- Abban-Zhuang's adjunction theory of stability threshold [1].
- Moduli methods, e.g, [16] Liu-Xu's work on K-stability of cubic threefolds. There are many problems in K-moduli of dimension 3.

Goal:

Try to understand K-moduli space of Fano 3-folds. Try to check K-stability of some interesting Fano varieties.

1. Talk 1: Introduction to K-stability: Fujita-Li's valuative criterion [14][10] and Fujita-Odaka's approximation [5].
2. Talk 2 and 3: Sketch the construction of K-moduli spaces: state the K-moduli functor $\mathfrak{M}^K$. Using Birkar's solution to BAB conjecture to show boundedness to show $\mathfrak{M}^K$ is Artin stack and admits good moduli space M^K. Survey Blum-Xu's work on separatedness [7], Xu-Zhuang's work on projectivity [20] and Liu-Xu-Zhuang's work on properness [17] of $\mathfrak{M}^K$. Wall-crossing behavior of K-moduli space of Fano pairs [3] [4].

   The same strategy [6] also works for the construction of moduli spaces of weighted K-stability (correspondding to the moduli space of Kahler- Ricci soliton).
3. Talk 4: Properties of K-semistable degenerations. Odaka's result on the singularities of K-semistable degenerations [18]. Fujita and Li-Liu's local to global volume inequality for K-semistable Fano pairs. Liu-Xu's [16] classification of normalised volume of 3-dimensional mild singularities. Liu's work [15] on K-moduli of cubic fourfold.

   properties of K-moduli: normal or not [12].
4. Talk 5: Equivariant K-stability and its application. State the equivariant K-stability criterion and sketch Zhuang's algebraic proof [21]. Application to an effective criterion of checking K-stability of Fano varieties of complexity 1 [11] .
5. Talk 6: Abban-Zhuang's adjunction theory of local stability threshold $\delta_Z(X, \Delta; V_\bullet)$ [1]
6. Talk 7: Degeneration methods and stratification of M^K.

Time and place

Thursday afternoon for each week at Quanzhai.

Some problems can be discussed

1. On K-moduli spaces

Xu's question [19]

Question 1. *Let $\overline{P}_g^K(c)$ be moduli space of K-Fano pairs (X, cS) where a general X is a smooth primitive Fano 3-fold and $S \in |-K_X|$ is a K3 surface. What is the relation of K-stability theoretic compactification $\overline{P}_g^K(c)$ with various compactification of moduli space $\mathcal{F}_g$ of K3 surface of genus g ?*

Remark 0.1. *ADL [4] has a nice answer for $X = \mathbb{P}^3$ and $S \subset \mathbb{P}^3$ a quartic K3.*

Calabi has proposed the problem of classify smooth Fano 3-fold admitting KE metric. From algebro-geometric perspective, a more ambitious problem is (I call it Algebraic Calabi problem)

Problem 1. *Understand the K-moduli space $M^K_{V,\rho}$ of Fano 3-folds X of fixed volume $V = (-K_X)^3$ and Picard number $\rho(X) = \rho$. More precisely, need to figue out which smooth Fano 3-fold is K-stable and what is the possible singular K-semistable degeneration ? (one may view it as bounaries of K-moduli space $M^K_{V,\rho}$)*

Remark 0.2. *Liu-Xu [16] solved the problem for cubic 3-fold case. We can try to discuss K-moduli of Fano 3-fold obtained via double cover or blowup of curves in a primitive Fano 3-fold. Recently, Cheltsov etc [2] made progress for smooth case.*

Question 2. *If a K-Fano variety has interesting Hodge structures, how to compare K-moduli space M^K of K-Fano varieties to the moduli space of Hodge structure ? e.g, Gushel-Mukai n-folds.*

The following is a cycle theoretic perspective on K-moduli space M^K

Question 3. *Can we construct some interesting cycles on K-moduli space M^K ?*

Remark 0.3. *A very natural cycle known on M^K so far is the CM line bundle λ_{CM}, which is ample line bundle. Maybe first we should try to understand the Picard group $\mathrm{Pic}(M^K)$ and the various cones of divisors on M^K, and the Kodaira dimension of M^K. It seems worth to look at the paper [8].*

Remark 0.4. *One may define a tautological ring $R^*(M^K) \subset A^*(M^K)$ generated by some interesting class, e.g, the kappa class*

$$\kappa_a := \pi_*(c_1(T_\pi)^{a_1} \cdots \cdot c_n(T_\pi)^{a_n}) \in A^{a_1+2a_2+\cdots+na_n-n}(M^K),\ a = (a_1, \cdots, a_n) \in \mathbb{N}^n$$

where $\pi : \mathfrak{X} \to M^K$ is universal family (at least in the stack sense). For the definitaion for Chow ring of Artin stack M^K, one may refer to [13].

Question 4 (Comparison with other moduli problem)**.** *Let $\sigma \in Stab(\mathbb{P}^3)$ be a stability condition and $M_v(\sigma)$ the moduli space of stable object with numberical invariant v. Assume the ideal sheaf $\mathcal{I}_Z$ of closed subscheme $Z \subset \mathbb{P}^3$ of $(2,n)$-complete intersection is genric object in $M_v(\sigma)$. Let $\overline{P}_n^K(c)$ be moduli space of c-K stable pairs with generic pairs is $(2,n)$-complete intersection in $\mathbb{P}^3$, how to relate the two moduli space $M_v(\sigma)$ and $\overline{P}_n^K(c)$? The wall-crossing behaviors of both moduli space is related ?*

Some arithmetic aspect: Try to show the Andreasson-Berman's conjectural height inequality for dimension 2 and 3. Odaka defined a height for Fano varieties. Maybe it is interesting to investigate. Is the K-moduli theory helpful to Shafarivich type problem for some Fano varieties ?

2. On detecting K-stability of a given variety

Donaldson's two question.

Question 5. *Is each smooth Mukai-Umemura Fano 3-fold K-stable [9] ? Is the moduli space $N_{L,r}$ of stable vector bundle of rank r with fixed determinant L on a curve K-stable ?*

Question 6. *Is there a similar effective criterion to checking K-stability of Fano varieties of complexity 2 ?*

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