优良极小模型的有界性

Boundedness of good minimal models

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摘 要

在本论文中，我们研究具备 Kawamata 对数终端（klt）奇点的优良极小模型，这些模型由相对于 Iitaka 纤维化的基底而言相对 nef 且 big 的 Weil 除子进行极化。我们证明，在固定适当的数值不变量之后，这类模型构成一个有界族。作为应用，我们构造了由线丛极化的 klt 优良极小模型的分离粗模空间。

我们还研究了在基底与一般纤维均有界的情形下，对数 Calabi–Yau 纤维化的有界性问题。我们证明，在固定若干自然不变量之后，此类对数 Calabi–Yau 纤维化的全空间在余维一意义下构成有界族，从而验证了 Birkar 和 Hacon 提出的一个猜想。随后，我们将该结果应用于 klt 优良极小模型的有界性问题。

由于广义对的结构会自然地出现在优良极小模型的 Iitaka 纤维化的基底上，我们同时研究了关于广义对的有界性问题。对于那些并不一定来源于 Iitaka 纤维化的抽象广义对，我们构造了反例：在固定合适不变量的情况下，其广义对数典范模型在曲面情形依然可能无界，而其底层簇的 Kodaira 维数可以任意。该结果回答了 Birkar 和 Hacon 提出的一个问题。

关键词：有界性；模空间；优良极小模型；对数 Calabi–Yau 纤维化；广义对

ABSTRACT

In this thesis, we consider good minimal models with klt singularities, polarized by Weil divisors that are relatively nef and big over the bases of the Iitaka fibration. We show that, after fixing appropriate numerical invariants, they form a bounded family. As an application, we construct separated coarse moduli spaces for klt good minimal models polarized by line bundles.

We also investigate the boundedness problem for log Calabi–Yau fibrations whose bases and general fibers are bounded. We prove that the total spaces of such fibrations are bounded in codimension one after fixing certain natural invariants, which confirms a conjecture of Birkar and Hacon. We then apply these results to the boundedness problem for klt good minimal models.

Since the structure of generalised pairs naturally appears on the base of the Iitaka fibration of good minimal models, we also study the boundedness problem for generalised pairs. For the abstract structure of generalised pairs, which may not arise from the Iitaka fibration, we construct counterexamples to the boundedness of generalised log canonical models of surfaces with suitably fixed invariants, where the underlying varieties can have arbitrary Kodaira dimension. This answers a question of Birkar and Hacon.

Keywords: boundedness; moduli space; good minimal models; log Calabi–Yau fibrations; generalised pairs

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LIST OF SYMBOLS AND ACRONYMS

Z	Projective variety Z
K_Z	The canonical divisor of Z
B_Z	The discriminant divisor on Z
M_Z	The moduli divisor on Z
$(Z, B_Z + M_Z)$	Generalised pair
Φ	DCC set
$\mathrm{vol}(D)$	The volume of divisor D
$\mathrm{vol}_{X\|S}(D)$	The restricted volume of divisor D along S
$\mathrm{Ivol}(D)$	The Iitaka volume of divisor D
$\kappa(Z)$	The Kodaira dimension of the variety Z
$a(D, Z, B_Z + M_Z)$	The generalised log discrepancy of D with respect to (Z, B_Z+M_Z)
$\mathcal{F}_{glc}(d, \Phi, v)$	The set of (d, Φ, v)-generalised log canonical pairs
$\mathcal{F}_{gklt}(d, \Phi, v)$	The set of (d, Φ, v)-generalised klt pairs
$\mathcal{G}_{klt}(d, \Phi, \leq u, v)$	The set of all $(d, \Phi, \leq u, v)$-polarized good minimal models
$\mathcal{G}_{klt}(d, \Phi, \leq u, \sigma)$	The set of all $(d, \Phi, \leq u, \sigma)$-polarized good minimal models
$\mathfrak{TS}_{klt}(d, \Phi, u, \sigma)$	The moduli functor for the set of (d, Φ, u, σ)-traditional stable minimal models
$TS_{klt}(d, \Phi, u, \sigma)$	The coarse moduli space for the set of (d, Φ, u, σ)-traditional stable minimal models

CHAPTER 1 INTRODUCTION

Throughout this thesis, we work over an algebraically closed field k of characteristic zero unless otherwise stated.

1.1 Background

The central problem in birational geometry is the classification of algebraic varieties. According to the standard minimal model conjecture and the abundance conjecture, any variety Y with mild singularities is birational to a variety X such that either X admits a Mori–Fano fibration $X \to Z$ or X is a good minimal model, that is, K_X is semiample. Therefore, canonically polarized varieties, Fano varieties, Calabi–Yau varieties, and their iterated fibrations play a central role in birational geometry.

One of the main problems in the classification of algebraic varieties is whether there are only finitely many families of such objects after fixing certain numerical invariants; in other words, whether they form a bounded family. Establishing the boundedness of a given class of varieties is a natural first step toward constructing the corresponding moduli space.

For canonically polarized varieties, boundedness was established in [1-3], and then for minimal models of general type, it was proved in [4]. For boundedness of Fano varieties, that is, the celebrated BAB conjecture, the proof was given in [5-6]. However, for Calabi–Yau varieties there is no natural choice of polarization, and in general they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces and abelian varieties of any fixed dimension are not bounded. Nevertheless, there has been recent progress toward the (birational) boundedness of fibered Calabi–Yau varieties and rationally connected Calabi–Yau varieties; see [7-14].

When studying the moduli of Calabi–Yau varieties, one typically fixes a polarization despite its non-uniqueness. Recently, Birkar established the following boundedness results for polarized Calabi–Yau varieties, which are crucial for constructing moduli spaces of such polarized varieties.

Theorem 1.1.1 ([15][Corollary 1.6 & 1.8]). *Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Consider Calabi–Yau pairs (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X. Then the following hold:*

(1) (klt case) If

- (X, B) *is a klt pair of dimension* d*,*
- *the coefficients of* B *are contained in* Φ*,*
- A *is a nef and big divisor on* X *such that* $\mathrm{vol}(A) \leq u$*,*

then the set of such (X, B) *forms a bounded family. If in addition* $A \geq 0$*, then the set of such* $(X, B + A)$ *also forms a bounded family.*

(2) (slc case) If

- (X, B) *is an slc pair of dimension* d*,*
- *the coefficients of* B *are contained in* Φ*,*
- A *is an ample divisor on* X *such that* $\mathrm{vol}(A) = u$*,*
- $A \geq 0$ *does not contain any non-klt center of* (X, B)*,*

then the set of such $(X, B + A)$ *forms a bounded family.*

1.2 Failure of boundedness for generalised log canonical surfaces

Since the boundedness results for good minimal models of maximal and minimal Kodaira dimension have been established, it remains to investigate good minimal models with intermediate Kodaira dimension. For a good minimal model X, there exists a contraction $f : X \to Z$ to a normal variety Z. The canonical bundle formula (see §2.5)

$$K_X \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

shows that the structure of $(Z, B_Z + M_Z)$ plays a fundamental role in understanding the geometry of X. We can then regard $(Z, B_Z + M_Z)$ as a *generalised pair* with ample $K_Z + B_Z + M_Z$, that is, a *generalised log canonical (lc) model*. We refer the reader to §2.3 for background on generalised pairs and their singularities.

In this thesis, to study the boundedness of good minimal models, we first investigate the boundedness of their bases, namely, the boundedness of generalised lc models. For the definition of boundedness for generalised pairs, see §2.11.

Definition 1.2.1. Fix $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ a DCC set, and $v \in \mathbb{Q}^{>0}$. Let $\mathcal{F}_{glc}(d, \Phi, v)$ be the set of projective generalised pairs $(X, B + M)$ with data $X' \xrightarrow{\phi} X$ and M' where

- $(X, B + M)$ is generalised lc of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M_i'$ where M_i' are nef Cartier and $\mu_i \in \Phi$,

- $K_X + B + M$ is ample, and
- $\mathrm{vol}(K_X + B + M) = v$.

When $M = 0$, that is, when (X, B) is a usual pair, $\mathcal{F}_{glc}(d, \Phi, v)$ forms a bounded family by [1-3]. When $(X, B + M)$ is generalised klt, boundedness is known by [16] and has applications to studying the boundedness and moduli of klt good minimal models in this thesis.

Although $\mathcal{F}_{glc}(d, \Phi, v)$ is log birationally bounded by [16][Proposition 5.2], Birkar and Hacon construct an unexpected counterexample to its boundedness for $d \geq 3$, see [17][§5.3]. Nevertheless, the generalised pairs given by the canonical bundle formula are bounded under certain conditions [17][Theorem 1.3], which plays a key role in the study of boundedness and moduli of slc good minimal models [18].

It is clear that $\mathcal{F}_{glc}(1, \Phi, v)$ forms a bounded family, and $\mathcal{F}_{glc}(d, \Phi, v)$ is not bounded for $d \geq 3$ by [17][§5.3]. It is natural to ask whether $\mathcal{F}_{glc}(2, \Phi, v)$ is bounded or not [17][Question 5.1]. In [19], Filipazzi shows that $\mathcal{F}_{glc}(2, \Phi, v)$ is bounded under the assumption that the Cartier index of M (rather than that of M') is bounded. However, as shown in [17][§5.2], the Cartier index of M is not bounded in general, even when the ambient variety X is fixed.

The first main result of this thesis is the construction of examples showing that $\mathcal{F}_{\mathrm{glc}}(2, \Phi, v)$ is not bounded in general, where $X \in \mathcal{F}_{\mathrm{glc}}(2, \Phi, v)$ can have arbitrary Kodaira dimension. This is based on joint work with Professor Christopher Hacon [20].

Theorem 1.2.2. *Fix $\kappa \in \{-\infty, 0, 1, 2\}$. Let $\mathcal{P}_\kappa \subset \mathcal{F}_{glc}(2, \Phi, v)$ be the subset of generalised pairs such that $\kappa(X) = \kappa$ for every $(X, B + M) \in \mathcal{P}_\kappa$. Then $\mathcal{P}_\kappa$ is not bounded.*

In [17], Birkar and Hacon construct a set of generalised pairs $(X, B + M)$ in $\mathcal{F}_{glc}(3, \Phi, v)$ such that each X is an lc Fano variety and the Cartier index of K_X is unbounded. However, after a flip, X becomes klt and bounded, while the failure of boundedness in dimension two implies that we cannot expect $\mathcal{F}_{glc}(d, \Phi, v)$ to be bounded in codimension one. On the other hand, for generalised lc surfaces $(X, B + M)$ in $\mathcal{F}_{glc}(2, \Phi, v)$, it can be shown that the Cartier index of K_X is bounded. Furthermore, if X is an lc Fano surface, then X is bounded family. We also prove the boundedness of $\mathcal{F}_{glc}(2, \Phi, v)$ under additional assumptions.

Theorem 1.2.3. *Let $\mathcal{Q} \subset \mathcal{F}_{glc}(2, \Phi, v)$ be a subset of generalised pairs. Then $\mathcal{Q}$ is bounded if one of the following holds for every $(X, B + M) \in \mathcal{Q}$:*

1. *X has only rational singularities,*

2. $X \to Z$ *is a minimal ruled fibration onto a nonsingular curve* Z*,*

3. $-K_X$ *is ample,*

4. K_X *is ample.*

By *minimal ruled fibration*, we mean a fibration whose general fiber is a nonsingular rational curve, with no exceptional curves of the first kind contained in any fiber. While boundedness in the Fano case fails in dimension three by [17], boundedness in the canonically polarized case holds in any dimension, see Corollary 3.1.2.

1.3 Boundedness of klt good minimal models

Birkar proved the following boundedness result for slc good minimal models polarized by effective Weil divisors that are relatively ample over the bases of the Iitaka fibration, and he constructed their projective coarse moduli spaces.

Theorem 1.3.1 ([18][Theorem 1.12]). *Let* $d \in \mathbb{N}$*,* $\Phi \subset \mathbb{Q}^{\geq 0}$ *be a DCC set,* $\Gamma \subset \mathbb{Q}^{>0}$ *be a finite set, and* $\sigma \in \mathbb{Q}[t]$ *be a polynomial. Consider pairs* (X, B) *and* $\mathbb{Q}$*-Cartier Weil divisors* A *on* X *satisfying the following conditions:*

- (X, B) *is an slc pair of dimension* d*,*
- *the coefficients of* B *are contained in* Φ*,*
- $K_X + B$ *is semiample, defining a contraction* $f : X \to Z$*,*
- A *is a divisor on* X *that is ample over* Z*,*
- $\mathrm{vol}(A|_F) \in \Gamma$*, where* F *is any general fiber of* $f : X \to Z$ *over any irreducible component of* Z*,*
- $(K_X + B + tA)^d = \sigma(t)$*, and*
- $A \geq 0$ *does not contain any non-klt center of* (X, B)*,*

then the set of such $(X, B + A)$ *forms a bounded family.*

Theorem 1.3.1 can be regarded as a relative version of Theorem 1.1.1(2). Note that we fix finitely many values for $\mathrm{vol}(A|_F)$. This is important for applications because on some stable minimal models, the volumes $\mathrm{vol}(A|_F)$ take different values over different irreducible components of Z. One naturally wonders whether a relative version of Theorem 1.1.1(1) exists; that is, for klt pairs (X, B), the polarization A need not be an effective divisor. This thesis addresses this question and uses it to construct the moduli space of klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of the Iitaka fibration (see Chapter 5).

Theorem 1.3.2. *Let* $d \in \mathbb{N}$*,* $\Phi \subset \mathbb{Q}^{\geq 0}$ *be a DCC set,* $u \in \mathbb{Q}^{>0}$*, and* $\sigma \in \mathbb{Q}[t]$ *be a* *polynomial. Let $\mathcal{G}_{klt}(d,\Phi,\leq u,\sigma)$ be the set of pairs (X,B) and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:*

- *(X,B) is a klt pair of dimension d,*
- *the coefficients of B are contained in Φ,*
- *K_X+B is semiample, defining a contraction $f: X\to Z$,*
- *A is a divisor on X that is nef and big over Z,*
- *$\mathrm{vol}(A|_F)\leq u$, where F is the general fiber of $f: X\to Z$, and*
- *$(K_X+B+tA)^d=\sigma(t)$.*

Then the set of such (X,B) forms a bounded family. If in addition $A\geq 0$, then the set of such $(X,B+A)$ also forms a bounded family.

Recently, there have been some other related results on the (birational) boundedness of klt good minimal models, see [4, 12-13, 21-26].

Remark 1.3.3. In Theorem 1.3.2, the condition that (X,B) is klt cannot be replaced by lc [20][§4.2]. While in Theorem 1.3.1 the condition on fixing $\sigma(t)$ cannot be replaced by fixing only $\mathrm{Ivol}(K_X+B)$ [17][§1], one expects that in the klt case of Theorem 1.3.2 such a replacement may be possible [13, 22, 24]; however, in this case the polarization A cannot be controlled as Theorem C.

Remark 1.3.4. After fixing $\sigma(t)$, in Theorems 1.3.1 and 1.3.2 it is expected that the condition on $\mathrm{vol}(A|_F)$ can be removed. Indeed, the effective b-semiampleness conjecture [27][Conjecture 7.13] predicts that $\mathrm{Ivol}(K_X+B)$ belongs to a DCC set Ψ depending only on d and Φ. In the slc case, by [18][Proposition 5.3], this implies that $\mathrm{vol}(A|_F)$ belongs to a finite set Γ. In the klt case, by (4.1), we obtain that $\mathrm{vol}(A|_F)\leq u$. Here both Γ and u depend only on (d,Φ,Ψ,σ).

A description of the proof of Theorem 1.3.2.

Theorem A (Boundedness of nef threshold)**.** *Under the same assumptions as Theorem 1.3.2, there exists a positive rational number τ, depending only on (d,Φ,u,σ), such that K_X+B+tA is nef and big for all $0<t<\tau$.*

Theorem B (Boundedness of pseudo-effective thereshold)**.** *Under the same assumptions as Theorem 1.3.2, there exists a positive rational number λ, depending only on (d,Φ,u,σ), such that K_X+B+tA is big for all $0<t<\lambda$.*

Theorem C. *Under the same assumptions as Theorem 1.3.2, there exist a natural number* *r depending only on* (d, Φ, u, σ) *and a very ample divisor* H *on* X *such that*

$$H^d \leq r, \quad (K_X + B) \cdot H^{d-1} \leq r, \quad \textit{and} \quad H - A \textit{ is pseudo-effective.}$$

In particular, by Lemma 2.11.3, the set of such (X, B) *forms a bounded family. If in addition* $A \geq 0$*, then the set of such* $(X, B + A)$ *also forms a bounded family.*

It is clear that Theorem A implies Theorem B, while Theorem C yields Theorem 1.3.2. The proof of Theorem A, B and C proceeds by induction on the dimension of X:

- Theorem B_d + Theorem $\mathrm{C}_{d-1} \implies$ Theorem A_d; cf. (4.2.1).
- Theorem $\mathrm{A}_{d-1} \implies$ Theorem B_d; cf. (4.3.1).
- Theorem $\mathrm{A}_d \implies$ Theorem C_d; cf. (4.4.1).

1.4 Boundedness of polarized log Calabi–Yau fibrations with bounded bases

Based on the predictions of the minimal model program and the abundance conjecture, it is important to extend boundedness results to Fano fibrations and Calabi-Yau fibrations. Such fibrations also frequently appear in inductive arguments. In [28], Jiang considered the birational boundedness of Fano fibrations under several conjectural assumptions. Later, Birkar used some of these arguments to obtain the birational boundedness of Fano fibrations and carried out further work to establish boundedness [29]. However, the boundedness of Calabi-Yau fibrations is not fully understood, although some literature addresses this direction [11-12, 16, 18, 21-22, 24, 26, 30-31].

Inspired by the study of Fano type fibrations in [28-29], we introduce a special structure for log Calabi-Yau fibrations equipped with polarizations on both the base and the general fiber. This section is based on joint work with Junpeng Jiao and Minzhe Zhu [13].

Definition 1.4.1. A *polarized log Calabi-Yau fibration* (resp. *weak polarized log Calabi–Yau fibration*) $f : ((X, B), A) \to (Z, H)$ consists of

1. a projective pair (X, B),
2. a fibration $f : X \to Z$ such that $K_X + B \sim_{\mathbb{R}} f^*N$ for some $\mathbb{R}$-divisor N on Z,
3. an integral divisor A on X that is ample over Z, and
4. a very ample divisor $H \geq 0$ on Z such that $H - N$ is ample (resp. pseudo-effective).

Note that H is a polarization on the base Z, and $A|_F$ is a polarization on the general fiber F of $f : X \to Z$. If f is constant, that is, if Z is a point, then $K_X + B \sim_{\mathbb{Q}} 0$ and A is an ample integral divisor on X. In this case, we call $((X, B), A)$ a *polarized log*

Calabi–Yau pair.

We now fix some invariants of a (weak) polarized log Calabi-Yau fibration.

Definition 1.4.2. Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{R}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{R}$ be a DCC set.

1. A *(weak)* (d, r, ϵ)*-polarized log Calabi-Yau fibration* is a (weak) polarized log Calabi-Yau fibration $f : ((X, B), A) \to (Z, H)$ satisfying
   - (X, B) is a projective ϵ-lc pair of dimension d, and
   - $H^{\dim Z} \leq r$.
2. If, additionally,
   - $\mathrm{vol}(A|_F) \leq v$, where F is a general fiber of $f : X \to Z$,

   then we call $f : ((X, B), A) \to (Z, H)$ a *(weak)* (d, v, r, ϵ)*-polarized log Calabi-Yau fibration*.
3. Furthermore, if
   - the coefficients of B belong to Φ,

   then we refer to $f : ((X, B), A) \to (Z, H)$ as a *(weak)* $(d, \Phi, v, r, \epsilon)$*-polarized log Calabi-Yau fibration*.

Our main boundedness statements in codimension one for weak polarized log Calabi–Yau fibrations concerns the case where the coefficients of B belong to a finite set Φ. In particular, this includes the special case $B = 0$, that is, $\Phi = \{0\}$, which was first conjectured by Birkar and Hacon.

Theorem 1.4.3. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Consider the set of all weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibrations $f : ((X, B), A) \to (Z, H)$. Then the set of such $(X, B + f^*H)$ is log bounded in codimension one.*

We apply our boundedness results on polarized log Calabi-Yau fibrations to klt good minimal models.

Corollary 1.4.4. *Let $d \in \mathbb{N}$, $u, v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a DCC set. Consider the set of $((X, B), A)$ such that*

1. *(X, B) is a projective klt pair of dimension d,*
2. *the coefficients of B are in Φ,*
3. *$K_X + B$ is semi-ample defining a contraction $f : (X, B) \to Z$,*
4. *$\mathrm{Ivol}(K_X + B) = u$,*
5. *A is an integral divisor on X that is ample over Z, and $\mathrm{vol}(A|_F) \leq v$, where F is the general fiber of $f : X \to Z$.*

Then (X, B) is log bounded in codimension one.

Sketch of proof of Theorem 1.4.3. Given a $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration $f : ((X, B), A) \to (Z, H)$, note that the base Z is bounded by assumption and the general fiber $((F, B_F), A_F)$ is bounded by [15]Corollary 1.6. We study the induced rational map from Z to a "moduli space" of the general fibers. For this purpose, we use the strongly embedded fine moduli space $\mathcal{S}$ of polarized log Calabi–Yau pairs constructed in [15, 18]. Since $\mathcal{S}$ also parametrizes the polarizations, the universal family $(\mathcal{X}, \mathcal{B}) \to \mathcal{S}$ is not necessarily of maximal variation, and the rational map $Z \dashrightarrow \mathcal{S}$ is not necessarily bounded. Applying [32], we obtain a new family $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ of maximal variation with $\mathbf{b}$-nef and big moduli part $\mathcal{M}^!$. Therefore, since the moduli part $\mathbf{M}_Z$ of $f : (X, B) \to Z$ is controlled by H, a volume argument shows that, up to a generically finite cover, the map $Z \dashrightarrow \mathcal{S}^!$ is bounded; see Theorem 6.2.1.

The traditional strategy for proving boundedness of polarized fibrations is to modify the vertical part of the polarization A so as to obtain a global ample divisor on X with bounded volume; see [18, 24, 26, 29-30]. However, this approach fails in our setting. Instead, we construct a new polarization L arising from the family $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ such that $L \equiv mA$ over the generic point of Z for some fixed $m \in \mathbb{N}$. More precisely, we define a polarization $\mathcal{L}$ on $(\mathcal{X}, \mathcal{B}) \to \mathcal{S}$ by pulling back $\mathcal{A}^!$ to a Galois cover of $\mathcal{X}$, taking the Galois sum, and then descending it to $\mathcal{X}$. For every $s \in \mathcal{S}$, we have $\mathcal{L}_s \equiv m\mathcal{A}_s$; see Theorem 6.1.2(6). Finally, we define L as the closure of the pullback of $\mathcal{L}$ via the moduli map $Z \dashrightarrow \mathcal{S}$. Since L arises from the fixed family $((\mathcal{X}^!, \mathcal{B}^!), \mathcal{A}^!) \to \mathcal{S}^!$ and the map $Z \dashrightarrow \mathcal{S}^!$ is bounded up to a generically finite cover, by the invariance of plurigenera and an argument about the descent of volume from a generically finite cover, we can show that, after modifying the vertical part of $((X, B), L) \to Z$, the volume of L can be controlled on a suitable birational model; see Lemma 6.2.2 and Theorem 6.3.2.

To proceed, we apply weak semistable reduction [33] and the minimal model program for lc pairs [34] in Theorem 6.3.1 to obtain a new birational model $((X', \Delta'), L') \to Z$ of $((X, B), L) \to Z$ satisfying that

1. $(X', \Delta' + \alpha L')$ is lc for some fixed positive real number α,
2. $K_{X'} + \Delta' + \alpha L'$ is big, and
3. $\mathrm{vol}(K_{X'} + \Delta' + \alpha L')$ is bounded from above.

Then, we apply [1-2] to obtain the log birational boundedness of (X', Δ'). Moreover, we can deduce that $\mathrm{Supp}(\Delta')$ contains both the strict transform of $\mathrm{Supp}(B)$ on X' and the exceptional divisors over X. We then apply the MMP in family [3] to bound (X, B) in codimension one.

The thesis is organized as follows.

- In Chapter 2, we introduce the necessary background used throughout this thesis.
- In Chapter 3, we first prove the boundedness of $\mathcal{F}_{\rm glc}(2, \Phi, v)$ under additional assumptions; see Theorem 1.2.3. Then, we construct counterexamples showing that $\mathcal{F}_{\rm glc}(2, \Phi, v)$ is not bounded in general, where the underlying varieties X can have arbitrary Kodaira dimension.
- In Chapter 4, we prove the boundedness of $(d, \Phi, \leq u, \sigma)$-polarized good minimal models; see Theorem 1.3.2.
- In Chapter 5, we construct the moduli space of (d, Φ, u, σ)-traditional stable minimal models; see Theorem 5.2.5.
- In Chapter 6, we prove boundedness in codimension one of weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibrations and apply this to klt good minimal models; see Theorem 1.4.3 and Corollary 1.4.4.

CHAPTER 2 PRELIMINARIES

2.1 Notations and conventions

We collect some notations and conventions used in this thesis.

1. For any scheme S and positive integer n, l, we denote $\mathbb{P}^n_k \times_{\operatorname{Spec} k} S$ and $\mathbb{P}^l_k \times_{\operatorname{Spec} k} S$ by $\mathbb{P}^n_S$ and $\mathbb{P}^l_S$, respectively. When $S = \operatorname{Spec} k$, we simply write $\mathbb{P}^n$ and $\mathbb{P}^l$ if there is no risk of confusion. Let $\mathbb{P}^n_S \times_S \mathbb{P}^l_S \cong \mathbb{P}^n \times \mathbb{P}^l \times S$ be the natural isomorphism, and
$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times S \xrightarrow{p_2} \mathbb{P}^l$$
be the projections. Then for any $a, b \in \mathbb{Z}$, we denote $p_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(b)$ by $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S}(a, b)$.
2. A projective morphism $f : X \to Y$ of schemes is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (f is not necessarily birational). In particular, f is surjective with connected fibers.
3. Let $f : X \to S$ be a morphism of schemes. Let L_1 and L_2 be line bundles on X. We say that L_1 and L_2 are *linearly equivalent over* S, denoted by $L_1 \sim_S L_2$, if there is a line bundle M on S such that $L_1 \cong L_2 \otimes f^*M$.
4. Suppose that X is a normal variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. We say that D_1 and D_2 are $\mathbb{Q}$-linear equivalent, denoted by $D_1 \sim_{\mathbb{Q}} D_2$, if there exists $m \in \mathbb{Z}_{>0}$ such that mD_1 and mD_2 are Cartier and $mD_1 \sim mD_2$. Moreover, fixed $l \in \mathbb{Z}_{>0}$, the notation $D_1 \sim_l D_2$ means that $lD_1 \sim lD_2$.
5. We say that a line bundle L on a variety X is *strongly ample* if L is very ample and $H^q(X, kL) = 0$ for any $k, q > 0$.
6. We say that a set $\Phi \subset \mathbb{Q}$ satisfies the *descending chain condition* (DCC, for short) if Φ does not contain any strictly decreasing infinite sequence. Similarly, we say that a set $\Phi \subset \mathbb{Q}$ satisfies the *ascending chain condition* (ACC, for short) if Φ does not contain any strictly increasing infinite sequence.
7. A *pair* (X, B) consists of a normal variety X and a boundary divisor B with rational coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Fixed $\epsilon \in \mathbb{Q}^{>0}$, singularities of (X, B) are defined by taking a log resolution $\phi : W \to X$ and writing
$$K_W + B_W = \phi^*(K_X + B).$$

We say (X, B) is *lc* (resp. *ϵ-lc*) (resp. *klt*) if every coefficient of B_W is ≤ 1 (resp. $\leq 1-\epsilon$) (resp. < 1).

2.2 b-divisors

Definition 2.2.1 (**b**-divisors)**.** Let X be a normal variety. A ***b**-divisor* $\mathbf{M}$ over X is a collection of $\mathbb{R}$-divisors $\mathbf{M}_Y$ on Y for each birational contraction $Y \to X$ from a normal variety that are compatible with respect to pushdown, that is, if $Y' \to X$ is another birational contraction and $\psi : Y' \dashrightarrow Y$ is a morphism, then $\psi_* \mathbf{M}_{Y'} = \mathbf{M}_Y$.

A **b**-divisor $\mathbf{M}$ is ***b**-$\mathbb{R}$-Cartier* if there is a birational contraction $Y \to X$ such that $\mathbf{M}_Y$ is $\mathbb{R}$-Cartier and $\mathbf{M}_{Y'}$ is the pullback of $\mathbf{M}_Y$ on Y' for any birational contraction $Y' \to Y$. In this case, we say that $\mathbf{M}$ descends to Y, and it is represented by $\mathbf{M}_Y$, we write $\mathbf{M} = \overline{\mathbf{M}_Y}$.

A **b**-$\mathbb{R}$-Cartier divisor $\mathbf{M}$ represented by $\mathbf{M}_Y$ for some birational model $Y \to X$ is ***b**-nef* if $\mathbf{M}_Y$ is nef. Similarly, $\mathbf{M}$ is ***b**-nef and big* if $\mathbf{M}_Y$ is nef and big.

Definition 2.2.2 (Discrepancy **b**-divisors)**.** The discrepancy **b**-divisor $\mathbf{A} = \mathbf{A}(X, B)$ of a sub-pair (X, B) is the **b**-$\mathbb{R}$-divisor of X with the trace $\mathbf{A}_Y = \sum a_i A_i$ defined by the formula

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, B)$ by

$$\mathbf{A}^*_Y = \sum_{a_i > -1} a_i A_i.$$

Note that $\mathbf{A}(X, B) = \mathbf{A}^*(X, B)$ when (X, B) is sub-klt. By the definition, we have $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) \simeq \mathcal{O}_X$ if (X, B) is lc. We also have $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq \mathcal{O}_X$ when (X, B) is klt.

2.3 Generalised pairs and singularities

We refer reader to [35] for a survey of generalised pairs.

Definition 2.3.1. A *generalised pair* (X, B, M) consists of

- a normal variety X,
- an effective $\mathbb{R}$-divisor B on X, and
- a b-$\mathbb{R}$-Cartier b-divisor M over X, represented by a projective birational morphism $f : X' \to X$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M' on X' such that M' is nef and

$K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := f_* M'$.

We will often refer to such a generalised pair by saying that $(X, B + M)$ is a generalised pair with data $X' \to X$ and M'.

Let D be a prime divisor over X. Replace X' with a log resolution of (X, B) such that D is a prime divisor on X'. We can write

$$K_{X'} + B' + M' = f^*(K_X + B + M).$$

We define the *generalised log discrepancy* of D to be $a(D, X, B, M) = 1 - \mathrm{mult}_D B'$.

We say that $(X, B+M)$ is *generalised klt* (resp. *generalised lc*) if $a(D, X, B, M) > 0$ (resp. $a(D, X, B, M) \geq 0$) for every prime divisor D over X.

Remark 2.3.2. If $(X, B + M)$ is a generalised log canonical surface, then the negativity lemma [36][Lemma 3.39] applied to M' and the numerical pullback of M implies that (X, B) is numerically log canonical. Thus, (X, B) is log canonical by [37][Proposition 3.5], and M is $\mathbb{Q}$-Cartier. Moreover, by [19][Remark 3.6], M is nef. Similarly, we can show that X is log canonical.

Definition 2.3.3 (Lc threshold of $\mathbb{Q}$-linear systems)**.** Let (X, B) be an lc pair. The *lc threshold* of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L \geq 0$ with respect to (X, B) is defined as

$$\mathrm{lct}(X, B, L) := \sup\{\, t \in \mathbb{R} \mid (X, B + tL) \text{ is lc } \}.$$

Now let H be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The $\mathbb{Q}$-linear system of H is

$$|H|_{\mathbb{Q}} := \{\, L \geq 0 \mid L \sim_{\mathbb{Q}} H \,\}.$$

We then define the *lc threshold* of $|H|_{\mathbb{Q}}$ with respect to (X, B) (also called the *global lc threshold* or *α-invariant*) as

$$\mathrm{lct}(X, B, |H|_{\mathbb{Q}}) := \inf\{\, \mathrm{lct}(X, B, L) \mid L \in |H|_{\mathbb{Q}} \,\},$$

which is equivalent to

$$\sup\{\, t \in \mathbb{R} \mid (X, B + tL) \text{ is lc for every } L \in |H|_{\mathbb{Q}} \,\}.$$

2.4 Minimal models

Suppose that $f : X \to Z$ and $f^m : X^m \to Z$ are projective morphisms, $\phi : X \dashrightarrow X^m$ is a projective birational contraction over Z and (X, B) and (X^m, B^m) are lc pairs, where $B^m = \phi_* B$. If $a(E, X, B) > a(E, X^m, B^m)$ (resp. $a(E, X, B) \geq a(E, X^m, B^m)$) for all prime ϕ-exceptional divisors $E \subset X$, X^m is $\mathbb{Q}$-factorial and $K_{X^m} + B^m$ is nef over Z, then we say that $\phi : X \dashrightarrow X^m$ is a *minimal model* (resp. *weak log canonical model*) of (X, B) over Z.

A minimal model (resp. weak log canonical model) $\phi : X \dashrightarrow X^m$ of (X, B) over Z is called a *good minimal model* (resp. *semi-ample model*) if $K_{X^m} + B^m$ is semi-ample over Z. In this case,

$$R(X/Z, K_{X^m} + B^m) := \bigoplus_{l \geq 0} f^m_* \mathcal{O}_{X^m}(l(K_{X^m} + B^m))$$

is a finitely generated $\mathcal{O}_Z$-algebra, and let

$$X^c = \operatorname{Proj} R(X/Z, K_{X^m} + B^m).$$

If $K_{X^m} + B^m$ is semi-ample and big over Z, then X^c is called the *log canonical model* of (X, B) over Z.

Lemma 2.4.1. *Assume that*

- (X, B) *is a lc pair and* $f : X \to Z$ *is a contraction,*
- $\mu : Z' \to Z$ *is a finite cover,*
- X' *is the normalization of* $X \times_Z Z'$ *and denote the natural finite cover* $X' \to X$ *by* π*, and the contraction* $X' \to Z'$ *by* f'*,*
- (X', B') *is a lc pair such that* $K_{X'} + B' = \pi^*(K_X + B)$*, and*
- $\eta : X'' \dashrightarrow X'/Z'$ *is an isomorphism in codimension one and* B'' *is the strict transform of* B' *on* X''*.*

$$\begin{tikzcd}
(X'', B'') \arrow[r, dashed, "\eta"] \arrow[dr] & (X', B') \arrow[r, "\pi"] \arrow[d, "f'"] & (X, B) \arrow[d, "f"] \\
& Z' \arrow[r, "\mu"] & Z
\end{tikzcd}$$

Then we have the following statements:

(1) If $(X, B) \dashrightarrow (X^m, B^m)$ *is a good minimal model of* $K_X + B$ *over* Z *and* $(X'', B'') \dashrightarrow (X''^m, B''^m)$ *is a good minimal model of* $K_{X''} + B''$ *over* Z'*, then* (X''^m, B''^m) *is isomorphic in codimension one to the normalization of* $(X^m, B^m) \times_Z Z'$*,*

(2) If furthermore $K_X + B$ *is big over* Z*, assume that* $(X, B) \dashrightarrow (X^c, B^c)$ *is the log canonical model of* $K_X + B$ *over* Z*, and* $(X'', B'') \dashrightarrow (X''^c, B''^c)$ *is the log canonical model of* $K_{X''} + B''$ *over* Z'*. Then* (X''^c, B''^c) *is isomorphic to the normalization of* $(X^c, B^c) \times_Z Z'$*.*

Proof (1). By [34][Lemma 2.4(1)], the set of exceptional divisors of $X \dashrightarrow X^m$ coincides with the support of $N_\sigma(K_X + B/Z)$, and the set of exceptional divisors of $X'' \dashrightarrow X''^m$

coincides the support of $N_\sigma(K_{X''}+B''/Z')$. Thus it suffices to prove

$$\mathrm{Supp}(N_\sigma(K_{X''}+B''/Z'))=\eta^{-1}\pi^{-1}\mathrm{Supp}(N_\sigma(K_X+B/Z)).$$

Since $X'\to X$ is a finite cover, by [38]$^{\S3,\text{ Theorem 5.16}}$, we have

$$\pi^{-1}\mathrm{Supp}(N_\sigma(K_X+B/Z))=\mathrm{Supp}(N_\sigma(K_{X'}+B'/Z')).$$

Since (X',B') is isomorphic in codimension one to (X'',B''), there is a one to one correspondence between $|m(K_{X'}+B')/Z'|$ and $|m(K_{X''}+B'')/Z'|$ for every $m\in\mathbb{N}$, hence

$$\eta^{-1}\mathrm{Supp}(N_\sigma(K_{X'}+B'/Z'))=\mathrm{Supp}(N_\sigma(K_{X''}+B''/Z')),$$

and we finish the proof.

