

Boundedness of traditional stable minimal models

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May 17, 2024

1 Background

- KSBA stable pairs
- Stable Calabi-Yau pairs
- Stable minimal models

2 Traditional stable minimal models

- Main results
- Sketch of proof
- Prop $A(d)$ implies Main(d)
- Prop $B(d)$ and Main($d-1$) implies Prop $A(d)$
- Prop $A(d-1)$ implies prop $B(d)$

1 Background

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- Stable Calabi-Yau pairs
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2 Traditional stable minimal models

We work over an algebraically closed field of characteristic 0.

The Minimal Model Program Conjecture and Abundance Conjecture predicts that, any projective variety Y with mild singularities is birational to either

- a **Mori fiber space** $X \rightarrow Z$ (i.e. $\rho(X/Z) = 1$, $-K_X$ is ample over Z and $\dim X > \dim Z$), or
- a **good minimal model** X (i.e. K_X is semiample, which defines a contraction $X \rightarrow Z$).

In this talk, we will focus on good minimal models.

We aim to construct moduli spaces for good minimal models. For this purpose, the first problem is whether there are only finitely many families for these objects after fixing certain numerical invariants.

Definition (Boundedness for pairs)

Let $d \in \mathbb{N}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let \mathcal{P} be a set of pairs (X, B) such that

- $\dim X = d$, and
- the coefficients of B are in Φ .

Then \mathcal{P} is said to be a **bounded family** if there is a fixed $r \in \mathbb{N}$ such that for any (X, B) in \mathcal{P} we can find a very ample divisor H on X satisfying

$$H^d \leq r \text{ and } (K_X + B) \cdot H^{d-1} \leq r.$$

Canonical polarized varieties: K_X ample

Canonical polarized varieties are ample models of good minimal models with maximal Kodaira dimension.

Definition

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. A (d, c, v) -**KSBA stable pair** is a connected projective pure dimensional pair (X, B) such that

- (X, B) is slc of dimension d ,
- $\frac{1}{c}B$ is an integral divisor,
- $K_X + B$ is ample with volume $\text{vol}(K_X + B) := (K_X + B)^d = v$.

Canonical polarized varieties: K_X ample

Theorem (Hacon-McKernan-Xu 2018)

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. Then (d, c, v) -KSBA stable pairs form a bounded family.

The higher dimensional analogue of Deligne-Mumford theorem for curves has been established by the contributions of many people over the past three decades.

Theorem (Kollár et al.)

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, v) -KSBA stable pairs.

Calabi-Yau varieties: $K_X \equiv 0$

For Calabi-Yau varieties (i.e. good minimal models with Kodaira dimension zero), there is no natural choice of polarization. In general, they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces and abelian varieties of any fixed dimension are not bounded.

It is widely open whether strict Calabi-Yau manifolds of dimension $d \geq 3$ are bounded or not, where a smooth proper variety X is strict Calabi-Yau if it is simply connected, $K_X \sim 0$, and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Calabi-Yau varieties: $K_X \equiv 0$

When studying moduli of Calabi-Yau varieties, a polarization (that is, ample divisor) is typically fixed despite the non-uniqueness of the choice.

Definition (Kollár-Xu, Birkar)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. A (d, c, u) -**stable Calabi-Yau pair** (X, B) , A is defined by the data:

- (X, B) is projective slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $\frac{1}{c}B$ is an integral divisor,
- $Ae \geq 0$ is an ample integral divisor with volume $\text{vol}(A) = u$,
- $(X, B + tA)$ is slc for some $t \in \mathbb{Q}^{>0}$.

The key point is that t is not fixed.

Calabi-Yau varieties: $K_X \equiv 0$

Theorem (Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. Then (d, c, u) -stable Calabi-Yau pairs form a bounded family.

The boundedness implies $(X, B + \tau A)$ is a KSBA stable pair, for some fixed $\tau \in \mathbb{Q}^{>0}$ depending only on d, c, u .

Theorem (in the first arxiv version of Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, u) -stable Calabi-Yau pairs.

