there exists a real number $\delta'_X \gg 0$ (δ'_X may depend on X a priori) such that $\vartheta(\delta'_X) = \varsigma(\delta'_X)$. By [Laz04, Corollary 2.2.45], $\vartheta(t)$ is a continuous function. Since both $\varsigma(t)$ and $\vartheta(t)$ are differentiable in (δ_X, δ'_X), by [Rud76, Theorem 5.9], there is a point $\gamma_X \in (\delta_X, \delta'_X)$ at which

$$[\vartheta(\delta'_X) - \vartheta(\delta_X)]\varsigma'(\gamma_X) = [\varsigma(\delta'_X) - \varsigma(\delta_X)]\vartheta'(\gamma_X).$$

Since $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, $\vartheta(\delta_X) = 0$ and $\varsigma(\delta_X) \geq \varsigma(\tau) > 0$, it follows that $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$, but by the previous paragraph, $\vartheta'(t) \leq \varsigma'(t)$ on $(\delta_X, +\infty)$, which is a contradiction. Therefore, $\vartheta(\tau) > 0$, and we can conclude that $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t > \tau$.

Step 3. From now on we assume that $\dim Z = m > 1$. Recall that in the Step 5 of the proof of Theorem 3.8, we pick a a general element T in the very ample linear system $|l(K_Z + B_Z + M_Z)|$ and let S be its pullback to X, then

$$S \sim_{\mathbb{Q}} l(K_X + B).$$

Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$, then

$$K_S + B_S \sim_{\mathbb{Q}} (\frac{1}{l} + 1)S|_S.$$

Moreover, $(S, B_S), A_S \to T$ is a traditional stable minimal model belonging to

$$\mathcal{TS}_{klt}(d-1, \Phi, \Gamma, \psi)$$

for some fixed polynomial $\psi(t) \in \mathbb{Q}[t]$ that depends on d, Φ, Γ, σ, and $\dim T = m - 1$. By induction, there is a positive rational number λ' depending only on d, Φ, Γ, σ such that $K_S + B_S + tA_S$ is big for all $0 < t < \lambda'$. Hence by Theorem 3.8, there is a positive rational number β depending only on d, Φ, Γ, σ such that $A_S + t(K_S + B_S)$ is ample for all $t > \beta$.

Step 4. We have $\varsigma(t) = (A + t(K_X + B))^d$. Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, if $t > \beta(l+1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$ is ample by Step 3, we have

$$\begin{aligned}\varsigma'(t) =& d(A + t(K_X + B))^{d-1} \cdot (K_X + B)\\ =& \frac{d}{l}(A + t(K_X + B))^{d-1} \cdot S\\ =& \frac{d}{l}(A_S + \frac{t}{l+1}(K_S + B_S))^{d-1}\\ >& 0.\end{aligned}$$

Hence $\varsigma(t)$ is an increasing function on $(\beta(l+1), +\infty)$.

If $\varsigma(t)$ has no roots (which occurs only when $\dim X$ is even), set $\tau = \beta(l+1) + 1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$, set $\tau = \max\{\beta(l+1), \lceil\alpha\rceil\} + 1$. Note that τ is a positive rational number. Moreover, on $[\tau, +\infty)$, $\varsigma(t)$ is a positive, increasing real function, and $\vartheta(t)$ is a non-negative, non-decreasing real function.

Step 5. In this step, we conclude the proof. We see that

$$\vartheta(t) = \mathrm{vol}(A + t(K_X + B)) = \frac{1}{l^d}\,\mathrm{vol}(lA + tS),$$

for any $S \sim_{\mathbb{Q}} l(K_X + B)$. For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose S_t as the pullback of a sufficiently general element $T_t \in |l(K_Z + B_Z + M_Z)|$ such that $S_t \nsubseteq \mathbf{B}_+(lA + tS_t)$. Then by Theorem 2.4, the function $s \mapsto \mathrm{vol}(lA + tS_t + sS_t)$ is differentiable at $s = 0$, let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t, we have

$$l^d\vartheta'(t) = l^d \frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{vol}(lA + tS_t + sS_t))\Big|_{s=0} = d\,\mathrm{vol}_{X|S_t}(lA + tS_t).$$