(2). Since $X'\to X$ is a finite cover, we have

$$\pi^{-1}\mathbb{E}(K_X+B/Z)=\mathbb{E}(K_{X'}+B'/Z')$$

by [39]$^{\text{Theorem 1.1}}$. Since (X'',B'') is isomorphic in codimension one to (X',B'), the divisorial part of $\mathbb{E}(K_{X''}+B''/Z')$ coincides with the strict transform of the divisorial part of $\mathbb{E}(K_{X'}+B'/Z')$. Let $(\widetilde{X}^c,\widetilde{B}^c)$ be the normalization of $(X^c,B^c)\times_Z Z'$. Since $X^m\to X^c$ contracts $\mathbb{E}(K_{X^m}+B^m/Z)$ and $X''^m\to X''^c$ contracts $\mathbb{E}(K_{X''^m}+B''^m/Z')$, we conclude that (X''^c,B''^c) is isomorphic in codimension one to $(\widetilde{X}^c,\widetilde{B}^c)$.

Note that $K_{\widetilde{X}^c}+\widetilde{B}^c$ is ample because $K_{X^c}+B^c$ is ample and $\widetilde{X}^c\to X^c$ is a finite cover. Since $K_{\widetilde{X}^c}+\widetilde{B}^c$ and $K_{X''^c}+B''^c$ are both ample, and since (X''^c,B''^c) and $(\widetilde{X}^c,\widetilde{B}^c)$ are isomorphic in codimension one, we conclude that (X''^c,B''^c) is isomorphic to $(\widetilde{X}^c,\widetilde{B}^c)$.

∎

2.5 Canonical bundle formula

Definition 2.5.1. An *lc-trivial fibration* (resp. *klt-trivial fibration*) $f:(X,B)\to Z$ consists of a projective surjective morphism $f:X\to Z$ with connected fibers between normal varieties and a pair (X,B) satisfying the following properties:

- (X, B) is sub-lc (resp. sub-klt) over the generic point of Z,
- $\mathrm{rank} f_*\mathcal{O}_X(\lceil\mathbf{A}^*(X,B)\rceil)=1$, and
- there exists a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L_Z on Z such that

$$K_X+B\sim_{\mathbb{R}} f^*L_Z.$$

In [32, 40], klt-trivial fibrations as in Definition 2.5.1 are called lc-trivial fibrations.

Let $f:(X,B)\to Z$ be an lc-trivial fibration such that $\dim Z>0$. Fix a prime divisor D on Z and let t_D be the lc threshold of f^*D with respect to (X, B) over the generic point of D. Now let $B_Z := \sum(1 - t_D)D$, where the sum runs over all the prime divisors on Z. Let $M_Z := L_Z - (K_Z + B_Z)$, then we have the following

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).$$

We call B_Z the *discriminant divisor* and M_Z the *moduli divisor* of adjunction. Note that B_Z is uniquely determined but M_Z is determined only up to $\mathbb{R}$-linear equivalence.

Now let $\phi : X' \to X$ and $\psi : Z' \to Z$ be birational morphisms from normal projective varieties and assume the induced map $f' : X' \dashrightarrow Z'$ is a morphism. Let $K_{X'} + B'$ be the pullback of $K_X + B$ on X' and similarly we can define a discriminant divisor $B_{Z'}$ and $L_{Z'} = \psi^* L_Z$ gives a moduli divisor $M_{Z'}$ so that

$$K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Z'} + B_{Z'} + M_{Z'}),$$

$B_Z = \psi_* B_{Z'}$ and $M_Z = \psi_* M_{Z'}$. In particular, the lc-trivial fibration $f : (X, B) \to Z$ induces **b**-$\mathbb{R}$-divisors $\mathbf{B}$ and $\mathbf{M}$ on Z, called the *discriminant* and *moduli* **b**-divisor respectively.

Theorem 2.5.2 ([40-41]). *With the above notation and assumptions, suppose that (X, B) is lc over the generic point of Z. If $Z' \to Z$ is a high resolution, then $\mathbf{M}_{Z'}$ is nef and for any birational morphism $Z'' \to Z'$ from a normal projective variety, $\mathbf{M}_{Z''}$ is the pullback of $\mathbf{M}_{Z'}$. In particular, we can view $(Z, B_Z + \mathbf{M}_Z)$ as a generalized pair with nef part $\mathbf{M}_{Z'}$.*

Proposition 2.5.3 ([32][Proposition 3.1]). *Let $f : (X, B) \to Z$ be a klt-trivial fibration over proper normal variety Z. Let $\tau : Z' \to Z$ be a surjective morphism from a proper normal variety Z', let X' be the normalization of the main component of $X \times_Z Z'$, and B' be the divisor on X' such that $K_{X'} + B' = \tau_X^*(K_X + B)$. Then we say that $f' : (X', B') \to Z'$ is the klt-trivial fibration induced by base change. Let $\mathbf{M}$ and $\mathbf{M}'$ be the corresponding moduli **b**-divisors of f and f' respectively. Then we have*

$$\tau^* \mathbf{M} = \mathbf{M}'.$$

Theorem 2.5.4 ([32]). *Let $f : (X, B) \to Z$ be a klt-trivial fibration over normal projective variety Z such that B is a $\mathbb{Q}$-divisor. Suppose that the geometric generic fiber $X_{\overline{\eta}} = X \times_Z$ $\mathrm{Spec}(\overline{k(Z)})$ is projective and $B_{\overline{\eta}}$ is effective. Then there exist non-singular* *projective varieties $\overline{Z}$, T and V, and a commutative diagram*

$$\begin{tikzcd}
(X,B) \arrow[d, "f"'] & & (X_T,B_T) \arrow[d, "f_T"'] & \\
Z \arrow[rrr, dashed, bend right, "\gamma"'] & \overline{Z} \arrow[l, "\tau"'] \arrow[r, "\rho"] & T \arrow[ll, dashed, bend right, "i"'] \arrow[r, "\pi"] & V
\end{tikzcd}$$

such that

1. $f_T : (X_T, B_T) \to T$ *is a klt-trivial fibration,*
2. τ *is generically finite and surjective, and* ρ *is surjective,*
3. *there exists a nonempty open subset* $U \subset \overline{Z}$ *and an isomorphism*

$$\begin{tikzcd}
(X,B)\times_Z U \arrow[rr, "\cong"'] \arrow[dr] & & (X_T,B_T)\times_T U \arrow[dl] \\
& U, &
\end{tikzcd}$$

4. *let* $\mathbf{M}$, $\mathbf{N}$ *be the corresponding moduli* $\mathbf{b}$*-divisors of* f *and* f_T*, then* $\mathbf{N}$ *is* $\mathbf{b}$*-nef and big, and* $\tau^*\mathbf{M} = \rho^*\mathbf{N}$*,*
5. π *is generically finite and surjective,* $\Phi : Z \dashrightarrow V$ *is bimeromorphic to the period map defined in [32]*[Proposition 2.1]*, and*
6. $i : T \dashrightarrow Z$ *is a rational map such that* $f_T : (X_{,}B_T) \to T$ *is equal to the pullback of* $f : (X, B) \to Z$ *via i over some open subset of* T*.*

Proof The assertions (1)–(4) are stated in [32][Theorem 3.3], while (5) and (6) can be derived from the proof of [32][Theorem 2.2]. Indeed, by algebraization theorem in [42][Theorem 11], the period map defined in [32][Proposition 2.1] is bimeromorphic to a morphism $\gamma^o : Z^o \to V^o$ from a non-empty open subset of Z to a non-singular quasi-projective variety V^o. Let $T^o \to V^o$ be a generically finite surjective morphism from a non-singular quasi-projective variety T^o such that if $\overline{Z}^o$ is the main part of $Z^o \times_{V^o} T^o$, then the induced morphism $\rho^o : \overline{Z}^o \to T^o$ has a section α. By base change via the section $i^o : T^o \xrightarrow{\alpha} \overline{Z}^o \xrightarrow{\tau^o} Z^o$, we induce a family $f_{T^o} : (X_{T^o}, B_{T^o}) \to T^o$. After replacing $\overline{Z}^o$ and T^o by generically finite covers from non-singular quasi-projective varieties, we have an isomorphism of pairs over $\overline{Z}^o$

$$(X,B)\times_Z \overline{Z}^o \xrightarrow{\sim} (X_{T^o}, B_{T^o}) \times_{T^o} \overline{Z}^o.$$

Let $\overline{Z}$, T and V be non-singular projective compactifications of $\overline{Z^o}$, T^o and V^o respectively, and let (X_T, B_T) be a normal projective compactification of (X_{T^o}, B_{T^o}) so that f_{T^o} induces a klt-trivial fibration $f_T : (X_T, B_T) \to T$. Then (5) and (6) are satisfied. ∎

2.6 Finite covers

Proposition 2.6.1. *Let $f : (X, B) \to Z$ be a projective isotrivial fibration between normal quasi-projective varieties with general fiber (F, B_F). Assume that there exists a Galois cover $Z' \to Z$ with Galois group G such that*

$$(X, B) \cong \big((F, B_F) \times Z'\big)/G,$$

where G acts on $(F, B_F) \times Z'$ and the action is G-equivariant with respect to the projection $(F, B_F) \times Z' \to Z'$. Assume further that (F, B_F) is klt and that $\kappa(F, K_F + B_F) \geq 0$.

Then there exists another Galois cover $\widetilde{Z} \to Z$ with the Galois group H such that

$$(\widetilde{X}, \widetilde{B}) := (X, B) \times_Z \widetilde{Z} \cong (F, B_F) \times \widetilde{Z}.$$

Moreover,

$$(X, B) \simeq \big((F, B_F) \times \widetilde{Z}\big)/H,$$

where H acts diagonally on $(F, B_F) \times \widetilde{Z}$, that is, there exists a homomorphism

$$\rho_F : H \to \mathrm{Aut}(F, B_F) := \{\sigma \in \mathrm{Aut}(F) \mid \sigma_* B_F = B_F\}$$

such that for every $h \in H$,

$$h \cdot ((f, b), \tilde{z}) = \big(\rho_F(h) \cdot (f, b),\ h \cdot \tilde{z}\big),$$

for all $(f, b) \in (F, B_F)$ and $\tilde{z} \in \widetilde{Z}$.

Remark 2.1: One may replace H by the quotient $H/\ker(\rho_F)$ and replace $\widetilde{Z}$ by the corresponding Galois cover, so that the induced action on (F, B_F) becomes faithful. However, we do not require this, since in applications we frequently replace $\widetilde{Z}$ by further Galois covers.

Proof We follow the argument of [43][Theorem 43].

Since $f : (X, B) \to Z$ becomes a product family after the Galois cover $Z' \to Z$, we have

$$\mathrm{Im}\big(G \to \mathrm{Aut}(F, B_F)\big) = \mathrm{Im}\big(\pi_1(Z) \to \mathrm{Aut}(F, B_F)\big).$$

Consider the discrete part of the monodromy representation

$$\rho^d : \pi_1(Z) \to \mathrm{Aut}(F, B_F)/\mathrm{Aut}^0(F, B_F),$$

where $\mathrm{Aut}^0(F, B_F)$ is the connected component of $\mathrm{Aut}(F, B_F)$ containing the identity. Then

$$H_d := \mathrm{Im}\big(\rho^d\big) = \mathrm{Im}\big(G \to \mathrm{Aut}(F, B_F)/\mathrm{Aut}^0(F, B_F)\big)$$

is a finite group. Let $Z^d \to Z$ be the corresponding Galois cover with Galois group H_d.

The trivialization of the translation part $\mathrm{Aut}^0(F, B_F)$ is more subtle and it depends on additional choices.

Since (F, B_F) is klt and $\kappa(F, K_F + B_F) \geq 0$, the group $\mathrm{Aut}^0(F, B_F)$ is an abelian variety by [32][Proposition 4.6]. A general $\mathrm{Aut}^0(F, B_F)$-orbit $A_F \subset F$ defines an isotrivial abelian family

$$X^d \supset A^d \to Z^d.$$

By assumption, there exists an f-ample line bundle L on X, whose pullback defines a relatively ample line bundle L_{A^d} on A^d. We may assume that the degree m of L_{A^d} on the general fiber is at least 3. By [43][Remark 44], there exists a closed subscheme $T^d \subset A^d$ such that $T^d \to Z^d$ is étale and every fiber $T^d_F \subset A_F$ is a translate of the subgroup of m-torsion points. Since L_{A^d} is H_d-invariant, the morphism $T^d \to Z^d$ is H_d-equivariant, hence it defines a monodromy representation

$$\pi_1(Z^d) \to \mathrm{Aut}^0(F, B_F).$$

Let $\Gamma \subset \pi_1(Z^d)$ be a finite-index subgroup that is normal in $\pi_1(Z)$, and let $\widetilde{Z} \to Z$ be the corresponding Galois cover with Galois group $H := \pi_1(Z)/\Gamma$.

By pullback, we obtain an isotrivial abelian fibration $\widetilde{A} \to \widetilde{Z}$ with a trivialization of the m-torsion points. For $m \geq 3$, by [44][p. 513, Lemma 2.9], we have $\widetilde{A} \simeq A_F \times \widetilde{Z}$. Consequently, the same pullback also trivializes the fibration

$$(\widetilde{X}, \widetilde{B}) \cong (F, B_F) \times \widetilde{Z} \to \widetilde{Z}.$$

The H-action on $(\widetilde{X}, \widetilde{B}) \cong (F, B_F) \times \widetilde{Z}$ is given by

$$h : ((f, b), \tilde{z}) \longmapsto \big(\rho_{F,\tilde{z}}(h) \cdot (f, b),\ h \cdot \tilde{z}\big),$$

where $\rho_{F,\tilde{z}} : H \to \mathrm{Aut}(F, B_F)$. Note that $\rho_{F,\tilde{z}}$ preserves the m-torsion points, and that the automorphisms of an abelian torsor (abelian variety without a specified origin) preserving a finite nonempty set form a discrete group. It follows that $\rho_{F,\tilde{z}}$ is in fact independent of $\tilde{z}$. Hence the H-action on $(\widetilde{X}, \widetilde{B})$ can be written as

$$h : ((f, b), \tilde{z}) \longmapsto \big(\rho_F(h) \cdot (f, b),\ h \cdot \tilde{z}\big).$$

∎

The following lemma allows us to modify a generically finite cover into a finite cover.

Lemma 2.6.2. *Let $\pi : T \to V$ be a generically finite cover between normal projective* *varieties. Then there exists a generically finite cover $S^! \to T$ from a smooth projective variety $S^!$ and a birational map $S^* \dashrightarrow V$ from a projective variety S^* such that $S^! \to S^*$ is a finite cover.*

Proof Let $S' \to T$ be a finite cover such that $S' \to V$ is Galois over an open subset of V with Galois group G. Let S'' be the closure of V in $K(S')$, then $S'' \dashrightarrow S'$ is birational and $V = S''/G$.

Let $S^! \to S''$ be a G-equivariant resolution such that $S^! \to S'$ is a morphism, and let S^* be the quotient of $S^!$ by G. Then, the map $S^* \dashrightarrow V$ is birational, $S^! \to T$ is a generically finite surjective morphism, and $S^! \to S^*$ is a finite cover.

$$\begin{tikzcd}
S' \arrow[d] & S'' \arrow[l, dashed] \arrow[d] & S^! \arrow[l] \arrow[ll, bend right] \arrow[d] \\
T \arrow[r, "\pi"] & V & S^* \arrow[l, dashed]
\end{tikzcd}$$

∎

The following lemma shows that relative $\mathbb{Q}$-linear triviality can descend under finite covers.

Lemma 2.6.3. *Assume that*

1. *$f : X \to Z$ is a contraction between two normal projective varieties,*
2. *D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
3. *$\mu : Z' \to Z$ is a finite cover,*
4. *X' is the normalization of $X \times_Z Z'$, and*
5. *denote the induced finite cover $X' \to X$ by π and the induced contraction $X' \to Z'$ by f'.*

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Z' \arrow[r, "\mu"] & Z
\end{tikzcd}$$

If $\pi^* D \sim_{\mathbb{Q},Z'} 0$, then $D \sim_{\mathbb{Q},Z} 0$.

Proof Replacing Z' with a finite cover and replacing X' accordingly, we can assume that $\mu : Z' \to Z$ is a Galois cover with Galois group G. Then G acts on X' by base change, hence $\pi : X' \to X$ is also a Galois cover. Since $\pi^* D \sim_{\mathbb{Q},Z'} 0$, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L' on Z' such that $\pi^* D \sim_{\mathbb{Q}} f'^* L'$. Since $\pi^* D$ is G-invariant, replacing L' with $\frac{1}{|G|} \sum_{g \in G} g^* L'$, we can assume that L' is G-invariant. Therefore, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on Z such that $L' = \mu^* L$. Then $\pi^* D \sim_{\mathbb{Q}} f'^* \mu^* L$, hence $D \sim_{\mathbb{Q}} f^* L$. This finishes the proof. ∎

2.7 Locally stable family

Lemma 2.7.1. *Let $f : X \to S$ be a projective flat morphism with integral fibers and of relative dimension d, and let L be a flat family of divisors over S. If there exists a point $0 \in S$ such that $L_0 \equiv 0$, then $L_s \equiv 0$ for all $s \in S$.*

Proof Let H be a relatively very ample line bundle on X. Take a closed point s on S. Choose $m \gg 0$ such that

$$\chi(X_s, n(mH_s + L_s)) = h^0(X_s, n(mH_s + L_s)) \quad \text{for } n \geq 1.$$

Since L is flat over S, it follows that

$$\chi(X_s, n(mH_s + L_s)) = \chi(X_0, n(mH_0 + L_0)).$$

Therefore, we have

$$h^0(X_s, n(mH_s + L_s)) = h^0(X_0, n(mH_0 + L_0)).$$

From the leading term in the polynomial expansion in n, we obtain

$$(mH_s + L_s)^d = (mH_0 + L_0)^d.$$

Similarly, we have

$$(mH_s)^d = (mH_0)^d.$$

Since $L_0 \equiv 0$, it follows that

$$(mH_s + L_s)^d = (mH_s)^d.$$

Expanding the left-hand side and canceling the dominant terms, we obtain

$$H_s^{d-1} \cdot L_s = H_s^{d-2} \cdot L_s^2 = 0.$$

Restricting to a surface by taking general hyperplane sections and applying the Hodge index theorem, we conclude that $L_s \equiv 0$. ∎

Definition 2.1 (Hodge line bundle): If $f : (X, B) \to Z$ is a locally stable family of pairs such that $N(K_{X/Z} + B) \sim_Z 0$, we set

$$\lambda_{\mathrm{Hodge}, f, N} := f_*(\mathcal{O}_X(N(K_{X/Z} + B)).$$

Proposition 2.7.2. *Let $f : (X, B) \to Z$ be a locally stable family of pairs such that $N(K_{X/Z} + B) \sim_Z 0$. Then we have the following statements:*

(1) $\lambda_{\mathrm{Hodge},f,N}$ *is the unique line bundle (up to isomorphism) satisfying*

$$\mathcal{O}_X(N(K_{X/Z}+B)) \cong f^*\lambda_{\mathrm{Hodge},f,N}.$$

(2) If $\varphi : Z' \to Z$ *is a morphism and* $f' : (X', B') \to Z'$ *denotes the pullback of* $(X, B) \to Z$ *by* φ*, then there is a canonical isomorphism*

$$\varphi^*\lambda_{\mathrm{Hodge},f,N} \xrightarrow{\sim} \lambda_{\mathrm{Hodge},f',N}.$$

(3) If Z *is smooth and the generic fiber of* $X \to Z$ *is normal, then* $f : (X, B) \to Z$ *is an lc-trivial fibration with* $\mathcal{O}_Z(N\mathbf{M}_Z) \cong \lambda_{\mathrm{Hodge},f,N}$*, and the moduli* ***b****-divisor* $\mathbf{M}$ *of* f *descends on* Z*.*

Proof This is [45][Proposition 14.7]. While the proposition is stated only for families of boundary polarized Calabi–Yau pairs, their proof also applies to families of general Calabi–Yau pairs. ∎

2.8 Positivity of line bundles on the blow ups of surfaces

Firstly, we recall some results on ampleness and very ampleness for blow ups of $\mathbb{P}^2$.

Theorem 2.8.1 ([46][Theorem 2.4]). *Let* $Y \to \mathbb{P}^2$ *be the blow up of* $\mathbb{P}^2$ *at* r *distinct points* $p_1, \dots, p_r$ *with exceptional divisors* $E_1, \dots, E_r$*. Let* H *denote the pull-back of* $\mathcal{O}_{\mathbb{P}^2}(1)$*. For* $d, m_1, \dots, m_r \in \mathbb{Z}^{>0}$*, let* $L = dH - \sum_{i=1}^r m_i E_i$ *be a line bundle on* Y*. If*

$$d \geq 1 + \sum_{i=1}^r m_i,$$

then L *is very ample.*

The bound is sharp and can be improved only if not all points $p_1, \dots, p_r$ lie on a line. If $d < 1 + \sum_{i=1}^r m_i$, determining whether L is very ample requires studying the positions of the points $p_1, \dots, p_r$.

Theorem 2.8.2 ([47][Theorem 2.1]). *Let* $C \subset \mathbb{P}^2$ *be an irreducible and reduced curve of degree e. Let* $Y \to \mathbb{P}^2$ *be the blow up of* $\mathbb{P}^2$ *at r distinct smooth points* $p_1, \dots, p_r \in C$ *with exceptional divisors* $E_1, \dots, E_r$*. Let* H *denote the pull-back of* $\mathcal{O}_{\mathbb{P}^2}(1)$*. For* $d, m_1, \dots, m_r \in \mathbb{Z}^{>0}$*, let* $L = dH - \sum_{i=1}^r m_i E_i$ *be a line bundle on* Y*. Let* F *denote the proper transform of* C *on* Y*. If*

$$L \cdot F = 3d - \sum_{i=1}^r m_i > 0$$

and

$$d > m_{i_1} + \cdots + m_{i_e}$$

for any e distinct indices $i_1, \dots, i_e \in \{1, \dots, r\}$*, then* L *is ample.*

Theorem 2.8.3 ([47][Theorem 3.6]). *Let* $C \subset \mathbb{P}^2$ *be an irreducible and reduced curve of degree e. Let* $Y \to \mathbb{P}^2$ *be the blow up of* $\mathbb{P}^2$ *at r distinct smooth points* $p_1, \dots, p_r \in C$ *with exceptional divisors* $E_1, \dots, E_r$*. Let* H *denote the pull-back of* $\mathcal{O}_{\mathbb{P}^2}(1)$*. For* $d, m \in \mathbb{Z}^{>0}$*, let* $L = dH - m\sum_{i=1}^{r} E_i$ *be a line bundle on* Y*. If*

$$(d+3)e > r(m+1) \quad \text{and} \quad r \geq e^2 + 2,$$

then L *is very ample.*

We also need the following result on the very ampleness of blow ups of ruled surfaces at arbitrary points.

Theorem 2.8.4 ([48][Theorem 1.3]). *Let* W *be a ruled surface with invariant* $d \geq 0$ *over a curve* C *of genus g, with a fiber* F *and a section* C^- *satisfying* $(C^-)^2 = -d$*. Let* $Y \to W$ *be the blow up of* W *at r distinct points* $p_1, \dots, p_r$ *with exceptional divisors* $E_1, \dots, E_r$*. For simplicity, we still denote by* C^- *and* F *the divisors on* Y *given by the pullbacks of the divisors* C^- *and* F *on* W*, with no confusion. For* $a, b, m_1, \dots, m_r \in \mathbb{Z}^{>0}$*, let*

$$L = aC^- + bF - \sum_{i=1}^{r} m_i E_i$$

be a line bundle on Y*. If*

$$b \geq ad + 2g + 1 + \sum_{i=1}^{r} m_i,$$

and for any effective curve $C' = F - \sum_{i=1}^{r} \alpha_i E_i$ *with* $0 \leq \alpha_i \leq 1$*, we have*

$$L \cdot C' = a - \sum_{i} \alpha_i m_i \geq 1,$$

then L *is very ample.*

2.9 Artin's theorem on contractions

Theorem 2.9.1 ([49][Corollary 6.10]). *Let* X' *be an algebraic space and* $Y' \subset X'$ *a closed subspace such that* $I' = \mathcal{I}(Y')$ *is locally principal, for instance assume that* X'*,* Y' *are regular and of dimensions* d*,* $d-1$*, respectively. Let* $f_0 : Y' \to Y$ *be a proper map. Suppose that*

1. For every coherent sheaf F on Y', if $n \gg 0$, then

$$R^1 f_{0*}(F \otimes (I'/I'^2)^{\otimes n}) = 0.$$

2. For every $n \geq 1$, the canonical map

$$f_{0*}(\mathcal{O}_{X'}/I'^n) \otimes_{f_{0*}\mathcal{O}_{Y'}} \mathcal{O}_Y \to \mathcal{O}_Y$$

is surjective.

Then there is a modification $f : X' \to X$, $Y \subset X$ whose set-theoretic restriction to Y is f_0.

By *modification*, we mean a pair consisting of a proper map $f : X' \to X$ of algebraic spaces and a closed subset $Y \subset X$, such that the restriction of f to $U := X \setminus Y$ is an isomorphism. We may also refer to the pair as a *contraction* of X', or as a *dilatation* of X, respectively.

Note that condition (2) is automatic if $R^1 f_{0*}(I'/I'^2)^{\otimes n} = 0$ for every $n \geq 1$.

2.10 Volume of divisors

We recall the definition of various types of volumes for divisors. In this paper, we mainly consider $\mathbb{Q}$-divisors. However, for the proof of Proposition 4.3.1, we need to deal with $\mathbb{R}$-divisors.

Definition 2.10.1 (Volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. The *volume* of D is

$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{d!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^d}.$$

Definition 2.10.2 (Iitaka volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X such that Iitaka dimension $\kappa(D)$ is non-negative. Then the *Iitaka volume* of D, denoted by $\mathrm{Ivol}(D)$, is

$$\mathrm{Ivol}(D) = \limsup_{m \to \infty} \frac{\kappa(D)!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$

Definition 2.10.3 (Restricted volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{Q}$-divisor on X. Let $S \subset X$ be a normal irreducible subvariety of dimension n. Suppose that S is not contained in the augmented base locus $\mathbf{B}_+(D)$. Then the *restricted volume of D along S* is

$$\mathrm{vol}_{X|S}(D) = \limsup_{m \to \infty} \frac{n!(\dim \mathrm{Im}(H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \to H^0(S, \mathcal{O}_S(\lfloor mD|_S \rfloor))))}{m^n}.$$

For the precise definition of the augmented base locus $\mathbf{B}_+(D)$, see [50]. In this paper, we only use the fact that $\mathbf{B}_+(D)$ is a Zariski-closed subset of X such that $\mathbf{B}_+(D) \subsetneq X$ if and only if D is big. The restricted volume $\mathrm{vol}_{X|S}(D)$ measures asymptotically the number of sections of the restriction $\mathcal{O}_S(\lfloor mD|_S \rfloor)$ that can be lifted to X. If D is ample, then the restriction maps are eventually surjective, and hence

$$\mathrm{vol}_{X|S}(D) = \mathrm{vol}(D|_S).$$

In general, it can happen that $\mathrm{vol}_{X|S}(D) < \mathrm{vol}(D|_S)$.

Theorem 2.10.4 ([51][Corollary 4.27])**.** *Let X be an irreducible projective variety of dimension d, and let $S \subset X$ be an irreducible (and reduced) Cartier divisor on X. Suppose that D is a big $\mathbb{R}$-divisor such that $S \not\subseteq \mathbf{B}_+(D)$. Then the function $t \mapsto \mathrm{vol}(D + tS)$ is di□erentiable at $t = 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{vol}(D + tS))\bigg|_{t=0} = d\ \mathrm{vol}_{X|S}(D).$$

By [51][Remark 4.29], volume function has continuous partials in all directions at any point $D \in \mathrm{Big}(X)$, i.e., the function $\mathrm{vol} : \mathrm{Big}(X) \to \mathbb{R}$ is C^1.

2.11 Bounded family

Definition 2.11.1 (Bounded families of couples and pairs)**.** A *couple* consists of a projective normal variety X and a reduced divisor D on X. We call (X, D) a couple rather than a pair because $K_X + D$ is not assumed to be $\mathbb{Q}$-Cartier and (X, D) is not assumed to have good singularities.

Two couples (X, D) and (X', D') are said to be *isomorphic* if there exists an isomorphism $X \to X'$ that maps D onto D'.

Let $\mathcal{P}$ be a set of couples. We say that $\mathcal{P}$ is *bounded* if the following conditions hold:

- There exist finitely many projective morphisms $V^i \to T^i$ of varieties,
- C^i is a reduced divisor on V^i, and
- for each $(X, D) \in \mathcal{P}$, there exist an index i, a closed point $t \in T^i$, and an isomorphism $\phi : V^i_t \to X$ such that (V^i_t, C^i_t) is a couple and $\phi_* C^i_t \geq D$.

A set of projective pairs (X, B) is said to be *bounded* if the set of couples $(X, \mathrm{Supp} B)$ forms a bounded family.

Definition 2.11.2 (Bounded families of generalised pairs)**.** A set $\mathcal{P}$ of generalised pairs is said to be a *bounded family* if there is a fixed $r \in \mathbb{N}$ such that for any $(X, B + M) \in \mathcal{P}$, we can find a very ample divisor A on X satisfying

$$A^{\dim X} \leq r \quad \text{and} \quad A^{\dim X-1} \cdot (K_X + B + M) \leq r.$$

The first condition implies that the underlying variety X is bounded. If the coefficients of B belong to a DCC set Φ and $K_X + B$ is $\mathbb{Q}$-Cartier, then the first two conditions imply that $(X, \operatorname{Supp} B)$ belongs to a log bounded family of couples by Lemma 2.11.3. However, since M is not necessarily effective, we cannot control $\operatorname{Supp} M$. In practice, we can only bound the intersection number $A^{\dim X-1} \cdot M$.

Boundedness for couples is equivalent to the following criterion.

Lemma 2.11.3 ([5][Lemma 2.20]). *Let $d, r \in \mathbb{N}$. Assume $\mathcal{P}$ is a set of couples (X, D) where X is of dimension d and there is a very ample divisor H on X with $H^d \leq r$ and $H^{d-1} \cdot D \leq r$. Then $\mathcal{P}$ is bounded.*

Lemma 2.11.4 ([18][Lemma 4.6]). *Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Then there exists $l \in \mathbb{N}$ satisfying the following. Assume*

- *X is a normal projective variety of dimension d,*
- *H is a very ample divisor,*
- *B is a divisor with coefficients in Φ, and*
- *$H^d \leq r$ and $B \cdot H^{d-1} \leq r$.*

Then $lH - B$ is pseudo-effective.

The following theorem is one of the main ingredients in the proof of Theorem A. We emphasise that it imposes no restriction on the coefficients of B and M.

Theorem 2.11.5 ([6][Theorem 1.8]). *Let $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{Q}^{>0}$. Then there is a positive rational number t depending only on d, r, ϵ, satisfying the following. Assume*

- *(X, B) is projective ϵ-lc of dimension d,*
- *H is a very ample divisor on X with $H^d \leq r$,*
- *$H - B$ is pseudo-effective, and*
- *$M \geq 0$ is an $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $H - M$ pseudo-e□ective.*

Then

$$\operatorname{lct}(X, B, |M|_{\mathbb{Q}}) \geq \operatorname{lct}(X, B, |H|_{\mathbb{Q}}) \geq t.$$

We will use the following boundedness result for polarized nef pairs to deduce Theorem C from Theorem A.

Theorem 2.11.6 ([15][Theorem 1.5]). *Let $d \in \mathbb{N}$, $\delta, v \in \mathbb{Q}^{>0}$. Consider pairs (X, B) and nef and big Weil divisors N on X such that*

- *(X, B) is projective ϵ-lc of dimension d,*
- *the coefficients of B are in $\{0\} \cup [\delta, \infty)$,*
- *$K_X + B$ is nef,*
- $\mathrm{vol}(K_X + B + N) \leq v$.

Then the set of such (X, B) forms a bounded family. If in addition $N \geq 0$, then the set of such $(X, B + N)$ forms a bounded family.

CHAPTER 3 FAILURE OF BOUNDEDNESS FOR GENERALISED LOG CANONICAL SURFACES

In this chapter, we will first prove the boundedness of $\mathcal{F}_{glc}(2, \Phi, v)$ under additional assumptions. Then, we construct counterexamples to the boundedness of $\mathcal{F}_{glc}(2, \Phi, v)$, where the underlying varieties X can have arbitrary Kodaira dimension. Since boundedness holds when X has only rational singularities by Theorem 1.2.3, in our examples, X has elliptic singularities. This chapter is based on the preprint [20].

3.1 Boundedness of generalised lc pairs under additional assumptions

We first recall the birational boundedness of $\mathcal{F}_{glc}(d, \Phi, v)$.

Theorem 3.1.1 ([16][Proposition 5.2]). *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. Then there exists a bounded set of couples $\mathcal{P}$ such that for each*

$$(X, B + M) \in \mathcal{F}_{glc}(d, \Phi, v)$$

with data $X' \xrightarrow{\phi} X$ and $M' = \sum \mu_i M_i'$, there is a log smooth couple $(\overline{X}, \overline{\Sigma}) \in \mathcal{P}$ and a birational map $\overline{X} \dashrightarrow X$ such that

- *$\overline{\Sigma} \geq \overline{B}$, where $\overline{B}$ is the sum of the reduced exceptional divisors of $\pi : \overline{X} \dashrightarrow X$ plus the birational transform of B,*
- *each M_i' descends to $\overline{X}$, say as $\overline{M}_i$, and*
- *letting $\overline{M} = \sum \mu_i \overline{M}_i$, we have*

$$\mathrm{vol}(K_{\overline{X}} + \overline{B} + \overline{M}) = v.$$

We can assume that $\psi : X' \dashrightarrow \overline{X}$ is a morphism. Since $K_X + B + M$ is ample,

$$\phi^*(K_X + B + M) \leq \psi^*(K_{\overline{X}} + \overline{B} + \overline{M}).$$

Thus, by [16][Lemma 2.17], $X \dashrightarrow \overline{X}$ does not contract any divisor, and $(X, B + M)$ is the generlised log canonical model of $(\overline{X}, \overline{B} + \overline{M})$.

Corollary 3.1.2. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. For each*

$$(X, B + M) \in \mathcal{F}_{\mathrm{glc}}(d, \Phi, v),$$

assume that K_X is pseudo-effective and that the MMP and Abundance conjectures hold.

Let Z be the log canonical model of X. Then Z belongs to a bounded family. Moreover, if K_X is ample, then the generalised pair $(X, B + M)$ is bounded.

Proof Let $(\overline{X}, \overline{\Sigma})$ be the birationally bounded model of $(X, B + M)$ given in Theorem 3.1.1, and let $\overline{E} \subset \overline{\Sigma}$ be the sum of the reduced exceptional divisors of the birational map $\pi : \overline{X} \dashrightarrow X$. If K_X is pseudo-effective, then since the rational map $\pi^{-1} : X \dashrightarrow \overline{X}$ does not contract any divisor, the variety Z is also the log canonical model of the pair $(\overline{X}, \overline{E})$. Moreover, since $(\overline{X}, \overline{E})$ is log smooth and belongs to a bounded family, its log canonical model also belongs to a bounded family, by [3]Theorem 1.2.

If K_X is ample, then X belongs to a bounded family by the previous paragraph. Therefore, there exists a positive integer $l \in \mathbb{N}$ depending only on (d, Φ, v) such that lK_X is very ample. Then we have

$$(lK_X)^d = l^d \operatorname{vol}(K_X) \leq l^d \operatorname{vol}(K_X + B + M) = l^d v,$$

and

$$(lK_X)^{d-1} \cdot (K_X + B + M) \leq l^{d-1}(K_X + B + M)^d = l^{d-1} v.$$

Hence, the generalised pair $(X, B + M)$ is bounded.

∎

Proof of Theorem 1.2.3. We use the notation in Theorem 3.1.1. Since $\pi^{-1} : X \dashrightarrow \overline{X}$ does not contract any divisor, we are dealing with surfaces, the map $\overline{X} \dashrightarrow X$ is an actual morphism.

