Boundedness of good minimal models

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Contents

MMP and Abundance

The central problem in birational geometry is the classification of algebraic varieties.

The Minimal Model Program Conjecture and Abundance Conjecture predicts that, any projective variety Y with mild singularities is birational to either

- a **Mori fiber space** $X \to Z$ (i.e. $\rho(X/Z) = 1$, $-K_X$ is ample over Z and $\dim X > \dim Z$), or
- a **good minimal model** X (i.e. K_X is semiample, which defines a contraction $X \to Z$).

Therefore, canonically polarized varieties, Fano varieties, Calabi–Yau varieties, and their iterated fibrations play a central role in birational geometry.

Boundedness

One of the main problems in the classification of algebraic varieties is whether there are only finitely many families of such objects after fixing certain numerical invariants; in other words, whether they form a bounded family.

Definition (Boundedness)

Let $d \in \mathbb{N}$. Let $\mathcal{P}$ be a set of varieties X of dimension d. We say that $\mathcal{P}$ is a **bounded family** if there exists a fixed $r \in \mathbb{N}$ such that for every $X \in \mathcal{P}$, one can find a very ample divisor H on X satisfying $H^d \leq r$.

Establishing the boundedness of a given class of varieties is a natural first step toward constructing the corresponding moduli space.

Definition (Singularities of pairs)

Let (X, B) be a pair, that is, X is a normal variety and $B \geq 0$ is an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier.
Let D be a prime divisor over X. Let $\pi : X' \to X$ be a log resolution of (X, B) such that D is a prime divisor on X'. We can write

$$K_{X'} + B' = \pi^*(K_X + B).$$

Then the *log discrepancy* of D is defined as

$$a(D, X, B) = 1 - \operatorname{mult}_D B'.$$

We say that (X, B) is **klt** (resp. **lc**, ϵ-**lc**) if $a(D, X, B) > 0$ (resp. $a(D, X, B) \geq 0$, $a(D, X, B) \geq \epsilon$) for every prime divisor D over X.

Varieties of general type

Theorem (Hacon-McKernan-Xu 2018)

Fix $d \in \mathbb{N}$ and $v \in \mathbb{Q}^{>0}$. Consider the set of varieties X such that

- *X is slc of dimension d,*
- *K_X is ample with volume $(K_X)^d = v$.*

Then X belongs to a bounded family.

Theorem (Martinelli-Schreieder-Tasin 2020)

Fix $d \in \mathbb{N}$ and $v \in \mathbb{Q}^{>0}$. Consider the set of varieties X such that

- *X is klt of dimension d,*
- *K_X is nef and big with volume $(K_X)^d = v$.*

Then X belongs to a bounded family.

Fano varieties

Theorem (Birkar 2021)

Fix $d \in \mathbb{N}$ and $\epsilon \in \mathbb{Q}^{>0}$. Consider the set of varieties X such that

- *X is ϵ-lc of dimension d,*
- *$-K_X$ is nef and big.*

Then X belongs to a bounded family.

Calabi-Yau varieties

For Calabi-Yau varieties, there is no natural choice of polarization. In general, they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces or abelian varieties of any fixed dimension are not bounded.

It is widely open whether strict Calabi-Yau manifolds of dimension $d \geq 3$ are bounded or not, where a smooth proper variety X is **strict Calabi-Yau** if it is simply connected, $K_X \sim 0$, and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Polarized Calabi-Yau varieties

When studying moduli of Calabi-Yau varieties, a polarization is typically fixed despite the non-uniqueness of the choice.

Theorem (Birkar 2023)

Let $d \in \mathbb{N}$ and $u \in \mathbb{Q}^{>0}$. Consider Calabi–Yau varieties X and $\mathbb{Q}$-Cartier Weil divisors A on X. Then the following hold:

(1) *(slc case) If*
- *X is slc of dimension d,*
- *A is an ample divisor on X such that $\mathrm{vol}(A) = u$,*
- *$A \geq 0$ does not contain any non-klt center of X,*

then the set of such (X, A) forms a bounded family.

(2) *(klt case) If*
- *X is klt of dimension d,*
- *A is a nef and big divisor on X such that $\mathrm{vol}(A) \leq u$,*

then X belongs to a bounded family. If in addition $A \geq 0$, then the set of such (X, A) also forms a bounded family.

Stable minimal models

For studing moduli of good minimal models, we need to choose a relative polarization.

Theorem (Birkar 2022)

Let $d \in \mathbb{N}$, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Consider varieties X and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:

- *X is slc of dimension d,*
- *K_X is semiample, defining a contraction $f : X \to Z$,*
- *A is a divisor on X that is ample over Z,*
- *$\mathrm{vol}(A|_F) \in \Gamma$, where F is any general fiber of $f : X \to Z$ over any irreducible component of Z,*
- *$(K_X + tA)^d = \sigma(t)$, and*
- *$A \geq 0$ does not contain any non-klt center of X,*

then the set of such (X, A) forms a bounded family.