Theorem (Birkar 2023)

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. Consider pairs (X, B) , A satisfying the following:

- (X, B) is a klt Calabi-Yau pair of dimension d ,
- $\frac{1}{c}B$ is an integral divisor,
- A is nef and big and integral with $\text{vol}(A) \leq v$.

Then the set of such (X, B) forms a bounded family.

Here, A is not necessarily an effective divisor. This theorem is crucially used in Odaka's work to get partial compactification of moduli spaces of Calabi-Yau varieties polarised by ample line bundles.

Good minimal models: arbitrary Kodaira dimension

For studying moduli of good minimal models, we need to choose a relative polarization.

Definition (Birkar)

A **stable minimal model** $(X, B), A$ consists of a projective connected pair (X, B) and an integral divisor $A \geq 0$ such that

- (X, B) is slc,
- $K_X + B$ is semi-ample defining a contraction $f : X \rightarrow Z$,
- $K_X + B + tA$ is ample for some $t > 0$, and
- $(X, B + tA)$ is slc for some $t > 0$.

Note that we do not assume that A is ample globally, the third condition is equivalent to A is ample over Z . Moreover, t is a priori not fixed.

Good minimal models: arbitrary Kodaira dimension

To get a good moduli theory we need to fix more invariants compared to the KSBA and stable Calabi-Yau cases.

Definition (Birkar)

Let $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, c, u, σ) -stable minimal model is a stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- $\frac{1}{c}B$ is an integral divisor,
- $\text{vol}(A|_F) = u$, where F is any general fiber of $f : X \rightarrow Z$ over any component of Z , and
- $(K_X + B + tA)^d = \sigma(t)$.

When t is sufficiently small, $(K_X + B + tA)$ is ample, in this case $\sigma(t) = \text{vol}(K_X + B + tA)$.

Good minimal models: arbitrary Kodaira dimension

Theorem (Birkar 2022)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. Then (d, c, u, σ) -stable minimal models form a bounded family.

The hardest part of the proof is to show that we can find a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on (d, c, u, σ) such that $(X, B + \tau A)$ is slc and $K_X + B + \tau A$ is ample, then we can apply [HMX18].

Theorem (Birkar 2022)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. There is a projective coarse moduli space for (d, c, u, σ) -stable minimal models.

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- Main results
- Sketch of proof
- $\text{Prop A}(d)$ implies $\text{Main}(d)$
- $\text{Prop B}(d)$ and $\text{Main}(d-1)$ implies $\text{Prop A}(d)$
- $\text{Prop A}(d-1)$ implies $\text{prop B}(d)$

Traditional stable minimal models

The reason for taking effective divisors as polarization is that one can obtain proper moduli spaces. However, this often leads to larger moduli spaces.

Definition (Birkar)

A **traditional stable minimal model** (X, B) , A consists of a projective connected pair (X, B) and an integral divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \rightarrow Z$,
- $K_X + B + tA$ is ample for some $t > 0$.

Again, A is not ample globally and t is a priori not fixed.

Definition

Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, Φ, u, σ) -**traditional stable minimal model** is a traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ ,
- $\text{vol}(A|_F) = u$, where F is any general fiber of $f : X \rightarrow Z$, and
- $(K_X + B + tA)^d = \sigma(t)$.

Let $\theta(t) = \text{vol}(K_X + B + tA)$ be a function on t . When t is sufficiently small, $(K_X + B + tA)$ is ample, in this case $\sigma(t) = \theta(t)$.

Theorem (J. 2023)

Fix $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$. Then (d, Φ, u, σ) -traditional stable minimal models form a bounded family.

When $\dim Z = 0$ or $A \geq 0$, it has already been proven by Birkar.

As an application, we can construct moduli spaces for klt good minimal models polarized by line bundles.

Corollary (J. 2023)

Fix $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. There is a separated coarse moduli space for (d, c, u, σ) -traditional stable minimal models.