(1). We aim to show that the Cartier index of M is bounded, and hence $(X, B + M)$ is bounded by [19]Theorem 1.14.

By Step 4 of [16]Proof of Theorem 1.5, p.44, the coefficients of B and the μ_i all belong to a fixed finite set depending only on $(2, \Phi, v)$. By Step 4 of [16]Proof of Theorem 1.3, p.35, there is a log smooth couple $(\overline{\mathcal{X}}, \overline{\Omega})$ and a log smooth projective morphism $(\overline{\mathcal{X}}, \overline{\Omega}) \to T$ onto a smooth variety such that for each $(X, B + M) \in \mathcal{F}_{glc}(2, \Phi, v)$, there is a closed point $t \in T$ so that $(\overline{X}, \overline{\Sigma}) \simeq (\overline{\mathcal{X}}_t, \overline{\Omega}_t)$. Moreover, for each i, there exist irreducible components $\overline{\mathcal{G}}_i, \overline{\mathcal{F}}_i$ of $\overline{\Omega}$ such that

$$\overline{\mathcal{F}}_{i,t} - \overline{\mathcal{G}}_{i,t} \sim n\overline{M}_i$$

for some $n \in \mathbb{N}$ depending only on $d = 2$. Therefore, there exist Cartier divisors $n\overline{\mathcal{M}}_i$ on $\overline{\mathcal{X}}$ such that $n\overline{\mathcal{M}}_{i,t} \sim n\overline{M}_i$ for each i. Let $\overline{\mathcal{M}} := \sum \mu_i \overline{\mathcal{M}}_i$, then $\operatorname{Supp}(\overline{\mathcal{M}}) \subset \overline{\Omega}$. Since μ_i belongs to a finite set, after replacing n by a bounded multiplier, $n\overline{\mathcal{M}}$ is Cartier and $n\overline{\mathcal{M}}_t \sim n\overline{M}$.

We now follow the idea in Step 5 of [19][Proof of Theorem 1.14, p.845]. For each $(X, B + M) \in \mathcal{F}_{glc}(2, \Phi, v)$ and the birational morphism $\pi : \overline{X} \to X$, we can write

$$\pi^* M = \overline{M} + \sum b_j E_j,$$

where $E = \sum_j E_j$ is the π-exceptional divisor. Since $\overline{M}$ and the exceptional divisors of $\pi : \overline{X} \to X$ deform in the family, and the numerical pullback is computed intersection-theoretically, it follows from the negative definiteness of intersection matrix $(E_j \cdot E_k)_{jk}$ that the system of linear equations

$$(\overline{M} + \sum_j b_j E_j) \cdot E_k = 0 \quad \text{for all } k$$

has a unique solution depending only on $(2, \Phi, v)$. Therefore, there exists a divisor

$$\overline{\Phi} \leq \overline{\Omega} + \overline{\mathcal{M}},$$

supported on $\overline{\Omega}$, such that $\pi^* M = \overline{\Phi}_t$. Let l be a positive integer depending only on $(2, \Phi, v)$ such that $l\overline{\Phi}$ is integral.

Since X has only rational singularities, by [36][Lemma 4.13], $l\overline{\Phi}_t \sim 0$ in an analytic neighborhood of any π-exceptional curve. Thus, $\pi_* \mathcal{O}_{\overline{X}_t}(l\overline{\Phi}_t)$ is locally free in an analytic neighborhood of any closed point $x \in X$. Then, lM is Cartier at $\widehat{\mathcal{O}}_{X,x}$ for every closed point $x \in X$. Therefore, lM is Cartier by [52][Lemma 5.12].

(2). If $X \to Z$ is a minimal ruled fibration onto a nonsingular curve Z, then X has only rational singularities by [53][Lemma 4.6]. It follows from (1) that $(X, B + M)$ is bounded.

(3). Since the exceptional divisors of $\pi : \overline{X} \to X$ belong to a bounded family, the dual graphs of the π-exceptional curves and the corresponding weights are determined. By [36][Remark 4.9], the analytic isomorphism type of the germ $x \in X$ is determined by the dual graph. Therefore, the Cartier index of $-K_X$ is bounded. Then, by the effective base point free theorem [54][Theorem 1.1] and the very ampleness lemma [55][Lemma 7.1] for lc pairs, we can find $l \in \mathbb{N}$ depending only on $(2, \Phi, v)$ such that $-lK_X$ is very ample.

For each $(X, B + M) \in \mathcal{F}_{glc}(2, \Phi, v)$ and the birational morphism $\pi : \overline{X} \to X$, we can write

$$\pi^* K_X = K_{\overline{X}} + \sum a_j E_j$$

and

$$\pi^*(K_X + B + M) = K_{\overline{X}} + \pi_*^{-1} B + \overline{M} + \sum c_j E_j,$$

where E_j are the π-exceptional divisors. Moreover, we have

$$(K_{\overline{X}} + \sum a_j E_j) \cdot E_k = 0$$

and

$$(K_{\overline{X}} + \pi_*^{-1} B + \overline{M} + \sum c_j E_j) \cdot E_k = 0$$

for all k. By the same proof as in (1), the coefficients a_j and c_j are determined by $(2, \Phi, v)$. Since $\pi_*^{-1} B$, M, and the exceptional divisors of the birational morphism $\pi : \overline{X} \to X$ deform in the family, it follows that

$$(-K_X)^2 = (-\pi^* K_X)^2$$

and

$$(-K_X) \cdot (K_X + B + M) = \pi^*(-K_X) \cdot \pi^*(K_X + B + M)$$

are bounded. Hence, $(X, B + M)$ belongs to a bounded family.

(4). It follows from Corollary 3.1.2. Alternatively, to bound $(X, B + M)$, we can bound the Cartier index of K_X and the intersection numbers

$$K_X^2 \quad \text{and} \quad K_X \cdot (K_X + B + M)$$

by the same argument as in (3). ∎

3.2 Calabi-Yau surfaces

In this section, we construct an unbounded set of generalised pairs $(X_i, B_i + M_i) \in \mathcal{P}_0 \subset \mathcal{F}_{glc}(2, \Phi, v)$ for $\Phi = \{0, 1\}$ and $v = 22$, such that $K_{X_i} \sim 0$, $B_i = 0$, and M_i is an ample Weil divisor with an unbounded Cartier index.

3.2.1 Construction

Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve, and let $p_{i,1}, \dots, p_{i,11} \in C$ be distinct points such that for each $1 \leq j \leq 11$, the divisor $p_{i,j} - p_0 \in \mathrm{Pic}^0(C)$ has order $n_{i,j}$, where p_0 is the identity element of the group structure on C. Let n_i be the order of the divisor $\sum_{j=1}^{11}(p_0 - p_{i,j})$. We may fix the points $p_{i,1}, \dots, p_{i,10}$ and vary $p_{i,11}$ so that $n_{i,j}$ is independent of i for $1 \leq j \leq 10$, and both $n_{i,11}$ and n_i tend to infinity as $i \to \infty$.

Let $f_i : Y_i \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $p_{i,1}, \dots, p_{i,11}$, and let $E_{i,1}, \dots, E_{i,11}$ be the exceptional divisors over $p_{i,1}, \dots, p_{i,11}$, respectively. Let F_i denote the birational transform of C on Y_i. Write $E_i = \sum_{j=1}^{11} E_{i,j}$, then we have

$$K_{Y_i} = f_i^* K_{\mathbb{P}^2} + E_i \quad \text{and} \quad F_i + E_i = f_i^* C,$$

hence we have

$$K_{Y_i} + F_i = f_i^*(K_{\mathbb{P}^2} + C) \sim 0.$$

Let $H_i = f_i^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $A_i = 5H_i - E_i$, then by Theorem 2.8.3, A_i is very ample on Y_i. We have

$$\mathrm{vol}(A_i) = A_i^2 = 25 - 11 = 14.$$

Hence, Y_i belongs to a bounded family.

Since $F_i \sim 3H_i - E_i$, an easy computation shows that $(A_i + 2F_i) \cdot F_i = 0$. Let $\widetilde{p_{i,j}}$ be the points on F_i lying over $p_{i,j}$ for $1 \leq j \leq 11$. Then we have

$$\begin{aligned}
(A_i + 2F_i)|_{F_i} =&(11H_i - 3E_i)|_{F_i} \\
=&f_i^*(\mathcal{O}_{\mathbb{P}^2}(11))|_{F_i} \otimes \mathcal{O}_{F_i}(-\sum_{j=1}^{11} \widetilde{p_{i,j}})^{\otimes 3} \\
=&f_i^*(\mathcal{O}_{\mathbb{P}^2}(11)|_C \otimes \mathcal{O}_C(-\sum_{j=1}^{11} p_{i,j})^{\otimes 3}) \\
=&f_i^*(\mathcal{O}_C(\sum_{j=1}^{11} (p_0 - p_{i,j}))^{\otimes 3}).
\end{aligned}$$

Let m_i be the order of $(A_i + 2F_i)|_{F_i} \in \mathrm{Pic}^0(F_i)$ so that $\mathcal{O}_{Y_i}(m_i(A_i + 2F_i)) \otimes \mathcal{O}_{F_i} \simeq \mathcal{O}_{F_i}$. Then we have $m_i = n_i$ if $3 \nmid n_i$, and $m_i = \frac{n_i}{3}$ if $3 \mid n_i$.

Lemma 3.2.1. *The linear system $|m_i(A_i + 2F_i)|$ is big and base point free on Y_i, and it defines a contraction $\pi_i : Y_i \to X_i$ to a projective Calabi-Yau surface, which contracts F_i to a simple elliptic singularity.*

Proof Since A_i is ample and F_i is effective, $A_i + 2F_i$ is big. Moreover, we have $(A_i + 2F_i) \cdot F_i' \geq 1$ for any irreducible curve $F_i' \neq F_i$, and $(A_i + 2F_i) \cdot F_i = 0$. Hence, $A_i + 2F_i$ is nef. We aim to show $|m_i(A_i + 2F_i)|$ is base point free.

We first claim that the restriction map $\mathrm{Pic}(2F_i) \to \mathrm{Pic}(F_i)$ is an isomorphism, which implies

$$\mathcal{O}_{Y_i}(m_i(A_i + 2F_i)) \otimes \mathcal{O}_{2F_i} \simeq \mathcal{O}_{2F_i}.$$

Indeed, the short exact sequence

$$0 \to \mathcal{O}_{Y_i}(-F_i) \otimes \mathcal{O}_{F_i} \to \mathcal{O}^*_{2F_i} \to \mathcal{O}^*_{F_i} \to 0$$

induces a long exact sequence

$$H^1(F_i, \mathcal{O}_{F_i}(-F_i|_{F_i})) \to \mathrm{Pic}(2F_i) \to \mathrm{Pic}(F_i) \to 0.$$

Since $-F_i^2 = 2$, we have $H^1(F_i, \mathcal{O}_{F_i}(-F_i|_{F_i})) = 0$, which yields the desired isomorphism.

Since $\mathcal{O}_{Y_i}(m_i(A_i + 2F_i)) \otimes \mathcal{O}_{2F_i} \simeq \mathcal{O}_{2F_i}$, we have the short exact sequence

$$0 \to \mathcal{O}_{Y_i}(m_iA_i + (2m_i - 2)F_i) \to \mathcal{O}_{Y_i}(m_i(A_i + 2F_i)) \to \mathcal{O}_{2F_i} \to 0. \tag{3.1}$$

By Theorem 2.8.2, the divisor

$$m_iA_i + (2m_i - 1)F_i \sim (11m_i - 3)H_i - (3m_i - 1)E_i$$

is ample. Hence, Kodaira vanishing gives

$$H^1(Y_i, m_iA_i + (2m_i - 2)F_i) = H^1(Y_i, K_{Y_i} + m_iA_i + (2m_i - 1)F_i) = 0.$$

Therefore, the map

$$H^0(Y_i, m_i(A_i + 2F_i)) \to H^0(\mathcal{O}_{2F_i})$$

induced by the exact sequence (3.1) is surjective.

Thus, there exists a divisor in the linear system $|m_i(A_i + 2F_i)|$ that does not intersect any point of $F_i = \mathrm{Supp}(2F_i)$. Moreover, since A_i is very ample, any base point of $|m_i(A_i+2F_i)|$ must be contained in F_i. Therefore, $|m_i(A_i+2F_i)|$ is base point free. Since it is also big, it defines a birational morphism $\pi_i : Y_i \to X_i$ onto a projective surface X_i with $K_{X_i} \sim 0$, which contracts F_i to a simple elliptic singularity. ∎

Now let $M_{Y_i} := A_i + 2F_i$ be the big and semi-ample Cartier divisor on Y_i, which defines the contraction $\pi_i : Y_i \to X_i$ that contracts F_i to a simple elliptic singularity. Then we have

$$K_{Y_i} + F_i = \pi_i^* K_{X_i} \quad \text{and} \quad \pi_i^* M_i = M_{Y_i},$$

where $M_i := \pi_{i*} M_{Y_i}$. Therefore, (X_i, M_i) is a generalised lc surface with data $Y_i \to X_i$ and M_{Y_i}. Moreover, $K_{X_i} + M_i$ is ample and

$$\mathrm{vol}(K_{X_i} + M_i) = \mathrm{vol}(K_{Y_i} + F_i + M_{Y_i}) = (11H_i - 3E_i)^2 = 22.$$

Thus,

$$(X_i, M_i) \in \mathcal{P}_0 \subset \mathcal{F}_{glc}(2, \Phi, v)$$

for $\Phi = \{0, 1\}$ and $v = 22$.

We claim that the Cartier index of M_i is not bounded. Indeed, assume that there exists $m \in \mathbb{N}$ such that mM_i is Cartier for all i. Then, by the effective base point free theorem [54]Theorem 1.1 and the very ampleness lemma [55]Lemma 7.1 for lc pairs, we can find $m' \in \mathbb{N}$ such that $m'M_i$ is very ample for all i. Hence, $m'M_{Y_i}$ is big and base point free for all i, which is a contradiction.

3.2.2 Failure of boundedness of $\mathcal{P}_0$

Proposition 3.2.2. *The set of generalised pairs in $\mathcal{P}_0$ is not bounded.*

Proof We will show that there is no fixed $r \in \mathbb{N}$ and a very ample divisor N_i on X_i satisfying $N_i \cdot (K_{X_i} + M_i) \leq r$ for all $(X_i, M_i) \in \mathcal{P}_0 \subset \mathcal{F}_{glc}(2, \Phi, v)$. Thus, $\mathcal{P}_0$ is not bounded in the sense of §2.11.

Let $N_{Y_i} := \pi_i^* N_i$. Then N_{Y_i} is a big and base point free Cartier divisor on Y_i with $N_{Y_i} \cdot F_i = 0$. Moreover, for any irreducible curve $F_i' \neq F_i$ on Y_i, we have $N_{Y_i} \cdot F_i' > 0$. Writing $N_{Y_i} \sim d_i H_i - \sum_{j=1}^{11} l_{i,j} E_{i,j}$ for some integers $d_i, l_{i,1}, \dots, l_{i,11}$, the condition $N_{Y_i} \cdot F_i = 0$ gives

$$3d_i - \sum_{j=1}^{11} l_{i,j} = 0. \tag{3.2}$$

Since $N_{Y_i} \cdot E_{i,j} = l_{i,j} > 0$ for all $1 \leq j \leq 11$, we have $d_i > 0$. For N_{Y_i} to be base point free on Y_i, we must have $N_{Y_i}|_{F_i} \simeq \mathcal{O}_{F_i}$. Since

$$\begin{aligned} N_{Y_i}|_{F_i} =& f_i^*(\mathcal{O}_{\mathbb{P}^2}(d_i))|_{F_i} \otimes \mathcal{O}_{F_i}(-\sum_{j=1}^{11} l_{i,j}\widetilde{p_{i,j}}) \\ =& f_i^*(\mathcal{O}_{\mathbb{P}^2}(d_i)|_C \otimes \mathcal{O}_C(-\sum_{j=1}^{11} l_{i,j}p_{i,j})) \\ =& f_i^*(\mathcal{O}_C(\sum_{j=1}^{11} l_{i,j}(p_0 - p_{i,j}))), \end{aligned}$$

it follows that

$$\mathcal{O}_C(\sum_{j=1}^{11} l_{i,j}(p_0 - p_{i,j})) \simeq \mathcal{O}_C. \tag{3.3}$$

Since the orders $n_{i,1}, \dots, n_{i,10}$ of $p_{i,1}, \dots, p_{i,10}$ are fixed, we can choose a strictly increasing subsequence of $n_{i,11}$, the order of $p_{i,11} - p_0$, such that $n_{i,11}$ is coprime to each of $n_{i,1}, \dots, n_{i,10}$. Thus, we have $l_{i,11} \to +\infty$ as $i \to +\infty$ by (3.3). Therefore, (3.2) implies that $d_i \to +\infty$ as $i \to +\infty$. Hence, we conclude that

$$N_i \cdot (K_{X_i} + M_i) = N_{Y_i} \cdot M_{Y_i} = 11d_i - 3\sum_{j=1}^{11} l_{i,j} = 2d_i \to +\infty$$

as $i \to +\infty$. ∎

Remark 3.2.3. If we choose $p_{i,11}$ such that $\sum_{j=1}^{11}(p_{i,j} - p_0) \in \mathrm{Pic}^0(C)$ is non-torsion, then $A_i + 2F_i$ is nef and big but not semiample in Lemma 3.2.1. In this case, X_i exists as a complex analytic space [56], or even as an algebraic space [49]. However, Y_i cannot be contracted to a projective surface X_i. Indeed, assume that X_i is projective, then there exists a very ample divisor N_i on X_i. By (3.3) in the proof of Proposition 3.2.2, we have $\mathcal{O}_C(\hat{n}_i l_{i,11}(p_0 - p_{i,11})) \simeq \mathcal{O}_C$, where $\hat{n}_i = \prod_{j=1}^{10} n_{i,j}$, which is a contradiction.

The failure of boundedness of the underlying varieties is more subtle. We also need to take non-projective surfaces into consideration. The following lemma is also mentioned in [57][Example 6], which shows that projectivity is not an open condition in the family of algebraic spaces with singularities slightly worse than rational singularities.

Lemma 3.2.4. *There exists a family of algebraic spaces $h : \mathcal{X} \to S$ such that the projective surfaces $X_i \in \mathcal{P}_0$ correspond to a countable dense subset S^τ of S.*

Proof Fix $C \subset \mathbb{P}^2$ a smooth elliptic curve and distinct points $p_1, \ldots, p_{10} \in C$. Consider the projection

$$\mathbb{P}^2 \times S \to S \simeq C \setminus \{p_1, \ldots, p_{10}\}.$$

Let $g : \mathcal{Y} \to S$ be the family of surfaces given by blowing up the constant sections $p_1 \times S, \ldots, p_{10} \times S$ and the diagonal section

$$\Gamma = \{(p_s, s) \subset \mathbb{P}^2 \times S \mid s \in S\}$$

in $\mathbb{P}^2 \times S$ where $p_s \in C$ is the point corresponding to $s \in S$ under the given isomorphism $S \simeq C \setminus \{p_1, \ldots, p_{10}\}$. Thus, for any $s \in S$, we have that $\mathcal{Y}_s$ is the blow up of $\mathbb{P}^2$ along $p_1, \ldots, p_{10}, p_s$. Let $\mathcal{E}$ be the exceptional divisor for $f : \mathcal{Y} \to \mathbb{P}^2 \times S$, $\mathcal{H}$ be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1) \times S$ via f, and $\mathcal{F}$ be the strict transform of $C \times S$.

Let $\mathcal{I}$ be the ideal sheaf for $\mathcal{F}$ in $\mathcal{Y}$, and $g_0 := g|_{\mathcal{F}} : \mathcal{F} \to S$. Then $\mathcal{I}/\mathcal{I}^2 \simeq -\mathcal{F}|_{\mathcal{F}}$ is ample over S. Therefore, we have

1. For every coherent sheaf $\mathcal{G}$ on $\mathcal{F}$, $R^1 g_{0*}(\mathcal{G} \otimes (\mathcal{I}/\mathcal{I}^2)^{\otimes n}) = 0$ for sufficiently large n, by relative Serre vanishing.
2. $R^1 g_{0*}(\mathcal{I}/\mathcal{I}^2)^{\otimes n} = 0$ for every $n \geq 1$, by relative Kodaira vanishing.

By Theorem 2.9.1, we have a contraction $\pi : \mathcal{Y} \to \mathcal{X}$ to an algebraic space $\mathcal{X}$ such that the restriction of π to $\mathcal{Y} \setminus \mathcal{F}$ is an isomorphism and it contracts $\mathcal{F}$ to a section of $\mathcal{X} \to S$. Then the induced family $h : \mathcal{X} \to S$ is a family of algebraic spaces. Let

$$S^\tau := \{s \in S \mid p_1 + \ldots + p_{10} + p_s \text{ is torsion on } C\}, \quad \text{and} \quad S^* := S \setminus S^\tau.$$

Here $p_1 + \ldots + p_{10} + p_s$ is the sum in C according to the group law on C. Then by Lemma 3.2.1, $\mathcal{M}_{\mathcal{Y}} := 11\mathcal{H} - 3\mathcal{E}$ is nef and big over S. Moreover, $(\mathcal{M}_{\mathcal{Y}})_s$ is semiample for $s \in S^\tau$, which contracts $\mathcal{Y}_s$ to a projective surface $\mathcal{X}_s$, while $(\mathcal{M}_{\mathcal{Y}})_s$ is not semiample for $s \in S^*$. Thus $\mathcal{X}_s$ is projective if and only if $s \in S^\tau$. ∎

Theorem 3.2.5. *The set of underlying varieties appearing in $\mathcal{P}_0$ is not bounded.*

Proof Let $g : \mathcal{Y} \xrightarrow{\pi} \mathcal{X} \xrightarrow{h} S$ be the two families obtained in Lemma 3.2.4, where $\mathcal{Y}$ is projective over S but $\mathcal{X}$ is not projective over S ($\mathcal{X}_s$ is projective if and only if $s \in S^\tau$). Suppose that the set $\{\mathcal{X}_s \mid s \in S^\tau\}$ is bounded. Then, there exists a projective family $h' : \mathcal{X}' \to T$ such that for every $s \in S^\tau$, there exists some $t = t(s) \in T$ with $\mathcal{X}_s \cong \mathcal{X}'_t$. We may assume that $\mathcal{X}' \to T$ is a locally closed subset of the corresponding Hilbert scheme. After replacing T with the closure of S^τ in T, we may assume that S^τ is dense in T.

After replacing T with a dense open subset and S^τ with the corresponding subset, we can take a common minimal resolution of $\mathcal{X}'_t$, and it is just the blowing up of a section of $\mathcal{X}' \to T$, and so we get a morphism $\pi' : \mathcal{Y}' \to \mathcal{X}'$ with an irreducible exceptional divisor $\mathcal{F}'$. Moreover, for every $s \in S^\tau$, there exists some $t = t(s) \in T$ such that $\mathcal{Y}'_t \supset \mathcal{F}'_t \cong \mathcal{F}_s \subset \mathcal{Y}_s$, where $\mathcal{F}_s$ is the strict transform of the elliptic curve $C \subset \mathbb{P}^2$.

$$\begin{tikzcd}
\mathcal{Y} \arrow[d, "\pi"] \arrow[rrrrrrrr, bend left] & & & & \mathcal{Y}_s \arrow[d] \arrow[r, phantom, "\simeq"] & \mathcal{Y}'_t \arrow[d] & & \mathcal{Y}' \arrow[r] \arrow[d, "\pi'"] & \mathcal{U} \arrow[dd] \\
\mathcal{X} \arrow[d, "h"] & & & & \mathcal{X}_s \arrow[d] \arrow[r, phantom, "\simeq"] & \mathcal{X}'_t \arrow[d] & & \mathcal{X}' \arrow[d, "h'"] & \\
S \arrow[rrrrrrrr, bend right] & \supset & S^\tau & \ni & s \arrow[r] & t = t(s) & \in & T \arrow[r] & H
\end{tikzcd}$$

Fix a relatively very ample divisor $\mathcal{A}'$ for $g' : \mathcal{Y}' \to T$ such that $H^1(\mathcal{Y}'_t, \mathcal{O}_{\mathcal{Y}'_t}(\mathcal{A}'_t)) = 0$ for $t \in T$. Let $\mathcal{U} \to H$ be the corresponding universal family over the Hilbert scheme H. Then there is a morphism $T \to H$ such that $\mathcal{Y}' = \mathcal{U}_T := \mathcal{U} \times_H T$. For every $s \in S^\tau$, we have a corresponding very ample divisor $\mathcal{A}_s$ of fixed degree on $\mathcal{Y}_s$. Since $H^i(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}) = 0$ for $i = 1, 2$ and $s \in S$, then $H^1(\mathcal{Y}_s, \mathcal{O}^*_{\mathcal{Y}_s}) \cong H^2(\mathcal{Y}_s, \mathbb{Z})$ for $s \in S$. Thus, after replacing S with an étale cover, there exists a line bundle $\mathcal{L}$ on $\mathcal{Y}$ such that $\mathcal{L}_s \cong \mathcal{O}_{\mathcal{Y}_s}(\mathcal{A}_s)$ for every $s \in S^\tau$. Since $H^1(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(\mathcal{A}_s)) = 0$ for $s \in S^\tau$, by [58]Proposition 7.7, after replacing S with an open neighborhood of some $s \in S^\tau$, we may assume that $\mathcal{L}_s$ is very ample with a fixed degree for all $s \in S$. This yields a morphism $S \to H$ such that $\mathcal{Y} = \mathcal{U}_S := \mathcal{U} \times_H S$.

Note that the closures of the images of S and T in H coincide, leading to a common cover $R \to S$ and $R \to T$ such that $\mathcal{Y}_R := \mathcal{Y} \times_S R$ coincides with $\mathcal{Y}'_R := \mathcal{Y}' \times_T R$. Consequently, for every $s \in S$, there exists some $t \in T$ with an isomorphism $\mathcal{Y}_s \cong \mathcal{Y}'_t$.

Let $\mathcal{N}'$ be the pullback of a relative very ample divisor on $\mathcal{X}'$ to $\mathcal{Y}'$. Then for every $s \in S$, there exists some $t \in T$ such that $\mathcal{N}'_t \subset \mathcal{Y}'_t$ gives a base point free divisor $\mathcal{N}_s \subset \mathcal{Y}_s$. However, $|\mathcal{N}_s|$ contracts $\mathcal{F}_s$, where $\mathcal{F}_s$ is the strict transform of the elliptic curve $C \subset \mathbb{P}^2$, which implies that $\mathcal{X}_s$ is projective, leading to a contradiction. ∎

3.2.3 Further remark

Remark 3.2.6. We say that X is a *polarized Calabi-Yau variety* if X is slc, $K_X \sim_{\mathbb{Q}} 0$, and there exists an ample $\mathbb{Q}$-Cartier Weil divisor M on X.

Let $d \in \mathbb{N}$ and $v \in \mathbb{Q}^{>0}$. Fix $\dim X = d$ and $\mathrm{vol}(M) = v$. In this subsection, we construct a set of polarized Calabi-Yau varieties with $d = 2$, and $v = 22$. Our example shows that to bound X, we cannot remove the klt condition in [15]Corollary 1.6, or the condition that $M \geq 0$ and (X, tM) is slc for some $0 < t \ll 1$ in [15]Corollary 1.8.

Example 3.2.7. We say that (W, D) is a *boundary polarized Calabi-Yau pair* [45, 59] if (W, D) is slc, $K_W + D \sim_{\mathbb{Q}} 0$, and D is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W. Note that $-K_W$ is ample and so W is an slc Fano variety.

Let $d \in \mathbb{N}$ and $v \in \mathbb{Q}^{>0}$. Fix $\dim W = d$ and $\mathrm{vol}(-K_W) = v$. For $(X_i, M_i) \in \mathcal{P}_0$ constructed in this subsection, let

$$W_i = C_p(X_i, M_i) := \mathrm{Proj} \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} H^0(X_i, \mathcal{O}_{X_i}(mM_i)) \cdot t^n$$

be the projective cone over X_i, and $X_i^\infty \simeq X_i$ be the divisor at infinity. Then by [45]Proposition 2.20.3, (W_i, X_i^∞) is a boundary polarized Calabi-Yau pair. Let $\gamma_i : V_i \to W_i$ be the blow up of the vertex with exceptional divisor X_i^-, set $X_i^+ = \gamma_i^* X_i^\infty$, and let $\beta_i : V_i \to X_i$ be the associated $\mathbb{P}^1$-bundle. Let $\tau_i : U_i \to V_i$ be the resolution of singularities with exceptional divisor $R_{F_i} \simeq \alpha_i^* F_i$ where $\alpha_i : U_i \to Y_i$ is the associated $\mathbb{P}^1$-bundle, and set $Y_i^\pm = \tau_i^* X_i^\pm$. We have the following commutative diagram.

$$\begin{tikzcd}
(U_i, R_{F_i} + Y_i^- + Y_i^+) \arrow[r, "\tau_i"] \arrow[d, "\alpha_i"] & (V_i, X_i^- + X_i^+) \arrow[r, "\gamma_i"] \arrow[d, "\beta_i"] & (W_i, X_i^\infty) \\
(Y_i, F_i + M_{Y_i}) \arrow[r, "\pi_i"] & (X_i, M_i) &
\end{tikzcd}$$

We claim that the Cartier index of $-K_{W_i}$ is unbounded, and hence (W_i, X_i^∞) is unbounded by the same argument as in [17][§5.3, p.20]. Indeed, if the Cartier index of $-K_{W_i}$ is bounded, then by [60][Corollary 1.6], there exsits $q \in \mathbb{N}$ such that $q(K_{W_i} + X_i^\infty) \sim 0$ for for all i. Hence, the Cartier index of X_i^∞ is also bounded. Since $X_i^\infty - (-K_{W_i})$ is ample, by the effective base point free theorem [54][Theorem 1.1] and the very ampleness lemma [55][Lemma 7.1] for lc pairs, we can find $l \in \mathbb{N}$ such that lX_i^∞ is very ample for all i. Therefore, $lX_i^+ = \gamma_i^*(lX_i^\infty)$ is big and base point free on V_i for all i. However, since $Y_i^+|_{Y_i^+} \sim M_{Y_i}$, it follows that $X_i^+|_{X_i^+} \sim M_i$, which implies that the Cartier index of X_i^+ is not bounded, leading to a contradiction.

Note that $(U_i, \alpha_i^* F_i + Y_i^- + Y_i^+)$ is a dlt modification of (W_i, X_i^∞) with minimal non-klt centers $F_i^- := R_{F_i} \cap Y_i^- \simeq F_i$ and $F_i^+ := R_{F_i} \cap Y_i^+ \simeq F_i$. Therefore, we obtain an unbounded set of boundary polarized Calabi-Yau 3-fold pairs (W_i, X_i^∞) such that $\mathrm{vol}(-K_{W_i}) = (X_i^\infty)^3 = M_i^2 = 22$ and $\mathrm{reg}(W_i, X_i^\infty) = 1$ (see [45][Definition 8.5]). This case is not treated in [45][§8].

3.3 Minimal surfaces of Kodaira dimension one

In this section, we construct an unbounded set of generalised pairs $(U_i, B_{U_i} + M_{U_i}) \in \mathcal{P}_1 \subset \mathcal{F}_{glc}(2, \Phi, v)$ for $\Phi = \{0, 1\}$ and $v = 68$, such that K_{U_i} is semiample with $\kappa(U_i) = 1$, $B_{U_i} = 0$, and M_{U_i} is an ample Weil divisor with unbounded Cartier index.

In the example of §3.2, let $D \subset \mathbb{P}^2$ be another smooth elliptic curve, and assume that $p_{i,1}, \ldots, p_{i,9} \in C \cap D$. We first blow up $\mathbb{P}^2$ at $p_{i,1}, \ldots, p_{i,9}$, giving a pencil spanned by the strict transforms of C and D. Then, we further blow up $p_{i,10}$ and $p_{i,11}$ on the strict transform of C, we obtain a surface Y_i with a morphsim $Y_i \to \mathbb{P}^1$. Let M_{Y_i} be the big and semiample Cartier divisor on Y_i that defines a contraction $\pi_i : Y_i \to X_i$ which contracts the strict transform F_i of C on Y_i. Let $M_i = \pi_{i*} M_{Y_i}$, and let $X_i \to \mathbb{P}^1$ be the induced morphism.

Let $G_{i,1}$ and $G_{i,2}$ be two general fibers of $Y_i \to \mathbb{P}^1$ that do not contain F_i. Let $\nu_i : V_i \to Y_i$ be the double cover of Y_i branched over $G_{i,1}$ and $G_{i,2}$, then $M_{V_i} := \nu_i^* M_{Y_i}$ is also a big and semiample Cartier divisor. Since Y_i is smooth and $G_{i,1}$, $G_{i,2}$ are disjoint smooth fibers, it follows that V_i is also smooth. Let $\mu_i : U_i \to X_i$ be the double cover of X_i branched over $\pi_{i*}G_{i,1}$ and $\pi_{i*}G_{i,2}$, then $M_{U_i} := \mu_i^* M_i$ is also ample with unbounded Cartier index. We have the following commutative diagram, where $\tau_i : \mathbb{P}^1 \to \mathbb{P}^1$ is a double cover branched over two points.

$$\begin{tikzcd}
(V_i, \nu_i^* F_i + M_{V_i}) \arrow[r, "\nu_i"] \arrow[d, "\gamma_i"] & (Y_i, F_i + M_{Y_i}) \arrow[d, "\pi_i"] \\
(U_i, M_{U_i}) \arrow[r, "\mu_i"] \arrow[d] & (X_i, M_i) \arrow[d] \\
\mathbb{P}^1 \arrow[r, "\tau_i"] & \mathbb{P}^1
\end{tikzcd}$$

Since $(Y_i, F_i + \frac{1}{2}(G_{i,1} + G_{i,2}) + M_{Y_i})$ is generalised lc, and

$$K_{V_i} + \nu_i^* F_i + M_{V_i} = \nu_i^*(K_{Y_i} + F_i + \frac{1}{2}(G_{i,1} + G_{i,2}) + M_{Y_i}),$$

by [36][Proposition 5.20], $(V_i, \nu_i^* F_i + M_{V_i})$ is also generalised lc. Since

$$\gamma_i^* K_{U_i} = K_{V_i} + \nu_i^* F_i \quad \text{and} \quad \gamma_i^* M_{U_i} = M_{V_i},$$

it follows that (U_i, M_{U_i}) is a generalised lc surface with data $V_i \to U_i$ and M_{V_i}. Moreover,

$$K_{U_i} = \mu_i^*(K_{X_i} + \frac{1}{2}(\pi_{i*}G_{i,1} + \pi_{i*}G_{i,2}))$$

is semiample with $\kappa(K_{U_i}) = 1$.

Recall that $M_{Y_i} \sim 11H_i - 3E_i$, and $G_{i,1} \sim G_{i,2} \sim 3H_i - (E_{i,1} + \cdots + E_{i,9})$. We have

$$M_i \cdot \pi_{i*}G_{i,k} = M_{Y_i} \cdot G_{i,k} = 33 - 3 \times 9 = 6$$

for $k = 1, 2$. Now, $K_{U_i} + M_{U_i}$ is ample and

$$\begin{aligned}
\operatorname{vol}(K_{U_i} + M_{U_i}) &= \mu_i^*(K_{X_i} + \frac{1}{2}(\pi_{i*}G_{i,1} + \pi_{i*}G_{i,2}) + M_i)^2 \\
&= 2(K_{X_i} + \frac{1}{2}(\pi_{i*}G_{i,1} + \pi_{i*}G_{i,2}) + M_i)^2 \\
&= 2(2M_i \cdot \pi_{i*}G_{i,2} + M_i^2) \\
&= 2(12 + 22) \\
&= 68.
\end{aligned}$$

Thus

$$(U_i, M_{U_i}) \in \mathcal{P}_1 \subset \mathcal{F}_{glc}(2, \Phi, v)$$

for $\Phi = \{0, 1\}$ and $v = 68$.

Remark 3.3.1. If we choose $p_{i,11}$ such that $\sum_{j=1}^{11}(p_{i,j} - p_0) \in \mathrm{Pic}^0(C)$ is non-torsion, then U_i is not projective. Indeed, assume that U_i is projective, then there exists an ample divisor N_{U_i} on U_i. Since the double cover $\mu_i : U_i \to X_i$ is always a Galois cover with group $G = \mathbb{Z}_2$, define $N_{U_i}^G := \frac{1}{|G|}\sum_{g\in G} g^* N_{U_i}$. Then $N_{U_i}^G$ is G-invariant, so there exists an ample divisor N_i on X_i such that $N_{U_i}^G = \mu_i^* N_i$. This implies that X_i is projective, which is a contradiction.

