Boundedness of traditional stable minimal models

Xiaowei Jiang

Yau Mathematical Sciences Center, Tsinghua University

May 17, 2024

Contents

1. Background
    - KSBA stable pairs
    - Stable Calabi-Yau pairs
    - Stable minimal models
2. Traditional stable minimal models

MMP and Abundance

We work over an algebraically closed field of characteristic 0.

The Minimal Model Program Conjecture and Abundance Conjecture predicts that, any projective variety Y with mild singularities is birational to either

- a **Mori fiber space** $X \to Z$ (i.e. $\rho(X/Z) = 1$, $-K_X$ is ample over Z and $\dim X > \dim Z$), or
- a **good minimal model** X (i.e. K_X is semiample, which defines a contraction $X \to Z$).

In this talk, we will focus on good minimal models.

Boundedness

We aim to construct moduli spaces for good mimimal models. For this purpose, the first problem is whether there are only finitely many families for these objects after fixing certain numerical invariants.

Definition (Boundedness for pairs)

Let $d \in \mathbb{N}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let $\mathcal{P}$ be a set of pairs (X, B) such that

- $\dim X = d$, and
- the coefficients of B are in Φ.

Then $\mathcal{P}$ is said to be a **bounded family** if there is a fixed $r \in \mathbb{N}$ such that for any (X, B) in $\mathcal{P}$ we can find a very ample divisor H on X satisfying

$$H^d \leq r \text{ and } (K_X + B) \cdot H^{d-1} \leq r.$$

Canonical polarized varieties: K_X ample

Canonical polarized varieties are ample models of good minimal models with maximal Kodaira dimension.

Definition

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. A (d, c, v)-**KSBA stable pair** is a connected projective pure dimensional pair (X, B) such that

- (X, B) is slc of dimension d,
- $\frac{1}{c}B$ is an integral divisor,
- $K_X + B$ is ample with volume $\mathrm{vol}(K_X + B) := (K_X + B)^d = v$.

Canonical polarized varieties: K_X ample

Theorem (Hacon-M^{c}Kernan-Xu 2018)

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. Then (d, c, v)-KSBA stable pairs form a bounded family.

The higher dimensional analogue of Deligne-Mumford theorem for curves has been established by the contributions of many people over the past three decades.

Theorem (Kollár, et al)

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, v)-KSBA stable pairs.

Calabi-Yau varieties: $K_X \equiv 0$

For Calabi-Yau varieties (i.e. good minimal models with Kodaira dimension zero), there is no natural choice of polarization. In general, they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces and abelian varieties of any fixed dimension are not bounded.

It is widely open whether strict Calabi-Yau manifolds of dimension $d \geq 3$ are bounded or not, where a smooth proper variety X is strict Calabi-Yau if it is simply connected, $K_X \sim 0$, and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Calabi-Yau varieties: $K_X \equiv 0$

When studying moduli of Calabi-Yau varieties, a polarization (that is, ample divisor) is typically fixed despite the non-uniqueness of the choice.

Definition (Kollár-Xu, Birkar)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. A (d, c, u)-**stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is projective slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $\frac{1}{c}B$ is an integral divisor,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = u$,
- $(X, B + tA)$ is slc for some $t \in \mathbb{Q}^{>0}$.

The key point is that t is not fixed.

Calabi-Yau varieties: $K_X \equiv 0$

Theorem (Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. Then (d, c, u)-stable Calabi-Yau pairs form a bounded family.

The boundedness implies $(X, B + \tau A)$ is a KSBA stable pair, for some fixed $\tau \in \mathbb{Q}^{>0}$ depending only on d, c, u.

Theorem (in the first arxiv version of Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, u)-stable Calabi-Yau pairs.

Calabi-Yau varieties: $K_X \equiv 0$

Theorem (Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. Consider pairs $(X, B), A$ satisfying the following:

- *(X, B) is a klt Calabi-Yau pair of dimension d,*
- *$\frac{1}{c}B$ is an integral divisor,*
- *A is nef and big and integral with $\mathrm{vol}(A) \leq v$.*

Then the set of such (X, B) forms a bounded family.

Here, A is not necessarily an effective divisor. This theorem is crucially used in Odaka's work to get partial compactification of moduli spaces of Calabi-Yau varieties polarised by ample line bundles.

Good minimal models: arbitrary Kodaira dimension

For studing moduli of good minimal models, we need to choose a relative polarization.