Strategy for the proof of boundedness

Proposition

Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $u \in \mathbb{Q}^{>0}$, $\sigma \in \mathbb{Q}[t]$ be a polynomial. Then for any (d, Φ, u, σ) -traditional stable minimal model $(X, B), A \rightarrow Z$, there are positive rational numbers λ, τ depending only on d, Φ, u, σ such that

- Ⓐ $K_X + B + tA$ is ample for all $0 < t < \tau$, and
- Ⓑ $K_X + B + tA$ is big for all $0 < t < \lambda$.

We outline the plan of the proof.

- ① $\text{Prop } A_d \implies \text{Main Thm}_d$
- ② $\text{Prop } B_d + \text{Main Thm}_{d-1} \implies \text{Prop } A_d$: induction on $\dim X$
- ③ $\text{Prop } A_{d-1} \implies \text{Prop } B_d$: induction on $\dim Z$

Boundedness for non-effective divisors

To ensure the validity of the induction argument, we need to extend the definition of boundedness to non-effective divisors. Our definition is inspired by [Birkar 2021, Theorem 1.8].

Definition

Let $d \in \mathbb{N}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let \mathcal{P} be a set of pairs (X, B) and integral divisor (not necessarily effective) A such that

- $\dim X = d$, and
- the coefficients of B are in Φ .

Then \mathcal{P} is said to be a **bounded family** if there is a fixed $r \in \mathbb{N}$ such that for any $(X, B), A$ in \mathcal{P} we can find a very ample divisor H on X satisfying

$$H^d \leq r, (K_X + B) \cdot H^{d-1} \leq r \text{ and } H - A \text{ is pseudo-effective.}$$

Prop $A_d \implies$ Main Thm $_d$

Step 1. By [Bir21, Lem8.2], we may assume that (X, B) is ϵ -lc and $N := I(K_X + B + \frac{\tau}{2}A)$ is an ample integral divisor. Then $\text{vol}(K_X + B + N) = (I + 1)^d \sigma(\frac{I\tau}{2(I+1)})$ is fixed. Thus by [Bir23, Thm 1.5], the set of (X, B) is bounded.

Theorem (Birkar 2023, Theorem 1.5)

Let $d \in \mathbb{N}$, $\delta, v \in \mathbb{Q}^{>0}$. Consider pairs (X, B) and nef and big integral divisors N on X such that

- (X, B) is projective ϵ -lc of dimension d ,
- the coefficients of B are in $\{0\} \cup [\delta, \infty)$,
- $K_X + B$ is nef,
- $\text{vol}(K_X + B + N) \leq v$.

Then the set of such (X, B) forms a bounded family.

Prop $A_d \implies$ Main Thm $_d$

Step 2. Let $D = 2I(K_X + B) + \frac{l\tau}{2}A$ be an ample integral divisor, then by [HLQ23, Thm 1.10], l_0D is an ample Cartier divisor for some l_0 depends only on d, Φ, u, σ .

By effective base point free theorem and very ampleness lemma, we can find a very ample divisor $H \in |l_0D|$ on X such that

$$H^d \leq r, (K_X + B) \cdot H^{d-1} \leq r \text{ and } H - A \text{ is pseudo-effective,}$$

where l, r depend only on d, Φ, u, σ .

Theorem (Han-Liu-Qi 2023, Theorem 1.10)

Let $\epsilon \in \mathbb{Q}^{>0}$. Suppose $\mathcal{C} := \{(X, B)\}$ is a set of ϵ -lc pairs that belongs to a bounded family \mathcal{P} . Then there exists $l_0 \in \mathbb{Z}^{>0}$ which only depends on ϵ, \mathcal{P} such that for any (X, B) in \mathcal{C} , and D a \mathbb{Q} -Cartier Weil divisor on X , l_0D is Cartier.

Prop B_d + Main Thm $_{d-1} \implies$ Prop A_d : induction on $\dim X$

Step 1. Replacing A, u, σ with $\alpha(K_X + B + \frac{\lambda}{2}A), (\frac{\alpha\lambda}{2})^{\dim F} u, \gamma$, we may assume that A is a big Weil divisor, where $\alpha \in \mathbb{N}$ and γ is a polynomial determined by $d, \Phi, u, \sigma, \lambda$.