Theorem 3.3.2. *The set of underlying varieties appearing in $\mathcal{P}_1$ is not bounded.*

Proof Suppose that U_i belongs to a bounded family. The induced fibration $U_i \to \mathbb{P}^1$ is just the Iitaka fibration and hence it also belongs to a bounded family. Let $\alpha_i : U_i \to U_i$ be the involution that fixes the ramification divisor of $\mu_i : U_i \to X_i$. From the construction in this subsection, we know that the involution α_i is determined by an involution $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ that fixes two points of $\mathbb{P}^1$, and hence α_i is bounded.

It then follows that $X_i = U_i / < \alpha_i >$ is also bounded, which contradicts Theorem 3.2.5. ∎

Remark 3.3.3. We say that U is a *stable minimal model* if U is slc, K_U is semiample defining a contraction $f : U \to Z$, and there exists a $\mathbb{Q}$-Cartier Weil divisor M_U on U which is ample over Z. In particular, F is a polarized Calabi-Yau variety with polarization $M_U|_F$, where F is the general fiber of $f : U \to Z$.

Let $d \in \mathbb{N}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial, and $u \in \mathbb{Q}^{>0}$. Fix $\dim U = d$, $(K_U + tM_U)^d = \sigma(t)$, and $\mathrm{vol}(M_U|_F) = u$. In this subsection, we have constructed a set of stable minimal models with $d = 2$, $\sigma(t) = 24t + 44t^2$, and $u = 12$. Our example shows that to bound U, we cannot remove the klt condition in [30]$^{\text{Theorem 1.3}}$, or the condition that $M_U \geq 0$ and (U, tM_U) is slc for some $0 < t \ll 1$ in [18]$^{\text{Theorem 1.12}}$. Moreover, in our example, both the general fiber F and the base Z are bounded, while previously known examples of the failure of boundedness of stable minimal models with fixed (d, σ, u) arise from the failure of boundedness of the general fibers.

3.4 Minimal surfaces of general type

In this section, we construct an unbounded set of generalised pairs $(X_i, B_i + M_i) \in \mathcal{P}_2 \subset \mathcal{F}_{glc}(2, \Phi, v)$ such that K_{X_i} is nef and big but not ample, $B_i = 0$, and M_i is an ample divisor with unbounded Cartier index.

Let $C \subset \mathbb{P}^2$ be a smooth cubic, and take r general lines $L_j \subset \mathbb{P}^2$. Let $h : W \to \mathbb{P}^2$ be the blow up of all singular points of $C+\sum L_j$. Denote the strict transform of $C+\sum L_j$ by $C'+\sum L'_j$. Define $H := h^*\mathcal{O}_{\mathbb{P}^2}(1)$. For $1 \le j \le r$, $1 \le l \le 3$, let E_{jl} be the exceptional divisor over $C \cap L_j$. Similarly, for $1 \le j < k \le r$, let F_{jk} be the exceptional divisor over $L_j \cap L_k$. Let $\tau : W \to Z$ be the contraction of $C'+\sum L'_j$. Then Z has one simple elliptic singularity (the image of C) and r quotient singularities (the image of the L_j). Moreover, K_Z is ample if $r \ge 4$. Since our computation does not coincide with [61][Example 5] (where it is stated that K_Z is ample if $r \ge 6$), we provide the details of the computation.

Lemma 3.1: If $r \ge 4$, then K_Z is ample.

Proof Note that we have the following relations:

$$K_W = h^*K_{\mathbb{P}^2} + \sum_{1\le j\le r, 1\le l\le 3} E_{jl} + \sum_{1\le j<k\le r} F_{jk}, \tag{3.4}$$

$$C' = 3H - \sum_{1\le j\le r, 1\le l\le 3} E_{jl}, \tag{3.5}$$

$$L'_j = H - \sum_{1\le l\le 3} E_{jl} - \sum_{1\le k\le r, k\ne j} F_{jk}. \tag{3.6}$$

From (3.5) and (3.6), we deduce that

$$2\sum_{1\le j<k\le r} F_{jk} = \sum_{1\le j\le r}(H - \sum_{1\le l\le 3} E_{jl} - L'_j) = rH - (3H - C') - \sum_{1\le j\le r} L'_j. \tag{3.7}$$

Then by (3.4), (3.5), and (3.7), we have

$$K_W = \frac{r-3}{2}H - \frac{1}{2}C' - \frac{1}{2}\sum_{1\le j\le r} L'_j.$$

Since L'_j is a rational curve with $(L'_j)^2 = 1-3-(r-1) = -1-r$, the log discrepancy is given by $a(L'_j, Z, 0) = \frac{2}{r+1}$. Thus, we have

$$\begin{aligned}\tau^*K_Z &= K_W + C' + \sum_{1\le j\le r}(1 - \frac{2}{r+1})L'_j \\ &= \frac{r-3}{2}H + \frac{1}{2}C' + \frac{r-3}{2(r+1)}\sum_{1\le j\le r} L'_j.\end{aligned}$$

If $r \ge 4$, then for any irreducible curve $\widetilde{C}$ such that $\widetilde{C} \ne C'$ and $\widetilde{C} \ne L'_j$ for all j, we have $\tau^*K_Z \cdot \widetilde{C} \ge \frac{r-3}{2} > 0$. Since $\tau^*K_Z \cdot C' = \tau^*K_Z \cdot L'_j = 0$, it follows that if $r \ge 4$, then

$$(\tau^*K_Z)^2 = (K_W + C' + \sum_{1\le j\le r}(1 - \frac{2}{r+1})L'_j) \cdot K_W$$

$$
\begin{aligned}
&= K_W^2 + -C'^2 + \sum_{1\le j\le r}(1-\frac{2}{r+1})(-2-L_j'^2)\\
&= (9-3r-\frac{r(r-1)}{2}) + (3r-9) + r(1-\frac{2}{r+1})(-2+r+1)\\
&= \frac{r(r-1)(r-3)}{2(r+1)} > 0.
\end{aligned}
$$

Thus, by the Nakai-Moishezon criterion, K_Z is ample if $r \ge 4$. ∎

Pick a point $p_i \in C' \setminus (C' \cap \sum L_j') \subset W$ such that p_i is torsion with order n_i. Let $g_i : Y_i \to W$ be the blow up of W at p_i with exceptional divisor E_i, and denote by $C_i' + \sum L_{ij}'$ the strict transform of $C' + \sum L_j'$ on Y_i. Then $g_i^*C' = C_i' + E_i$. Define $H_i := g_i^*H$, $E_{ijl} := g_i^*E_{jl}$ for $1 \le j \le r$, $1 \le l \le 3$, and $F_{ijk} := g_i^*F_{jk}$ for $1 \le j < k \le r$. Let $\pi_i : Y_i \to X_i$ be the contraction of $C_i' + \sum L_{ij}'$. Define $E_{X_i} := \pi_{i*}E_i$. Then $f_i : X_i \to Z$ is a contraction with a single exceptional divisor E_{X_i}. We have the following commutative diagram.

$$
\begin{tikzcd}
(Y_i \supset H_i, C_i', L_{ij}', E_i, E_{ijl}, F_{ijk}) \arrow[r, "g_i"] \arrow[d, "\pi_i"] & (W \supset H, C', L_j', E_{jl}, F_{jk}) \arrow[r, "h"] \arrow[d, "\tau"] & \mathbb{P}^2 \\
(X_i \supset E_{X_i}) \arrow[r, "f_i"] & Z &
\end{tikzcd}
$$

Lemma 3.4.1. *K_{X_i} is nef and big but not ample.*

Proof We may write

$$K_{X_i} = f_i^*K_Z + mE_{X_i}$$

for some $m \in \mathbb{Q}$. Then, together with the following relations:

$$K_W + C' + \sum_{1\le j\le r}(1-\frac{2}{r+1})L_j' = \tau^*K_Z,$$

$$K_{Y_i} = g_i^*K_W + E_i,$$

$$K_{Y_i} + C_i' + \sum_{1\le j\le r}(1-\frac{2}{r+1})L_{ij}' = \pi_i^*K_{X_i}, \tag{3.8}$$

$$g_i^*C' = E_i + C_i',$$

$$g_i^*L_j' = L_{ij}',$$

we deduce that $m = 0$, hence $K_{X_i} = f_i^*K_Z$ is nef and big. Moreover, since $K_{X_i} \cdot E_{X_i} = 0$, it follows that K_{X_i} is not ample. ∎

We aim to construct an ample $\mathbb{Q}$-Cartier divisor M_i on X_i such that its Cartier index is unbounded. The argument is similar to Lemma 3.2.1.

Fix $r \geq 4$ and let $d \geq 2(3r + \binom{r}{2} + 2)$ be a fixed positive integer. Then, by Theorem 2.8.1,

$$A_i := dH_i - \sum E_{ijl} - \sum F_{ijk} - E_i$$

is very ample on Y_i. Let (a, b) be the unique solution of the system of linear equations

$$\begin{cases} (A_i + aC_i' + b\sum L_{ij}') \cdot C_i' = 0, \\ (A_i + aC_i' + b\sum L_{ij}') \cdot \sum L_{ij}' = 0, \end{cases}$$

and let $q \in \mathbb{N}$ such that $M_{Y_i} := q(A_i + aC_i' + b\sum L_{ij}')$ is an integral divisor. Since $a = -A_i \cdot C_i'/(C_i')^2 > 0$ and $b = -A_i \cdot L_{ij}'/(L_{ij}')^2 > 0$ for any j, it follows that $M_{Y_i} \cdot \widetilde{C}_i > 0$ for any irreducible curve $\widetilde{C}_i$ on Y_i with $\widetilde{C}_i \neq C_i'$ and $\widetilde{C}_i \neq L_{ij}'$ for all j, and hence M_{Y_i} is nef. Moreover, since M_{Y_i} is the sum of an ample divisor and effective divisors, it follows that M_{Y_i} is big. Let $m_i \in \mathbb{N}$ such that $\mathcal{O}_{Y_i}(m_i M_{Y_i}) \otimes \mathcal{O}_{C_i'} \simeq \mathcal{O}_{C_i'}$, where m_i depends on the torsion order of p_i.

Lemma 3.4.2. *The linear system $|m_i M_{Y_i}|$ is base point free.*

Proof Let D_i be either the curve C_i' or L_{ij}', since $D_i^2 < 0$ and $g(D_i) \leq 1$, we have $H^1(D_i, \mathcal{I}_{D_i}^t/\mathcal{I}_{D_i}^{t+1}) = 0$ for $t \geq 1$, where $\mathcal{I}_{D_i} := \mathcal{O}_{Y_i}(-D_i)$ is the ideal sheaf of D_i in Y_i. Thus, the cohomology sequence induced by the truncated exponential exact sequences

$$0 \to \mathcal{I}_{D_i}^t/\mathcal{I}_{D_i}^{t+1} \to \mathcal{O}_{(t+1)D_i}^* \to \mathcal{O}_{tD_i}^* \to 0$$

yields the fact that the restriction maps

$$\mathrm{Pic}((t+1)D_i) \to \mathrm{Pic}(tD_i)$$

are isomorphisms for all $t \geq 1$. Define $G_i := m_i q(aC_i' + b\sum L_{ij}')$. Since

$$\mathcal{O}_{Y_i}(m_i M_{Y_i}) \otimes \mathcal{O}_{\mathrm{Supp}(G_i)} \simeq \mathcal{O}_{\mathrm{Supp}(G_i)},$$

we have

$$\mathcal{O}_{Y_i}(m_i M_{Y_i}) \otimes \mathcal{O}_{G_i} \simeq \mathcal{O}_{G_i}.$$

Thus, we obtain the following short exact sequence

$$0 \to \mathcal{O}_{Y_i}(m_i M_{Y_i} - G_i) \to \mathcal{O}_{Y_i}(m_i M_{Y_i}) \to \mathcal{O}_{G_i} \to 0.$$

This gives the long exact sequence

$$H^0(Y_i, \mathcal{O}_{Y_i}(m_i M_{Y_i})) \to H^0(G_i, \mathcal{O}_{G_i}) \to H^1(Y_i, \mathcal{O}_{Y_i}(m_i q A_i)).$$

Since $m_i q A_i - K_{Y_i}$ is very ample by Theorem 2.8.1, it follows that $H^1(Y_i, \mathcal{O}_{Y_i}(m_i q A_i)) = 0$ by Kodaira's vanishing. Therefore, $H^0(Y_i, \mathcal{O}_{Y_i}(m_i M_{Y_i})) \to H^0(G_i, \mathcal{O}_{G_i})$ is surjective, and hence the base points of $|m_i M_{Y_i}|$ are not contained in $\mathrm{Supp}(G_i)$. Moreover, since A_i is very ample, any base point of $|m_i M_{Y_i}|$ must be contained in C'_i or L'_{ij}. Thus, $|m_i M_{Y_i}|$ is base point free. ∎

Now, let $M_i := \pi_{i*} M_{Y_i}$ be the corresponding ample divisor on X_i, where the Cartier index of M_i is unbounded. Then $M_{Y_i} = \pi^* M_i$, and it follows from by (3.8) that (X_i, M_i) is a generalised lc surface with data $Y_i \to X_i$ and M_{Y_i}. Moreover, $K_{X_i} + M_i$ is ample and $\mathrm{vol}(K_{X_i} + M_i)$ is fixed. Thus,

$$(X_i, M_i) \in \mathcal{P}_2 \subset \mathcal{F}_{glc}(2, \Phi, v)$$

for $\Phi = \{0, 1\}$ and some fixed v.

Theorem 3.4.3. *The set of underlying varieties appearing in $\mathcal{P}_2$ is not bounded.*

Proof If we choose a non-torsion point $p_i \in C' \setminus (C' \cap \sum L'_j) \subset W$, then by the same argument as in Remark 3.2.3, X_i is non-projective. Moreover, we have

$$H^1(Y_i, \mathcal{O}_{Y_i}) = H^2(Y_i, \mathcal{O}_{Y_i}) = 0$$

for every choice of $p_i \in C' \setminus (C' \cap \sum L'_j) \subset W$. Therefore, we can use the same argument as in Theorem 3.2.5 to show that $\mathcal{P}_2$ is not bounded. ∎

Remark 3.4.4. Given a variety Z and a birational contraction $f : X \to Z$, we can write

$$K_X + B = f^* K_Z$$

for some uniquely determined B. We say (X, B) is a *crepant model* of Z if $B \geq 0$. By the birational case of [29][Theorem 1.2], we know that if Z is ϵ-lc for some fixed $\epsilon \in \mathbb{Q}^{>0}$ and it belongs to a bounded family, then the underlying varieties of all the crepant models of Z form a bounded family. Our example in this subsection shows that the ϵ-lc condition is necessary.

3.5 Weak Fano surfaces

In this section, we construct an unbounded set of generalised pairs $(X_i, B_i + M_i) \in \mathcal{P}_{-\infty} \subset \mathcal{F}_{glc}(2, \Phi, v)$ for $\Phi = \{0, \frac{1}{8}\}$ and $v = \frac{45}{16}$, such that $-K_{X_i}$ is nef and big but not ample, $B_i = 0$, and M_i is an ample $\mathbb{Q}$-divisor with unbounded Cartier index. Our example is inspired by [62][2.19].

Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve, and Z be the projective cone over C. Then Z is a Fano surface with $\rho(Z) = 1$, and $K_Z + C^\infty \sim_{\mathbb{Q}} 0$, where C^∞ is the section of Z at infinity. Let $\tau : W \to Z$ be the blow up of the vertex with exceptional divisor C^-, and $W \to C$ be the associated $\mathbb{P}^1$-bundle. Fix $p_0 \in C^-$, then we can embed C^- into $\mathbb{P}^2$ such that p_0 serves as the identity element of the group structure on C^-. Let C^+ be the positive section of $W \to C$, and F be a fiber of $W \to C$. By [63]Proposition V.2.3, we have $\mathrm{Pic}(W) = \mathbb{Z}C^- \oplus \mathbb{Z}F$, and $C^+ = C^- + 3F$. Moreover, We have

$$-(C^-)^2 = (C^+)^2 = 3, \quad C^+ \cdot F = C^- \cdot F = 1, \quad \text{and} \quad F^2 = 0.$$

Pick a point $p_i \in C^- \subset W$ such that the divisor $p_i - p_0 \in \mathrm{Pic}^0(C^-)$ has order n_i. Let $g_i : Y_i \to W$ be the blow up of W at p_i with exceptional divisor E_i. Then Y_i is bounded by [64]Lemma 3.8. Denote $C_i^\pm := g_i^* C^\pm$ and $F_i := g_i^* F$ on Y_i. Then we have $C_i^- = E_i + G_i^-$, where G_i^- is the strict transform of C^- on Y_i. Since

$$-3 = (C_i^-)^2 = (E_i + G_i^-)^2 = -1 + 2 + (G_i^-)^2,$$

it follows that $(G_i^-)^2 = -4$. By [63]Proposition V.3.2, we have

$$\mathrm{Pic}(Y_i) = \mathbb{Z}C_i^- \oplus \mathbb{Z}F_i \oplus \mathbb{Z}E_i.$$

Let $\pi_i : Y_i \to X_i$ be the contraction that contracts G_i^-. Define $E_{X_i} := \pi_{i*}E_i$. Then $f_i : X_i \to Z$ is a contraction with a single exceptional divisor E_{X_i}. Write $\pi_i^* E_{X_i} = E_i + nG_i^-$. From $\pi_i^* E_{X_i} \cdot G_i^- = 0$, we deduce that $1 + n(-4) = 0$, hence $n = \frac{1}{4}$. Therefore,

$$(E_{X_i})^2 = (E_i + \frac{1}{4}G_i^-)^2 = (-1 + \frac{1}{2} + \frac{1}{16}(-4)) = -\frac{3}{4}.$$

Let $C_{X_i}^+ := f_i^* C^\infty$. We have the following commutative diagram.

$$\begin{tikzcd}
(Y_i \supset C_i^+, C_i^-, F_i, E_i, G_i^-) \arrow[r, "g_i"] \arrow[d, "\pi_i"] & (W \supset C^+, C^-, F) \arrow[r] \arrow[d, "\tau"] & C \\
(X_i \supset C_{X_i}^+, E_{X_i}) \arrow[r, "f_i"] & (Z \supset C^\infty) &
\end{tikzcd}$$

Lemma 3.5.1. *$-K_{X_i}$ is nef and big but not ample.*

Proof Assume that

$$K_{X_i} = f_i^* K_Z + mE_{X_i}$$

for some $m \in \mathbb{Q}$. Then, together with the following relations:

$$K_W + C^- = \tau^* K_Z,$$

$$K_{Y_i} = g_i^* K_W + E_i,$$

$$K_{Y_i} + G_i^- = \pi_i^* K_{X_i},$$

$$g_i^* C^- = E_i + G_i^-,$$

we deduce that $m = 0$, hence $-K_{X_i} = -f_i^* K_Z$ is nef and big. Moreover, since $-K_{X_i} \cdot E_{X_i} = 0$, it follows that $-K_{X_i}$ is not ample. ∎

Lemma 3.2: $N_i := C^+_{X_i} - \frac{1}{2}E_{X_i}$ is ample on X_i.

Proof Let

$$N_{Y_i} := \pi_i^* N_i = (C_i^- + 3F_i) - \frac{1}{2}(E_i + \frac{1}{4}(C_i^- - E_i)) = \frac{7}{8}C_i^- + 3F_i - \frac{3}{8}E_i.$$

Note that

$$N_i^2 = (N_{Y_i})^2 = (C^+_{X_i})^2 + \frac{1}{4}(E_{X_i})^2 = 3 + \frac{1}{4}(-\frac{3}{4}) = \frac{45}{16} > 0.$$

Thus, to show that N_i is ample, it suffices to check that $N_{Y_i} \cdot C_i' > 0$ for any irreducible curve $C_i' \neq G_i^-$ on Y_i by the Nakai-Moishezon criterion.

Let $L_i := 5C_i^- + 24F_i - E_i$, then L_i is very ample by Theorem 2.8.4. Moreover, we have

$$L_i + 2G_i^- = 5C_i^- + 24F_i - E_i + 2(C_i^- - E_i) = 8N_{Y_i}.$$

It follows that

$$8N_{Y_i} \cdot C_i' = (L_i + 2G_i^-) \cdot C_i' > 0$$

for any irreducible curve $C_i' \neq G_i^-$ on Y_i. ∎

Remark 3.5.2. The Cartier index of E_{X_i} is unbounded. Indeed, if it were bounded, then, since the Cartier index of $C^+_{X_i}$ is bounded, it follows that the Cartier index of $N_i = C^+_{X_i} - \frac{1}{2}E_{X_i}$ is also bounded. Since $N_i - K_{X_i}$ is ample, by the effective base point free theorem [54][Theorem 1.1] and the very ampleness lemma [55][Lemma 7.1] for lc pairs, there exists $l \in \mathbb{N}$ such that lN_i is very ample for all i. Then $\pi_i^*(8lN_i) \sim 8lC_i^+ - 4lE_i - lG_i^-$ is base point free on Y_i, and thus $\mathcal{O}_{Y_i}(8lC_i^+ - 4lE_i - lG_i^-)|_{G_i^-} \simeq \mathcal{O}_{G_i^-}$. It follows that $\mathcal{O}_{C^-}(3lp_i) \simeq \mathcal{O}_{\mathbb{P}^2}(l)|_{C^-} \simeq \mathcal{O}_{C^-}(3lp_0)$ for all i, which is a contradiction. Similarly, we can show that if we choose $p_i \in C^- \subset W$ such that the divisor $p_i - p_0 \in \mathrm{Pic}^0(C^-)$ is non-torsion, then no multiple of E_{X_i} is Cartier.

Note that in our example, Z is fixed and Y_i is bounded. However, since the Cartier index of E_{X_i} is unbounded, by the proof of [62][Claim 2.19.1], we deduce that X_i cannot be obtained from Z by blowing up a zero-dimensional subscheme of bounded length. Thus, we cannot apply the "sandwich" argument in [64][Proof of Theorem 6.9] to get the boundedness of X_i.

Now, $M_i := 2C^+_{X_i} - \frac{1}{2}E_{X_i}$ is an ample $\mathbb{Q}$-divisor on X_i with unbounded Cartier index.

Then,

$$M_{Y_i} := \pi_i^* M_i = 2C_i^+ - \frac{1}{2}(E_i + \frac{1}{4}G_i^-) = \frac{1}{8}(16C_i^+ - 4E_i - G_i^-)$$

is nef and big, and hence $16C_i^+ - 4E_i - G_i^-$ is a nef Cartier divisor on Y_i. Moreover, $(Y_i, G_i^- + M_{Y_i})$ is a generalised lc pair. Therefore, (X_i, M_i) is a generalised lc surface with data $Y_i \to X_i$ and M_{Y_i}. Moreover, $K_{X_i} + M_i \sim_{\mathbb{Q}} C_{X_i}^+ - \frac{1}{2}E_{X_i}$ is ample and

$$\mathrm{vol}(K_{X_i} + M_i) = (C_{X_i}^+)^2 + \frac{1}{4}(E_{X_i})^2 = \frac{45}{16}.$$

Thus,

$$(X_i, M_i) \in \mathcal{P}_{-\infty} \subset \mathcal{F}_{glc}(2, \Phi, v)$$

for $\Phi = \{0, \frac{1}{8}\}$ and $v = \frac{45}{16}$.

Theorem 3.5.3. *The set of underlying varieties appearing in $\mathcal{P}_{-\infty}$ is not bounded.*

Proof We will show that there is no fixed $r \in \mathbb{N}$ and a very ample divisor H_i on X_i satisfying $H_i^2 \leq r$ for all i. Thus, X_i is not bounded.

Let $H_{Y_i} := \pi_i^* H_i$, then H_{Y_i} is a big and base point free Cartier divisor on Y_i, which contracts only G_i^-. Since $\mathrm{Pic}(Y_i) = \mathbb{Z}C_i^+ \oplus \mathbb{Z}F_i \oplus \mathbb{Z}E_i$, write

$$H_{Y_i} \sim a_i C_i^+ + b_i F_i + c_i E_i,$$

then we have

$$H_{Y_i} \cdot G_i^- = 0, \quad H_{Y_i} \cdot E_i > 0, \quad \text{and} \quad H_{Y_i} \cdot C_i^+ > 0,$$

which implies

$$b_i = -c_i, \quad -c_i > 0, \quad \text{and} \quad 3a_i + b_i > 0. \tag{3.9}$$

Moreover, we have

$$\begin{aligned} H_i^2 &= (a_i C_i^+ + b_i F_i - b_i E_i)^2 \\ &= (a_i C_i^+ + b_i F_i)^2 + b_i^2 E_i^2 \\ &= 3a_i^2 + 2a_i b_i - b_i^2 \\ &= (a_i + b_i)(3a_i - b_i) > 0. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtain $a_i + b_i > 0$ and $3a_i - b_i > 0$. For H_{Y_i} to be base point free on Y_i, we must have $H_{Y_i}|_{G_i^-} \simeq \mathcal{O}_{G_i^-}$, which gives $\mathcal{O}_{Y_i}(3(a_i C_i^+ + b_i F_i - b_i E_i))|_{G_i^-} \simeq \mathcal{O}_{G_i^-}$, and hence

$$\mathcal{O}_{C^-}(3b_i(p_0 - p_i)) \simeq \mathcal{O}_{C^-}.$$

Thus, we have $b_i \to +\infty$ as $i \to +\infty$. Therefore, $H_i^2 \to +\infty$ as $i \to +\infty$. ∎

Remark 3.5.4. If we choose $p_i \in C^- \subset W$ such that the divisor $p_i - p_0 \in \mathrm{Pic}^0(C^-)$ is non-torsion, then X_i is not projective. Indeed, if X_i is projective, then there exists a very ample divisor H_i on X_i. Hence, by the proof of Theorem 3.5.3, we have $\mathcal{O}_{C^-}(3b_i(p_0 - p_i)) \simeq \mathcal{O}_{C^-}$. It follows that $p_i - p_0 \in \mathrm{Pic}(C^-)$ is torsion, which leads to a contradiction.

Proof of Theorem 1.2.2. It follows from Theorem 3.2.5, Theorem 3.3.2, Theorem 3.4.3 and Theorem 3.5.3

∎

CHAPTER 4 BOUNDEDNESS OF KLT GOOD MINIMAL MODELS

In this chapter, we prove boundedness for good minimal models with klt singularities, polarized by Weil divisors that are relatively nef and big over the bases of the Iitaka fibration. This chapter is based on the preprint [30].

4.1 Boundedness of generalised pairs on bases of fibrations

In this section, we consider the set of good minimal models whose general fibers of the Iitaka fibration belong to a bounded family and whose Iitaka volume is fixed.

Definition 4.1.1. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Let $\mathcal{G}_{klt}(d, \Phi, \leq u, v)$ be the set of (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:

- (X, B) is a klt pair of dimension d,
- the coefficients of B are contained in Φ,
- $K_X + B$ is semiample, defining a contraction $f : X \to Z$,
- A is a divisor on X that is nef and big over Z,
- $\mathrm{vol}(A|_F) \leq u$, where F is the general fiber of $f : X \to Z$, and
- $\mathrm{Ivol}(K_X + B) = v$.

Since $K_X + B$ is semiample, there exists a contraction $f : X \to Z$ onto a normal variety Z. By the canonical bundle formula in §2.5, we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),$$

and we may then regard $(Z, B_Z + M_Z)$ as a generalised pair with ample $K_Z + B_Z + M_Z$, that is, a *generalised log canonical (lc) model*.

Lemma 4.1.2. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exist $p, q \in \mathbb{N}$ depending only on (d, Φ, u), and $l \in \mathbb{N}$, $\epsilon \in \mathbb{Q}^{>0}$ depending only on (d, Φ, u, v), such that for any*

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \leq u, v),$$

the following hold:

1. *We have an adjunction formula*

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where $pM_{Z'}$ is Cartier on some high resolution $Z' \to Z$.

2. *The pair (X, B) is ϵ-lc, and lB is a Weil divisor.*

Proof Let (F, B_F) be a general fiber of $f : (X, B) \to Z$, and $A_F := A|_F$. Replacing X with the ample model of A over Z, we may assume that A is ample over Z. Applying [15][Corollary 1.4] to (F, B_F) and A_F, there exists $m \in \mathbb{N}$, depending only on d and Φ, such that $H^0(F, \mathcal{O}_X(mA_F)) \neq 0$. Hence $mA \sim G$ for some Weil divisor G. Replacing A with the horizontal part of G, we may assume that A is effective. By Theorem 1.1.1 (1), the set of $(F, B_F + A_F)$ forms a bounded family, hence we can assume that $\mathrm{vol}(A_F)$ is fixed.

Applying [16][Lemma 7.4] yields integers p, q satisfying (1). Moreover, by [16][Lemma 8.2], the set of log discrepancies

$$\{a(D, X, B) \leq 1 \mid D \text{ a prime divisor over } X\}$$

is finite, and hence (2) holds. Note that the proof of [16][Lemma 8.2] uses A only in the relative sense over Z. ∎

Definition 4.1.3 ([16][Definition 1.1]). Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. Let $\mathcal{F}_{gklt}(d, \Phi, v)$ be the set of projective generalised pairs $(X, B+M)$ with nef part M' such that

- $(X, B + M)$ is generalised klt of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M_i'$ where $\mu_i \in \Phi$ and M_i' are nef Cartier, and
- $K_X + B + M$ is ample with volume $\mathrm{vol}(K_X + B + M) = v$.

Now we can prove the boundedness of bases of Iitaka fibrations with their induced generalised pair structure under natural assumptions.

Theorem 4.1.4 ([16]). *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exists $l \in \mathbb{N}$ depending only on (d, Φ, u, v) such that for any*

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \leq u, v),$$

we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

such that the corresponding set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. Moreover, $l(K_Z + B_Z + M_Z)$ is very ample.

Proof By Lemma 4.1.2 (1), there exist $p, q \in \mathbb{N}$ depending only on (d, Φ, u) such that we can write an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where $pM_{Z'}$ is Cartier on some higher resolution $Z' \to Z$.

By definition of the discriminant part of the canonical bundle formula and the ACC for lc thresholds [2][Theorem 1.1], we see that the coe□cients of B_Z belong to a DCC subset of $\mathbb{Q}^{>0}$ depending only on d and Φ, which we denote by Ψ. Moreover, $(Z, B_Z + M_Z)$ is generalised klt pair and

$$\mathrm{Ivol}(K_X + B) = \mathrm{vol}(K_Z + B_Z + M_Z) = v.$$

Adding $\frac{1}{p}$, we can assume $\frac{1}{p} \in \Psi$, we see that

$$(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, \Psi, v).$$

In the proof of [16][Theorem 1.4], a divisor Θ is constructed such that

$$l(K_X + \Theta) \sim l(1 + t)(K_X + B + M)$$

is ample, (X, Θ) is ϵ-lc, and the coefficients of Θ belong to a fixed DCC set Ψ'. Here $l \in \mathbb{N}$, $t, \epsilon \in \mathbb{Q}^{>0}$, and $\Psi' \subset \mathbb{Q}^{>0}$ depend only on (d, Φ, u, v). Moreover, (X, Θ) is log birationally bounded. By [2][Theorem 1.6], (X, Θ) belongs to a bounded family. Thus, we may replace l so that both $l(K_X + \Theta)$ and $l(K_X + B + M)$ are very ample. Hence, the set of generalised pairs $(Z, B_Z + M_Z)$ forms a bounded family. Replacing q, l with ql, we conclude the proof. ∎

4.2 Boundedness of nef threshold

In this section, we show that the nef threshold of $K_X + B$ with respect to A is bounded for all

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \leq u, \sigma).$$

We follow the argument of [18][Theorem 4.1] with some modifications. The main difference is that, since A may not be an effective divisor in our situation, we cannot directly apply the cone theorem to bound the nef threshold.

Therefore, we first assume that $K_X + B + \lambda A$ is big for some rational number $\lambda \in [0, 1]$. We can then replace $K_X + B + \lambda A$ by an effective $\mathbb{Q}$-divisor E, but this loses control of the coefficients of E. For this reason, we require a stronger boundedness result on singularities in Theorem 2.11.5 compared to [18][Lemma 4.7]. To make the induction argument go through, we also need to show that $H - A$ is pseudo-effective, as in Theorem C.

Proposition 4.2.1. *Theorem B_d and Theorem C_{d-1} imply Theorem A_d.*

Proof We proceed by induction on the dimension of X.

Step 1. For each

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \leq u, \sigma),$$

we have

$$\sigma(t) = (K_X + B + tA)^d = \sum_{i=0}^{d} \binom{d}{i} (K_X + B)^{d-i} \cdot A^i t^i,$$

so the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ are determined by d and σ for each $0 \leq i \leq d$. In particular, for a general fiber F of $X \to Z$,

$$\operatorname{Ivol}(K_X + B) \cdot \operatorname{vol}(A|_F) = (K_X + B)^{\dim Z} \cdot A^{d-\dim Z} \tag{4.1}$$

is a fixed number depending only on d and σ. Let $B_F := B|_F$, and $A_F := A|_F$. Applying [15][Theorem 1.3] to $(F, B_F), A_F$, it follows that the linear system $|mA_F|$ defines a birational map for some m depending only on d and Φ. Replacing A by mA, we may assume that A_F is effective. By Theorem 1.1.1 (1), the set of $(F, B_F + A_F)$ forms a bounded family, hence we may assume that $\operatorname{vol}(A_F)$ is fixed, which we again denote by u. Therefore, $v := \operatorname{Ivol}(K_X + B)$ is also fixed by (4.1).

Step 2. By Theorem B and Lemma 4.1.2 (2), we may choose $\alpha \in \mathbb{N}$ depending only on (d, Φ, u, v, λ) such that $\alpha(K_X + B + \frac{\lambda}{2}A)$ is a big Weil divisor. Moreover,

$$\begin{aligned} &\operatorname{vol}\left(K_X + B + t\alpha(K_X + B + \tfrac{\lambda}{2}A)\right) \\ =&\operatorname{vol}\left((1 + t\alpha)(K_X + B) + \tfrac{t\alpha\lambda}{2}A\right) \\ =&\left((1 + t\alpha)(K_X + B) + \tfrac{t\alpha\lambda}{2}A\right)^d \end{aligned}$$

is a polynomial γ in t whose coe□cients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$, α and λ. Therefore, γ is determined by $(d, \Phi, u, \sigma, \lambda)$.

Replacing A, u, σ with $\alpha(K_X + B + \frac{\lambda}{2}A), (\frac{\alpha\lambda}{2})^{\dim F}u, \gamma$, we may assume that A is a big Weil divisor.

Step 3. Since when $\dim X = 1$, $K_X + B + tA$ is always ample, and when $\dim Z = 0$, $K_X + B + tA$ is nef and big for all $0 < t < 1$, we may assume that $\dim X \geq 2$ and $\dim Z \geq 1$.

We claim that it suffices to find $\tau \in (0, 1]$, depending only on (d, Φ, u, σ), such that

$K_X + B + \tau A$ is nef. Indeed, once such a τ is found, $K_X + B + tA$ is nef and big for any $t \in (0, \tau)$. Since A is nef and big over Z, by the base point free theorem it is semiample over Z, so we may pick $0 < t' \ll t$ such that $K_X+B+t'A$ is nef and big. Then K_X+B+tA is a positive linear combination of $K_X + B + t'A$ and $K_X + B + \tau A$, and hence is nef and big.

We aim to find such a τ in the subsequent steps.

Step 4. By Theorem 4.1.4, there exists $l \in \mathbb{N}$ depending only on (d, Φ, u, v) such that we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

and the generalised klt pair $(Z, B_Z + M_Z)$ belongs to a bounded family. Moreover,

$$L := l(K_Z + B_Z + M_Z)$$

is very ample.

Let T be a general member of $|L|$, and let S be its pullback to X. Define

$$K_S + B_S := (K_X + B + S)|_S$$

and set $A_S := A|_S$. Then

$$(S, B_S),\ A_S \to T \in \mathcal{G}_{klt}\big(d-1, \Phi, \leq u, \psi\big)$$

for some polynomial $\psi(t)$ depending only on (d, Φ, u, σ).

Indeed, we may choose a general $T \in |L|$ such that $A|_S$ is nef and big over T and $(X, B + S)$ is plt. Hence (S, B_S) is a projective klt pair, and $K_S + B_S$ is semi-ample, defining the contraction $g\colon S \to T$. If G is a general fibre of $S \to T$, then

$$\mathrm{vol}(A_S|_G) = \mathrm{vol}(A|_G) = u,$$

since G is among the general fibres of $X \to Z$. Moreover,

$$\begin{aligned}
\psi(t) &= (K_S + B_S + tA_S)^{d-1} \\
&= \big((K_X + B + S + tA)|_S\big)^{d-1} \\
&= (K_X + B + S + tA)^{d-1} \cdot S \\
&= \big((l+1)(K_X + B) + tA\big)^{d-1} \cdot S \\
&= \big((l+1)(K_X + B) + tA\big)^{d-1} \cdot l(K_X + B),
\end{aligned}$$

which is a polynomial in t whose coefficients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$ and by l, and hence depend only on d, σ, and l.