Definition (Birkar)

A **stable minimal model** $(X, B), A$ consists of a projective connected pair (X, B) and an integral divisor $A \geq 0$ such that

- (X, B) is slc,
- $K_X + B$ is semi-ample defining a contraction $f\colon X \to Z$,
- $K_X + B + tA$ is ample for some $t > 0$, and
- $(X, B + tA)$ is slc for some $t > 0$.

Note that we do not assume that A is ample globally, the third condition is equivalent to A is ample over Z. Moreover, t is a priori not fixed.

Good minimal models: arbitrary Kodaira dimension

To get a good moduli theory we need to fix more invariants compared to the KSBA and stable Calabi-Yau cases.

Definition (Birkar)

Let $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, c, u, σ)-stable minimal model is a stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- $\frac{1}{c}B$ is an integral divisor,
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f: X \to Z$ over any component of Z, and
- $(K_X + B + tA)^d = \sigma(t)$.

When t is sufficiently small, $(K_X + B + tA)$ is ample, in this case $\sigma(t) = \mathrm{vol}(K_X + B + tA)$.

Good minimal models: arbitrary Kodaira dimension

Theorem (Birkar 2022)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. Then (d, c, u, σ)-stable minimal models form a bounded family.

The hardest part of the proof is to show that we can find a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on (d, c, u, σ) such that $(X, B + \tau A)$ is slc and $K_X + B + \tau A$ is ample, then we can apply [HMX18].

Theorem (Birkar 2022)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. There is a projective coarse moduli space for (d, c, u, σ)-stable minimal models.

Traditional stable minimal models

The reason for taking effective divisors as polarization is that one can obtain proper moduli spaces. However, this often leads to larger moduli spaces. Traditionally, for example, Viehweg considers ample line bundles as polarization for treating the moduli of smooth good minimal models.

Definition (Birkar)

A **traditional stable minimal model** $(X, B), A$ consists of a projective connected pair (X, B) and an integral divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \to Z$,
- $K_X + B + tA$ is ample for some $t > 0$.

Again, A is not ample globally and t is a priori not fixed.

Traditional stable minimal models

Definition

Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, Φ, u, σ)-traditional stable minimal model is a traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ,
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$, and
- $(K_X + B + tA)^d = \sigma(t)$.

Let $\theta(t) = \mathrm{vol}(K_X + B + tA)$ be a function on t. When t is sufficiently small, $(K_X + B + tA)$ is ample, in this case $\sigma(t) = \theta(t)$.

Main results

Theorem (J. 2023)

Fix $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$. Then (d, Φ, u, σ)-traditional stable minimal models form a bounded family.

When $\dim Z = 0$ or $A \geq 0$, it has already been proven by Birkar.

As an application, we can construct moduli spaces for klt good minimal models polarized by line bundles, where $\Phi = \{c\}$ in the following theorem.

Corollary (J. 2023)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. There is a separated coarse moduli space for (d, c, u, σ)-traditional stable minimal models.

Strategy for the proof of boundedness

Proposition

Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then for any (d, Φ, u, σ)-traditional stable minimal model $(X, B), A \to Z$, there are positive rational numbers λ, τ depending only on d, Φ, u, σ such that

- (A) *$K_X + B + tA$ is big for all $0 < t < \lambda$, and*
- (B) *$K_X + B + tA$ is ample for all $0 < t < \tau$.*

If A is an effective divisor, we can take $\lambda = 1$. Clearly, Prop B $\implies$ Prop A. We outline the plan of the proof.

1. Prop B $\implies$ Main Thm
2. Prop A_d $\implies$ Prop B_d: induction on $\dim X$
3. Prop B_{d-1} $\implies$ Prop A_d: induction on $\dim Z$

Boundedness for non-effective divisors

To ensure the validity of the induction argument, we need to extend the definition of boundedness to non-effective divisors. Our definition is inspired by [Birkar 2021, Theorem 1.8].

Definition

Let $d \in \mathbb{N}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let $\mathcal{P}$ be a set of pairs (X, B) and integral divisor (not necessarily effective) A such that

- $\dim X = d$, and
- the coefficients of B are in Φ.

Then $\mathcal{P}$ is said to be a **bounded family** if there is a fixed $r \in \mathbb{N}$ such that for any $(X, B), A$ in $\mathcal{P}$ we can find a very ample divisor H on X satisfying

$$H^d \leq r, (K_X + B) \cdot H^{d-1} \leq r \text{ and } H - A \text{ is pseudo-effective}.$$

Prop B $\implies$ Main Thm

Step 1. By [Bir21, Lem8.2], we may assume that (X, B) is ϵ-lc and $N := l(K_X + B + \frac{\tau}{2}A)$ is an ample integral divisor. Then $\mathrm{vol}(K_X + B + N) = (l+1)^d \sigma(\frac{l\tau}{2(l+1)})$ is fixed. Thus by [Bir23, Thm 1.5], the set of (X, B) is bounded.

