BOUNDEDNESS OF POLARIZED LOG CALABI-YAU FIBRATIONS WITH BOUNDED BASES

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Abstract. We investigate the boundedness problem for log Calabi-Yau fibrations whose bases and general fibers are bounded. We prove that the total spaces of log Calabi-Yau fibrations are bounded in codimension one after fixing some natural invariants, which confirms a conjecture of Birkar and Hacon. We also prove that the total spaces are bounded if, in addition, the irregularity of the general fibers vanishes. Then we apply our results to the boundedness problem for stable minimal models and fibered Calabi-Yau varieties.

Contents

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1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic zero.

According to the minimal model program conjecture and the abundance conjecture, every projective variety Y is birational to a projective variety X with mild singularities such that either X is canonically polarized, or X admits a Mori–Fano fibration $X \to Z$, or X admits a Calabi–Yau fibration $X \to Z$. For this reason, canonically polarized varieties, Fano varieties, Calabi–Yau varieties, and their fibrations play a central role in birational geometry. From the perspective of constructing a moduli space for a given class of varieties, the first step is to determine whether they form a bounded family. For the definition of boundedness for varieties, see §2.7.

The boundedness of canonically polarized varieties is established in [HMX14, HMX18], and the boundedness of Fano varieties with mild singularities, known as the famous BAB conjecture, is proved by Birkar [Bir19,Bir21b]. However, for Calabi-Yau varieties, due to the lack of a natural polarization, the question of boundedness remains widely open even in dimension three for strict Calabi-Yau manifolds. Nevertheless, for polarized Calabi-Yau varieties, Birkar shows that boundedness holds under certain conditions [Bir23a]: either one allows a non-effective polarization while requiring the underlying variety to be klt, or, if the underlying variety is slc, the polarization must be an effective divisor that does not contain the non-klt center of the variety.

Based on the predictions of the minimal model program and the abundance conjecture, it is important to extend boundedness results to Fano fibrations and Calabi-Yau fibrations. Such fibrations also frequently appear in inductive arguments. In [Jia18], Jiang considered the birational boundedness of Fano fibrations under several conjectural assumptions. Later, Birkar used some of these arguments to obtain the birational boundedness of Fano fibrations and carried out further work to establish boundedness [Bir24]. However, the boundedness of Calabi-Yau fibrations is not fully understood, although some literature addresses this direction [FS20, Bir21a, Bir22, Jia23, BDCS24, FHS24, Fil24, Jia25, HH25, Zhu25].

Polarized log Calabi-Yau fibration. In this paper, we investigate the following guiding question: If the base and general fiber of a Calabi-Yau fibration belong to a bounded family, under what conditions does the total space belong to a bounded family? Inspired by the study of Fano type fibrations in [Jia18, Bir24], we introduce a special structure for log Calabi-Yau fibrations with polarizations on both the base and the general fiber.

Definition 1.1. A *polarized log Calabi-Yau fibration* $f : (X, B), A \to (Z, H)$ consists of

- a normal projective pair (X, B),
- a fibration $f : X \to Z$ such that $K_X + B \sim_{\mathbb{R}} f^*N$ for some $\mathbb{R}$-divisor N on Z,

- an integral divisor A on X that is ample over Z, and
- a very ample divisor $H \geq 0$ on Z such that $H - N$ is ample.

We call $f : (X, B), A \to (Z, H)$ a *weak polarized log Calabi-Yau fibration* if $H - N$ is only assumed to be pseudo-effective. Moreover, if $A = -K_X$ which is only big over Z, then we omit A and call $f : (X, B) \to (Z, H)$ a *(weak) Fano type fibration.*

Note that H is a polarization on the base Z, and $A|_F$ is a polarization on the general fiber F of $f : X \to Z$. The positivity condition on $H - N$ means that the "degree" of $K_X + B$ with respect to A is bounded in some sense. When Z is a point, the last condition in the definition is vacuous: in this case, the fibration is simply a *polarized log Calabi-Yau pair.*

We now fix some invariants of a (weak) polarized log Calabi-Yau fibration.

Definition 1.2. Let $d \in \mathbb{Z}^{>0}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$, and let $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a DCC set.

(1) A *(weak) (d, r, ϵ)-polarized log Calabi-Yau fibration* is a (weak) polarized log Calabi-Yau fibration $f : (X, B), A \to (Z, H)$ satisfying
   - (X, B) is a projective ϵ-lc pair of dimension d, and
   - $H^{\dim Z} \leq r$.

   Similarly, if $A = -K_X$ which is only big over Z, then we omit A and call $f : (X, B) \to (Z, H)$ a *(weak) (d, r, ϵ)-Fano type fibration.*

(2) If, additionally,
   - $\mathrm{vol}(A|_F) \leq v$, where F is a general fiber of $f : X \to Z$,

   then we call $f : (X, B), A \to (Z, H)$ a *(weak) (d, v, r, ϵ)-polarized log Calabi-Yau fibration.*

(3) Furthermore, if
   - the coefficients of B belong to Φ,

   then we refer to $f : (X, B), A \to (Z, H)$ as a *(weak) $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration.*

Boundedness of polarized log Calabi-Yau fibration. Our first result on the boundedness in codimension one for weak polarized log Calabi-Yau fibrations concerns the case when the coefficients of B belong to a finite set Φ. In particular, a special case of this occurs when $B = 0$, i.e., when $\Phi = \{0\}$. This result was first conjectured by Birkar and Hacon.