Step 5. By Theorem C in lower dimension, there exists a fixed $r \in \mathbb{N}$ such that for any $(S, B_S), A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1} \leq r, \quad (K_S + B_S) \cdot H_S^{d-2} \leq r, \quad \text{and} \quad H_S - A_S \text{ is pseudo-effective.}$$

By Lemma 2.11.4, we may further assume that $H_S - B_S$ is pseudo-effective.

Since A is big, there exists an effective $\mathbb{Q}$-divisor E such that $A \sim_{\mathbb{Q}} E$. As S is the pullback of a general element of a very ample linear system, we have $E_S := E|_S$ effective and $A_S \sim_{\mathbb{Q}} E_S$. Moreover,

$$H_S - E_S \sim_{\mathbb{Q}} H_S - A_S$$

is also pseudo-effective.

By the same argument as in Step 1, $v' := \mathrm{Ivol}(K_S + B_S)$ is fixed. Therefore, (S, B_S) is ϵ-lc for some $\epsilon \in \mathbb{Q}^{>0}$ depending only on $(d-1, \Phi, u, v')$ by Lemma 4.1.2 (2).

Thus by Theorem 2.11.5, there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $(d-1, \epsilon, r)$ such that

$$\mathrm{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt. Then by inversion of adjunction [36][Theorem 5.50], $(X, B + S + \tau E)$ is plt near S. Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z: otherwise, the non-lc locus of $(X, B + \tau E)$ maps onto a closed subset of Z positive dimension which intersects T, hence S intersects the non-lc locus of $(X, B + \tau E)$, a contradiction.

Step 6. In this step, we assume that $K_X + B + \tau E$ is not nef. Otherwise, $K_X + B + \tau A \sim_{\mathbb{Q}} K_X + B + \tau E$ is nef, and we are done by Step 3.

Let R be a $(K_X + B + \tau E)$–negative extremal ray, since $K_X + B + \tau E$ is nef and big over Z, R is not contained in the fibers of $X \to Z$. By Step 5, the non-lc locus of $(X, B + \tau E)$ maps to finitely many points of Z, so R is not contained in the image

$$\mathrm{Im}(\overline{\mathrm{NE}}(\Pi) \to \overline{\mathrm{NE}}(X)),$$

where Π is the non-lc locus of $(X, B + \tau E)$.

Then by the length of extremal ray [65] [66][Theorem 1.1], R is generated by a curve C with

$$(K_X + B + \tau E) \cdot C \geq -2d.$$

Since $L \in |l(K_Z + B_Z + M_Z)|$ is very ample, $f^*L \cdot C = L \cdot f_*C \geq 1$, we see that

$$(K_X + B + 2df^*L + \tau E) \cdot C \geq 0.$$

It follows that

$$K_X + B + 2df^*L + \tau E$$

is nef. Since $f^*L \sim l(K_X + B)$, we see that

$$K_X + B + \frac{\tau}{1+2dl}E \sim_{\mathbb{Q}} \frac{1}{1+2dl}\Big(K_X + B + 2df^*L + \tau E\Big)$$

is nef. Hence after replacing τ with $\frac{\tau}{1+2dl}$, we can assume that $K_X + B + \tau E$ is nef. ∎

4.3 Boundedness of pseudo-effective thereshold

In this section, we show that the pseudo-effective threshold of $K_X + B$ with respect to A is bounded for all

$$(X, B),\ A \to Z\ \in\ \mathcal{G}_{klt}(d, \Phi, \leq u, \sigma).$$

Proposition 4.3.1. *Theorem A_{d-1} implies Theorem B_d.*

Proof *Step 0.* In this step, we introduce the top self-intersection function $\varsigma(t)$ and the volume function $\vartheta(t)$, and then outline the main idea of the proof using these functions.

Let

$$\varsigma(t) \in \mathbb{Q}[t], \quad \varsigma(t) := \big(A + t(K_X + B)\big)^d = \sum_{i=0}^{d} \binom{d}{i} A^{d-i} \cdot (K_X + B)^i t^i,$$

be the top self-intersection function. It is easy to see that fixing ς is equivalent to fixing σ. Let

$$\vartheta(t) := \operatorname{vol}\big(A + t(K_X + B)\big)$$

be the volume function. Then $\vartheta(t)$ is a non-negative, non-decreasing real function of t, and $\vartheta(t) = \varsigma(t)$ for $t \gg 0$.

It is enough to show that there exists a positive rational number τ, depending only on (d, Φ, u, σ), such that

$$A + t(K_X + B) \text{ is big for all } t > \tau.$$

In other words, it suffices to show that $\vartheta(t) > 0$ for all $t > \tau$.

We will prove the proposition by showing:

- There exists a positive rational number τ, such that $\varsigma(t) > 0$ and strictly increasing for all $t \geq \tau$.

- Since $\varsigma(t) = \vartheta(t)$ for $t \gg 0$, a comparison of their derivatives shows that $\vartheta(t)$ decreases no faster than $\varsigma(t)$ as t decreases. Hence, $\vartheta(t) \geq \varsigma(t) > 0$ for all $t \geq \tau$.

Since the volume function $\vartheta(t)$ is differentiable only in the big cone, and our goal is to find $\tau \in \mathbb{Q}^{>0}$ such that $\vartheta(t) > 0$ for all $t > \tau$, we avoid a circular argument by applying the elementary result Lemma 4.3.2.

Step 1. We prove this proposition by induction on the dimension of Z. Since $A^{d-i} \cdot (K_X + B)^i = 0$ for $i > \dim Z$, the dimension of Z is determined by $\varsigma(t)$. Thus, we may assume $\dim Z = m$ is fixed. If $\dim Z = 0$, then clearly $A + t(K_X + B)$ is big. Hence, we may assume $\dim Z > 0$. By Step 1 of the proof of Proposition 4.2.1, we may assume that $\operatorname{vol}(A|_F)$ is fixed, which we again denote by u, and consequently $v := \operatorname{Ivol}(K_X + B)$ is also fixed, where F is a general fiber of $X \to Z$.

If $\dim Z = 1$, then

$$\varsigma(t) = \big(A + t(K_X + B)\big)^d = A^d + dA^{d-1} \cdot (K_X + B)t = A^d + duvt.$$

Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, it follows that $\varsigma'(t) = duv$. Since

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z) \sim_{\mathbb{Q}} vF,$$

we have

$$\vartheta(t) = \operatorname{vol}\big(A + t(K_X + B)\big) = v^d \operatorname{vol}(\frac{1}{v}A + tF).$$

For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose a sufficiently general fiber F_t of $X \to Z$ such that $F_t \not\subseteq \mathbf{B}_+(\frac{1}{v}A + tF_t)$. Then by Theorem 2.10.4, the function $s \mapsto \operatorname{vol}(\frac{1}{v}A + tF_t + sF_t)$ is differentiable at $s = 0$. Let $\vartheta'(t)$ denote the derivative of $\vartheta(t)$ with respect to t. This derivative is well-defined for all t such that $\vartheta(t) > 0$. By Theorem 2.10.4, we have

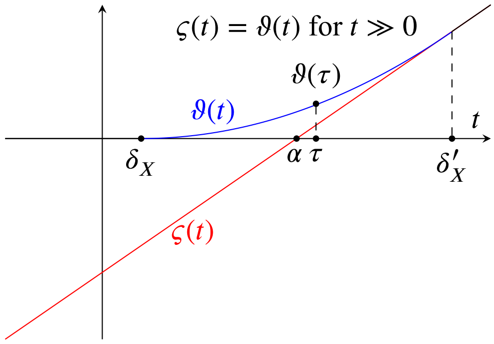
$$\frac{1}{v^d}\vartheta'(t) = \frac{1}{v^d}\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\operatorname{vol}(\frac{1}{v}A + tF_t + sF_t)\Big)\Big|_{s=0} = d\operatorname{vol}_{X|F_t}(\frac{1}{v}A + tF_t).$$

It follows that for all t such that $\vartheta(t) > 0$,

$$\vartheta'(t) = dv^d \operatorname{vol}_{X|F_t}\Big(\frac{1}{v}A + tF_t\Big) \leq dv^d \operatorname{vol}\Big(\big(\frac{1}{v}A + tF_t\big)|_{F_t}\Big) = dv^d \frac{1}{v^{d-1}}u = \varsigma'(t). \tag{4.2}$$

Let δ_X be the largest real number such that $\vartheta(\delta_X) = 0$, where δ_X may *a priori* depend on X. Since $\vartheta(t) = \varsigma(t)$ for all $t \gg 0$, there exists $\delta'_X \gg 0$ (possibly depending on X) such that $\vartheta(\delta'_X) = \varsigma(\delta'_X)$. By [67]$^{\text{Corollary 2.2.45}}$, the function $\vartheta(t)$ is continuous on $[\delta_X, \delta'_X]$, and since both $\varsigma(t)$ and $\vartheta(t)$ are differentiable on (δ_X, δ'_X), Lemma 4.3.2 yields

Figure 4.1 The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z = 1$



some $\gamma_X \in (\delta_X, \delta'_X)$ such that

$$\left(\vartheta(\delta'_X) - \vartheta(\delta_X)\right) \varsigma'(\gamma_X) = \left(\varsigma(\delta'_X) - \varsigma(\delta_X)\right) \vartheta'(\gamma_X). \tag{4.3}$$

Let α be the root of $\varsigma(t)$ and set $\tau := \max\{\lceil \alpha \rceil + 1, 1\}$, so that τ is a positive rational number with $\varsigma(t) > 0$ for all $t \geq \tau$. We claim that $\vartheta(\tau) > 0$. Suppose, for a contradiction, that $\vartheta(\tau) = 0$. Then $\tau \leq \delta_X$, hence $\varsigma(\delta_X) \geq \varsigma(\tau) > 0$. Since $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, $\vartheta(\delta_X) = 0$, and $\varsigma(\delta_X) > 0$, it follows that $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$, contradicting the inequality $\vartheta'(t) \leq \varsigma'(t)$ for all $t > \delta_X$ stated in (4.2). Therefore $\vartheta(\tau) > 0$, and hence $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t \geq \tau$.

Step 2. From now on we assume that $\dim Z = m > 1$. Recall that in Step 4 of the proof of Proposition 4.2.1, we pick a general element T in the very ample linear system $|l(K_Z + B_Z + M_Z)|$ and let S be its pullback to X, so that

$$S \sim_{\mathbb{Q}} l(K_X + B).$$

Define

$$K_S + B_S := (K_X + B + S)|_S \quad \text{and} \quad A_S := A|_S,$$

so that

$$K_S + B_S \sim_{\mathbb{Q}} \left(\frac{1}{l} + 1\right) S|_S.$$

Moreover,

$$(S, B_S), A_S \to T \in \mathfrak{G}_{klt}(d-1, \Phi, \leq u, \psi)$$

for some fixed polynomial $\psi(t) \in \mathbb{Q}[t]$ depending only on (d, Φ, u, σ), with $\dim T = m-1$. By Theorem A in lower dimension, there exists a positive rational number β, depending only on (d, Φ, u, σ), such that $A_S + t(K_S + B_S)$ is nef and big for all $t > \beta$.

Step 3. Recall that $\varsigma(t) = (A + t(K_X + B))^d$. If $t > \beta(l+1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$

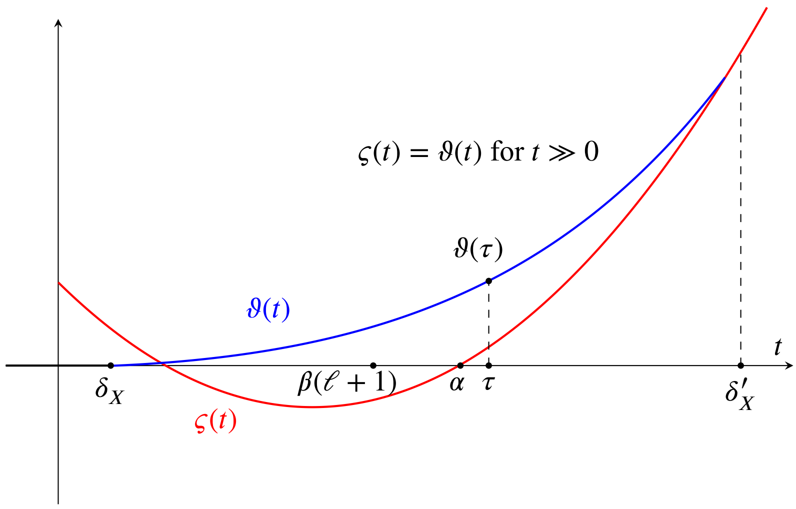
is nef and big by Step 2. We have

$$\begin{aligned}\varsigma'(t) &= d\big(A+t(K_X+B)\big)^{d-1}\cdot(K_X+B)\\ &= \frac{d}{l}\big(A+t(K_X+B)\big)^{d-1}\cdot S\\ &= \frac{d}{l}\left(A_S+\frac{t}{l+1}(K_S+B_S)\right)^{d-1}\\ &> 0.\end{aligned}$$

Hence $\varsigma(t)$ is an increasing function on $\big(\beta(l+1),+\infty\big)$.

If $\varsigma(t)$ has no roots (which occurs only when $\dim Z$ is even), set $\tau=\beta(l+1)+1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$ and set $\tau=\max\{\beta(l+1),\lceil\alpha\rceil\}+1$. Note that τ is a positive rational number. Moreover, on $[\tau,+\infty)$, $\varsigma(t)$ is a positive, increasing real function, and $\vartheta(t)$ is a non-negative, non-decreasing real function.

Figure 4.2 The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z>1$



Step 4. In this step, we conclude the proof. We see that

$$\vartheta(t)=\operatorname{vol}(A+t(K_X+B))=\frac{1}{l^d}\operatorname{vol}(lA+tS),$$

for any $S\sim_{\mathbb{Q}} l(K_X+B)$. For each t such that $A+t(K_X+B)$ is big, i.e., $\vartheta(t)>0$, we may choose S_t as the pullback of a sufficiently general element $T_t\in|l(K_Z+B_Z+M_Z)|$ such that $S_t\not\subseteq \mathbf{B}_+(lA+tS_t)$. Then by Theorem 2.10.4, the function $s\mapsto\operatorname{vol}(lA+tS_t+sS_t)$ is differentiable at $s=0$. Let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t. This derivative is well-defined for all t such that $\vartheta(t)>0$. By Theorem 2.10.4, we have

$$l^d\vartheta'(t)=l^d\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}(\operatorname{vol}\big(lA+tS_t+sS_t)\big)\Big|_{s=0}=d\operatorname{vol}_{X|S_t}(lA+tS_t).$$

It follows that for all $t \geq \tau$ such that $\vartheta(t) > 0$, we have

$$\begin{aligned}\vartheta'(t) =&\frac{d}{l^d}\operatorname{vol}_{X|S_t}(lA + tS_t)\\ \leq&\frac{d}{l}\operatorname{vol}\big((A + \frac{t}{l}S_t)|_{S_t}\big)\\ =&\frac{d}{l}\operatorname{vol}\big(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})\big)\\ =&\frac{d}{l}\big(A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})\big)^{d-1}\\ =&\varsigma'(t),\end{aligned} \tag{4.4}$$

where the second-to-last equality follows from the fact that $A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})$ is nef on $[\tau, +\infty)$.

Let δ_X and δ'_X be the real numbers as in Step 1. We claim that $\vartheta(\tau) > 0$. Suppose, for a contradiction, that $\vartheta(\tau) = 0$. Then $\tau \leq \delta_X$, so $\varsigma(\delta_X) \geq \varsigma(\tau) > 0$ since $\varsigma(t)$ is a positive, increasing function on $[\tau, +\infty)$ by Step 3. Since $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, $\vartheta(\delta_X) = 0$, and $\varsigma(\delta_X) > 0$, it follows that $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$ for some $\gamma_X \in (\delta_X, \delta'_X)$ satisfying (4.3), contradicting the inequality $\vartheta'(t) \leq \varsigma'(t)$ for all $t > \tau$ such that $\vartheta(t) > 0$ stated in (4.4). We conclude that $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t > \tau$. ∎

We use the following elementary result in the proof of Proposition 4.3.1. Note that differentiability at the endpoints is not required.

Lemma 4.3.2 ([68][Theorem 5.9]). *Let f and g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Then there exists a point $x \in (a, b)$ such that*

$$\big(f(b) - f(a)\big)g'(x) = \big(g(b) - g(a)\big)f'(x).$$

Remark 4.3.3. In the case $\dim X = 2$, by the Zariski decomposition for normal surfaces [69][Corollary 7.5], the volume of a big divisor is greater than or equal to its self-intersection. Thus, when $\dim X = 2$, Proposition 4.3.1 follows immediately from this fact without using restricted volumes. However, this property does not necessarily hold in higher dimensions. For example, let Y be a smooth 3-fold with K_Y ample, and let $\pi : X = \mathrm{Bl}_P Y \to Y$ be the blow-up of Y at a closed point P. Then $K_X = \pi^* K_Y + 2E$, where E is the exceptional divisor over $P \in Y$, and K_X is big. It follows that $\operatorname{vol}(K_X) = \operatorname{vol}(K_Y) = (K_Y)^3$, while $(K_X)^3 = (K_Y)^3 + 8E^3 = (K_Y)^3 + 8 > \operatorname{vol}(K_X)$.

4.4 Boundedness of klt good minimal models

In this section, we prove the boundedness of klt good minimal models.

Proposition 4.4.1. *Theorem A_d implies Theorem C_d.*

Proof For each

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \le u, \sigma),$$

by Step 1 of the proof of Proposition 4.2.1, we may fix $v := \mathrm{Ivol}(K_X + B)$. By Lemma 4.1.2 (2), (X, B) is ϵ-lc and lB is a Weil divisor for some $\epsilon > 0$ and $l \in \mathbb{N}$ depending only on (d, Φ, u, v). Replacing l by a bounded multiple, Theorem A implies that

$$L := l\left(K_X + B + \frac{\tau}{2}A\right)$$

is a nef and big $\mathbb{Q}$-Cartier Weil divisor. Let

$$L' := l(K_X + B) + L,$$

which is also a nef and big Weil divisor. Then $L' - K_X = (l-1)(K_X + B) + B + L$ is pseudo-effective. By [15]Theorem 1.1, there exists $m \in \mathbb{N}$, depending only on d and ϵ, such that the linear system $|mL'|$ defines a birational map. Picking a general member $N \in |mL'|$, we have that $N \ge 0$ is a nef and big Weil divisor. It then follows that

$$\mathrm{vol}(K_X+B+N) = (2ml+1)^d \,\mathrm{vol}\left(K_X+B+\frac{l\tau}{2(2ml+1)}A\right) = (2ml+1)^d\, \sigma\left(\frac{l\tau}{2(2ml+1)}\right),$$

which is fixed. Consequently, by Theorem 2.11.6, the set of $(X, B+N)$ forms a bounded family.

Therefore, there exist a fixed $r \in \mathbb{N}$ and a very ample divisor H on X such that

$$H^d \le r \quad \text{and} \quad H^{d-1} \cdot (K_X + B + N) \le r.$$

By Lemma 2.11.4, $H - N$ is pseudo-effective. Since

$$N - \frac{\tau}{2}A = (ml+1)(K_X + B) + (ml-1)\left(K_X + B + \frac{\tau}{2}A\right)$$

is also pseudo-effective, it follows that $\frac{2}{\tau}H - A$ is pseudo-effective. Replacing H by a bounded multiple, we may assume that $H - A$ is pseudo-effective. ∎

Proof of Theorem 1.3.2 This directly follows from Theorem C. ∎

CHAPTER 5 MODULI SPACE OF TRADITIONAL STABLE MINIMAL MODELS

In this chapter, we apply the boundedness results obtained in Chapter 4 to construct the moduli space of klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of their respective Iitaka fibrations. This chapter is based on the appendix of the preprint [30].

We refer readers to [70] for the notions of stacks, algebraic stacks, Deligne-Mumford stacks and algebraic spaces.

Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $u \in \mathbb{Q}^{\geq 0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. In chapter, we will fix these data.

5.1 Background on the moduli of good minimal models

According to the standard Minimal Model Conjecture and Abundance Conjecture, any variety with non-negative Kodaira dimension is birational to a good minimal model. Therefore, the next main goal of birational geometry is to construct moduli spaces for these objects.

For canonical polarized varieties, i.e., ample models of good minimal models with maximal Kodaira dimension, the projective coarse moduli spaces of these varieties have been established in any dimension by the contributions of many people over the past three decades, [64, 71-76]. see Kollár's book [76] and the references therein for details. Notably, advancements in the higher-dimensional minimal model program [1-3, 34, 77-78] have played a crucial role in these developments, particularly in the boundedness and properness of the relevant moduli functors.

For Calabi-Yau varieties, i.e., good minimal models with Kodaira dimension zero, there is no natural choice of polarization. When studying moduli of Calabi-Yau varieties, a polarization is typically fixed despite the non-uniqueness of the choice. Various techniques, such as GIT, Hodge theory, minimal model program and mirror symmetry, have been applied to study the moduli of polarized Calabi-Yau varieties and have been successful in specific examples; refer to [45, 79-84] for a partial list. Recently, Birkar establishes a boundedness result for slc (resp. klt) Calabi-Yau varieties polarized by effective (resp. possibly non-effective) ample Weil divisors with fixed volume in any dimension, and it is crucial for constructing their projective (resp. separated) coarse moduli spaces [15].

For good minimal models with arbitrary non-negative Kodaira dimension, Birkar has recently proven a boundedness result for slc good minimal models polarized by effective Weil divisors that are relatively ample over the bases of Iitaka fibration, and he has constructed their projective coarse moduli spaces [18]. This achievement is based on a series of recent works [5-6, 15-17, 85]. When restricted to good minimal models with maximal Kodaira dimension, he also obtains a new moduli functor for varieties whose canonical divisors are nef and big but not ample, a similar construction in this case has appeared in the work of Filipazzi-Inchiostro [86].

The reason for taking effective divisors as polarization in [18] is that one can obtain proper or even projective moduli spaces. However, this often leads to larger moduli spaces than desired, as it also parametrizes all the additional effective divisors. Traditionally, people use line bundles as polarization, although it is difficult to meaningfully compactify these moduli spaces. For example, Viehweg considers ample line bundles as polarization for treating the moduli of smooth good minimal models [87]. Then Taji constructs moduli of good minimal models with canonical singularities [88]. More recently, Hashizume-Hattori consider klt good minimal models of Kodaira dimension one, polarized by line bundles that are relatively ample over the bases of Iitaka fibration [24].

This chapter studies the existence of a moduli space for klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles relatively ample over the bases of their Iitaka fibrations.

5.2 Moduli functor of traditional stable minimal models

Let $\Bbbk$ be an algebraically closed field of characteristic zero. We define the main object studied in this chapter, as introduced in Birkar's survey note [89]§10.

Definition 5.2.1 (Traditional stable minimal models)**.** A *traditional stable minimal model* $(X, B), A$ over $\Bbbk$ consists of a projective connected pair (X, B) and a Cartier divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \to Z$, and
- $K_X + B + tA$ is ample for some $t > 0$.

A (d, Φ, u, σ)-*traditional stable minimal model* is a traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ,
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$, and
- $(K_X + B + tA)^d = \sigma(t)$.

We recall the notion of relative Mumford divisor from [76][Definition 4.68].

Definition 5.2.2 (Relative Mumford divisor)**.** Let $f : X \to S$ be a flat finite type morphism with S_2 fibers of pure dimension d. A subscheme $D \subset X$ is a *relative Mumford divisor* if there is an open set $U \subset X$ such that

- $\mathrm{codim}_{X_s}(X_s \setminus U_s) \geq 2$ for each $s \in S$,
- $D|_U$ is a relative Cartier divisor,
- D is the closure of $D|_U$, and
- X_s is smooth at the generic points of D_s for every $s \in S$.

By $D|_U$ being relative Cartier we mean that $D|_U$ is a Cartier divisor on U and that its support does not contain any irreducible component of any fiber U_s.

If $D \subset X$ is a relative Mumford divisor for $f : X \to S$ and $T \to S$ is a morphism, then the *divisorial pullback* D_T on $X_T := X \times_S T$ is the relative Mumford divisor defined to be the closure of the pullback of $D|_U$ to U_T. In particular, for each $s \in S$, we define $D_s = D|_{X_s}$ to be the closure of $D|_{U_s}$ which is the divisorial pullback of D to X_s.

Definition 5.2.3 (Locally stable family)**.** A *locally stable family of klt pairs* $(X, B) \to S$ over a reduced Noetherian scheme S is a flat finite type morphism $X \to S$ with S_2 fibers and a $\mathbb{Q}$-divisor B on X satisfying

- each prime component of B is a relative Mumford divisor,
- $K_{X/S} + B$ is $\mathbb{Q}$-Cartier, and
- (X_s, B_s) is a klt pair for any point $s \in S$.

We define families of traditional minimal models and the corresponding moduli functor.

Definition 5.2.4. Let S be a reduced scheme over $\mathbb{k}$.

1. When $S = \mathrm{Spec}\, \mathbb{K}$ for a field $\mathbb{K}$, we define a traditional stable minimal model over $\mathbb{K}$ as in Definition 5.2.1 by replacing $\mathbb{k}$ with $\mathbb{K}$ and replacing connected with geometrically connected. Similarly we can define (d, Φ, u, σ)-traditional stable minimal models over $\mathbb{K}$.
2. For general S, a *family of traditional stable minimal models* over S consists of a projective morphism $X \to S$ of schemes, a $\mathbb{Q}$-divisor B and a line bundle A on X

such that

- $(X, B) \to S$ is a locally stable family,
- $(X_s, B_s), A_s$ is a traditional stable minimal model over $k(s)$ for every $s \in S$.

Here X_s is the fiber of $X \to S$ over s and B_s is the divisorial pullback of B to X_s. Moreover, $K_{X_s} + B_s$ is semi-ample which defines a contration $X_s \to Z_s$, and A_s is a line bundle on X_s which is ample over Z_s. We will denote this family by $(X, B), A \to S$.

3. Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \dots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A *family of* (d, Φ, u, σ)*-marked traditional stable minimal models* over S is a family of traditional stable minimal models $(X, B), A \to S$ such that
   - $B = \sum_{i=1}^{m} a_i D_i$, where $D_i \geq 0$ are relative Mumford divisors, and
   - $(X_s, B_s), A_s$ is a (d, Φ, u, σ)-traditional stable minimal model over $k(s)$ for every $s \in S$, where $B_s = \sum_{i=1}^{m} a_i D_{i,s}$.
4. We define the moduli functor $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ of (d, Φ, u, σ)-traditional stable minimal models from the category of reduced $\mathbb{k}$-schemes to the category of groupoids by choosing:
   - On objects: for a reduced $\mathbb{k}$-scheme S, one take
   $$\begin{aligned}&\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(S)\\ =&\{\text{family of } (d, \Phi, u, \sigma)\text{-traditional stable minimal models over } S\}.\end{aligned}$$
   We define an isomorphism $(f' : (X', B'), A' \to S) \to (f : (X, B), A \to S)$ of any two objects in $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(S)$ to be an isomorphism $\alpha_X : (X', B') \to (X, B)$ over S such that $A' \sim_S \alpha_X^* A$.
   - On morphisms: $(f_T : (X_T, B_T), A_T \to T) \to (f : (X, B), A \to S)$ consists of morphisms of reduced $\mathbb{k}$-schemes $\alpha : T \to S$ such that the natural map $g : X_T \to X \times_S T$ is an isomorphism, B_T is the divisorial pullback of B and $A_T \sim_T g^* \alpha_X^* A$. Here $\alpha_X : X \times_S T \to X$ is the base change of α.

Now we can state our main result on moduli.

Theorem 5.2.5. *$\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ is a separated Deligne-Mumford stack of finite type, which admits a coarse moduli space $TS_{klt}(d, \Phi, u, \sigma)$ as a separated algebraic space.*

A conjecture of Campana predicts that a smooth family of varieties admitting good minimal models over a special (in the sense of Campana) quasi-projective base is isotrivial, see [88] and the references therein for recent developments in this direction. We hope that our moduli spaces constructed in this paper will play a role in extending this conjecture to the setting of log smooth family of pairs in the future.

5.3 Moduli stack of traditional stable minimal models

The general strategy for constructing moduli stack of varieties is to embed the varieties into a single projective space and then employ Hilbert scheme arguments. Moreover, the theory of relative Mumford divisors developed in [76] also works for varieties polarized by effective divisors. However, for varieties polarized by non-canonical line bundles, to obtain a universal object for these line bundles, we consider embedding the varieties into the product of two projective spaces, following the approach in [87][§1.7].

For any $(X, B), A \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(k)$, we will use the following lemma to produce two different very ample line bundles $\mathcal{O}_X(L_1)$, $\mathcal{O}_X(L_2)$ on X such that $\mathcal{O}_X(L_1) \sim \mathcal{O}_X(L_2) \otimes A$.

Lemma 5.3.1. *Let $\mathbb{K}$ be a field of characteristic zero. Then there exist natural number τ and I depending only on (d, Φ, u, σ) such that $\tau\Phi \subset \mathbb{N}$ and they satisfy the following. For any $(X, B), A \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(K)$ and nef Cartier divisor M on X, we have*

- *$\tau(K_X + B)$ is a base point free divisor, $A + \tau(K_X + B)$ is an ample Cartier divisor,*
- *Let $L_M := I(A + \tau(K_X + B)) + M$, then L_M is strongly ample, i.e. L_M is very ample and $H^q(X, kL_M) = 0$ for any $k, q > 0$,*

Proof By the same argument as [18][Proof of Lemma 10.2], it is enough to find τ and I when $\mathbb{K} = \mathbb{C}$. Note that A is a line bundle in our setting. Hence, by the proof of Theorem 1.3.2, there exists $\tau \in \mathbb{N}$ such that $\tau(K_X + B)$ is base point free, and both $A + (\tau - 1)(K_X + B)$ and $A + \tau(K_X + B)$ are ample Cartier divisors. Applying the effective base point free theorem [90][Theorem 1.1] and the very ampleness lemma [55][Lemma 7.1] to $A + \tau(K_X + B)$, we obtain $I_0 \in \mathbb{N}$ such that $L_0 := I_0(A + \tau(K_X + B))$ is very ample.

After replacing I_0 with a bounded multiple, we may assume that $L_0 - (K_X + B)$ is nef and big. Let $I = (d + 2)I_0$ and $\mathcal{F} := L_M - I_0(A + \tau(K_X + B))$, then

$$H^i(X, \mathcal{F} \otimes L_0^{\otimes(-i)}) = 0$$

for all $i > 0$ by Kawamata-Viehweg vanishing theorem. Thus $\mathcal{F}$ is 0-regular with respect to L_0 ([67][Definition 1.8.4]), and hence $\mathcal{F}$ is base point free by [67][Theorem 1.8.5]. Therefore,

$$L_M = L_0 + \mathcal{F}$$

is very ample by [63][Exercise II 7.5(d)]. Again we have $L_M - (K_X + B)$ is nef and big, hence

$H^q(X, kL_M) = 0$ for any $k, q > 0$. ∎

Notation 5.3.2. From now on, we will fix the positive natural numbers I and τ obtained in Lemma 5.3.1. Let S be a reduced scheme, for any $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(S)$, we define

$$L_{1,S} := I(A + \tau(K_{X/S} + B)) + I(A + \tau(K_{X/S} + B)) = 2IA + 2I\tau(K_{X/S} + B),$$

$$\begin{aligned} L_{2,S} :=& I(A + \tau(K_{X/S} + B)) + (I - 1)(A + \tau(K_{X/S} + B)) + \tau(K_{X/S} + B) \\ =& (2I - 1)A + 2I\tau(K_{X/S} + B) \end{aligned}$$

and $L_{3,S} := L_{1,S} + L_{2,S}$ to be the divisorial sheaves on X. Then $L_{1,S} - L_{2,S} = A$, and $L_{j,S}$ are strongly ample line bundles over S for $j = 1, 2, 3$ by Lemma 5.3.1 and the proof of Lemma 5.3.3.

Lemma 5.3.3. *Let $(X, B = \sum_{i=1}^{m} a_i D_i), A \to S$ be a family of (d, Φ, u, σ)-marked traditional stable minimal models over reduced Noetherian scheme S. For $j = 1, 2, 3$, let $L_{j,S}$ be the divisorial sheaves on X as Notation 5.3.2. Then for every $k \in \mathbb{Z}_{>0}$, the functions $S \to \mathbb{Z}$ by sending*

1. *$s \mapsto h^0(X_s, kL_{j,s})$ for $j = 1, 2, 3$ and*
2. *$s \mapsto \deg_{L_{3,s}}(D_{i,s})$ for $i = 1, 2, \dots, m$*

are locally constant on S, where $L_{j,s} = L_{j,S}|_{X_s}$ and $D_{i,s} = D_i|_{X_s}$ are the divisorial pullbacks to X_s, and $\deg_{L_{3,s}}(D_{i,s}) := D_{i,s} \cdot L_{3,s}^{d-1}$.

Proof (1). For $j = 1, 2, 3$, it is enough to show that $L_{j,S}$ are flat over S: since then $\chi(X_s, kL_{j,s})$ are locally constant, and $L_{j,S}$ are strongly ample over S by Lemma 5.3.1, hence $h^0(X_s, kL_{j,s})$ are locally constant. Since $X \to S$ is flat, it suffices to show that $\mathcal{O}_X(L_{j,S})$ are line bundles by [63][Proposition III 9.2(c)(e)].

Since $(X, B) \to S$ is a locally stable family, B is a relative Mumford divisor over S, we see that $\tau(K_{X/S} + B)$ is $\mathbb{Q}$-Cartier, and it is mostly flat ([76][Definition 3.26]) over S. Moreover, since $\mathcal{O}_{X_s}(\tau(K_{X_s} + B_s))$ is a base point free line bundle for any $s \in S$ by Lemma 5.3.1, $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a mostly flat family of line bundles. Therefore, by [76][Corollary 4.34 and Proposition 5.29], $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a line bundle on X. Furthermore, since A is a line bundle on X, $\mathcal{O}_X(L_{j,S})$ are line bundles for $j = 1, 2, 3$.

(2). It follows from [76][Theorem 4.3.5]. ∎

Let $n, l \in \mathbb{Z}_{>0}$, $\mathbf{c} = (c_1, c_2, \dots, c_m) \in \mathbb{N}^m$, and $h \in \mathbb{Q}[k]$ be a polynomial. Let S be a reduced scheme, for any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(S)$ and $j = 1, 2, 3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation 5.3.2. We define

$\mathfrak{TS}_{h,n,l,\mathbf{c}}$ to be a full subcategory of $\mathfrak{TS}_{klt}(d,\Phi,u,\sigma)$ such that $\mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$ is a groupoid whose objects consist of families of (d,Φ,u,σ)-traditional stable minimal models over S satisfying:

- the Hilbert polynomial of X_s with respect to $L_{3,s}$ is h,
- $h^0(X_s, L_{1,s}) - 1 = n$,
- $h^0(X_s, L_{2,s}) - 1 = l$, and
- $(\deg_{L_{3,s}}(D_{1,s}), \deg_{L_{3,s}}(D_{2,s}), \dots, \deg_{L_{3,s}}(D_{m,s})) = \mathbf{c}$

for every $s \in S$.

Lemma 5.3.4. *We can write*

$$\mathfrak{TS}_{klt}(d,\Phi,u,\sigma) = \bigsqcup_{h,n,l,c} \mathfrak{TS}_{h,n,l,c}$$

as disjoint union, and each $\mathfrak{TS}_{h,n,l,c}$ is a union of connected components of $\mathfrak{TS}_{klt}(d,\Phi,u,\sigma)$. Moreover, there are only finitely many $n,l \in \mathbb{Z}_{>0}$, $\boldsymbol{c} = (c_1, c_2, \dots, c_m) \in \mathbb{N}^m$ and $h \in \mathbb{Q}[k]$ such that $\mathfrak{TS}_{h,n,l,c}$ is not empty.

Proof Given any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{TS}_{klt}(d,\Phi,u,\sigma)(S)$. By Lemma 5.3.3, the Hilbert functions

$$h_s(k) = \chi(X_s, kL_{3,s}) = h^0(X_s, kL_{3,s})$$

of X_s with respect to $L_{3,s}$, and the numbers

$$n_s = h^0(X_s, L_{1,s}) - 1, \quad l_s = h^0(X_s, L_{2,s}) - 1 \quad \text{and} \quad c_{i,s} = \deg_{L_{3,s}}(D_{i,s})$$

are locally constant on $s \in S$ for all $1 \le i \le m$. The first assertion follows from this fact.