Moreover, if $t \geq \tau$ and if $A + t(K_X + B)$ is big, we have

$$\begin{aligned}\vartheta'(t) =& \frac{d}{l^d}\,\mathrm{vol}_{X|S_t}(lA + tS_t)\\ \leq& \frac{d}{l}\,\mathrm{vol}((A + \frac{t}{l}S_t)|_{S_t})\\ =& \frac{d}{l}\,\mathrm{vol}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t}))\\ =& \frac{d}{l}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t}))^{d-1}\\ =& \varsigma'(t),\end{aligned}$$

where the third equality follows from the fact that $A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})$ is ample on $[\tau, +\infty)$.

We claim that $\vartheta(\tau) > 0$. Otherwise, $\vartheta(\tau) = 0$, employing the same argument as in the last paragraph of Step 2, we can get a contradiction. We conclude that $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t > \tau$. □

Remark 3.10. In the case when $\dim X = 2$, by Zariski decomposition [Laz04, Theorem 2.3.19] [Eno20, Theorem 3.1], the volume of a big divisor should be greater than or equal to its self-intersection. Thus when $\dim X = 2$, Proposition 3.9 quickly follows from this fact. However, this property does not necessarily hold for higher-dimensional varieties. For example, let Y be a smooth 3-fold such that K_Y is ample, and $\pi : X = \mathrm{Bl}_P Y \to X$ be the blowing up of Y at a closed point P. Then $K_X = \pi^* K_Y + 2E$, where E is the exceptional divisor over $P \in Y$, and K_X is big. Therefore, $\mathrm{vol}(K_X) = \mathrm{vol}(K_Y) = K_Y^3$, but $K_X^3 = K_Y^3 + 8E^3 = K_Y^3 + 8 > \mathrm{vol}(K_X)$.

***Proof of Theorem** 1.5.* This directly follows from Theorem 3.8 and Proposition 3.9. $\square$

4. Moduli spaces

In this section, we prove Theorem 1.7, that is, we show that there is a separated Deligne-Mumford stack of finite type parametrizing families of traditional stable minimal models and it admits a coarse moduli space. We refer readers to [Alp24] for the notions of stacks, algebraic stacks, Deligne-Mumford stacks and algebraic spaces.

The general strategy for constructing moduli stacks of varieties is to embed the varieties into a single projective space and then employ Hilbert scheme arguments. Moreover, the theory of relative Mumford divisors developed in [Kol23] also works for varieties polarized by effective divisors. However, for varieties polarized by non-canonical line bundles, to obtain a universal object for these line bundles, we consider embedding the varieties into the product of two projective spaces, following the approach in [Vie95, Section 1.7].

Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. In this section, we will fix these data.

Lemma 4.1. *Let K be a field of characteristic zero. Then there exist natural number τ and I depending only on d, Φ, Γ, σ such that $\tau\Phi \subset \mathbb{N}$ and they satisfy the following. For any $(X, B), A \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(K)$ and nef Cartier divisor M on X, we have*

- *$\tau(K_X + B)$ is a base point free divisor, $A + \tau(K_X + B)$ is an ample Cartier divisor,*
- *Let $L_M := I(A + \tau(K_X + B)) + M$, then L_M is strongly ample, i.e. L_M is very ample and $H^q(X, kL_M) = 0$ for any $k, q > 0$,*

Proof. By the same argument as [Bir22, Proof of Lemma 10.2], it is enough to find τ and I when $K = \operatorname{Spec} \mathbb{C}$. Note that A is a line bundle in our setting, hence by the proof of Theorem 1.5, there exist τ, I_0 such that $\tau(K_X + B)$ is base point free, $A + \tau(K_X + B)$ is ample Cartier, and $L_0 := I_0(A + \tau(K_X + B))$ is very ample.