Theorem (Birkar 2023, Theorem 1.5)

Let $d \in \mathbb{N}$, $\delta, v \in \mathbb{Q}^{>0}$. Consider pairs (X, B) and nef and big integral divisors N on X such that

- *(X, B) is projective ϵ-lc of dimension d,*
- *the coefficients of B are in $\{0\} \cup [\delta, \infty)$,*
- *$K_X + B$ is nef,*
- *$\mathrm{vol}(K_X + B + N) \leq v$.*

Then the set of such (X, B) forms a bounded family.

Prop B $\implies$ Main Thm

Step 2. Let $D = 2l(K_X + B) + \frac{l_T}{2}A$ be an ample integral divisor, then by [HLQ23, Thm 1.10], $I_0 D$ is an ample Cartier divisor for some I_0 depends only on d, Φ, u, σ.

By effective base point free theorem and very ampleness lemma, we can find a very ample divisor $H \in |ID|$ on X such that $H^d \leq r$, $(K_X + B) \cdot H^{d-1} \leq r$, and $H - A$ is pseudo-effective, where I, r depend only on d, Φ, u, σ. Hence, $(X, B), A$ belongs to a bounded family.

Theorem (Han-Liu-Qi 2023, Theorem 1.10)

Let $\epsilon \in \mathbb{Q}^{>0}$. Suppose $\mathcal{C} := \{(X, B)\}$ is a set of ϵ-lc pairs that belongs to a bounded family $\mathcal{P}$. Then there exists $I_0 \in \mathbb{Z}^{>0}$ which only depends on $\epsilon, \mathcal{P}$ such that for any (X, B) in $\mathcal{C}$, and D a $\mathbb{Q}$-Cartier Weil divisor on X, $I_0 D$ is Cartier.

Prop $A_d \implies$ Prop B_d: induction on $\dim X$

When $\dim X = 1$ or $\dim Z = 0$, $K_X + B + \frac{\lambda}{2}A$ is ample automatically. From now on we assume that $\dim X \geq 2$ and $\dim Z \geq 1$.

Step 1. Replacing A, u, σ with $\alpha(K_X + B + \frac{\lambda}{2}A), (\frac{\alpha\lambda}{2})^{\dim F}u, \gamma$, we may assume that A is a big Weil divisor, where $\alpha \in \mathbb{N}$ and γ is a polynomial determined by $d, \Phi, u, \sigma, \lambda$.

Step 2. By [Bir21], we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

for $l \in \mathbb{N}$ depending only on d, Φ, u, σ such that the set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. Moreover, $L := l(K_Z + B_Z + M_Z)$ is very ample.

Let T be a general member of $|L|$ and let S be its pullback to X. Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$. Then one can check $(S, B_S), A_S \to T$ is a traditional stable minimal model of dimension $d - 1$.

Step 3. By induction, $(S, B_S), A_S$ belongs to a bounded family. Hence there is a fixed $r \in \mathbb{N}$ such that for any $(S, B_S), A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1} \leq r, (K_S + B_S) \cdot H_S^{d-2} \leq r, \text{ and } H_S - A_S \text{ is pseudo-effective.}$$

By [Bir22, Lem 4.6], we may assume that $H_S - B_S$ is pseudo-effective. By [Bir21, Lem 8.2], we may assume that (S, B_S) is ϵ-lc.

Since A is big, there is an effective $\mathbb{Q}$-divisor E such that $A \sim_{\mathbb{Q}} E$, we lose control of coefficients of E. Let $E_S := E|_S$ be an effective $\mathbb{Q}$-divisor, then $H_S - E_S \sim_{\mathbb{Q}} H_S - A_S$ is also pseudo-effective.

Thus by [Bir21, Thm 1.8], there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $d-1, \epsilon, r$ such that

$$\operatorname{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt.

Prop $A_d \implies$ Prop B_d: inversion of adjunction

Theorem (Birkar 2021, Theorem 1.8)

Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number t depending only on d, r, ϵ, satisfying the following. Assume

- *(X, B) is projective ϵ-lc of dimension d,*
- *A is a very ample divisor on X with $A^d \leq r$,*
- *$A - B$ is pseudo-effective, and*
- *$M \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor with $A - M$ pseudo-effective.*

Then

$$\operatorname{lct}(X, B, |M|_{\mathbb{R}}) \geq \operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Then by inversion of adjunction, $(X, B + S + \tau E)$ is plt near S. Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z.