The second assertion follows from the fact that n_s, l_s, $c_{i,s}$ and h_s belong to a finite set for all $1 \le i \le m$ by Theorem 1.3.2 (these finiteness results can be reduced to the case when $s = \operatorname{Spec}\mathbb{C}$ by the same argument as [18][Proof of Lemma 10.2]). ∎

Lemma 5.3.5. *$\mathfrak{TS}_{h,n,l,c}$ is a stack.*

Proof Since our argument follows the same strategy as in [70][Proposition 2.5.14 and Example 2.5.9], we only sketch the proof here.

Axiom (1) of [70][Definition 2.5.1] follows from descent [70][Proposition 2.1.7, Proposition 2.1.19, Proposition 2.1.4(1) and Proposition 2.1.16(2)].

To verify Axiom (2) of [70][Definition 2.5.1], i.e., given any descent datum (f', ξ) with respect to a covering $S' \to S$ (see [24][Remark 2.10] for notions of covering and descent datum), where $(f' : (X', B'), A' \to S') \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S')$, we need to show that f' descends to a family $(f : (X, B), A \to S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$. We use the strongly f'-

ample line bundles $\mathcal{O}_{X'}(L'_{1,S'})$ and $\mathcal{O}_{X'}(L'_{2,S'})$ as Notation 5.3.2 instead of $\omega_{C'/S'}^{\otimes 3}$ in [70][Proposition 2.5.14], then the same argument as in *loc.cit.* implies that $(X', B') \to S'$ descends to $(X, B) \to S$. Moreover, by applying [70][Proposition 2.1.4(2) and Proposition 2.1.16(2)] to the covering $X' \to X$, we see that A' descends to a line bundle A on X. Since every geometric fiber of $f : (X, B), A \to S$ is identified with a geometric fiber of $f' : (X', B'), A' \to S'$, $(f : (X, B), A \to S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$. ∎

For any scheme S and positive integer n, l, Let $\mathbb{P}^n_S \times_S \mathbb{P}^l_S \cong \mathbb{P}^n \times \mathbb{P}^l \times S$ be the natural isomorphism, and

$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times S \xrightarrow{p_2} \mathbb{P}^l$$

be the projections. Then for any $a, b \in \mathbb{Z}$, we denote $p_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(b)$ by $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S}(a, b)$.

Theorem 5.3.6. *$\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is an algebraic stack of finite type.*

Proof *Step 1.* In this step, we consider a suitable Hilbert scheme parametrizing the total spaces of interest.

For any $(f : (X, B), A \to S) \in \mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$ and for $j = 1, 2, 3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation 5.3.2. We get an embedding

$$X \hookrightarrow \mathbb{P}(f_*\mathcal{O}_X(L_{1,S})) \times_S \mathbb{P}(f_*\mathcal{O}_X(L_{2,S})).$$

We proceed to parametrize such embedding.

Let $H = \mathrm{Hilb}_h(\mathbb{P}^n \times \mathbb{P}^l)$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n \times \mathbb{P}^l$ with Hilbert polynomial h. Let $X_H = \mathrm{Univ}_h(\mathbb{P}^n \times \mathbb{P}^l) \overset{i}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^l \times H$ be the universal family over H, and

$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times H \xrightarrow{p_2} \mathbb{P}^l.$$

be the natural projections. Note that the $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on $\mathbb{P}^n \times \mathbb{P}^l$ induces a $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on H. Let $M_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, 1)$ and $N_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, -1)$ be the universal line bundles on X_H.

Step 2. In this step, we parametrize the boundary divisors in the moduli problem.

By [91][Theorem 12.2.1 and Theorem 12.2.4], the locus $s \in H$ such that X_s is geometrically connected and reduced, equidimensional, and geometrically normal is an open subscheme H_1 of H.

Since $f_1 : X_{H_1} \to H_1$ is equidimensional, and over reduced bases relative Mumford divisors are the same as K-flat divisors [76][Definition 7.1 and comment 7.4.2], there is a separated H_1-scheme $\mathrm{MDiv}_c(X_{H_1}/H_1)$ of finite type which parametrizes relative Mumford divisors of degree c with respect to M_{H_1} by [76]Theorem 7.3. Fixing $\mathbf{c} = (c_1, c_2, \dots, c_m) \in \mathbb{N}^m$, let

$$H_2 := \mathrm{MDiv}_{c_1}(X_{H_1}/H_1) \times_{H_1} \mathrm{MDiv}_{c_2}(X_{H_1}/H_1) \times_{H_1} \cdots \times_{H_1} \mathrm{MDiv}_{c_m}(X_{H_1}/H_1)$$

be the m-fold fiber product, we denote the universal family by

$$(X_{H_2}, B_{H_2} = \sum_{i=1}^{m} a_i D_{i,H_2}), N_{H_2} \to H_2,$$

where D_{i,H_2} are the universal families of relative Mumford divisors on X_{H_2} of degree c_i with respect to M_{H_2} for $1 \le i \le m$.

Step 3. By [76]Theorem 4.8, there is a locally closed partial decomposition $H_3 \to H_2$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_2$, then the family obtained by base change $f_W : (X_W, B_W) \to W$ is locally stable iff q factors as $q : W \to H_3 \to H_2$.

Since $f_3 : (X_{H_3}, B_{H_3}) \to H_3$ is locally stable, By [76]Theorem 4.28, there is a locally closed partial decomposition $H_4 \to H_3$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_3$, the divisorial pullback of $\tau(K_{X_{H_3}/H_3} + B_{H_3})$ to $W \times_{H_3} X_{H_3}$ is Cartier iff q factors as $q : W \to H_4 \to H_3$.

Step 4. Since the fibers X_s of $f_4 : X_{H_4} \to H_4$ are reduced and connected by Step 2, we have $h^0(X_s, \mathcal{O}_{X_s}) = 1$. Since $\tau(K_{X_{H_4}/H_4} + B_{H_4})$ is Cartier by Step 3, by [87]Lemma 1.19, there is a locally closed subscheme $H_5 \subset H_4$ with the following property: for any scheme W and morphism $q : W \to H_4$,

$$\mathcal{O}_{X_W}(1, 0) \sim_W N_W^{2I} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W) \text{ and}$$

$$\mathcal{O}_{X_W}(0, 1) \sim_W N_W^{2I-1} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W)$$

iff q factors as $q : W \to H_5 \to H_4$, where $\mathcal{O}_{X_W}(1, 0)$ and $\mathcal{O}_{X_W}(0, 1)$ are the pullbacks of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(1, 0)$ and $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(0, 1)$ to X_W, respectively.

Step 5. In this step, we cut the locus parametrizing (d, Φ, u, σ)-traditional stable minimal models.

(1). By [18]Lemma 8.5, there is a locally closed subscheme $H_6 \subset H_5$ such that for any $s \in H_6$, $K_{X_s} + B_s$ is semi-ample defining a contraction $X_s \to Z_s$.

(2). Since ampleness and klt are open conditions, there is an open subscheme $H_7 \subset H_6$ such that $N_s + \tau(K_{X_s} + B_s)$ is ample and (X_s, B_s) is klt for any $s \in H_7$.

(3). By [18]Lemma 8.7 (the condition of N_s being effective is not required in the proof), there is a locally closed subscheme $H_8 \subset H_7$ such that for any $s \in H_8$, $\mathrm{vol}(N_s|_F) = u$ for the general fibres F of $X_s \to Z_s$.

(4). For each $s \in H_8$, since $K_{X_s} + B_s$ is semi-ample and $N_s + \tau(K_{X_s} + B_s)$ is ample, $K_{X_s} + B_s + tN_s$ is ample for each $t \in (0, \frac{1}{\tau}]$, then

$$\theta_s(t) = \mathrm{vol}(K_{X_s} + B_s + tN_s) = (K_{X_s} + B_s + tN_s)^d$$

is a polynomial in t of degree $\leq d$ on the interval $(0, \frac{1}{\tau}]$. By Step 3(iv) of [18][Proof of Proposition 9.5], there is an open and closed subscheme $H_9 \subset H_8$ such that $\theta_s(t) = \sigma(t)$ on the interval $(0, \frac{1}{\tau}]$.

Therefore, $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is a family of (d, Φ, u, σ)-traditional stable minimal models. For $j = 1, 2$, let L_{j,H_9} be the strongly ample line bundles over H_9 as Notation 5.3.2. Then $f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})$ and $f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})$ are locally free sheaves of rank $n+1$ and $l+1$, respectively. Shrinking H_9, we may assume that they are free sheaves, and hence

$$\mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})) \cong \mathbb{P}^n_{H_9} \text{ and } \mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})) \cong \mathbb{P}^l_{H_9}.$$

Step 6. In this step, we will prove that

$$\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}} \cong [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}].$$

Then since H_9 is a finite type scheme and $[H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$ is an algebraic stack, $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is a finite type algebraic stack.

We follow the arguments of [70][Theorem 3.1.17] and [45][Proposition 3.9]. By our construction, the universal family $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is an object in $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(H_9)$, which induces a morphism $H_9 \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$, where this morphism just forgets the projective embeddings. Moreover, this morphism is $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$-invariant, hence descends to a morphism $\Psi^{\mathrm{pre}} : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]^{\mathrm{pre}} \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ of prestacks. Since $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is a stack by Lemma 5.3.5, the universal property of stackification [70][Theorem 2.5.18] yields a morphism $\Psi : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}] \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$.

To construct the inverse, consider $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$, since $f_*\mathcal{O}_X(L_{1,S})$ and $f_*\mathcal{O}_X(L_{2,S})$ are locally free by Step 1, there exists an open cover $S = \cup_i S_i$ over which their restrictions are free. Choosing trivializations induce embeddings $g_i : (X_{S_i}, B_{S_i}) \hookrightarrow \mathbb{P}^n \times \mathbb{P}^l \times S_i$. Moreover, we have $A_{S_i} \sim_{S_i} N_{S_i} := g_i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S_i}(1, -1)$. Hence by our construction of H_9, we have morphisms $\Phi_i : S_i \to H_9$. Over the intersections $S_i \cap S_j$, the trivializations differ by a section $s_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1})$. Therefore the Φ_i glue to a morphism $\Phi : S \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, which induces a morphism $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}} \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, that is the inverse of Ψ. ∎

5.4 Moduli space of traditional stable minimal models

We need the following separatedness result to obtain the coarse moduli space of traditional stable minimal models.

Theorem 5.4.1. *Let $f : (X, B), A \to C$ and $f' : (X', B'), A' \to C$ be two families of (d, Φ, u, σ)-traditional stable minimal models over a smooth curve C. Let $0 \in C$ be a closed point and $C^o := C \setminus \{0\}$ the punctured curve. Assume there exists an isomorphism*

$$g^o : ((X, B), A) \times_C C^o \to ((X', B'), A') \times_C C^o$$

over C^o, then g^o can be extended to an isomorphism $g : (X, B), A \to (X', B'), A'$ over C.

Proof Consider $L := A' + \tau(K_{X/C} + B)$ and $L' := A' + \tau(K_{X'/C} + B')$, where τ is the positive natural number as Lemma 5.3.1. By the proof of Lemma 5.3.3, L is an f-ample Cartier divisor on X (resp. L' is an f'-ample Cartier divisor on X'). Let $g : X \dashrightarrow X'$ be the birational map induced by g^o, then by the same argument as in [24][Proof of Proposition 4.4], g is an isomorphism over C. ∎

Corollary 5.4.2. *For any $(X, B), A \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)(\Bbbk)$,* Aut$((X, B), A)$ *is finite.*

Proof It follows from Theorem 5.4.1 and the argument of [92][Proof of Corollary 3.5]. ∎

Proof of Theorem 5.2.5 By Theorem 5.3.6 and Lemma 5.3.4, $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ is an algebraic stack of finite type. By Corollary 5.4.2 and [70][Theorem 3.6.4], $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ is a Deligne-Mumford stack. Moreover, Theorem 5.4.1 and [70][Theorem 3.8.2(3)] imply that $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ is a separated Deligne-Mumford stack of finite type. Therefore, we may apply the Keel–Mori's theorem [73][70][Theorem 4.3.12] to see that $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, u, \sigma)$ has a coarse moduli space $TS_{klt}(d, \Phi, u, \sigma)$, which is a separated algebraic space. ∎

CHAPTER 6 BOUNDEDNESS OF POLARIZED LOG CALABI–YAU FIBRATIONS WITH BOUNDED BASES

In this chapter, we prove boundedness of polarized log Calabi–Yau fibrations with bounded bases. We will prove the following more general form of Theorem 1.4.3. This chapter is based on the preprint [13].

Theorem 6.0.1. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer l and a bounded set of couples $\mathcal{P}$ depending only on d, Φ, v, r, ϵ satisfying the following.*

Assume that $f : ((X, B), A) \to (Z, H)$ is a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration, and $H_Z \geq 0$ is a general element of $|6dH|$. Then there exists a couple (V, Θ) and an effective integral divisor J on V such that

1. *there is a contraction $h : V \to Z$ and V is $\mathbb{Q}$-factorial,*
2. *$V \dashrightarrow X/Z$ is an isomorphism in codimension one,*
3. *$(V, \Theta + \mathrm{Supp}(J))$ belongs to $\mathcal{P}$,*
4. *Θ contains h^*H_Z and the strict transform of B, and*
5. *$J \equiv lA_V$ over the generic point of Z, where A_V is the strict transform of A on V.*

Lemma 6.0.2. *Assume that Theorem 6.0.1 holds when A is an effective integral divisor and $\mathrm{vol}(A|_F) = v$ for some fixed $v \in \mathbb{Q}^{>0}$, where F is the general fiber of $f : X \to Z$. Then the theorem holds in general.*

Proof If (F, B_F) is the general fiber of $f : (X, B) \to Z$ and $A_F := A|_F$, then by [15][Theorem 1.3], there exists a positive integer m depending only on $\dim F$ and ϵ such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. By upper-semicontinuity of cohomology and [77][Lemma 3.2.1], $mA \sim_Z G$ for some effective integral divisor G on X. Replacing A and v with G and $m^{\dim F} v$ respectively, we may assume that $A \geq 0$. Moreover, by [15][Corollary 1.6], the pair $(F, \mathrm{Supp}(B_F + A_F))$ belongs to a log bounded family. Hence, we can assume that $\mathrm{vol}(A_F)$ is fixed. ∎

From now until the end of this section, we will assume that A is an effective integral divisor and that $\mathrm{vol}(A|_F) = v$ for some fixed $v \in \mathbb{Q}^{>0}$.

6.1 Family of polarized log Calabi-Yau pairs

Definition 6.1.1. ([15, 18]) Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. A

(d, Φ, v)*-polarized log Calabi-Yau pair* $((X, B), A)$ is defined by the data:

1. (X, B) is projective slc pair of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
2. the coefficients of B are in Φ,
3. $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
4. $(X, B + tA)$ is slc for some $t \in \mathbb{Q}^{>0}$.

If (X, B) is klt, then $((X, B), A)$ is called a *klt* (d, Φ, v)*-polarized log Calabi-Yau pair.*

Given a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration $f : ((X, B), A) \to (Z, H)$, it follows that the general fiber $((F, B_F), A_F)$ of f is a klt $(\dim F, \Phi, v)$-polarized log Calabi–Yau pair, hence it is bounded by [15][Corollary 1.6].

In the following theorem, we use the moduli theory for polarized log Calabi–Yau pairs [18] to construct a locally stable family of polarized log Calabi-Yau pairs $f_{\mathcal{S}} : ((\mathcal{X}, \mathcal{B}), \mathcal{A}) \to \mathcal{S}$ such that, over an open subset of Z, the fibration $f : ((X, B), A) \to Z$ is obtained as the pullback of $f_{\mathcal{S}}$. We then apply [32] to $f_{\mathcal{S}}$ to produce a new family $f_{\mathcal{S}^!} : (\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ of maximal variation. As a consequence, the moduli **b**-divisor $\mathcal{M}^!$ of $f_{\mathcal{S}^!}$ descends to a nef and big divisor $\mathcal{M}_{\mathcal{S}^!}$ on $\mathcal{S}^!$, which plays a crucial role in establishing the boundedness of the moduli map in Theorem 6.2.1. A key step in the proof of Theorem 6.1.2 is the construction of a new polarization $\mathcal{L}$ on $\mathcal{X}$ induced from $\mathcal{X}^!$, such that $\mathcal{L}_s \equiv m\mathcal{A}_s$ for some bounded integer $m \in \mathbb{N}$ and all $s \in \mathcal{S}$. This construction allows us to prove the boundedness of the log canonical volume of a certain log general type pair in Theorem 6.3.2. Finally, we establish several auxiliary results that will be used in later subsections.

Theorem 6.1.2. *Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Let $f : (X, B) \to Z$ be a klt-trivial fibration, and A be an effective integral divisor on X. Assume that the general fiber $((F, B_F), A_F)$ of f is a klt (d, Φ, v)-polarized log Calabi-Yau pair. Then there exists a commutative diagram*

$$\begin{tikzcd}
((X,B),A) \arrow[d, "f"] & ((X_U,B_U),A_U) \arrow[l, hook'] \arrow[r] \arrow[d, "f_U"] & ((\mathcal{X},\mathcal{B}),\mathcal{A},\mathcal{L}) \arrow[d, "f_{\mathcal{S}}"] & \overline{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & ((\mathcal{X}^!,\mathcal{B}^!),\mathcal{L}^!) \arrow[d, "f_{\mathcal{S}^!}"] \\
Z & U \arrow[l, hook'] \arrow[r, "\phi"] & \mathcal{S} \arrow[drr, bend right, "\gamma"'] & \overline{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & (\mathcal{S}^!,\mathcal{M}^!) \arrow[d, "\pi"] \\
 & & & & (\mathcal{S}^*,\mathcal{H})
\end{tikzcd}$$

satisfying the following:

1. *$\mathcal{S}$, $\overline{\mathcal{S}}$, and $\mathcal{S}^!$ are smooth schemes.*
2. *$\mathcal{S}^!$ and $\mathcal{S}^*$ are projective schemes.*

3. *The morphisms* $\tau : \overline{\mathcal{S}} \to \mathcal{S}$ *and* $\pi : \mathcal{S}^! \to \mathcal{S}^*$ *are finite covers,* $\rho : \overline{\mathcal{S}} \to \mathcal{S}^!$ *is a dominant morphism, and* $\gamma : \mathcal{S} \to \mathcal{S}^*$ *is a morphism.*
4. *The generic fiber of the base change of* $(\mathcal{X}, \mathcal{B}) \to \mathcal{S}$ *to* $\overline{\mathcal{S}}$ *is isomorphic to the generic fiber of the base change of* $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ *to* $\overline{\mathcal{S}}$*.*
5. $\overline{\mathcal{X}}$ *is a common resolution of the main components of* $\mathcal{X} \times_{\mathcal{S}} \overline{\mathcal{S}}$ *and* $\mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}$*.*
6. *There exist* $\mathbb{Q}$*-Cartier integral divisors* $\mathcal{A}$ *and* $\mathcal{L}$ *on* $\mathcal{X}$*, and a* $\mathbb{Q}$*-Cartier integral divisor* $\mathcal{L}^!$ *on* $\mathcal{X}^!$*, such that for some* $m \in \mathbb{N}$*, depending only on* (d, Φ, v)*, the following hold:*
   - $\mathcal{L}_s \equiv m\mathcal{A}_s$ *for all* $s \in \mathcal{S}$*, and*
   - $\tau_{\mathcal{X}}^* \mathcal{L} = \rho_{\mathcal{X}}^* \mathcal{L}^!$*.*
7. *The morphisms* $(\mathcal{X}, \mathcal{B} + \alpha\mathcal{L}) \to \mathcal{S}$ *and* $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ *are locally stable for some* $\alpha \in \mathbb{Q}^{>0}$ *depending only on* (d, Φ, v)*.*
8. *There exists a very ample divisor* $\mathcal{H} \geq 0$ *on* $\mathcal{S}^*$ *such that:*
   - *the morphism* π *is étale and Galois over* $\mathcal{S}^* \setminus \mathrm{Supp}(\mathcal{H})$*, and*
   - *every fiber of* $((\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^!) \to \mathcal{S}^!$ *over* $\mathcal{S}^! \setminus \mathrm{Supp}(\pi^*\mathcal{H})$ *is a klt* $(d, \Phi, m^d v)$*-polarized log Calabi–Yau pair.*
9. *The moduli* **b***-divisor* $\mathcal{M}^!$ *of* $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ *descends to* $\mathcal{S}^!$*. Moreover, there exists an effective divisor* $\mathcal{M}^! \sim_{\mathbb{Q}} \mathcal{M}^!_{\mathcal{S}^!}$ *such that* $l\mathcal{M}^!$ *is Cartier and* $l\mathcal{M}^! \geq \pi^*\mathcal{H}$ *for some* $l \in \mathbb{N}$ *depending only on* (d, Φ, v)*.*
10. *There exists an open subset* $U \subset Z$ *and a morphism* $\phi : U \to \mathcal{S}$ *such that* $((X_U, B_U), A_U) \to U$ *is isomorphic to the base change of* $((\mathcal{X}, \mathcal{B}), \mathcal{A}) \to \mathcal{S}$ *via* ϕ*.*
11. *If the composition* $\gamma \circ \phi$ *extends to a morphism* $\psi : Z \to \mathcal{S}^*$*, then*

$$\psi(Z) \not\subset \pi(\mathrm{Supp}(\mathcal{M}^!)).$$

Proof *Step* 1. In this step, we construct a universal family parametrizing the general fibers of $f : ((X, B), A) \to Z$.

By [18]$^{\text{Lemma 10.2}}$, there exists $\alpha \in \mathbb{Q}^{>0}$ and $r \in \mathbb{N}$, depending only on (d, Φ, v), such that the following hold:

- for the general fiber $((F, B_F), A_F)$ of f, the pair $(F, B_F + \alpha A_F)$ is klt, and
- the divisor $r(K_F + B_F + \alpha A_F)$ is very ample and has no higher cohomology.

Set

$$n := h^0\big(r(K_F + B_F + \alpha A_F)\big) - 1.$$

Then $r(K_F + B_F + \alpha A_F)$ defines an embedding $F \hookrightarrow \mathbb{P}^n$. Since this divisor is very ample without higher cohomology, there exists a non-empty open subset $U \subset Z$ such that $r(K_{X_U} + B_U + \alpha A_U)$ defines an embedding $X_U \hookrightarrow \mathbb{P}^n_U$.

By [18]Proposition 9.5, there exists a finite type scheme $\mathcal{S}_{(1)}$ representing the functor of strongly embedded $(d, \Phi_{1/c}, v, \alpha, r, \mathbb{P}^n)$-polarized log Calabi–Yau families (see [18]Definition 9.3) over reduced schemes, where $c \in \mathbb{N}^{>0}$ satisfies $c\Phi \subset \mathbb{N}$. After replacing $\mathcal{S}_{(1)}$ by a suitable locally closed subscheme, we may assume that $\mathcal{S}_{(1)}$ parametrizes klt (d, Φ, v)-polarized log Calabi–Yau pairs.

Let

$$\big((\mathcal{X}_{(1)} \subset \mathbb{P}^n_{\mathcal{S}_{(1)}}, \mathcal{B}_{(1)}), \mathcal{A}_{(1)}\big) \to \mathcal{S}_{(1)}$$

be the corresponding universal family. Then the morphism $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)} + \alpha\mathcal{A}_{(1)}) \to \mathcal{S}_{(1)}$ is locally stable and satisfies

$$K_{\mathcal{X}_{(1)}} + \mathcal{B}_{(1)} \sim_{\mathbb{Q}, \mathcal{S}_{(1)}} 0.$$

Moreover, there exists a moduli morphism $\phi \colon U \to \mathcal{S}_{(1)}$ such that the restriction $((X_U, B_U), A_U) \to U$ is isomorphic to the pullback of the universal family via ϕ.

Step 2. In this step, we apply Theorem 2.5.4 to the universal family obtained in Step 1.

By applying Theorem 2.5.4 to a projective compactification of $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \to \mathcal{S}_{(1)}$, we have a non-singular quasi-projective variety $\overline{\mathcal{S}}_{(1)}$, non-singular projective varieties $\mathcal{T}$ and $\mathcal{V}$, and a commutative diagram

$$\begin{tikzcd}
(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \arrow[d, "f_{\mathcal{S}_{(1)}}"'] & & (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \arrow[d, "f_{\mathcal{T}}"'] & \\
\mathcal{S}_{(1)} \arrow[rr, dashed, bend left, "i"] \arrow[rrr, dashed, bend right, "\gamma"'] & \overline{\mathcal{S}}_{(1)} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{T} \arrow[r, "\pi"] & \mathcal{V},
\end{tikzcd}$$

such that

- $(\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \to \mathcal{T}$ is a klt-trivial fibration,
- $\tau : \overline{\mathcal{S}}_{(1)} \to \mathcal{S}_{(1)}$ and $\pi : \mathcal{T} \to \mathcal{V}$ are generically finite, surjective morphisms, $\rho : \overline{\mathcal{S}}_{(1)} \to \mathcal{T}$ is a dominant morphism,

- there exists a nonempty open subset $\mathcal{U} \subset \overline{\mathcal{S}}_{(1)}$ and an isomorphism

$$\begin{tikzcd}
(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \times_{\mathcal{S}_{(1)}} \mathcal{U} \arrow[rr, "\cong"] \arrow[dr] & & (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \times_{\mathcal{T}} \mathcal{U} \arrow[dl] \\
& \mathcal{U} &
\end{tikzcd}$$

- the moduli **b**-divisor of $f_{\mathcal{T}}$ is **b**-nef and big,
- $\gamma : \mathcal{S}_{(1)} \dashrightarrow \mathcal{V}$ is bimeromorphic to the period map defined in [32][Proposition 2.1], and
- $i : \mathcal{T} \dashrightarrow \mathcal{S}_{(1)}$ is a generically finite rational map such that $f_{\mathcal{T}} : (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \to \mathcal{T}$ is equal to the pullback of $f_{\mathcal{S}_{(1)}}$ via i over some open subset of $\mathcal{T}$.

Step 3. In this step, we shrink $\mathcal{S}_{(1)}$ and construct a smooth projective variety $\mathcal{S}^!$ over which $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is a locally stable family of maximal variation. Then, we verify (1)–(4).

Let $\mathcal{S}_{(2)}$ be an open subset of $\mathcal{S}_{(1)}$ and $\overline{\mathcal{S}}_{(2)}$ be an open subset of $\mathcal{U}$ such that

- $\mathcal{S}_{(2)}$ is smooth,
- γ is a morphism on $\mathcal{S}_{(2)}$,
- $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is a finite cover, and
- $i|_{\mathcal{T}^o} : \mathcal{T}^o \to \mathcal{S}_{(2)}$ is a finite morphism for some open subset $\mathcal{T}^o$ of $\mathcal{T}$.

Let $((\mathcal{X}_{(2)}, \mathcal{B}_{(2)}), \mathcal{A}_{(2)}) \to \mathcal{S}_{(2)}$ be the corresponding base change. Then, the pullback of $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)} + \alpha\mathcal{A}_{(2)}) \to \mathcal{S}_{(2)}$ via i defines a locally stable morphism $(\mathcal{X}_{\mathcal{T}^o}, \mathcal{B}_{\mathcal{T}^o} + \alpha\mathcal{A}_{\mathcal{T}^o}) \to \mathcal{T}^o$.

By [93][Lemma 4], there exists a generically finite cover $\overline{\mathcal{T}}^o \to \mathcal{T}^o$ and a compactification $\overline{\mathcal{T}}^o \hookrightarrow \mathcal{S}^!$ such that the pullback of $(\mathcal{X}_{\mathcal{T}^o}, \mathcal{B}_{\mathcal{T}^o} + \alpha\mathcal{A}_{\mathcal{T}^o}) \to \mathcal{T}^o$ on $\overline{\mathcal{T}}^o$ extends to a locally stable morphism $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$.

By Lemma 2.6.2, after replacing $\mathcal{S}^!$ with a generically finite cover from a smooth projective variety and $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$ with the corresponding base change, we may assume that there exists a birational map $\mathcal{S}^* \dashrightarrow \mathcal{V}$ such that $\mathcal{S}^! \to \mathcal{S}^*$ is a finite cover. Replacing $\mathcal{S}_{(2)}$ by an open subset and shrinking $\overline{\mathcal{S}}_{(2)}$ accordingly, we may assume that $\gamma : \mathcal{S}_{(2)} \to \mathcal{S}^*$ is a morphism.

After replacing $\overline{\mathcal{S}}_{(2)}$ by a finite cover, we may assume that $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}^!$ is a dominant morphism. In this case, we have an isomorphism

$$(\mathcal{X}_{(2)}, \mathcal{B}_{(2)}) \times_{\mathcal{S}_{(2)}} \overline{\mathcal{S}}_{(2)} \cong (\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)}.$$

Next, after replacing $\overline{\mathcal{S}}_{(2)}$ by another finite cover, we may assume that $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is a Galois cover with Galois group G. Replacing $\mathcal{S}_{(2)}$ by an open subset and shrinking $\overline{\mathcal{S}}_{(2)}$

accordingly, we may assume that $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is an étale Galois cover. Therefore, $\overline{\mathcal{S}}_{(2)}$ is smooth.

Step 4. In this step, we construct new polarizations $\mathcal{L}_{(2)}$ and $\mathcal{L}^!$ on $\mathcal{X}_{(2)}$ and $\mathcal{X}^!$ respectively that satisfy (6).

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{X}_{(2)} \arrow[d] & \mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \overline{\mathcal{S}}_{(2)} \overset{\Psi}{\cong} \mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & \mathcal{X}^! \arrow[d] & \\
\mathcal{S}_{(2)} \arrow[rrr, bend right, "\gamma"'] & \overline{\mathcal{S}}_{(2)} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{S}^! \arrow[r, "\pi"] & \mathcal{S}^*
\end{tikzcd}$$

Recall that $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is an étale Galois cover with Galois group G. For each $g \in G$ acting on $\overline{\mathcal{S}}_{(2)}$, let

$$\sigma'_g : \mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \overline{\mathcal{S}}_{(2)} \longrightarrow \mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \overline{\mathcal{S}}_{(2)}$$

be the morphism induced by the base change $\mathrm{id}_{\mathcal{X}_{(2)}} \times_{\mathcal{S}_{(2)}} g$, and let

$$\sigma_g : \mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)} \longrightarrow \mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)}$$

be the morphism $\Psi \circ \sigma'_g \circ \Psi^{-1}$. Then the action of G on $\mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)}$ is G-equivariant with respect to the projection $\mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)} \to \overline{\mathcal{S}}_{(2)}$, and $\mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}_{(2)} \to \mathcal{X}_{(2)}$ is also an étale Galois cover with Galois group G. Let

$$\overline{\mathcal{L}}_{(2)} := \sum_{g \in G} g^* \rho_{\mathcal{X}}^* \mathcal{A}^!,$$

then $\overline{\mathcal{L}}_{(2)}$ is G-invariant, and hence there exists an effective $\mathbb{Q}$-Cartier integral divisor $\mathcal{L}_{(2)}$ on $\mathcal{X}_{(2)}$ such that $\overline{\mathcal{L}}_{(2)} = \tau_{\mathcal{X}}^* \mathcal{L}_{(2)}$.

Denote by $\mathcal{S}^!_{(2)}$ and $\mathcal{S}^*_{(2)}$ the images of $\overline{\mathcal{S}}_{(2)}$ in $\mathcal{S}^!$ and $\mathcal{S}^*$, respectively, and let $\mathcal{X}_{\mathcal{S}^!_{(2)}}$ be the base change of $\mathcal{X}^!$ over $\mathcal{S}^!_{(2)}$. Let $v \in \mathcal{S}^*_{(2)}$ be a closed point and let $s \in \pi^{-1}(v)$ be a closed point. Set $S := \gamma^{-1}(v)$ and $\overline{S} := \tau^{-1}(S)$. Let $(\mathcal{X}_s, \mathcal{B}_s) \to s$, $(\mathcal{X}_S, \mathcal{B}_S) \to S$, and $(\mathcal{X}_{\overline{S}}, \mathcal{B}_{\overline{S}}) \to \overline{S}$ be the families obtained by base change, where

$$(\mathcal{X}_{\overline{S}}, \mathcal{B}_{\overline{S}}) := (\mathcal{X}_S, \mathcal{B}_S) \times_S \overline{S} \overset{\Psi}{\cong} (\mathcal{X}_s, \mathcal{B}_s) \times \overline{S}.$$

Now the group G acts on $(\mathcal{X}_s, \mathcal{B}_s) \times \overline{S}$, and this action is G-equivariant with respect to the projection $(\mathcal{X}_s, \mathcal{B}_s) \times \overline{S} \to \overline{S}$.

By Proposition 2.6.1, there exists a Galois cover $\tilde{\tau} : \widetilde{S} \to S$ with Galois group H, which also acts on $\mathcal{X}_s$, such that

$$(\mathcal{X}_{\widetilde{S}}, \mathcal{B}_{\widetilde{S}}) := (\mathcal{X}_S, \mathcal{B}_S) \times_S \widetilde{S} \cong (\mathcal{X}_s, \mathcal{B}_s) \times \widetilde{S}.$$

Moreover,

$$(\mathcal{X}_S, \mathcal{B}_S) \simeq \big((\mathcal{X}_s, \mathcal{B}_s) \times \widetilde{S}\big)/H,$$

where H acts diagonally on $(\mathcal{X}_s, \mathcal{B}_s) \times \widetilde{S}$. After replacing $\widetilde{S}$ by a higher Galois cover, we may assume that $\tilde{\tau}$ factors through $\overline{S}$. We have the following diagram:

$$\begin{tikzcd}
 & & (\mathcal{X}_{\widetilde{S}}, \mathcal{B}_{\widetilde{S}}) \arrow[dll, "\tilde{\tau}_{\mathcal{X}}"'] \arrow[dl, "\widetilde{\pi}_{\mathcal{X}}"] \arrow[dd] \arrow[dr, "\tilde{\rho}_{\mathcal{X}}"] & \\
(\mathcal{X}_S, \mathcal{B}_S) \arrow[dd] & (\mathcal{X}_{\overline{S}}, \mathcal{B}_{\overline{S}}) \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[rr, "\rho_{\mathcal{X}}"] \arrow[dd] & & \bigcup_{s\in\pi^{-1}(v)}(\mathcal{X}_s, \mathcal{B}_s) =: (\mathcal{X}_v, \mathcal{B}_v) \arrow[dd] \\
 & & \widetilde{S} \arrow[dll, "\tilde{\tau}"'] \arrow[dl, "\widetilde{\pi}"] \arrow[dr, "\tilde{\rho}"] & \\
S & \overline{S} \arrow[l, "\tau"'] \arrow[rr, "\rho"] & & \pi^{-1}(v)
\end{tikzcd}$$

Denote $\mathcal{A}^!_v := \mathcal{A}^!|_{\mathcal{X}_v}$. Then, for each $g \in G$, there exists $h \in H$, which is a lift of g, such that

$$\widetilde{\pi}^*_{\mathcal{X}} g^* \rho^*_{\mathcal{X}} \mathcal{A}^!_v = h^* \widetilde{\pi}^*_{\mathcal{X}} \rho^*_{\mathcal{X}} \mathcal{A}^!_v = h^* \tilde{\rho}^*_{\mathcal{X}} \mathcal{A}^!_v = \tilde{\rho}^*_{\mathcal{X}} h^* \mathcal{A}^!_v,$$

which is vertical over $\mathcal{X}_v$ via $\tilde{\rho}_{\mathcal{X}}$. Here the last equality follows from the fact that H acts diagonally on $(\mathcal{X}_s, \mathcal{B}_s) \times \widetilde{S}$. Therefore, $(\widetilde{\pi}_{\mathcal{X}})_* \widetilde{\pi}^*_{\mathcal{X}} g^* \rho^*_{\mathcal{X}} \mathcal{A}^!_v$ is also vertical over $\mathcal{X}_v$ via $\rho_{\mathcal{X}}$. Hence,

$$\overline{\mathcal{L}}_{(2),v} := \overline{\mathcal{L}}_{(2)}|_{\mathcal{X}_{\overline{S}}} = \sum_{g\in G} g^* \rho^*_{\mathcal{X}} \mathcal{A}^!_v$$

is vertical over $\mathcal{X}_v$ via $\rho_{\mathcal{X}}$. Since v can be any closed point of $S^*_{(2)}$, it follows that $\overline{\mathcal{L}}_{(2)}$ is vertical over $\mathcal{X}_{S^!_{(2)}}$. Let

$$\overline{S}_{(2)} \longrightarrow \overline{S}^!_{(2)} \longrightarrow S^!_{(2)}$$

be the Stein factorization of $\rho : \overline{S}_{(2)} \to S^!_{(2)}$. Replacing $S^!_{(2)}$ by $\overline{S}^!_{(2)}$, we may assume that ρ has connected fibers. After shrinking $S^!_{(2)}$, and then replacing $\mathcal{X}_{S^!_{(2)}}$ accordingly, we may assume that

$$\rho_{\mathcal{X}} : \mathcal{X}^! \times_{S^!} \overline{S}_{(2)} \to \mathcal{X}_{S^!_{(2)}}$$

is equidimensional with integral fibers. Therefore, there exists a divisor $\mathcal{L}^!_{(2)}$ on $\mathcal{X}_{S^!_{(2)}}$ such that $\overline{\mathcal{L}}_{(2)} = \rho_{\mathcal{X}}^*\mathcal{L}^!_{(2)}$. Let $\mathcal{L}^!$ be the closure of $\mathcal{L}^!_{(2)}$ on $\mathcal{X}^!$, then we have

$$\tau_{\mathcal{X}}^*\mathcal{L}_{(2)} = \rho_{\mathcal{X}}^*\mathcal{L}^!.$$

Note that i is a morphism on $\mathcal{S}^!_{(2)}$, and let $\overline{\mathcal{T}}$ be the preimage of $i(\mathcal{S}^!_{(2)})$ in $\overline{\mathcal{S}}$. Since $\mathcal{A}^! = i^*\mathcal{A}_{(2)}$, we have

$$\rho_{\mathcal{X}}^*\mathcal{A}^!|_{\overline{\mathcal{T}}} = \rho_{\mathcal{X}}^* i^*\mathcal{A}_{(2)}|_{\overline{\mathcal{T}}} = \tau_{\mathcal{X}}^*\mathcal{A}_{(2)}|_{\overline{\mathcal{T}}}.$$

By Lemma 2.7.1, it follows that

$$(\rho_{\mathcal{X}}^*\mathcal{A}^!)_s \equiv (\tau_{\mathcal{X}}^*\mathcal{A}_{(2)})_s \quad \text{for all } s \in \overline{\mathcal{S}}_{(2)}.$$

Moreover, since $\tau_{\mathcal{X}}$ is the quotient morphism by G, the divisor $\tau_{\mathcal{X}}^*\mathcal{A}_{(2)}$ is G-invariant. Hence,

$$\tau_{\mathcal{X}}^*\mathcal{A}_{(2)} = \frac{1}{|G|}\sum_{g\in G} g^*\tau_{\mathcal{X}}^*\mathcal{A}_{(2)}.$$

Therefore, for any $s \in \overline{\mathcal{S}}_{(2)}$, we obtain

$$(\tau_{\mathcal{X}}^*\mathcal{L}_{(2)})_s = (\overline{\mathcal{L}}_{(2)})_s \equiv \Big(\sum_{g\in G} g^*\rho_{\mathcal{X}}^*\mathcal{A}^!\Big)_s \equiv \Big(\sum_{g\in G} g^*\tau_{\mathcal{X}}^*\mathcal{A}_{(2)}\Big)_s = |G|\,(\tau_{\mathcal{X}}^*\mathcal{A}_{(2)})_s.$$

Since the morphism $\overline{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is surjective, we conclude that

$$(\mathcal{L}_{(2)})_s \equiv |G|\,(\mathcal{A}_{(2)})_s \quad \text{for all } s \in \mathcal{S}_{(2)}.$$

Step 5. In this step, we verify (7)–(9).