Step 2. By [Bir21], we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

for $l \in \mathbb{N}$ depending only on d, Φ, u, σ such that $L := l(K_Z + B_Z + M_Z)$ is very ample.

Let T be a general member of $|L|$ and let S be its pullback to X . Define

$$K_S + B_S = (K_X + B + S)|_S$$

and $A_S = A|_S$. Then one can check $(S, B_S), A_S \rightarrow T$ is a traditional stable minimal model of dimension $d - 1$.

Prop $B_d + \text{Main Thm}_{d-1} \implies \text{Prop } A_d$: Control singularities in $\dim d - 1$

Step 3. By induction, there is a fixed $r \in \mathbb{N}$ such that for any $(S, B_S), A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1} \leq r, (K_S + B_S) \cdot H_S^{d-2} \leq r, \text{ and } H_S - A_S \text{ is pseudo-effective.}$$

By [Bir22, Lem 4.6], we may assume that $H_S - B_S$ is pseudo-effective. By [Bir21, Lem 8.2], we may assume that (S, B_S) is ϵ -lc.

Since A is big, there is an effective \mathbb{Q} -divisor E such that $A \sim_{\mathbb{Q}} E$, we lose control of coefficients of E . Let $E_S := E|_S$ be an effective \mathbb{Q} -divisor, then $H_S - E_S \sim_{\mathbb{Q}} H_S - A_S$ is also pseudo-effective.

Thus by [Bir21, Thm 1.8], there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $d - 1, \epsilon, r$ such that

$$\text{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt.

Prop B_d + Main Thm $_{d-1} \implies$ Prop A_d : inversion of adjunction

Theorem (Birkar 2021, Theorem 1.8)

Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number t depending only on d, r, ϵ , satisfying the following. Assume

- (X, B) is projective ϵ -lc of dimension d ,
- A is a very ample divisor on X with $A^d \leq r$,
- $A - B$ is pseudo-effective, and
- $M \geq 0$ is an \mathbb{R} -Cartier \mathbb{R} -divisor with $A - M$ pseudo-effective.

Then

$$\mathrm{lct}(X, B, |M|_{\mathbb{R}}) \geq \mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Then by inversion of adjunction, $(X, B + S + \tau E)$ is plt near S . Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z .

Prop $B_d + \text{Main Thm}_{d-1} \implies \text{Prop } A_d$: length of extremal ray

Step 4. If $K_X + B + \tau E$ is nef, replacing τ with $\frac{\tau}{2}$ we can assume that $K_X + B + \tau A$ is ample.

If $K_X + B + \tau E$ is not nef, Let R be a $(K_X + B + \tau E)$ -negative extremal ray, since $K_X + B + \tau E$ is ample over Z , R is not contained in fibers of $X \rightarrow Z$. By Step 3, R is not contained the non-lc locus of $(X, B + \tau E)$.

Then by Cone Theorem [Amb03][Fuj11], R is generated by a curve C with $(K_X + B + \tau E) \cdot C \geq -2d$. Since $L = l(K_Z + B_Z + M_Z)$ is very ample, $l(K_X + B) \cdot C = f^*L \cdot C \geq 1$, we see that

$$((K_X + B) + 2dl(K_X + B) + \tau E) \cdot C \geq 0.$$

It follows that $K_X + B + \frac{\tau}{1+2dl}E$ is nef.

Prop $A_{d-1} \implies$ Prop B_d : variations of the volume function $\vartheta(t)$ and the top self-intersection function $\varsigma(t)$

Step 0. Let $\varsigma \in \mathbb{Q}[t]$ be a polynomial function such that

$$\varsigma(t) = (A + t(K_X + B))^d = \sum_{i=0}^d \binom{d}{i} A^{d-i} \cdot (K_X + B)^i t^i,$$