After replacing I_0 with a bounded multiple, we may assume that $L_0 - (K_X + B)$ is nef and big. Let $I = (d+2)I_0$ and $\mathcal{F} := L_M - I_0(A + \tau(K_X + B))$, then

$$H^i(X, \mathcal{F} \otimes L_0^{\otimes(-i)}) = 0$$

for all $i > 0$ by Kawamata-Viehweg vanishing theorem. Thus $\mathcal{F}$ is 0-regular with respect to L_0 ([Laz04, Definition 1.8.4]), and hence $\mathcal{F}$ is base point free by [Laz04, Theorem 1.8.5]. Therefore,

$$L_M = L_0 + \mathcal{F}$$

is very ample by [Har77, Exercise II 7.5(d)]. Again we have $L_M - (K_X + B)$ is nef and big, hence $H^q(X, kL_M) = 0$ for any $k, q > 0$. $\square$

Notation 4.2. From now on, we will fix the positive natural numbers I and τ obtained in Lemma 4.1. Let S be a reduced scheme, for any $(f : (X,B), A \to S) \in \mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)(S)$, we define

$$\begin{aligned} L_{1,S} &:= I(A+\tau(K_{X/S}+B)) + I(A+\tau(K_{X/S}+B)) = 2IA + 2I\tau(K_{X/S}+B), \\ L_{2,S} &:= I(A+\tau(K_{X/S}+B)) + (I-1)(A+\tau(K_{X/S}+B)) + \tau(K_{X/S}+B) \\ &= (2I-1)A + 2I\tau(K_{X/S}+B) \end{aligned}$$

and $L_{3,S} := L_{1,S} + L_{2,S}$ to be the divisorial sheaves on X. Then $L_{1,S} - L_{2,S} = A$, and $L_{j,S}$ are strongly ample line bundles over S for $j = 1,2,3$ by Lemma 4.1 and the proof of Lemma 4.3.

Lemma 4.3. *Let $(X, B = \sum_{i=1}^{m} a_i D_i), A \to S$ be a family of (d,Φ,Γ,σ)-marked traditional stable minimal models over reduced Noetherian scheme S. For $j = 1,2,3$, let $L_{j,S}$ be the divisorial sheaves on X as Notation 4.2. Then for every $k \in \mathbb{Z}_{>0}$, the functions $S \to \mathbb{Z}$ by sending*

(1) $s \mapsto h^0(X_s, kL_{j,s})$ for $j = 1,2,3$ and
(2) $s \mapsto \deg_{L_{3,s}}(D_{i,s})$ for $i = 1,2,\dots,m$

are locally constant on S, where $L_{j,s} = L_{j,S}|_{X_s}$ and $D_{i,s} = D_i|_{X_s}$ are the divisorial pullbacks to X_s, and $\deg_{L_{3,s}}(D_{i,s}) := D_{i,s} \cdot L_{3,s}^{d-1}$.

Proof. (1). For $j = 1,2,3$, it is enough to show that $L_{j,S}$ are flat over S: since then $\chi(X_s, kL_{j,s})$ are locally constant, and $L_{j,S}$ are strongly ample over S by Lemma 4.1, hence $h^0(X_s, kL_{j,s})$ are locally constant. Since $X \to S$ is flat, it suffices to show that $\mathcal{O}_X(L_{j,S})$ are line bundles by [Har77, Proposition III 9.2(c)(e)].

Since $(X,B) \to S$ is a locally stable family, B is a relative Mumford divisor over S, we see that $\tau(K_{X/S}+B)$ is $\mathbb{Q}$-Cartier, and it is mostly flat ([Kol23, Definition 3.26]) over S. Moreover, since $\mathcal{O}_{X_s}(\tau(K_{X_s}+B_s))$ is a base point free line bundle for any $s \in S$ by Lemma 4.1, $\mathcal{O}_X(\tau(K_{X/S}+B))$ is a mostly flat family of line bundles. Therefore, by [Kol23, Corollary 4.34 and Proposition 5.29], $\mathcal{O}_X(\tau(K_{X/S}+B))$ is a line bundle on X. Furthermore, since A is a line bundle on X, $\mathcal{O}_X(L_{j,S})$ are line bundles for $j = 1,2,3$.