By the construction in the previous step, the general fiber of $((\mathcal{X}_{(2)}, \mathcal{B}_{(2)}), \mathcal{L}_{(2)}) \to \mathcal{S}_{(2)}$ is a $(d, \Phi, |G|^d v)$-polarized Calabi-Yau pair. After replacing $\mathcal{S}_{(2)}$ with an open subset and decreasing α, we may assume that $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)} + \alpha\mathcal{L}_{(2)}) \to \mathcal{S}_{(2)}$ is locally stable. Applying [93]Lemma 4 to an open subset $(\mathcal{S}^!)^o$ of $\mathcal{S}^!$ over which $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ is locally stable, and then repeating the same arguments as in step 3, we may assume that $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ is locally stable. In the process, we may have lost the local stability of $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$, but this will not be used later. Therefore, (7) holds.

For (8), let $\mathcal{H} \geq 0$ be a very ample divisor on $\mathcal{S}^*$. Because $\mathcal{S}^! \to \mathcal{S}^*$ is a finite cover, $\pi^*\mathcal{H}$ is a big divisor on $\mathcal{S}^!$. Then we can choose $\mathcal{H}$ general such that

- π is étale and Galois over $\mathcal{S}^* \setminus \mathrm{Supp}(\mathcal{H})$, and
- every fiber of $((\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^!) \to \mathcal{S}^!$ over $\mathcal{S}^! \setminus \mathrm{Supp}(\pi^*\mathcal{H})$ is a klt (d, Φ, v)-polarized log Calabi–Yau pair.

Now, we address (9). Since $\mathcal{S}^!$ is smooth and $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is locally stable of maximal variation, by Proposition 2.7.2, the moduli **b**-divisor $\mathcal{M}^!$ descends to a nef and big divisor $\mathcal{M}^!_{S^!}$ on $S^!$. We can choose a general member $0 \le \mathcal{M}^! \in |\mathcal{M}^!_{S^!}|_{\mathbb{Q}}$ such that $l\mathcal{M}^!$ is Cartier and $\pi^*\mathcal{H} \le l\mathcal{M}^!$ for some $l \in \mathbb{N}$ depending only on (d, Φ, v).

Step 6. In this step, we construct $\mathcal{S}$ and verify (10) and (11). Let

$$\mathcal{S}_{(3)} := \gamma^{-1}(\mathcal{S}^* \setminus \pi(\mathrm{Supp}(\mathcal{M}^!))) \cap \mathcal{S}_{(2)},$$

and $\overline{\mathcal{S}}_{(3)}$ be the preimage of $\mathcal{S}_{(3)}$. Let $((\mathcal{X}_{(3)}, \mathcal{B}_{(3)}), \mathcal{A}_{(3)}, \mathcal{L}_{(3)}) \to \mathcal{S}_{(3)}$ and $\overline{\mathcal{X}}_{(3)} \to \overline{\mathcal{S}}_{(3)}$ be the corresponding base change.

Note that $\mathcal{S}_{(3)}$ is an open subset of $\mathcal{S}_{(1)}$, and the moduli map $\phi : U \to \mathcal{S}_{(1)}$ obtained in Step 1 may map onto $\mathcal{S}_{(1)} \setminus \mathcal{S}_{(3)}$. Thus, we repeat the same arguments on $\mathcal{S}_{(1)} \setminus \mathcal{S}_{(3)}$, obtaining a stratification of $\mathcal{S}_{(1)}$, denoted by $\mathcal{S}$. Let $\overline{\mathcal{S}}$ be the preimage of $\mathcal{S}$, and replace $\mathcal{S}^!$ and $\mathcal{S}^*$ accordingly. Let $\overline{\mathcal{X}}$ be a common resolution of the main components of $\mathcal{X} \times_{\mathcal{S}} \overline{\mathcal{S}}$ and $\mathcal{X}^! \times_{\mathcal{S}^!} \overline{\mathcal{S}}$. Then, we have the following diagram

$$\begin{tikzcd}
((\mathcal{X}, \mathcal{B}), \mathcal{A}, \mathcal{L}) \arrow[d] & \overline{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & ((\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^!) \arrow[d] & \\
\mathcal{S} \arrow[rrr, bend right, "\gamma"'] & \overline{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{S}^! \arrow[r, "\pi"] & \mathcal{S}^*,
\end{tikzcd}$$

that satisfies the requirements (1)–(9).

Recall that $((X_U, B_U), A_U) \to U$ is isomorphic to the pullback of $((\mathcal{X}_{(1)}, \mathcal{B}_{(1)}), \mathcal{A}_{(1)}) \to \mathcal{S}_{(1)}$ via the moduli morphism $\phi : U \to \mathcal{S}_{(1)}$. After replacing U by an open subset, we may assume ϕ induces an morphism $\phi : U \to \mathcal{S}$, then $((X_U, B_U), A_U) \to U$ is isomorphic to the pullback of $((\mathcal{X}, \mathcal{B}), \mathcal{A}) \to \mathcal{S}$ via $U \to \mathcal{S}$. Therefore, (10) follows.

Finally, we deal with (11). Suppose that $\gamma \circ \phi$ extends to a morphism $\psi : Z \to \mathcal{S}^*$. By the construction of $\mathcal{S}_{(3)}$, we have $\gamma^{-1}(\pi(\mathrm{Supp}(\mathcal{M}^!))) = \emptyset$. Since $\psi|_U$ factor through $\mathcal{S}$, we have $\psi(Z) \not\subset \pi(\mathrm{Supp}(\mathcal{M}^!))$.

∎

6.2 Boundedness of moduli map

In this section, we construct a birational model (W, D) of Z such that (W, D) is log bounded and the map $W \dashrightarrow \mathcal{S}^*$ induced by the moduli map $Z \dashrightarrow \mathcal{S}^*$ is a bounded morphism.

Theorem 6.2.1. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Let*

$f : ((X, B), A) \to (Z, H)$ be a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration. Let

$$\begin{tikzcd}
((X,B),A) \arrow[r, dashed] \arrow[d] & ((\mathcal{X},\mathcal{B}),\mathcal{A},\mathcal{L}) \arrow[d] & \overline{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & ((\mathcal{X}^!,\mathcal{B}^!),\mathcal{L}^!) \arrow[d] & \\
(Z,H) \arrow[r, dashed, "\phi"'] & \mathcal{S} \arrow[rrr, bend right, "\gamma"'] & \overline{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & (\mathcal{S}^!,\mathcal{M}^!) \arrow[r, "\pi"] & (\mathcal{S}^*,\mathcal{H})
\end{tikzcd}$$

be the commutative diagram obtained in Theorem 6.1.2. Then there exists a birational morphism $h : W \to Z$ from a normal projective variety W and a reduced divisor D on W such that

1. *the induced rational map $\psi_W : W \dashrightarrow \mathcal{S}^*$ is a morphism,*
2. *$D \supset \mathrm{Supp}(h_*^{-1}B_Z + E + \psi_W^*\mathcal{H})$, where B_Z is the discriminant $\mathbb{Q}$-divisor with respect to $f : (X, B) \to Z$, and E is the sum of reduced exceptional divisors of h, and*
3. *$K_W + D - h^*H$ is big.*

Moreover, the set of (W, D) forms a log bounded family, and the morphism $\psi_W : W \to \mathcal{S}^*$ is bounded.

Proof *Step* 1. In this step, we construct a birational model W of Z such that $W \dashrightarrow Z$ and $W \dashrightarrow \mathcal{S}^*$ are morphisms.

Since the coefficients of B belong to the finite set Φ, by [17]$^{\text{Lemma 6.7}}$, there exists $q \in \mathbb{N}$ and $\delta \in \mathbb{Q}^{>0}$ depending only on d, Φ, v, ϵ, such that we can write the canonical bundle formula

$$q(K_X + B) \sim qf^*(K_Z + B_Z + \mathbf{M}_Z)$$

such that $(Z, B_Z, \mathbf{M})$ is a δ-lc generalized pair, qB_Z is integral, and $q\mathbf{M}_{Z'}$ is Cartier, where $\mathbf{M}_{Z'}$ is the moduli divisor on any sufficiently high resolution $Z' \to Z$. In particular, The coefficients of B_Z belong to a fixed finite set $\mathcal{I}$. Replacing $\mathcal{I}$ by $\mathcal{I} \cup \{1 - \frac{\delta}{2}\}$, we may assume that $1 - \frac{\delta}{2} \in \mathcal{I}$.

Let $g : Z' \to Z$ be a log resolution of (Z, B_Z) such that the moduli **b**-divisor $\mathbf{M}$ of f descends to Z', and the rational map $\gamma \circ \phi : Z \dashrightarrow \mathcal{S}^*$ extends to a morphism $\psi' : Z' \to \mathcal{S}^*$. In particular, $\mathbf{M}_{Z'}$ is nef. Define

$$B_{Z'} := g_*^{-1}B_Z + (1 - \frac{\delta}{2})E_{Z'},$$

where $E_{Z'}$ is the sum of all reduced g-exceptional divisors. Since $(Z, B_Z, \mathbf{M})$ is δ-lc, it follows that

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective and has the same support as $E_{Z'}$. Moreover, the coefficients of $B_{Z'}$ belong to the finite set $\mathcal{I}$.

By the boundedness of the length of extremal rays, $K_Z + B_Z + \mathbf{M}_Z + 3dH$ is ample. Since

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective, it follows that

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$$

is big. Consider $(Z', B_{Z'}, \mathbf{M} + 3d\overline{g^*H} + 3d\overline{\psi'^*\mathcal{H}})$ as a $\frac{\delta}{2}$-lc generalized pair with nef part $\mathbf{M} + 3d\overline{g^*H} + 3d\overline{\psi'^*\mathcal{H}}$. By [85][Lemma 4.4], the divisor

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$$

admits a generalized log canonical model $Z' \dashrightarrow W$. In particular, $Z' \dashrightarrow W$ is a birational contraction. Since $d \geq \dim Z'$, the boundedness of the length of extremal rays ensures that the birational contraction $Z' \dashrightarrow W$ is automatically over both Z and $\mathcal{S}^*$, inducing morphisms $h : W \to Z$ and $\psi_W : W \to \mathcal{S}^*$. Let B_W be the pushforward of $B_{Z'}$. Then

$$\operatorname{Supp}(B_W) \supset \operatorname{Supp}(h_*^{-1}B_Z + E),$$

where E is the sum of reduced exceptional divisors of h.

$$\begin{tikzcd}
\overline{Z} \arrow[r, "\pi_{Z'}"'] \arrow[d] \arrow[dr, "\overline{\psi}"'] & Z' \arrow[r, "g"'] \arrow[dr, "\psi'"'] \arrow[rr, dashed, bend left] & Z \arrow[d, dashed, "\psi"] & W \arrow[l, "h"] \arrow[dl, "\psi_W"] \\
\overline{\mathcal{S}} \arrow[r] & \mathcal{S}^! \arrow[r, "\pi"'] & \mathcal{S}^* &
\end{tikzcd}$$

Step 2. In this step, we show that $l\mathbf{M}_{Z'} - \psi'^*\mathcal{H}$ is pseudo-effective for some $l \in \mathbb{N}$ depending only on (d, Φ, v).

Let $\pi_{Z'} : \overline{Z} \to Z'$ be a generically finite cover from a smooth variety $\overline{Z}$ such that

- $\psi' : Z' \to \mathcal{S}^*$ lifts to a morphism $\overline{\psi} : \overline{Z} \to \mathcal{S}^!$ which factors through $\overline{\mathcal{S}}$, and
- the generic fiber of $(X, B) \times_Z \overline{Z} \to \overline{Z}$ is isomorphic to the generic fiber of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \overline{Z} \to \overline{Z}$.

Since $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is locally stable over the smooth base $\mathcal{S}^!$, the morphism

$$(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \overline{Z} \to \overline{Z}$$

is also locally stable over the smooth base $\overline{Z}$. By parts (2) and (3) of Proposition 2.7.2, the moduli **b**-divisor $\overline{\mathbf{M}}$ of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \overline{Z} \to \overline{Z}$ descends to $\overline{Z}$ and satisfies

$$\overline{\psi}^* \mathcal{M}^!_{\mathcal{S}^!} \sim_{\mathbb{Q}} \overline{\mathbf{M}}_{\overline{Z}}.$$

Let $\widetilde{\mathbf{M}}$ be the moduli **b**-divisor of $(X, B) \times_Z \overline{Z} \to \overline{Z}$. Since the generic fiber of $(X, B) \times_Z \overline{Z} \to \overline{Z}$ is isomorphic to the generic fiber of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \overline{Z} \to \overline{Z}$, it follows from [5][Lemma 3.5] that $\widetilde{\mathbf{M}}_{\overline{Z}} \simeq \overline{\mathbf{M}}_{\overline{Z}}$. We may still denote the moduli **b**-divisor of $(X, B) \times_Z \overline{Z} \to \overline{Z}$ by $\overline{\mathbf{M}}$ without confusion, and it descends to $\overline{Z}$.

Since $\pi_{Z'} : \overline{Z} \to Z'$ is a generically finite cover and $\mathbf{M}$ descends to Z', it follows from Proposition 2.5.3 that

$$\overline{\mathbf{M}}_{\overline{Z}} \sim_{\mathbb{Q}} \pi^*_{Z'} \mathbf{M}_{Z'}.$$

By parts (9) and (11) of Theorem 6.1.2, there exists $l \in \mathbb{N}$ depending only on (d, Φ, v) such that

$$\pi^* \mathcal{H} \leq l \mathcal{M}^!,$$

and $\psi'(Z') \not\subset \pi(\mathrm{Supp}(\mathcal{M}^!))$. Then, we obtain

$$\pi^*_{Z'} l \mathbf{M}_{Z'} \sim_{\mathbb{Q}} l \overline{\mathbf{M}}_{\overline{Z}} \sim_{\mathbb{Q}} \overline{\psi}^* l \mathcal{M}^!_{\mathcal{S}^!} \geq \overline{\psi}^* \pi^* \mathcal{H} \sim_{\mathbb{Q}} \pi^*_{Z'} \psi'^* \mathcal{H}.$$

Therefore, $l\mathbf{M}_{Z'} - \psi'^* \mathcal{H}$ is pseudo-effective.

Step 3. In this step, we show that $\mathrm{vol}(K_W + B_W + \mathbf{M}_W + 3dh^* H + 3d\psi^*_W \mathcal{H})$ is bounded from above.

Since $Z' \dashrightarrow W$ is the generalized log canonical model of

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^* H + 3d\psi'^* \mathcal{H},$$

and $l\mathbf{M}_{Z'} - \psi'^* \mathcal{H}$ is pseudo-effective by Step 2, we have

$$\begin{aligned} &\mathrm{vol}(K_W + B_W + \mathbf{M}_W + 3dh^* H + 3d\psi^*_W \mathcal{H}) \\ \leq &\mathrm{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^* H + 3d\psi'^* \mathcal{H}) \\ \leq &\mathrm{vol}(K_{Z'} + B_{Z'} + (3dl + 1)\mathbf{M}_{Z'} + 3dg^* H). \end{aligned} \tag{6.1}$$

By Step 1, $q\mathbf{M}_{Z'}$ is Cartier. Hence, replacing l with ql, we may assume that

$$l(\mathbf{M}_{Z'} + 3dg^* H)$$

is Cartier.

Since the coefficients of $B_{Z'}$ belong to the finite set $\mathcal{I}$, by [85][Theorem 8.1], there exists $e \in (0,1)$ depending only on $d, \mathcal{I}, l$ such that

$$K_{Z'} + B_{Z'} + e\mathbf{M}_{Z'} + 3dg^*H$$

is big. Choose $\lambda \in (0,1)$ such that

$$\lambda e + (1-\lambda)(3dl+1) = 1.$$

Then, we have

$$\begin{aligned}
&\lambda(K_{Z'} + B_{Z'} + e\mathbf{M}_{Z'} + 3dg^*H) \\
+&(1-\lambda)(K_{Z'} + B_{Z'} + (3dl+1)\mathbf{M}_{Z'} + 3dg^*H) \\
=&K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\operatorname{vol}(K_{Z'} + B_{Z'} + (3dl+1)\mathbf{M}_{Z'} + 3dg^*H) \\
\leq&\frac{1}{(1-\lambda)^d}\operatorname{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H).
\end{aligned} \tag{6.2}$$

By the definition of the weak polarized log Calabi–Yau fibration, $H-(K_Z+B_Z+\mathbf{M}_Z)$ is pseudo-effective. Since

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective and exceptional over Z, and $\dim Z \leq d$, we have

$$\begin{aligned}
&\operatorname{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H) \\
=&\operatorname{vol}(K_Z + B_Z + \mathbf{M}_Z + 3dH) \\
\leq&(3d+1)^d H^{\dim Z} \\
\leq&(3d+1)^d r.
\end{aligned} \tag{6.3}$$

Combining equations (3.1)–(3.3), we conclude that

$$\operatorname{vol}(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}) \leq \frac{(3d+1)^d}{(1-\lambda)^d}r.$$

Step 4. In this step, we show that W belongs to a bounded family. Moreover, the morphism $\psi_W : W \to S^*$ is also bounded.

Since Z' is smooth and $\mathbf{M}$ descends to Z', after replacing Z' with a higher model so that $Z' \to W$ is a morphism, $(W, B_W, \mathbf{M}_W + 3d\overline{h^*H} + 3d\overline{\psi_W^*\mathcal{H}})$ is a $\frac{\delta}{2}$-lc generalized pair with nef part $\mathbf{M} + 3d\overline{g^*H} + 3d\overline{\psi'^*\mathcal{H}}$, satisfying the following conditions:

- The coefficients of B_W belong to the finite set $\mathcal{I}$,

- $l(\mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H})$ is Cartier, and
- $K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}$ is ample with bounded volume,

it follows from [17]Lemma 6.6 that $(W, B_W, \mathbf{M} + 3d\overline{h^*H} + 3d\overline{\psi_W^*\mathcal{H}})$ is log bounded. In particular, there exists $m \in \mathbb{N}$ depending only on d, Φ, v, ϵ, r such that

$$H_W := m(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H})$$

is very ample and $\mathrm{vol}(H_W)$ is bounded from above.

Let $\Gamma_{\psi_W} \subset W \times \mathcal{S}^*$ be the graph of the morphism $\psi_W : W \to \mathcal{S}^*$. Since H_W and $\mathcal{H}$ are very ample, the product $W \times \mathcal{S}^*$ can be embedded into a projective space via the Segre embedding $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \subset \mathbb{P}^N$. Moreover, the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\Gamma_{\psi_W} \cong W$ is given by $H_W + \psi_W^*\mathcal{H}$. Note that

$$\begin{aligned}\mathrm{vol}(H_W + \psi_W^*\mathcal{H}) &= \mathrm{vol}\Big(m(K_W + B_W + \mathbf{M}_W + 3d\,h^*H) + (3dm+1)\psi_W^*\mathcal{H}\Big)\\ &\le \mathrm{vol}\Big((m+1)(K_W + B_W + \mathbf{M}_W + 3d\,h^*H) + (3dm+1)\psi_W^*\mathcal{H}\Big)\\ &\le \mathrm{vol}\Big((m+1)\big(K_W + B_W + \mathbf{M}_W + 3d\,h^*H + 3d\psi_W^*\mathcal{H}\big)\Big),\end{aligned}$$

which is bounded from above by Step 3. The first inequality follows from the fact that $K_W + B_W + \mathbf{M}_W + 3d\,h^*H$ is big (see Step 1). Consequently, Γ_{ψ_W} is bounded. Since every morphism $\psi_W : W \to \mathcal{S}^*$ is determined by its graph Γ_{ψ_W}, it follows that the morphism $\psi_W : W \to \mathcal{S}^*$ is bounded.

Step 5. In this step, we define a reduced divisor D on W and conclude the proof.

Since h^*H and $\psi_W^*\mathcal{H}$ are base point free, it follows that $3d(H_W + h^*H + \psi_W^*\mathcal{H})$ is very ample. We can find a general reduced divisor

$$0 \le D \in |3d(H_W + h^*H + \psi_W^*\mathcal{H})|$$

such that D contains the support of $h_*^{-1}B_Z + E + \psi_W^*\mathcal{H}$, where E is the sum of the reduced exceptional divisors of h. Moreover, by the boundedness of the length of extremal rays, the divisor $K_W + D - h^*H$ is big.

Let $d' = \dim W$, by [1]Lemma 3.2, we have

$$\begin{aligned}((4d'+2)\,H_W)^{d'-1} \cdot D &\le 2^{d'}\,\mathrm{vol}\big(K_W + D + (4d'+2)H_W\big)\\ &\le 2^{d'}\,\mathrm{vol}\big(K_W + B_W + \mathbf{M}_W + D + (4d'+2)H_W\big)\\ &= \alpha^{d'}\,\mathrm{vol}\big(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}\big),\end{aligned}$$

which is bounded from above by Step 3, where $\alpha = 2 + 2m(4d' + 3d + 2)$. Hence, (W, D) is log bounded, completing the proof.

∎

Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. Let α be the positive rational number defined in Theorem 6.1.2. Let $f : ((X, B), A) \to (Z, H)$ be a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration. By Theorem 6.2.1, there exists a family of pairs $(\mathcal{W}, \mathcal{D}) \to T$ over a finite type scheme T, and a projective morphism $\Theta : \mathcal{W} \to \mathcal{S}^*$ such that $(W, D) \cong (\mathcal{W}_t, \mathcal{D}_t)$, and $\psi_W : W \to \mathcal{S}^*$ is equivalent to $\Theta_t : \mathcal{W}_t \to \mathcal{S}^*$ for some closed point $t \in T$.

Let $\overline{\mathcal{W}}$ be the normalization of the main component of $\mathcal{W} \times_{\mathcal{S}^*} \mathcal{S}^!$, and let $\overline{\mathcal{D}}_{\overline{\mathcal{W}}}$ denote the preimage of $\mathcal{D}$ via the map $\overline{\mathcal{W}} \to \mathcal{W}$. After replacing $(\overline{\mathcal{W}}, \overline{\mathcal{D}}_{\overline{\mathcal{W}}})$ with its log resolution and passing to a stratification of T, we may assume that the pair $(\overline{\mathcal{W}}, \overline{\mathcal{D}}_{\overline{\mathcal{W}}})$ is log smooth over T. Let $\overline{\Theta} : (\overline{\mathcal{W}}, \overline{\mathcal{D}}_{\mathcal{W}}) \to \mathcal{S}^!$ be the induced morphism, and let $\overline{\mathcal{F}} : ((\overline{\mathcal{X}}_{\overline{\mathcal{W}}}, \overline{\mathcal{B}}_{\overline{\mathcal{W}}}), \overline{\mathcal{L}}_{\overline{\mathcal{W}}}) \to \overline{\mathcal{W}}$ be the pullback of $((\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^!) \to \mathcal{S}^!$ via $\overline{\Theta}$. We have the following commutative diagram.

$$\begin{tikzcd}
(\overline{\mathcal{X}}_{\overline{\mathcal{W}}}, \overline{\mathcal{B}}_{\overline{\mathcal{W}}}), \overline{\mathcal{L}}_{\overline{\mathcal{W}}} \arrow[r, "\overline{\mathcal{F}}"] \arrow[d] & (\overline{\mathcal{W}}, \overline{\mathcal{D}}_{\overline{\mathcal{W}}}) \arrow[r] \arrow[d, "\overline{\Theta}"] & (\mathcal{W}, \mathcal{D}) \arrow[r] \arrow[d, "\Theta"] & T \\
(\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \arrow[r] & \mathcal{S}^! \arrow[r, "\pi"] & \mathcal{S}^* &
\end{tikzcd}$$

Lemma 6.2.2. *There exists $w \in \mathbb{N}$ depending only on d, Φ, v, r, ϵ such that*

$$\operatorname{vol}(K_{\overline{\mathcal{X}}_{\overline{\mathcal{W}}_t}} + \overline{\mathcal{B}}_{\overline{\mathcal{W}}_t} + \alpha\overline{\mathcal{L}}_{\overline{\mathcal{W}}_t} + \overline{\mathcal{F}}_t^* \overline{\mathcal{D}}_{\overline{\mathcal{W}}_t}) \le w$$

for every closed point $t \in T$.

Proof Since $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ is locally stable, it follows that

$$(\overline{\mathcal{X}}_{\overline{\mathcal{W}}}, \overline{\mathcal{B}}_{\overline{\mathcal{W}}} + \alpha\overline{\mathcal{L}}_{\overline{\mathcal{W}}}) \to \overline{\mathcal{W}}$$

is also locally stable. Since $(\overline{\mathcal{W}}, \overline{\mathcal{D}}_{\overline{\mathcal{W}}})$ is log smooth, it follows from [76][Corollary 4.55] that

$$(\overline{\mathcal{X}}_{\overline{\mathcal{W}}}, \overline{\mathcal{B}}_{\overline{\mathcal{W}}} + \alpha\overline{\mathcal{L}}_{\overline{\mathcal{W}}} + \overline{\mathcal{F}}^* \overline{\mathcal{D}}_{\overline{\mathcal{W}}})$$

is lc. After passing to a stratification of T, we may assume that

$$(\overline{\mathcal{X}}_{\overline{\mathcal{W}}}, \overline{\mathcal{B}}_{\overline{\mathcal{W}}} + \alpha\overline{\mathcal{L}}_{\overline{\mathcal{W}}} + \overline{\mathcal{F}}^* \overline{\mathcal{D}}_{\overline{\mathcal{W}}}) \to T$$

admits a fiberwise log resolution $(\overline{\mathcal{Y}}_{\overline{\mathcal{W}}}, \overline{\mathcal{R}}_{\overline{\mathcal{W}}}) \to T$. Then, by [1][Theorem 1.8 (3)],

$$\operatorname{vol}(K_{\overline{\mathcal{X}}_{\overline{\mathcal{W}}_t}} + \overline{\mathcal{B}}_{\overline{\mathcal{W}}_t} + \alpha\overline{\mathcal{L}}_{\overline{\mathcal{W}}_t} + \overline{\mathcal{F}}_t^* \overline{\mathcal{D}}_{\overline{\mathcal{W}}_t}) = \operatorname{vol}(K_{\overline{\mathcal{Y}}_{\overline{\mathcal{W}}_t}} + \overline{\mathcal{R}}_{\overline{\mathcal{W}}_t, >0})$$

is independent of $t \in T$. ∎

6.3 Log birational boundedness

In this section, we do some preparation for the proof of log birational boundedness of weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration $f : ((X, B), A) \to (Z, H)$.

In the following theorem, we construct a special birational model $((X', \Delta'), A') \to (Z', D')$ of $((X, B), A) \to Z$, where $(Z', D') \to Z$ factors through the log bounded birational model (W, D) of Z constructed in Theorem 6.2.1.

Theorem 6.3.1. *Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Let α be the rational number defined in Theorem 6.1.2. Assume that*

- *$f : (X, B) \to Z$ is a log Calabi–Yau fibration such that (X, B) is klt, and A is an effective integral divisor on X,*
- *the general fiber $((X_g, B_g), A_g)$ is a (d, Φ, v)-polarized log Calabi–Yau pair,*
- *there is a canonical bundle formula $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z)$,*
- *$W \to Z$ is a birational morphism, and*
- *D is a reduced divisor on W containing the strict transform of $\mathrm{Supp}(B_Z)$ together with the exceptional divisors over Z.*

Then we can construct a commutative diagram

$$\begin{tikzcd}
((\overline{X}, \overline{B}), \overline{A}) \arrow[r] \arrow[d, "\overline{f}"] & ((X', B'), A') \arrow[rr, dashed] \arrow[d, "f'"] & & ((X, B), A) \arrow[d, "f"] \\
(\overline{Z}, \overline{D}) \arrow[r, "\pi'"] & (Z', D') \arrow[r] & (W, D) \arrow[r] & Z
\end{tikzcd}$$

satisfying the following properties:

(i) $Z' \to W$ is a birational morphism,

(ii) $f' : X' \to Z'$ is a contraction, and B', A' are horizontal $\mathbb{Q}$-divisors on X',

(iii) the generic fiber of $(X', B' + \alpha A') \to Z'$ is isomorphic to the generic fiber of $(X, B + \alpha A) \to Z$,

(iv) $\pi' : \overline{Z} \to Z'$ is a finite cover,

(v) (Z', D') and $(\overline{Z}, \overline{D})$ are log smooth, where D' is the sum of the strict transform of D and all exceptional divisors over W, and $\overline{D}$ is the preimage of D' under π',

(vi) $\overline{X}$ is the normalization of $X' \times_{Z'} \overline{Z}$, and $\overline{B}, \overline{A}$ are horizontal $\mathbb{Q}$-divisors equal to the pullback of B', A' on $\overline{X}$ over the generic point of Z', and

(vii) $\overline{f} : ((\overline{X}, \overline{B}), \overline{A}) \to \overline{Z}$ is a family of (d, Φ, v)-polarized log Calabi–Yau pairs.

Furthermore, if $\overline{f} : (\overline{X}, \overline{B}) \to \overline{Z}$ has klt fibers over codimension one points of $\overline{Z} \setminus \overline{D}$, then, setting $\Delta' := B' + \mathrm{red}(f'^* D')$, we have:

1. f' has integral fibers over codimension one points of $Z' \setminus D'$,

2. *$(X', \Delta' + \alpha A')$ is lc,*
3. *$K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + D' + \mathbf{M}_{Z'})$, and*
4. *$\mathrm{Supp}(\Delta')$ contains the strict transform of $\mathrm{Supp}(B)$ together with all exceptional divisors over X.*

Proof *Step* 1. In this step we construct a birational morphism $Z' \to W$ and a finite cover $\overline{Z} \to Z'$.

Let Y be a log resolution of $(X, B + \alpha A)$. Let B_Y be the strict transform of B plus the reduced horizontal exceptional divisors over Z, and let A_Y be the strict transform of A. Let $Z^o \subset Z$ be an open subset such that

- $W \to Z$ is an isomorphism over Z^o,
- $f : ((X, B), A) \to Z$ is a family of (d, Φ, v)-polarized log Calabi-Yau pair over Z^o, and
- $f_Y : (Y, B_Y + A_Y) \to Z$ is log smooth over Z^o.

Then B_Y and A_Y are effective $\mathbb{Q}$-divisors which are horizontal over Z^o. By [33][Theorem 2.1 and Proposition 4.4], there is an extension $Z^o \hookrightarrow Z'$ such that

- Z' is a log resolution of (W, D),
- there is an equidimensional toroidal morphism $f'_Y : Y' \to Z'$,
- if B'_Y, A'_Y are the closures of $B_Y|_{Z^o}$ and $A_Y|_{Z^o}$, respectively, then they are contained in the toroidal boundary of Y', and
- $((Y', B'_Y), A'_Y) \to Z'$ is an extension of $((Y, B_Y), A_Y) \times_Z Z^o \to Z^o$.

Let D' be the sum of the strict transform of D on Z' and all reduced exceptional divisors over W. By [33][Proposition 5.1], there exists a finite cover $\pi' : \overline{Z} \to Z'$ such that $\overline{f}_Y : \overline{Y} \to \overline{Z}$ is an equidimensional toroidal morphism with reduced fibers, where $\overline{Y}$ is the normalization of $Y' \times_{Z'} \overline{Z}$.

Note that the finite cover $\overline{Z} \to Z'$ is a Kawamata covering. To ensure the smoothness of $\overline{Z}$ in the construction, we add extra branch loci artificially (see [94][Theorem 1.8.2]). Let R' be the divisor on Z' whose support contains the union of $\mathrm{Supp}(D')$ and the branch divisors of π'. Define $\overline{R} := \mathrm{red}(\pi'^* R')$. Then $(\overline{Z}, \overline{R})$ is log smooth by [33][Lemma 5.9].

Let $\overline{D} := \mathrm{red}(\pi'^* D')$ be the reduced divisor on $\overline{Z}$. By construction, $\overline{D} \subset \overline{R}$. Let $\overline{B}_Y$ and $\overline{A}_Y$ be the pullbacks of B'_Y and A'_Y to $\overline{Y}$. Then they are contained in the toroidal boundary of $\overline{Y}$.

By [95][Proposition 2.16], $(\overline{Y}, \overline{B}_Y + \mu \overline{A}_Y + \overline{f}_Y^* \Sigma)$ is lc for any reduced simple normal crossing divisor Σ on $\overline{Z}$, where $\mu \in (0,1)$ is sufficiently small. It follows that

$$\overline{f}_Y : (\overline{Y}, \overline{B}_Y + \mu\overline{A}_Y) \to \overline{Z}$$

is a locally stable morphism by [76]Corollary 4.55.

Step 2. In this step we construct a family of (d, Φ, v)-polarized log Calabi–Yau pairs $((\overline{X}, \overline{B}), \overline{A}) \to \overline{Z}$.