What's next?

The previous boundedness result can be regarded as a relative version of the boundedness result for slc polarized Calabi–Yau varieties.

One naturally wonders whether a relative version of the boundedness result for klt polarized Calabi–Yau varieties exists; that is, the polarization need not be an effective divisor.

We address this question and use it to construct the moduli space of klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of the Iitaka fibration.

Failure of boundedness for generalised log canonical surfaces

Since the structure of generalised pairs naturally appears on the base of the Iitaka fibration of good minimal models, we also study the boundedness problem for generalised pairs.

For the abstract structure of generalised pairs, which may not arise from the Iitaka fibration, in the joint work with Hacon, we construct counterexamples to the boundedness of generalised log canonical models of surfaces with fixed suitable invariants, where the underlying varieties can have arbitrary Kodaira dimension. This answers a question of [Birkar–Hacon 2022]

Main Theorem: Boundedness of klt good minimal models

Theorem (J. 2023)

Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Consider the set of varieties X and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:

- *X is klt of dimension d,*
- *K_X is semiample, defining a contraction $f : X \to Z$,*
- *A is a divisor on X that is nef and big over Z,*
- *$\mathrm{vol}(A|_F) \leq u$, where F is the general fiber of $f : X \to Z$, and*
- *$(K_X + tA)^d = \sigma(t)$.*

Then the set of such X forms a bounded family. If in addition $A \geq 0$, then the set of such (X, A) also forms a bounded family.

Further remarks

Remark

If we replace the condition $(K_X + tA)^d = \sigma(t)$ with the condition that $\mathrm{Ivol}(K_X) = v$ for some fixed $v \in \mathbb{Q}^{>0}$, then in later joint work with Junpeng Jiao and Minzhe Zhu, we proved that X is bounded in codimension one.

Remark

After fixing $\sigma(t)$ in the Main Theorem, it is expected that the condition on $\mathrm{vol}(A|_F)$ can be removed. Indeed, the effective b-semiampleness conjecture of Prokhorov-Shokurov predicts that $\mathrm{Ivol}(K_X)$ belongs to a DCC set Ψ depending only on d. Then by the following formula

$$\mathrm{Ivol}(K_X + B) \cdot \mathrm{vol}(A|_F) = (K_X + B)^{\dim Z} \cdot A^{d - \dim Z},$$

we obtain that $\mathrm{vol}(A|_F) \leq u$.

Traditional stable minimal models

The reason for taking effective divisors as polarization in [Birkar 2022] is that one can obtain proper moduli spaces. However, this often leads to larger moduli spaces. Traditionally, people use line bundles as polarization.

Definition (Birkar)

A **traditional stable minimal model** (X, A) consists of a projective connected variety X and a line bundle A such that

- X is klt,
- K_X is semi-ample defining a contraction $f : X \to Z$,
- A is ample over Z.

Moduli of traditional stable minimal models

Definition

Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, u, σ)-**traditional stable minimal model** is a traditional stable minimal model (X, A) such that

- $\dim X = d$,
- $\mathrm{vol}(A|_F) = u$, where F is any general fiber of $f : X \to Z$, and
- $(K_X + tA)^d = \sigma(t)$.

Corollary (J. 2023)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. There is a separated coarse moduli space for (d, u, σ)-traditional stable minimal models.

Strategy for the proof of boundedness

Proposition

Under the same assumptions as Main Theorem, there are positive rational numbers λ, τ and a natural number r depending only on d, u, σ such that

- A *(Boundedness of nef threshold). $K_X + tA$ is nef and big for all $0 < t < \tau$,*
- B *(Boundedness of pseudo-effective threshold). $K_X + tA$ is big for all $0 < t < \lambda$, and*
- C *we can find a very ample divisor H on X satisfying*

$$H^d \leq r, \text{ and } H - A \text{ is pseudo-effective}.$$

The proof of Prop A, B and C proceeds by induction on the dim X:

1. Prop B_d + Porp $C_{d-1} \implies$ Prop A_d
2. Prop $A_{d-1} \implies$ Prop B_d
3. Prop $A_d \implies$ Porp C_d

Novel idea

Prop B is entirely new. To bound the pseudo-effective threshold, we introduce a variant of the volume function for $A + tK_X$ and compare its graph with that of the self-intersection function of $A + tK_X$ by analyzing their derivatives.

Prop A modifies Birkar's proof in the lc case. Since our polarization may not be effective, we cannot use the length of extremal rays to bound the nef threshold. We first bound the pseudo-effective threshold and may assume the polarization is an effective $\mathbb{Q}$-divisor, but lose control of its coefficients. This requires a stronger boundedness result on singularities from Birkar's BAB paper. For the induction to work, we also need to show that $H - A$ is pseudo-effective.

Thank you!