(2). It follows from [Kol23, Theorem 4.3.5]. $\square$

Let $n, l \in \mathbb{Z}_{>0}$, $\mathbf{c} = (c_1, c_2, \dots, c_m) \in \mathbb{N}^m$, and $h \in \mathbb{Q}[k]$ be a polynomial. Let S be a reduced scheme, for any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)(S)$ and $j = 1,2,3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation 4.2. We define $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ to be a full subcategory of $\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)$ such that $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$ is a groupoid whose objects consist of families of (d,Φ,Γ,σ)-traditional stable minimal models over S satisfying:

- the Hilbert polynomial of X_s with respect to $L_{3,s}$ is h,
- $h^0(X_s, L_{1,s}) - 1 = n$,
- $h^0(X_s, L_{2,s}) - 1 = l$, and
- $(\deg_{L_{3,s}}(D_{1,s}), \deg_{L_{3,s}}(D_{2,s}), \dots, \deg_{L_{3,s}}(D_{m,s})) = \mathbf{c}$

for every $s \in S$.

Lemma 4.4. *We can write*

$$\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma) = \bigsqcup_{h,n,l,\boldsymbol{c}} \mathfrak{T}\mathfrak{S}_{h,n,l,\boldsymbol{c}}$$

as disjoint union, and each $\mathfrak{T}\mathfrak{S}_{h,n,l,\boldsymbol{c}}$ *is a union of connected components of* $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)$. *Moreover, there are only finitely many* $n, l \in \mathbb{Z}_{>0}$, $\boldsymbol{c} = (c_1, c_2, \dots, c_m) \in \mathbb{N}^m$ *and* $h \in \mathbb{Q}[k]$ *such that* $\mathfrak{T}\mathfrak{S}_{h,n,l,\boldsymbol{c}}$ *is not empty.*

Proof. Given any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(S)$. By Lemma 4.3, the Hilbert functions

$$h_s(k) = \chi(X_s, kL_{3,s}) = h^0(X_s, kL_{3,s})$$

of X_s with respect to $L_{3,s}$, and the numbers

$$n_s = h^0(X_s, L_{1,s}) - 1, \quad l_s = h^0(X_s, L_{2,s}) - 1 \quad \text{and} \quad c_{i,s} = \deg_{L_{3,s}}(D_{i,s})$$

are locally constant on $s \in S$ for all $1 \le i \le m$. The first assertion follows from this fact.

The second assertion follows from the fact that n_s, l_s, $c_{i,s}$ and h_s belong to a finite set for all $1 \le i \le m$ by Theorem 1.5 (these finiteness results can be reduced to the case when $s = \operatorname{Spec} \mathbb{C}$ by the same argument as [Bir22, Proof of Lemma 10.2]). □

Lemma 4.5. $\mathfrak{T}\mathfrak{S}_{h,n,l,\boldsymbol{c}}$ *is a stack.*

Proof. Since our argument follows the same strategy as in [Alp24, Proposition 2.5.14 and Example 2.5.9], we only sketch the proof here.

Axiom (1) of [Alp24, Definition 2.5.1] follows from descent [Alp24, Proposition 2.1.7, Proposition 2.1.19, Proposition 2.1.4(1) and Proposition 2.1.16(2)].

To verify Axiom (2) of [Alp24, Definition 2.5.1], i.e., given any descent datum (f', ξ) with respect to a covering $S' \to S$ (see [HH23, Remark 2.10] for notions of covering and descent datum), where $(f' : (X', B'), A' \to S') \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S')$, we need to show that f' descends to a family $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$. We use the strongly f'-ample line bundles $\mathcal{O}_{X'}(L'_{1,S'})$ and $\mathcal{O}_{X'}(L'_{2,S'})$ as Notation 4.2 instead of $\omega_{C'/S'}^{\otimes 3}$ in [Alp24, Proposition 2.5.14], then the same argument as in *loc.cit.* implies that $(X', B') \to S'$ descends to $(X, B) \to S$. Moreover, by applying [Alp24, Proposition 2.1.4(2) and Proposition 2.1.16(2)] to the covering $X' \to X$, we see that A' descends to a line bundle A on X. Since every geometric fiber of $f : (X, B), A \to S$ is identified with a geometric fiber of $f' : (X', B'), A' \to S'$, $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$. □

Theorem 4.6. $\mathfrak{T}\mathfrak{S}_{h,n,l,\boldsymbol{c}}$ *is an algebraic stack of finite type.*

Proof. Step 1. In this step, we consider a suitable Hilbert scheme parametrizing the total spaces of interest.