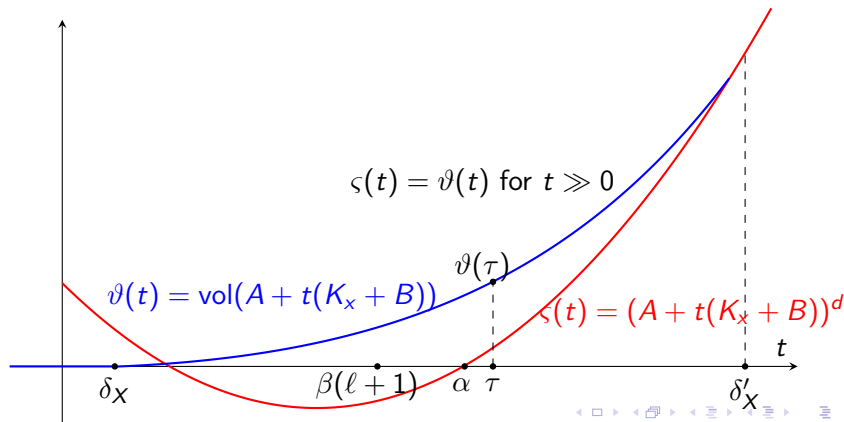
so ς is determined by σ . Let

$$\vartheta(t) = \text{vol}(A + t(K_X + B))$$

be a function on t , which is a non-negative non-decreasing real function on t , which agrees with $\varsigma(t)$ for $t \gg 0$. It suffices to show that $\vartheta(t) > 0$ for all $t > \tau = \tau(d, \Phi, u, \sigma)$.

Prop $A_{d-1} \implies$ Prop B_d : idea of proof

- Find a positive rational number τ depending only on $\varsigma(t)$ such that $\varsigma(t) > 0$ is strictly increasing for $t \geq \tau$.
- Show that $\vartheta(t) \geq \varsigma(t) > 0$ for all $t \geq \tau$. Since $\varsigma(t) = \vartheta(t)$ for $t \gg 0$, we establish that $\vartheta(t)$ decreases at a slower rate than $\varsigma(t)$ as t decreases by comparing their derivatives.



Prop $A_{d-1} \implies$ Prop B_d : induction on $\dim Z$

Step 1. we pick a general element T in $|I(K_Z + B_Z + M_Z)|$ and let S be its pullback to X , then $S \sim_{\mathbb{Q}} I(K_X + B)$. Define $K_S + B_S = (K_X + B + S)|_S$ and $A_S = A|_S$, then $(S, B_S), A_S \rightarrow T$ is a traditional stable minimal model with $\dim T = m - 1$. By Prop A_{d-1} , there is a positive rational number $\beta = \beta(d, \Phi, u, \sigma)$ such that $A_S + t(K_S + B_S)$ is ample for all $t > \beta$. In particular, if $t > \beta(l + 1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$ is ample, and

$$\varsigma'(t) = \frac{d}{l} \left(A_S + \frac{t}{l+1} (K_S + B_S) \right)^{d-1} > 0.$$

If $\varsigma(t)$ has no roots (which occurs only when $\dim Z$ is even), set $\tau = \beta(l + 1) + 1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$, set $\tau = \max\{\beta(l + 1), \lceil \alpha \rceil\} + 1$. Note that $\tau > 0$ and on $[\tau, +\infty)$, $\varsigma(t)$ is a positive, increasing real function.

Prop $A_{d-1} \implies$ Prop B_d : derivative of volume $\vartheta(t)$

Step 2. We see that $\vartheta(t) = \text{vol}(A + t(K_X + B)) = \frac{1}{d!} \text{vol}(IA + tS)$. By [LM09, Cor 4.27], we have

$$\begin{aligned} I^d \vartheta'(t) &= I^d \frac{d}{ds} \vartheta(t+s) \Big|_{s=0} \\ &= \frac{d}{ds} (\text{vol}(IA + tS + sS)) \Big|_{s=0} \\ &= d \text{vol}_{X|S}(IA + tS) \\ &\leq \text{vol}((IA + tS)|_S) \\ &= I^d \varsigma'(t). \end{aligned}$$

Therefore, if $t \geq \tau$, $\vartheta'(t) \leq \varsigma'(t)$.

Thank you!