Since $(\overline{Y}, \overline{B}_Y + \mu\overline{A}_Y) \to \overline{Z}$ is locally stable and $\overline{Z}$ is smooth, every lc center of $(\overline{Y}, \overline{B}_Y + \mu\overline{A}_Y)$ dominates $\overline{Z}$ by [76]Corollary 4.56. As a general fiber $(Y_g', B_g' + \mu A_g')$ is klt, we conclude that $(\overline{Y}, \overline{B}_Y + \mu\overline{A}_Y)$ is klt. The general fiber $(\overline{Y}_g, \overline{B}_{Y_g})$ has a semi-ample model (X_g, B_g), and hence it admits a good minimal model by [3]Lemma 2.9.1. Thus, by [34]Theorem 1.1, running an MMP on $K_{\overline{Y}}+\overline{B}_Y$ over $\overline{Z}$ yields a good minimal model $(\overline{X}', \overline{B}')$ over $\overline{Z}$, and let $\overline{A}'$ be the pushforward of $\overline{A}_Y$.

By [76]Corollary 4.57.1, $(\overline{X}', \overline{B}') \to \overline{Z}$ is also locally stable. Since $K_{\overline{X}'} + \overline{B}'$ is semi-ample over $\overline{Z}$ and has Kodaira dimension 0 on the generic fiber, and since $\overline{X}' \to \overline{Z}$ is equidimensional, upper semi-continuity of fiber dimensions gives

$$K_{\overline{X}'} + \overline{B}' \sim_{\mathbb{Q},\overline{Z}} 0.$$

Define $(\overline{X}', \overline{B}' + \mu\overline{A}') \dashrightarrow (\overline{X}, \overline{B} + \mu\overline{A})$ to be the log canonical model of $K_{\overline{X}'} + \overline{B}' + \mu\overline{A}'$ over $\overline{Z}$. As $\overline{X}' \dashrightarrow \overline{X}$ is a birational contraction,

$$K_{\overline{X}} + \overline{B} \sim_{\mathbb{Q},\overline{Z}} 0. \tag{6.4}$$

Finally, by [76]Corollary 4.57.2, $((\overline{X}, \overline{B}), \overline{A}) \to \overline{Z}$ is a stable family of polarized log Calabi–Yau pairs. Since the general fiber is (d, Φ, v)-polarized, by the definition of α the family $(\overline{X}, \overline{B} + \alpha\overline{A}) \to \overline{Z}$ is locally stable.

Step 3. In this step we construct a contraction $f' : X' \to Z'$ together with horizontal $\mathbb{Q}$-divisors B', A' on X', and show that the generic fiber of $(X', B' + \alpha A') \to Z'$ is isomorphic to that of $(X, B + \alpha A) \to Z$.

By the Hurwitz formula [96]§2.41.4 we have

$$K_{\overline{Z}} + \overline{R} = \pi^*(K_{Z'} + R'),$$

where both $(\overline{Z}, \overline{R})$ and (Z', R') are log smooth by construction. By [76]Corollary 4.55, $(\overline{Y}, \overline{B}_Y + \mu\overline{A}_Y + \overline{f}_Y^*\overline{R})$ is lc. Let $\pi_Y : \overline{Y} \to Y'$ be the natural finite cover. Since étale morphisms are stable under base change, the ramification divisor of π_Y is contained in $\mathrm{Supp}(\overline{f}_Y^*\overline{R})$. Thus, by [96][§2.41.4],

$$K_{\overline{Y}} + \overline{B}_Y + \mu\overline{A}_Y + \overline{f}_Y^*\overline{R} = \pi_Y^*\big(K_{Y'} + B'_Y + \mu A'_Y + \mathrm{red}(f_Y'^* R')\big),$$

and $(Y', B'_Y + \mu A'_Y + \mathrm{red}(f_Y'^* R'))$ is lc.

Since the general fiber (Y'_g, B'_{Y_g}) admits a semi-ample model (X_g, B_g), by [34][Theorem 1.1] we can run an MMP on

$$K_{Y'} + B'_Y + \mathrm{red}(f_Y'^* R')$$

over Z' (equivalently, on $K_{Y'} + B'_Y + \mathrm{red}(f_Y'^* R') - a f_Y'^* R'$ for $a \ll 1$). This yields a good minimal model

$$(X'', B'' + \mathrm{red}(f''^* R'))$$

over Z', with morphism $f'' : X'' \to Z'$. Let A'' be the pushforward of A'_Y.

By Lemma 2.4.1(1), $\overline{X}'$ is isomorphic in codimension one to the normalization of $X'' \times_{Z'} \overline{Z}$. Now let

$$(X', B' + \mathrm{red}(f'^* R') + \mu A')$$

be the log canonical model of $K_{X''} + B'' + \mathrm{red}(f''^* R') + \mu A''$ over Z', where $f' : X' \to Z'$. Then the generic fiber of

$$(X', B' + \mathrm{red}(f'^* R') + \alpha A') \to Z'$$

coincides with that of

$$(X, B + \alpha A) \to Z.$$

As both B'_Y and A'_Y are horizontal over Z', so are B' and A'. By Lemma 2.4.1(2), $\overline{X}$ is isomorphic to the normalization of $X' \times_{Z'} \overline{Z}$ and

$$K_{\overline{X}} + \overline{B} + \alpha\overline{A} + \overline{f}^*\overline{R} = \pi_X^*\big(K_{X'} + B' + \alpha A' + \mathrm{red}(f'^* R')\big),$$

where $\pi_X : \overline{X} \to X'$. Using $\overline{A} = \pi_X^* A'$, this simplifies to

$$K_{\overline{X}} + \overline{B} + \overline{f}^*\overline{R} = \pi_X^*(K_{X'} + B' + \mathrm{red}(f'^* R')).$$

Combining (6.4) with Lemma 2.6.3, we conclude that

$$K_{X'} + B' + \mathrm{red}(f'^* R') \sim_{\mathbb{Q}, Z'} 0.$$

Finally, since $(\overline{X}, \overline{B} + \alpha\overline{A} + \overline{f}^*\overline{R})$ is lc, [96][Corollary 2.43] implies that $(X', B' + \mathrm{red}(f'^* R') + \alpha A')$ is also lc.

Step 4. In this step we prove the furthermore part. From now on we assume that $(\overline{X}, \overline{B}) \to \overline{Z}$ has klt fibers over codimension one points in $\overline{Z} \setminus \overline{D}$, and denote $\Delta' := B' + \mathrm{red}(f'^*D')$.

Let P be a prime divisor on Z' not contained in $\mathrm{Supp}(D')$, and let $\widetilde{B_Z}$ be the strict transform of B_Z on W. Since $\mathrm{Supp}(\widetilde{B_Z}) \subseteq \mathrm{Supp}(D)$ and $\mathrm{Supp}(D')$ contains both the strict transform of D and all exceptional divisors over Z, by the definition of the discriminant part in the canonical bundle formula, we conclude that $f' : X' \to Z'$ has a reduced fiber over the generic point of P.

Let $\overline{P}$ be an irreducible component of the preimage of P on $\overline{Z}$. By assumption, $(\overline{X}, \overline{B})$ has a klt fiber over the generic point of $\overline{P}$. By inversion of adjunction, $(\overline{X}, \overline{B} + \overline{f}^*\overline{P})$ is plt near the fiber over the generic point of $\overline{P}$. By [96][§2.41.4], over the generic point of $\overline{P}$ the divisor $K_{\overline{X}} + \overline{B} + \overline{f}^*\overline{P}$ is equivalent to the pullback of $K_{X'} + B' + f'^*P$. Hence, near the fiber of the generic point of P, the pair $(X', B' + f'^*P)$ is plt by [96][Corollary 2.43]. Therefore, f'^*P is irreducible over the generic point of P. This proves (1).

Because f' is equidimensional and has reduced fibers over codimension one points of $Z' \setminus D'$, we obtain

$$\mathrm{red}(f'^*R') = \mathrm{red}(f'^*D) + f'^*(R' - D').$$

Since

$$K_{X'} + B' + \mathrm{red}(f'^*R') \sim_{\mathbb{Q},Z'} 0$$

and $(X', B' + \mathrm{red}(f'^*R') + \alpha A')$ is lc, it follows that

$$K_{X'} + \Delta' = K_{X'} + B' + \mathrm{red}(f'^*D) \sim_{\mathbb{Q},Z'} 0$$

and $(X', \Delta' + \alpha A')$ is also lc. This proves (2).

Next, observe that if P is a prime divisor on Z' not contained in $\mathrm{Supp}(D')$, then $(X', \Delta' + f'^*P)$ is plt over the generic point of P, which implies that the discriminant divisor of $f' : (X', \Delta') \to Z'$ is contained in $\mathrm{Supp}(D')$. If P is a prime divisor contained in $\mathrm{Supp}(D')$, then $\mathrm{lct}(X', \Delta'; P) = 0$. Thus,

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + D' + \mathbf{M}_{Z'}),$$

where $\mathbf{M}$ is the moduli $\mathbf{b}$-divisor corresponding to $f : (X, B) \to Z$. This proves (3).

Finally, we prove (4). First, we show that $\mathrm{Supp}(f'^*D')$ contains all exceptional divisors over X. Suppose E' is a prime divisor on X' exceptional over X but not contained in $\mathrm{Supp}(f'^*D')$. Since $(X, B) \to Z$ and $(X', B') \to Z'$ have the same generic fiber, E' is vertical over Z'. As $f' : X' \to Z'$ is equidimensional, $P' := f'(E')$ is a prime divisor on Z' not contained in $\mathrm{Supp}(D')$. Because $\mathrm{Supp}(D')$ contains all exceptional divisors over Z, the image of P' on Z is also a prime divisor P. Let F be a component of $f^{-1}P$ dominating P. Then F is a non-klt center of $(X', B' + f'^*P')$ over the generic point of P', distinct from E', since E' is exceptional over X. This contradicts the fact that $(X', B' + f'^*P')$ is plt near the fiber over the generic point of P'.

Let Q' be an irreducible component of the strict transform of $\mathrm{Supp}(B^v)$ in X', and let Q be the image of Q' in X. Then $Q \subseteq \mathrm{Supp}(B^v)$. Since $f' : X' \to Z'$ is equidimensional, $f'(Q')$ is a prime divisor on Z'. By [31][Lemma 2.6.(b)], every f-vertical log center of (X, B) dominates a generalized log center of $(Z, B_Z, \mathbf{M})$. It follows that $f(Q)$ is a generalized log center of $(Z, B_Z, \mathbf{M})$, and hence $f'(Q')$ is a generalized log place of $(Z, B_Z, \mathbf{M})$. By construction, the divisor D' contains the strict transform of $\mathrm{Supp}(B_Z)$ together with all the exceptional divisors over Z. Therefore, $f'(Q') \subseteq \mathrm{Supp}(D')$, and consequently $Q' \subseteq \mathrm{Supp}(f'^*D')$. Hence, $\mathrm{Supp}(f'^*D')$ contains the strict transform of $\mathrm{Supp}(B^v)$. On the other hand, since the fibrations $(X, B) \to Z$ and $(X', B') \to Z'$ have the same generic fiber, $\mathrm{Supp}(B')$ contains the strict transform of $\mathrm{Supp}(B^h)$. Combining the above, we conclude that $\mathrm{Supp}(\Delta')$ contains the strict transform of $\mathrm{Supp}(B)$ and all exceptional divisors over X. This proves (4).

∎

In the following theorem, we aim to bound the log canonical volume of the special birational model constructed in Theorem 6.3.1.

Theorem 6.3.2. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists a rational number $\alpha \in (0, 1)$ and positive numbers m, w depending only on d, Φ, v, r, ϵ satisfying the following:*

If $f : ((X, B), A) \to (Z, H)$ is a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration, then there exists a polarized log Calabi-Yau fibration $f' : ((X', \Delta'), L') \to Z'$ such that

1. *$X' \dashrightarrow X$ is a birational map, and $Z' \to Z$ is a birational morphism,*
2. *the generic fiber of $f : (X, B) \to Z$ is isomorphic to the generic fiber of $f' : (X', \Delta') \to Z'$,*
3. *$L'_g := L'|_{X'_g}$ is numerically equivalent to $mA'_g := mA'|_{X'_g}$ on X'_g, where A' is the strict transform of A on X', and*
4. *The coefficients of Δ' are in $\Phi \cup \{1\}$.*

Moreover, we have

5. Δ' *contains the strict transform of* $\operatorname{Supp}(B)$ *on* X' *and all exceptional divisors over* X*,*
6. $(X', \Delta' + \alpha L')$ *is lc,*
7. $K_{X'} + \Delta' + \alpha L' - h'^* H$ *is big, where* $h' : X' \to Z$*, and*
8. $\operatorname{vol}(K_{X'} + \Delta' + \alpha L') \le w$.

Proof *Step* 1. In this step we construct a polarized log Calabi–Yau fibration

$$f' : ((X', \Delta'), L') \to Z'$$

by Theorem 6.3.1.

By Theorem 6.2.1, there exists a birational morphism $h : W \to Z$ and a reduced divisor D on W such that

- (W, D) is log bounded,
- the induced rational map $\psi_W : W \dashrightarrow \mathcal{S}^*$ is a bounded morphism,
- $D \supseteq \operatorname{Supp}(h_*^{-1} B_Z + E + \psi_W^* \mathcal{H})$, where E is the sum of the reduced exceptional divisors of h, and $\mathcal{H}$ is a very ample divisor on $\mathcal{S}^*$,
- $K_W + D - h^* H$ is big.

Let $\overline{W}$ be the normalization of the main component of $W \times_{\mathcal{S}^*} \mathcal{S}^!$, and let $D_{\overline{W}}$ denote the preimage of D via $\overline{W} \to W$. After replacing $(\overline{W}, D_{\overline{W}})$ with its log resolution, we may assume that $(\overline{W}, D_{\overline{W}})$ is log smooth. Then $\overline{W} \to W$ is generically finite. Let

$$f_{\overline{W}} : ((X_{\overline{W}}, B_{\overline{W}}), L_{\overline{W}}) \to \overline{W}$$

be the pullback of $((\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^!) \to \mathcal{S}^!$ via $\overline{W} \to \mathcal{S}^!$.

Let L on X be the closure of the pullback of $\mathcal{L}$ via the moduli map $U \to \mathcal{S}$ for some open subset $U \subset Z$. By Theorem 6.1.2, the general fiber $((X_g, B_g), L_g)$ is a $(\dim X_g, \Phi, v')$-polarized log Calabi–Yau pair, where v' depends only on d, Φ, v.

Applying Theorem 6.3.1, we obtain a family of $(\dim X_g, \Phi, v')$-polarized log Calabi–Yau pairs

$$\overline{f} : ((\overline{X}, \overline{B}), \overline{L}) \to \overline{Z},$$

and a polarized log Calabi–Yau fibration

$$f' : ((X', \Delta'), L') \to Z'$$

satisfying (1)–(4). We may assume that Z' is the log resolution of (W, D) extracting all exceptional divisors of $\overline{W} \to W$.

Step 2. In this step we prove (5)–(7).

By Theorem 6.3.1 (2)(4), to show that Δ' contains the strict transform of $\mathrm{Supp}(B)$ on X' and all exceptional divisors over X, and that $(X', \Delta' + \alpha L')$ is lc, it suffices to prove that $(\overline{X}, \overline{B}) \to \overline{Z}$ has klt fibers in $\overline{Z} \setminus \overline{D}$.

Let $\widetilde{Z} \to \overline{Z}$ be a generically finite morphism such that

- $\widetilde{Z} \to Z \dashrightarrow \mathcal{S}$ is a morphism and factors through $\overline{\mathcal{S}} \to \mathcal{S}$, and
- $\widetilde{Z} \to Z' \to W$ factors through $\overline{W} \to W$.

We have the following commutative diagram.

$$\begin{tikzcd}
 & & (Z', D') \arrow[d, "g"] & (\overline{Z}, \overline{D}) \arrow[l] & (\widetilde{Z}, \widetilde{D}) \arrow[l] \arrow[dl] \arrow[dd] \\
U \arrow[r, hook] \arrow[dr] & Z & (W, D) \arrow[l, "h"'] \arrow[d] & (\overline{W}, D_{\overline{W}}) \arrow[l] \arrow[d] & \\
 & \mathcal{S} \arrow[r, hook] & (\mathcal{S}^*, \mathcal{H}) & \mathcal{S}^! \arrow[l] & \overline{\mathcal{S}} \arrow[l] \arrow[lll, bend left]
\end{tikzcd}$$

Let $((\widetilde{X}, \widetilde{B}), \widetilde{L})$ be the normalization of the main component of the base change of $((\overline{X}, \overline{B}), \overline{L})$ by $\widetilde{Z} \to \overline{Z}$. Let $((\widetilde{X}_W, \widetilde{B}_W), \widetilde{L}_W)$ be the normalization of the main component of the base change of $((X_{\overline{W}}, B_{\overline{W}}), L_{\overline{W}})$ by $\widetilde{Z} \to \overline{W}$.

By Theorem 6.1.2 (4) and (6), the generic fiber of $((\widetilde{X}, \widetilde{B}), \widetilde{L}) \to \widetilde{Z}$ is isomorphic to the generic fiber of $((\widetilde{X}_W, \widetilde{B}_W), \widetilde{L}_W) \to \widetilde{Z}$. Moreover, since both $((\widetilde{X}, \widetilde{B}), \widetilde{L}) \to \widetilde{Z}$ and $((\widetilde{X}_W, \widetilde{B}_W), \widetilde{L}_W) \to \widetilde{Z}$ are families of polarized log Calabi–Yau pairs, by the separatedness of the moduli of polarized log Calabi–Yau pairs, we obtain

$$((\widetilde{X}, \widetilde{B}), \widetilde{L}) \cong ((\widetilde{X}_W, \widetilde{B}_W), \widetilde{L}_W).$$

By Theorem 6.1.2 (8), Theorem 6.2.1 (2), and the fact that $D_{\overline{W}}$ is the preimage of D, we conclude that $(X_{\overline{W}}, B_{\overline{W}}) \to \overline{W}$ has klt fibers over $\overline{W} \setminus D_{\overline{W}}$. Therefore, $(\widetilde{X}, \widetilde{B}) \to \widetilde{Z}$ has klt fibers over $\widetilde{Z} \setminus \widetilde{D}'$, where $\widetilde{D}'$ is the preimage of $D_{\overline{W}}$. Since $\overline{D}$ contains the preimage of D on $\overline{Z}$, it follows that $\mathrm{Supp}(\widetilde{D}') \subseteq \mathrm{Supp}(\widetilde{D})$, where $\widetilde{D}$ is the preimage of $\overline{D}$. Hence $(\overline{X}, \overline{B})$ has klt fibers over $\overline{Z} \setminus \overline{D}$.

We now show that

$$K_{X'} + \Delta' + \alpha L' - h'^* H$$

is big. By Theorem 6.2.1 (3), $K_W + D - h^* H$ is big. Since D' contains the strict transform of D plus the reduced exceptional divisors over W, it follows that $K_{Z'} + D' - (K_W + D)$ is effective. Hence $K_{Z'} + D' - g^* h^* H$ is big. Let $0 < a \ll 1$. By Theorem 6.3.1 (3), we have

$$K_{X'} + \Delta' + \alpha L' - h'^* H = f'^*(K_{Z'} + D' + \mathbf{M}_{Z'} - g^* h^* H) + aL' + (\alpha - a)L'.$$

Since L' is big over Z' and $L' \geq 0$, it follows that $K_{X'} + \Delta' + \alpha L' - h'^* H$ is the sum of a big $\mathbb{Q}$-divisor and an effective $\mathbb{Q}$-divisor, and hence big.

Step 3. In this step we prove that $\mathrm{vol}(K_{X'} + \Delta' + \alpha L')$ is bounded from above.

Consider the following commutative diagram:

$$\begin{tikzcd}
(X', B' + \alpha L') \arrow[d, "f'"'] & (\widetilde{X}, \widetilde{B} + \alpha \widetilde{L}) \arrow[l, "\mu"'] \arrow[r, "\eta"] \arrow[d, "\widetilde{f}"'] & \widetilde{X}' \arrow[r, "\nu"] & (X_{\overline{W}}, B_{\overline{W}} + \alpha L_{\overline{W}}) \arrow[d, "f_{\overline{W}}"] \\
(Z', D') & (\widetilde{Z}, \widetilde{D}) \arrow[l, "\pi"'] \arrow[rr, "\tau"] & & (\overline{W}, D_{\overline{W}})
\end{tikzcd}$$

Here $\widetilde{X} \xrightarrow{\eta} \widetilde{X}' \xrightarrow{\nu} X_{\overline{W}}$ is the Stein factorization of $\widetilde{X} \to X_{\overline{W}}$, hence ν is a finite morphism and η is a birational morphism. Now we claim that

$$\eta_* \mu^*(K_{X'} + \Delta' + \alpha L') = \nu^*(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}}).$$

Since the generic fiber of

$$(X', \Delta' + \alpha L') \times_{Z'} \widetilde{Z} \to \widetilde{Z}$$

is equal to the generic fiber of

$$(X_{\overline{W}}, B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_W) \times_{\overline{W}} \widetilde{Z} \to \widetilde{Z},$$

it suffices to compare vertical divisors. Let $\widetilde{P}$ be a prime divisor on $\widetilde{X}$ which is vertical over $\widetilde{Z}$, and assume that its image $\widetilde{P}' := \eta(\widetilde{P})$ is also a prime divisor on $\widetilde{X}'$. Let $P_{\overline{W}}$ denote the image of $\widetilde{P}$ on $X_{\overline{W}}$. We claim that the image of $\widetilde{P}$ on X' is a prime divisor as well.

Indeed, since $P_{\overline{W}}$ is a prime divisor and $f_{\overline{W}}$ is equidimensional, its image $f_{\overline{W}}(P_{\overline{W}})$ is a prime divisor on $\overline{W}$. By Step 1, the morphism $Z' \to W$ is a log resolution of (W, D) extracting all exceptional divisors of $\overline{W} \to W$. It follows that $\pi \circ \widetilde{f}(\widetilde{P})$ is a prime divisor on Z'. Since both f' and $\widetilde{f}$ are equidimensional with fibers of the same dimension, the image $\mu(\widetilde{P})$ is a prime divisor on X', which we denote by P'.

We now distinguish two cases:

Case (1): $\mathrm{coeff}_{P'} \Delta' = 1$. In this case, $f'(P')$ is a prime divisor contained in $\mathrm{Supp}(D')$. By construction, $\pi^{-1}(D')$ is the union of $\tau^{-1}(D_{\overline{W}})$ and some τ-exceptional divisors. Therefore, $f_{\overline{W}}(P_{\overline{W}})$ is a prime divisor contained in $\mathrm{Supp}(D_{\overline{W}})$. Hence, by [96]§2.41.4, over the generic point of $\widetilde{P}'$, we have

$$\nu^*(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}}) = K_{\widetilde{X}'} + \widetilde{P}' = \eta_* \mu^*(K_{X'} + \Delta' + \alpha L').$$

Case (2): $\mathrm{coeff}_{P'} \Delta' = 0$. In this case, $f'(P')$ is not contained in $\mathrm{Supp}(D')$ and $f_{\overline{W}}(P_{\overline{W}})$ is not contained in $\mathrm{Supp}(D_{\overline{W}})$. Hence, by Theorem 6.3.1 (1), f' has reduced fibers over the generic point of $f'(P')$. Since the ramification locus of $\overline{W} \to W$ is contained in $\mathrm{Supp}(D_{\overline{W}})$ by Theorem 6.1.2 (8), the map $\overline{W} \dashrightarrow Z'$ is étale over the generic point of $f'(P')$. Therefore, the ramification index of μ along $\widetilde{P}$ equals that of ν along $\widetilde{P}$. By the Hurwitz formula, we have

$$\nu^*(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}}) = \eta_* \mu^*(K_{X'} + \Delta' + \alpha L')$$

over the generic point of $\widetilde{P}$. This proves the claim.

By the claim, we obtain

$$\mathrm{vol}(\mu^*(K_{X'} + \Delta' + \alpha L')) \leq \mathrm{vol}(\nu^*(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}})).$$

By [97][Lemma 4.3], it follows that

$$\mathrm{vol}(\mu^*(K_{X'} + \Delta' + \alpha L')) = \deg(\mu)\ \mathrm{vol}(K_{X'} + \Delta' + \alpha L'),$$
$$\mathrm{vol}(\nu^*(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}})) = \deg(\nu)\ \mathrm{vol}(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}}).$$

Since $\deg(\nu) \cdot \deg(\overline{W}/Z) = \deg(\mu)$, we conclude

$$\mathrm{vol}(K_{X'} + \Delta' + \alpha L') \leq \frac{1}{\deg(\overline{W}/Z)}\ \mathrm{vol}(K_{X_{\overline{W}}} + B_{\overline{W}} + \alpha L_{\overline{W}} + f_{\overline{W}}^* D_{\overline{W}}) \leq w$$

by Lemma 6.2.2, where w is a positive integer depending only on d, Φ, v, r, ϵ.

∎

6.4 Log boundedness in codimension one

We now proceed to establish the main theorem of this section.

Proof of Theorem 6.0.1 *Step* 1. Let

$$h' : ((X', \Delta'), L') \to Z' \to Z$$

be the fibration constructed in Theorem 6.3.2. By [2][Theorem 1.3], there exists a fixed positive integer n such that the linear system

$$|n(K_{X'} + \Delta' + \alpha L')|$$

defines a birational map. Let $\pi : Y' \to X'$ be a log resolution of $(X', \Delta' + L')$ such that

$$|n\pi^*(K_{X'} + \Delta' + \alpha L')| = |M| + F,$$

where $|M|$ is the free part and F is the fixed part. Set

$$G := M + \pi^* h'^* H.$$

Then $|G|$ is base point free and defines a birational morphism $\mu : Y' \to Y$ such that $\mu_* G$ is very ample on Y. Moreover, every curve contracted by μ intersects $\pi^* h'^* H$ trivially, hence the induced map $g : Y \dashrightarrow Z$ is in fact a morphism.

By construction, we have

$$G + F \sim_{\mathbb{Q}} n\pi^*(K_{X'} + \Delta' + \alpha L') + \pi^* h'^* H.$$

Let η_Z denote the generic point of Z. Since $K_{X'} + \Delta' \sim_{\mathbb{Q},\eta_Z} 0$, we obtain

$$G + F \sim_{\mathbb{Q},\eta_Z} n\alpha\pi^* L'.$$

Step 2. Define

$$\Sigma' := \mathrm{red}(\pi_*^{-1}\Delta') + G + F + \pi^* h'^* H_Z + E',$$

where E' denotes the reduced exceptional divisor of $\pi : Y' \to X'$, and set $\Sigma = \mu_* \Sigma'$. In this step, we prove that (Y, Σ) belongs to a log bounded family and that the morphism $g : Y \to Z$ is bounded.

Since $K_{X'} + \Delta' + \alpha L' - h'^* H$ is big by Theorem 6.3.2 (7), it follows that

$$\mathrm{vol}(G) \le \mathrm{vol}\left((n+1)(K_{X'} + \Delta' + \alpha L')\right) \le (n+1)^d w.$$

By [2][Lemma 7.3], there exists a fixed positive number $\beta < 1$ such that $K_{X'} + \beta(\Delta' + \alpha L')$ is big.

Define

$$c := \frac{1}{\min\{\, c_i \in \Phi \cup \{1\} \mid c_i \neq 0 \,\}},$$

and choose a fixed positive number t satisfying

$$\frac{c + t\beta}{1 + t} \le 1, \quad \text{equivalently,} \quad t \ge \frac{c-1}{1-\beta}.$$

Then we conclude that

$$\begin{aligned}
&\mathrm{vol}(K_{Y'} + \Sigma' + (4d+2)G) \\
\le\, &\mathrm{vol}(K_{X'} + \pi_*\Sigma' + (4d+2)\pi_* G) \\
\le\, &\mathrm{vol}(K_{X'} + c\Delta' + (10d+3)(n+1)(K_{X'} + \Delta' + \alpha L'))
\end{aligned}$$

$$\begin{aligned}&\leq \mathrm{vol}(K_{X'} + c\Delta' + t(K_{X'} + \beta(\Delta' + \alpha L')) + (10d+3)(n+1)(K_{X'} + \Delta' + \alpha L'))\\ &\leq \mathrm{vol}((1+t+(10d+3)(n+1))(K_{X'} + \Delta' + \alpha L'))\\ &\leq (1+t+(10d+3)(n+1))^d w,\end{aligned}$$

where the second inequality holds since $K_{X'} + \Delta' + \alpha L' - h'^* H$ is big. Therefore, by [1][Lemma 3.2],

$$\begin{aligned}\Sigma \cdot ((4d+2)\mu_* G)^{d-1} &= \Sigma' \cdot ((4d+2)G)^{d-1}\\ &\leq 2^d \,\mathrm{vol}(Y', K_{Y'} + \Sigma' + (4d+2)G)\\ &\leq 2^d (1+t+(10d+3)(n+1))^d w.\end{aligned}$$

Thus by [1][Lemma 2.4.2 (4)], (Y, Σ) forms a log bounded family. By [98][Lemma 2.8], $g : Y \to Z$ is a bounded morphism.

Step 3. There exists a family of contractions

$$\mathcal{Y} \to \mathcal{Z} \to T$$

and three effective divisors Ω, $\mathcal{G}$, and $\mathcal{F}$ on $\mathcal{Y}$ such that for some closed point $t \in T$, the fiber $\mathcal{Y}_t \to \mathcal{Z}_t$ is isomorphic to $g : Y \to Z$, with

$$\Omega_t \simeq \Sigma, \quad \mathcal{G}_t \simeq \mu_* G, \quad \mathcal{F}_t \simeq \mu_* F.$$

Since $\mu_* G$ is a very ample divisor on Y and ampleness is an open condition, after passing to a stratification we may assume that $\mathcal{G}$ is ample over the generic point of $\mathcal{Z}$. If we write

$$\mathcal{J}_{\mathcal{Y}} = \mathcal{G} + \mathcal{F},$$

then $\mathcal{J}_{\mathcal{Y}}$ is big over $\mathcal{Z}$. Setting

$$J_Y = \mu_* G + \mu_* F,$$

we have

$$J_Y \sim_{\mathbb{Q}, \eta_Z} n\alpha \mu_* \pi^* L'.$$

After taking a log resolution of $(\mathcal{Y}, \Omega)$ and passing to a stratification of T, we may assume that T is smooth and $(\mathcal{Y}, \Omega)$ is log smooth over T. Replacing $\mathcal{J}_{\mathcal{Y}}$ by its pullback, it remains big over $\mathcal{Z}$. After passing to a finite étale cover of a stratification of T (see [96][Claim 4.38.1]), we may assume that every prime component of Ω restricts to a prime divisor fiberwise. Moreover, after replacing $(\mathcal{Y}, \Omega)$ by a sequence of blowups of strata, we extract all divisors whose log discrepancies with respect to $(\mathcal{Y}, (1-\epsilon)\Omega)$ are at most one. Up to a further stratification of T, this process can be assumed to be fiberwise. Therefore, the induced birational map

$$\mathcal{Y}_t \dashrightarrow X' \dashrightarrow X$$

is a birational contraction.

Since H is very ample and $H_Z \in |6dH|$ is general, we may assume that $(X, B + \frac{1}{2}f^*H_Z)$ is ϵ-lc. By the canonical bundle formula,

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z).$$

By the boundedness of the length of extremal rays, $K_Z + B_Z + \mathbf{M}_Z + 3dH$ is ample, and hence

$$K_X + B + \tfrac{1}{2}f^*H_Z \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z + \tfrac{1}{2}H_Z)$$

is semi-ample.

Let $\Gamma_{\mathcal{Y}_t}$ be the strict transform of $B + \frac{1}{2}f^*H_Z$ on $\mathcal{Y}_t$, together with $(1-\frac{1}{2}\epsilon)E$, where E is the reduced exceptional divisor of $\mathcal{Y}_t \dashrightarrow X$. Define $\Gamma_{\mathcal{Y}}$ to be the divisor supported on Ω whose restriction to $\mathcal{Y}_t$ is $\Gamma_{\mathcal{Y}_t}$. Since the coefficients of $B + \frac{1}{2}f^*H_Z$ lie in a finite set, the possible coefficients appearing in $\Gamma_{\mathcal{Y}_t}$ also belong to a finite set $\Phi \cup \{\frac{1}{2}, 1 - \frac{1}{2}\epsilon\}$. Therefore, without loss of generality, we may assume that $\Gamma_{\mathcal{Y}}$ is fixed on $\mathcal{Y}$.

By construction, $(X, B + \frac{1}{2}f^*H_Z)$ is a good minimal model of $(\mathcal{Y}_t, \Gamma_{\mathcal{Y}_t})$. By [3]Theorem 1.2, the pair $(\mathcal{Y}, \Gamma_{\mathcal{Y}})$ admits a relative good minimal model $(\mathcal{V}, \Gamma)$ over T, which, up to a stratification of T, induces good minimal models fiberwise. By the boundedness of the length of extremal rays, the induced map $\mathcal{V} \dashrightarrow \mathcal{Z}$ is a morphism. If we denote the pushforward of $\mathcal{J}_{\mathcal{Y}}$ by $\mathcal{J}$, then $\mathcal{J}$ is big over $\mathcal{Z}$. By [34]Lemma 2.4, the pair $(\mathcal{V}_t, \Gamma_t)$ is isomorphic in codimension one to $(X, B + \frac{1}{2}f^*H_Z)$.

Since L' is numerically equivalent to the strict transform of mA on the generic fiber of $X' \to Z$, and since $J_Y \sim_{\mathbb{Q},\eta_Z} n\alpha\mu_*\pi^*L'$, we conclude that $\mathcal{J}_t$ is numerically equivalent to the strict transform of $mn\alpha A$ on the generic fiber of $\mathcal{V}_t \to Z$. Replacing n by a bounded multiple, we may assume that $l := mn\alpha$ is an integer. Thus, the pair $(\mathcal{V}_t, \Gamma_t)$ and the integral divisor $\mathcal{J}_t$ are what we need.

∎

6.5 Application to klt good minimal models

In this section, we apply our boundedness results on polarized log Calabi-Yau fibrations to klt good minimal models.

Definition 6.5.1 ([16][Definition 1.1]). Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let $\mathcal{F}_{gklt}(d, \Phi, u)$ be the set of projective generalized pairs $(X, B, \mathbf{M})$ such that

1. $(X, B, \mathbf{M})$ is klt of dimension d ,
2. the coefficients of B are in Φ,
3. there is a birational morphism $X' \to X$ such that $\mathbf{M}$ descends to X' and $\mathbf{M}_{X'} = \sum \mu_i M_i'$ where M_i' is nef Cartier and $\mu_i \in \Phi$ for any i,
4. $K_X + B + \mathbf{M}_X$ is ample, and
5. $\mathrm{vol}(K_X + B + \mathbf{M}_X) = u$.

Proof of Corollary 1.4.4 By Lemma 6.0.2, we may assume that A is an effective integral divisor and $\mathrm{vol}(A|_F) = v$ is fixed. By [16][Lemma 8.2], all log discrepancies of (X, B) that are smaller than 1 are in a fixed finite set Φ', depending only on d, u, v, Φ. In particular, the coefficients of B belong to Φ', and there exists a positive number ϵ, depending only on d, u, v, Φ, such that (X, B) is ϵ-lc.

By [16][Lemma 7.4], there exists a positive integer p, depending only on d, u, Φ, such that we can write a canonical bundle formula

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z),$$

where $p\mathbf{M}_{Z'}$ is Cartier on some high resolution $Z' \to Z$.

By [2][Theorem 1.1], the coefficients of B_Z are in a fixed DCC set Ψ, depending only on d, Φ. Replacing Ψ by $\Psi \cup \{\frac{1}{p}\}$, we obtain

$$(Z, B_Z, \mathbf{M}) \in \mathcal{F}_{gklt}(d', \Psi, u),$$

where $d' = \dim Z$. By [16][Theorem 1.4], $(Z, B_Z, \mathbf{M})$ belongs to a bounded family. Furthermore, by the remark following [18][Theorem 4.3], there exists a fixed positive integer l such that

$$H := l(K_Z + B_Z + \mathbf{M}_Z)$$

is very ample. Consequently,

$$f : ((X, B), A) \to (Z, H)$$

is a $(d, \Phi', v, l^{d'} u, \epsilon)$-polarized log Calabi-Yau fibration. Therefore, the corollary follows from Theorem 1.4.3. ∎

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RESUME

Resume

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Achievements during the PhD

Preprints

[1] Failure of Boundedness for Generalised Log Canonical Surfaces (joint with Christopher Hacon), arXiv:2504.06482

[2] Boundedness of polarized log Calabi-Yau fibrations with bounded bases (joint with Junpeng Jiao and Minzhe Zhu), arXiv:2504.05243

[3] Boundedness of klt good minimal models, arXiv:2312.03313

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2. ……

3. ……

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答辩委员会表决，（× 票/一致）同意通过论文答辩，并建议授予 ×××（姓名）×××（门类）学博士/硕士学位。