For any $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$ and for $j = 1, 2, 3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation 4.2. We get an embedding

$$X \hookrightarrow \mathbb{P}(f_*\mathcal{O}_X(L_{1,S})) \times_S \mathbb{P}(f_*\mathcal{O}_X(L_{2,S})).$$

We proceed to parametrize such embedding.

Let $H = \mathrm{Hilb}_h(\mathbb{P}^n \times \mathbb{P}^l)$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n \times \mathbb{P}^l$ with Hilbert polynomial h. Let $X_H = \mathrm{Univ}_h(\mathbb{P}^n \times \mathbb{P}^l) \overset{i}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^l \times H$ be the universal family over H, and

$$\mathbb{P}^n \overset{p_1}{\leftarrow} \mathbb{P}^n \times \mathbb{P}^l \times H \overset{p_2}{\rightarrow} \mathbb{P}^l.$$

be the natural projections. Note that the $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on $\mathbb{P}^n \times \mathbb{P}^l$ induces a $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on H. Let $M_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1,1)$ and $N_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1,-1)$ be the universal line bundles on X_H.

Step 2. In this step, we parametrize the boundary divisors in the moduli problem.

By [Gro66, Theorem 12.2.1 and Theorem 12.2.4], the locus $s \in H$ such that X_s is geometrically connected and reduced, equidimensional, and geometrically normal is an open subscheme H_1 of H.

Since $f_1 : X_{H_1} \to H_1$ is equidimensional, and over reduced bases relative Mumford divisors are the same as K-flat divisors [Kol23, Definition 7.1 and comment 7.4.2], there is a separated H_1-scheme $\mathrm{MDiv}_c(X_{H_1}/H_1)$ of finite type which parametrizes relative Mumford divisors of degree c with respect to M_{H_1} by [Kol23, Theorem 7.3]. Fixing $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$, let

$$H_2 := \mathrm{MDiv}_{c_1}(X_{H_1}/H_1) \times_{H_1} \mathrm{MDiv}_{c_2}(X_{H_1}/H_1) \times_{H_1} \cdots \times_{H_1} \mathrm{MDiv}_{c_m}(X_{H_1}/H_1)$$

be the m-fold fiber product, we denote the universal family by

$$(X_{H_2}, B_{H_2} = \sum_{i=1}^{m} a_i D_{i,H_2}), N_{H_2} \to H_2,$$

where D_{i,H_2} are the universal families of relative Mumford divisors on X_{H_2} of degree c_i with respect to M_{H_2} for $1 \le i \le m$.

Step 3. By [Kol23, Theorem 4.8], there is a locally closed partial decomposition $H_3 \to H_2$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_2$, then the family obtained by base change $f_W : (X_W, B_W) \to W$ is locally stable iff q factors as $q : W \to H_3 \to H_2$.

Since $f_3 : (X_{H_3}, B_{H_3}) \to H_3$ is locally stable, By [Kol23, Theorem 4.28], there is a locally closed partial decomposition $H_4 \to H_3$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_3$, the divisorial pullback of $\tau(K_{X_{H_3}/H_3} + B_{H_3})$ to $W \times_{H_3} X_{H_3}$ is Cartier iff q factors as $q : W \to H_4 \to H_3$.

Step 4. Since the fibers X_s of $f_4 : X_{H_4} \to H_4$ are reduced and connected by Step 2, we have $h^0(X_s, \mathcal{O}_{X_s}) = 1$. Since $\tau(K_{X_{H_4}/H_4} + B_{H_4})$ is Cartier by Step 3, by [Vie95, Lemma 1.19], there is a locally closed subscheme $H_5 \subset H_4$ with the following property: for any scheme W and morphism $q : W \to H_4$,

$$\mathcal{O}_{X_W}(1,0) \sim_W N_W^{2I} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W) \text{ and}$$

$$\mathcal{O}_{X_W}(0,1) \sim_W N_W^{2I-1} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W)$$

iff q factors as $q : W \to H_5 \to H_4$, where $\mathcal{O}_{X_W}(1,0)$ and $\mathcal{O}_{X_W}(0,1)$ are the pullbacks of $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^l\times H_4}(1,0)$ and $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^l\times H_4}(0,1)$ to X_W, respectively.