Prop $A_d \implies$ Prop B_d: length of extremal ray

Step 4. If $K_X + B + \tau E$ is nef, replacing τ with $\frac{\tau}{2}$ we can assume that $K_X + B + \tau A$ is ample.

If $K_X + B + \tau E$ is not nef, Let R be a $(K_X + B + \tau E)$-negative extremal ray, since $K_X + B + \tau E$ is ample over Z, R is not contained in fibers of $X \to Z$. By Step 3, the non-lc locus of $(X, B + \tau E)$ maps to finitely many points of Z, so R is not contained in the image $\mathrm{Im}(\overline{\mathrm{NE}}(\Pi) \to \overline{\mathrm{NE}}(X))$, where Π is the non-lc locus of $(X, B + \tau E)$.

Then by the general Cone Theorem [Amb03][Fuj11], R is generated by a curve C with $(K_X + B + \tau E) \cdot C \geq -2d$. Since $L = l(K_Z + B_Z + M_Z)$ is very ample, $l(K_X + B) \cdot C = f^*L \cdot C = L \cdot f_*C \geq 1$, we see that

$$((K_X + B) + 2dl(K_X + B) + \tau E) \cdot C \geq 0.$$

It follows that $K_X + B + \frac{\tau}{1+2dl}E$ is nef. After replacing τ with $\frac{\tau}{1+2dl}$, we finish the proof.

Prop $B_{d-1} \implies$ Prop A_d: variations of the volume function $\vartheta(t)$ and the top self-intersection function $\varsigma(t)$

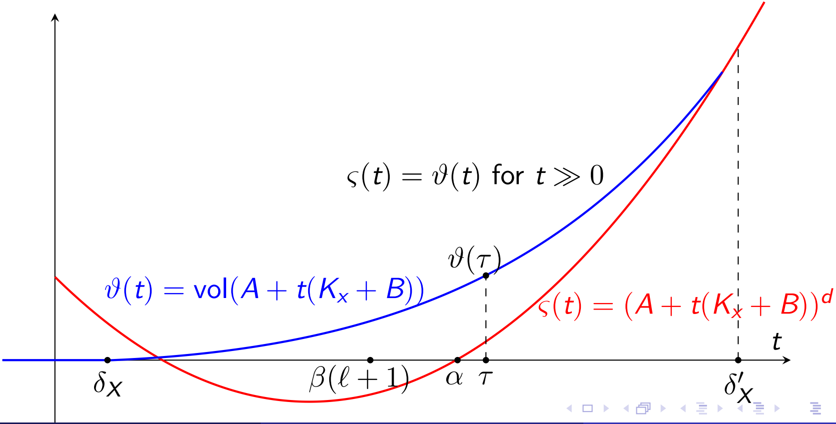
It is clear if $\dim Z = 0$. We may assume that $\dim Z = m > 0$.
Step 0. Let $\varsigma \in \mathbb{Q}[t]$ be a polynomial function such that

$$\varsigma(t) = (A + t(K_X + B))^d = \sum_{i=0}^{d} \binom{d}{i} A^{d-i} \cdot (K_X + B)^i t^i,$$

so ς is determined by σ. It is enough to show that there exists a positive rational number τ depending only on d, Φ, u, σ such that $A + t(K_X + B)$ is big for all $t > \tau$. Let $\vartheta(t) = \mathrm{vol}(A + t(K_X + B))$ be a function on t, then $\vartheta(t)$ is a non-negative non-decreasing real function on t, which agrees with $\varsigma(t)$ for $t \gg 0$. It suffices to show that $\vartheta(t) > 0$ for all $t > \tau$.

Prop $B_{d-1} \implies$ Prop A_d: idea of proof

First, we find a positive rational number τ depending only on $\varsigma(t)$ such that $\varsigma(t) > 0$ is strictly increasing for $t \geq \tau$. Next, we aim to show that $\vartheta(t) > 0$ for all $t > \tau$. Since $\varsigma(t) = \vartheta(t)$ for $t \gg 0$, our goal is to establish that $\vartheta(t)$ decreases at a slower rate than $\varsigma(t)$ as t decreases, achieved by comparing their derivatives.