Step 5. In this step, we cut the locus parametrizing (d,Φ,Γ,σ)-traditional stable minimal models.

(1). By [Bir22, Lemma 8.5], there is a locally closed subscheme $H_6 \subset H_5$ such that for any $s \in H_6$, $K_{X_s} + B_s$ is semi-ample defining a contraction $X_s \to Z_s$.

(2). Since ampleness and klt are open conditions, there is an open subscheme $H_7 \subset H_6$ such that $N_s + \tau(K_{X_s} + B_s)$ is ample and (X_s, B_s) is klt for any $s \in H_7$.

(3). By [Bir22, Lemma 8.7] (the condition of N_s being effective is not required in the proof), there is a locally closed subscheme $H_8 \subset H_7$ such that for any $s \in H_8$, $\mathrm{vol}(N_s|_F) \in \Gamma$ for the general fibres F of $X_s \to Z_s$.

(4). For each $s \in H_8$, since $K_{X_s} + B_s$ is semi-ample and $N_s + \tau(K_{X_s} + B_s)$ is ample, $K_{X_s} + B_s + tN_s$ is ample for each $t \in (0, \frac{1}{\tau}]$, then

$$\theta_s(t) = \mathrm{vol}(K_{X_s} + B_s + tN_s) = (K_{X_s} + B_s + tN_s)^d$$

is a polynomial in t of degree $\leq d$ on the interval $(0, \frac{1}{\tau}]$. By Step 3(iv) of [Bir22, Proof of Proposition 9.5], there is an open and closed subscheme $H_9 \subset H_8$ such that $\theta_s(t) = \sigma(t)$ on the interval $(0, \frac{1}{\tau}]$.

Therefore, $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is a family of (d,Φ,Γ,σ)-traditional stable minimal models. For $j = 1, 2$, let L_{j,H_9} be the strongly ample line bundles over H_9 as Notation 4.2. Then $f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})$ and $f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})$ are locally free sheaves of rank $n+1$ and $l+1$, respectively. Shrinking H_9, we may assume that they are free sheaves, and hence

$$\mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})) \cong \mathbb{P}^n_{H_9} \text{ and } \mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})) \cong \mathbb{P}^l_{H_9}.$$

Step 6. In this step, we will prove that

$$\mathfrak{TS}_{h,n,l,\mathbf{c}} \cong [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}].$$

Then since H_9 is a finite type scheme and $[H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$ is an algebraic stack, $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is a finite type algebraic stack.

We follow the arguments of [Alp24, Theorem 3.1.17] and [ABB+23, Proposition 3.9]. By our construction, the universal family $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is an object in $\mathfrak{TS}_{h,n,l,\mathbf{c}}(H_9)$, which induces a morphism $H_9 \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$, where this morphism just forgets the projective embeddings. Moreover, this morphism is $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$-invariant, hence descends to a morphism $\Psi^{\mathrm{pre}} : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]^{\mathrm{pre}} \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$ of prestacks. Since $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is a stack by Lemma 4.5, the universal property of stackification [Alp24, Theorem 2.5.18] yields a morphism $\Psi : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}] \to \mathfrak{TS}_{h,n,l,\mathbf{c}}$.

To construct the inverse, consider $(f : (X,B), A \to S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$, since $f_*\mathcal{O}_X(L_{1,S})$ and $f_*\mathcal{O}_X(L_{2,S})$ are locally free by Step 1, there exists an open cover $S = \cup_i S_i$ over which their restrictions are free. Choosing trivializations induce embeddings $g_i : (X_{S_i}, B_{S_i}) \hookrightarrow \mathbb{P}^n \times \mathbb{P}^l \times S_i$. Moreover, we have $A_{S_i} \sim_{S_i} N_{S_i} := g_i^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^l\times S_i}(1,-1)$. Hence by our construction of H_9, we have morphisms $\Phi_i : S_i \to H_9$. Over the intersections $S_i \cap S_j$, the trivializations differ by a section $s_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1})$. Therefore the Φ_i glue to a morphism $\Phi : S \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, which induces a morphism $\mathfrak{TS}_{h,n,l,\mathbf{c}} \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, that is the inverse of Ψ. □

We need the following separatedness result to obtain the coarse moduli spaces of traditional stable minimal models.

Theorem 4.7. *Let $f : (X, B), A \to C$ and $f' : (X', B'), A' \to C$ be two families of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over a smooth curve C. Let $0 \in C$ be a closed point and $C^o := C \setminus \{0\}$ the punctured curve. Assume there exists an isomorphism*

$$g^o : ((X, B), A) \times_C C^o \to ((X', B'), A') \times_C C^o$$

over C^o, then g^o can be extended to an isomorphism $g : (X, B), A \to (X', B'), A'$ over C.

Proof. Consider $L := A' + \tau(K_{X/C} + B)$ and $L' := A' + \tau(K_{X'/C} + B')$, where τ is the positive natural number as Lemma 4.1. By the proof of Lemma 4.3, L is a f-ample Cartier divisor on X (resp. L' is a f'-ample Cartier divisor on X'). Let $g : X \dashrightarrow X'$ be the birational map induced by g^o, then by the same argument as in [HH23, Proof of Proposition 4.4], g is a C-isomorphism. □

Corollary 4.8. *Let $(X, B), A$ be a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model. Then* $\mathrm{Aut}((X, B), A)$ *is finite.*

Proof. It follows from Theorem 4.7 and the argument of [BX19, Proof of Corollary 3.5]. □

***Proof of Theorem** 1.7.* By Theorem 4.6 and Lemma 4.4, $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is an algebraic stack of finite type. By Corollary 4.8 and [Alp24, Theorem 3.6.4], $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a Deligne-Mumford stack. Moreover, Theorem 4.7 and [Alp24, Theorem 3.8.2(3)] imply that $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ is a separated Deligne-Mumford stack of finite type. Therefore, we may apply the Keel–Mori Theorem [KM97][Alp24, Theorem 4.3.12] to see that $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ has a coarse moduli space $TS_{klt}(d, \Phi, \Gamma, \sigma)$, which is a separated algebraic space. □

References

[ABB+23] Kenneth Ascher, Dori Bejleri, Harold Blum, Kristin DeVleming, Giovanni Inchiostro, Yuchen Liu, and Xiaowei Wang. Moduli of boundary polarized Calabi-Yau pairs, 2023. arXiv:2307.06522. 23

[Alp24] Jarod Alper. Stacks and Moduli. *Book available online, Feb 1/2014*, 2024. 19, 21, 23, 24

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. 2

[BDCS20] Caucher Birkar, Gabriele Di Cerbo, and Roberto Svaldi. Boundedness of elliptic Calabi-Yau varieties with a rational section, 2020. arXiv:2010.09769. 2

[Bir12] Caucher Birkar. Existence of log canonical flips and a special LMMP. *Publ. Math. Inst. Hautes Études Sci.*, 115:325–368, 2012. 2

[Bir21a] Caucher Birkar. Boundedness and volume of generalised pairs, 2021. arXiv:2103.14935. 4, 9, 10

[Bir21b] Caucher Birkar. Generalised pairs in birational geometry. *EMS Surv. Math. Sci.*, 8(1-2):5–24, 2021. 9

[Bir21c] Caucher Birkar. Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)*, 193(2):347–405, 2021. 12, 15

[Bir22] Caucher Birkar. Moduli of algebraic varieties, 2022. arXiv:2211.11237. 2, 3, 4, 5, 10, 11, 12, 15, 19, 21, 23

[Bir23a] Caucher Birkar. Boundedness and moduli of algebraic varieties. *Personal communication*, 2023. 2, 6

[Bir23b] Caucher Birkar. Geometry of polarised varieties. *Publ. Math. Inst. Hautes Études Sci.*, 137:47–105, 2023. 2, 5, 9, 10, 11

[Bir23c] Caucher Birkar. Singularities on Fano fibrations and beyond, 2023. arXiv:2305.18770. 2

[BX19] Harold Blum and Chenyang Xu. Uniqueness of K-polystable degenerations of Fano varieties. *Ann. of Math. (2)*, 190(2):609–656, 2019. 24

[BZ16] Caucher Birkar and De-Qi Zhang. Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. *Publ. Math. Inst. Hautes Études Sci.*, 123:283–331, 2016. 9