Prop $\mathrm{B}_{d-1} \implies$ Prop A_d: induction on $\dim Z$

Step 1. we pick a general element T in the very ample linear system $|l(K_Z + B_Z + M_Z)|$ and let S be its pullback to X, then $S \sim_{\mathbb{Q}} l(K_X + B)$. Define $K_S + B_S = (K_X + B + S)|_S$ and $A_S = A|_S$, then $(S, B_S), A_S \to T$ is a traditional stable minimal model with $\dim T = m - 1$. By Prop B_{d-1}, there is a positive rational number β depending only on d, Φ, u, σ such that $A_S + t(K_S + B_S)$ is ample for all $t > \beta$. In particular, if $t > \beta(l + 1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$ is ample.

Step 2. We have $\varsigma(t) = (A + t(K_X + B))^d$. Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, if $t > \beta(l+1)$,

$$\begin{aligned}
\varsigma'(t) =& d(A + t(K_X + B))^{d-1} \cdot (K_X + B) \\
=& \frac{d}{l}(A + t(K_X + B))^{d-1} \cdot S \\
=& \frac{d}{l}(A_S + \frac{t}{l+1}(K_S + B_S))^{d-1} \\
>& 0.
\end{aligned}$$

If $\varsigma(t)$ has no roots (which occurs only when $\dim Z$ is even), set $\tau = \beta(l+1) + 1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$, set $\tau = \max\{\beta(l+1), \lceil\alpha\rceil\} + 1$. Note that τ is a positive rational number. Moreover, on $[\tau, +\infty)$, $\varsigma(t)$ is a positive, increasing real function, and $\vartheta(t)$ is a non-negative, non-decreasing real function.

Prop $\mathrm{B}_{d-1} \implies$ Prop A_d: derivative of volume $\vartheta(t)$

Step 3. We see that $\vartheta(t) = \mathrm{vol}(A + t(K_X + B)) = \frac{1}{l^d}\mathrm{vol}(lA + tS)$. For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose general S_t such that $S_t \not\subseteq \mathbf{B}_+(lA + tS_t)$. Then by [LM09, Cor 4.27], we have

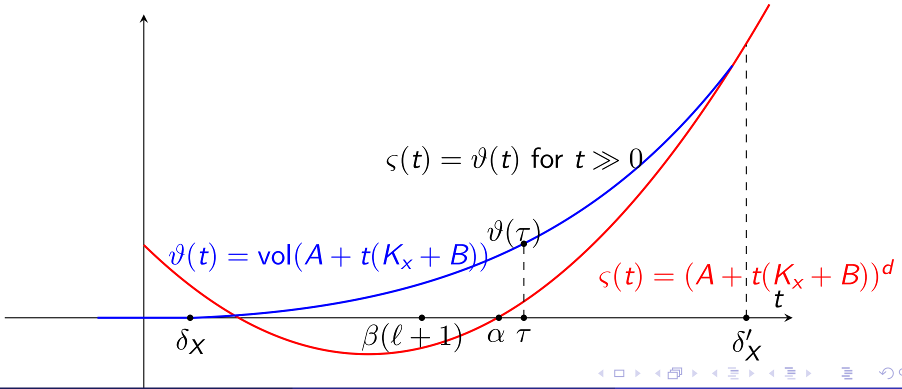
$$l^d\vartheta'(t) = l^d\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{vol}(lA+tS_t+sS_t))\Big|_{s=0} = d\,\mathrm{vol}_{X|S_t}(lA+tS_t).$$

Moreover, if $t \geq \tau$ and $\vartheta(t) > 0$, we have

$$\begin{aligned}\vartheta'(t) =& \frac{d}{l^d}\mathrm{vol}_{X|S_t}(lA + tS_t) \leq \frac{d}{l}\mathrm{vol}((A + \frac{t}{l}S_t)|_{S_t}) \\ =& \frac{d}{l}\mathrm{vol}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})) \\ =& \frac{d}{l}(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t}))^{d-1} = \varsigma'(t).\end{aligned}$$

Prop $\mathrm{B}_{d-1} \implies$ Prop A_d

Step 4. We claim that $\vartheta(\tau) > 0$, and finish the proof. Indeed, let δ_X be the largest real number such that $\vartheta(\delta_X) = 0$, where δ_X may depend on X. If $\vartheta(\tau) = 0$, then $\tau \leq \delta_X$. There exists a real number $\delta'_X \gg 0$ such that $\vartheta(\delta'_X) = \varsigma(\delta'_X)$. Since both $\varsigma(t)$ and $\vartheta(t)$ are differentiable in (δ_X, δ'_X), there is a real number $\gamma_X \in (\delta_X, \delta'_X)$ at which $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$, but by Step 3, $\vartheta'(t) \leq \varsigma'(t)$ on $(\delta_X, +\infty)$, a contradiction.



Thank you!