[ELM+06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Asymptotic invariants of base loci. *Ann. Inst. Fourier (Grenoble)*, 56(6):1701–1734, 2006. 8

[Eno20] Makoto Enokizono. An integral version of Zariski decompositions on normal surfaces, 2020. arXiv:2007.06519. 18

[FHS21] Stefano Filipazzi, Christopher D. Hacon, and Roberto Svaldi. Boundedness of elliptic Calabi-Yau threefolds, 2021. arXiv:2112.01352. 2, 4

[FI21] Stefano Filipazzi and Giovanni Inchiostro. Moduli of $\mathbb{Q}$-Gorenstein pairs and applications, 2021. arXiv:2108.07988. 2

[Fil20] Stefano Filipazzi. On the boundedness of n-folds with $\kappa(X) = n-1$, 2020. arXiv:2005.05508. 4

[FS20] Stefano Filipazzi and Roberto Svaldi. Invariance of plurigenera and boundedness for generalized pairs. *Mat. Contemp.*, 47:114–150, 2020. 4

[Fuj11] Osamu Fujino. Fundamental theorems for the log minimal model program. *Publ. Res. Inst. Math. Sci.*, 47(3):727–789, 2011. 15

[Fuj17] Osamu Fujino. Effective basepoint-free theorem for semi-log canonical surfaces. *Publ. Res. Inst. Math. Sci.*, 53(3):349–370, 2017. 12

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966. 22

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977. 19, 20

[HH23] Kenta Hashizume and Masafumi Hattori. On boundedness and moduli spaces of K-stable Calabi-Yau fibrations over curves, 2023. arXiv:2305.01244. 2, 4, 21, 24

[HLQ23] Jingjun Han, Yuchen Liu, and Lu Qi. ACC for local volumes and boundedness of singularities. *J. Algebraic Geom.*, 32(3):519–583, 2023. 5, 12

[HMX13] Christopher D. Hacon, James McKernan, and Chenyang Xu. On the birational automorphisms of varieties of general type. *Ann. of Math. (2)*, 177(3):1077–1111, 2013. 2

[HMX14] Christopher D. Hacon, James McKernan, and Chenyang Xu. ACC for log canonical thresholds. *Ann. of Math. (2)*, 180(2):523–571, 2014. 2, 10

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu. Boundedness of moduli of varieties of general type. *J. Eur. Math. Soc. (JEMS)*, 20(4):865–901, 2018. 2, 5

[HX13] Christopher D. Hacon and Chenyang Xu. Existence of log canonical closures. *Invent. Math.*, 192(1):161–195, 2013. 2

[Jia22] Junpeng Jiao. Boundedness of polarized Calabi-Yau fibrations, 2022. arXiv:2202.07238. 4

[KM97] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997. 24

[KM98] János Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. 15

[Kol93] János Kollár. Effective base point freeness. *Math. Ann.*, 296(4):595–605, 1993. 12

[Kol23] János Kollár. *Families of Varieties of General Type*, volume 231 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2023. 2, 8, 19, 20, 22

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry. I*, volume 48 of *Ergebnisse Der Mathematik Und Ihrer Grenzgebiete.* Springer-Verlag, Berlin, 2004. 17, 18, 19

[LM09] Robert Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009. 8

[PS09] Yu. G. Prokhorov and V. V. Shokurov. Towards the second main theorem on complements. *J. Algebraic Geom.*, 18(1):151–199, 2009. 4

[Rud76] Walter Rudin. *Principles of Mathematical Analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. 17

[She23] Junchao Shentu. Arakelov Type Inequalities and Deformation Boundedness of polarized varieties, 2023. arXiv:2302.10200. 6

[Taj20] Behrouz Taji. Birational geometry of smooth families of varieties admitting good minimal models, 2020. arXiv:2005.01025. 2

[Vie95] Eckart Viehweg. *Quasi-Projective Moduli for Polarized Manifolds*, volume 30 of *Ergebnisse Der Mathematik Und Ihrer Grenzgebiete.* Springer-Verlag, Berlin, 1995. 2, 19, 22

[Zhu23] Minzhe Zhu. Boundedness of stable minimal models with klt singularities, 2023. arXiv:2311.12665. 4

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