Theorem 1.3. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Consider the set of all weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibrations $f : (X, B), A \to (Z, H)$. Then the set of such $(X, B + f^*H)$ is log bounded in codimension one.*

For a more general version of this result, see Theorem 3.1. By combining Theorem 1.3 with the technique from [Bir23b], we establish the boundedness in codimension one for polarized log Calabi-Yau fibrations with arbitrary real coefficients for B. However, we need to assume the ampleness of $H - N$ due to a technical reason.

Theorem 1.4. *Let $d \in \mathbb{N}$ and $v, r, \epsilon, \delta \in \mathbb{R}^{>0}$. Consider the set of all (d, v, r, ϵ)-polarized log Calabi-Yau fibrations $(X, B), A \to (Z, H)$ and $\mathbb{R}$-divisors $0 \leq \Delta \leq$*

B where the non-zero coefficients of Δ are greater than δ. Then the set of such $(X, \Delta + f^*H)$ is log bounded in codimension one.

Question 1.5. With the same notation as Theorem 1.3, is the set of such pairs $(X, B + f^*H)$ log bounded?

When $\dim Z = 1$, boundedness is studied in [HH25]. For $\mathbb{Q}$-factorial terminal minimal threefolds of Kodaira dimension two, [FHS24] derives boundedness from boundedness in codimension one by studying the Kawamata–Morrison cone conjecture and the liftability of flops. In this paper, we propose an alternative approach. Under the additional condition that $\operatorname{Supp} R^1 f_*\mathcal{O}_X \subsetneq Z$, we obtain the actual boundedness of $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibrations.

Theorem 1.6. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Consider the set of all weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibrations $f : (X, B), A \to (Z, H)$ such that $\operatorname{Supp} R^1 f_*\mathcal{O}_X \subsetneq Z$. Then the set of such $(X, B + f^*H)$ is log bounded.*

Boundedness of stable minimal models and fibered Calabi-Yau varieties. We now apply these general boundedness results to some special cases of polarized log Calabi-Yau fibrations. First, we consider the case where $K_X + B$ is semi-ample. It turns out that under some natural conditions, we can choose $H = lN$ for some bounded positive integer $l > 1$. Then $H - N$ is automatically ample. In this case, $(X, B), A$ is a so-called *stable minimal model* [Bir22, Jia23, Zhu25]. Jiao [Jia25] shows that (X, B) is crepant birationally bounded. When $\dim F = 1$, the log boundedness in codimension one of (X, B) follows from [Fil24].

Corollary 1.7. *Let $d \in \mathbb{N}$, $u, v \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Consider the set of $(X, B), A$ such that*

- *(X, B) is a projective klt pair of dimension d,*
- *the coefficients of B are in Φ,*
- *$K_X + B$ is semi-ample defining a contraction $f : (X, B) \to Z$,*
- *$\operatorname{Ivol}(K_X + B) = u$,*
- *A is an integral divisor on X that is ample over Z, and $\operatorname{vol}(A|_F) \leq v$, where F is the general fiber of $f : X \to Z$.*

Then (X, B) is log bounded in codimension one. Moreover, if $\operatorname{Supp} R^1 f_*\mathcal{O}_X \subsetneq Z$, then (X, B) forms a log bounded family.

Next we consider another important case of polarized log Calabi-Yau fibration, where the total space is a Calabi-Yau variety. Such a fibration is called a *fibered Calabi-Yau variety*. In this case, we have $N \sim_{\mathbb{Q}} 0$, so $H - N$ is automatically ample. Furthermore, we assume that the base Z is rationally connected. Note that if X is a strict Calabi-Yau manifold, then by [BDCS24, Corollary 5.1], this condition is automatically satisfied.

Corollary 1.8. *Let $d \in \mathbb{N}$ and $\epsilon, v \in \mathbb{R}^{>0}$. Assume that*

- *(X, B) is a projective ϵ-lc pair of dimension d,*

- $K_X + B \sim_{\mathbb{R}} 0$,
- *$f : X \to Z$ is a contraction to a rationally connected variety Z, and*
- *A is an integral divisor on X such that $0 < \mathrm{vol}(A|_F) \le v$, where F is the general fiber of $f : X \to Z$.*

Then the set of such X is bounded in codimension one.

By [Bir23b], we do not need to assume the boundedness of the torsion index of $K_X + B$. If $\dim F = 1$, this result is proved by [BDCS24, Theorem 1.4]. The case where X has terminal singularities is given by [Jia25, Theorem 8.2].

Sketch of proof. We sketch the proofs of our main theorems, starting with Theorem 1.3. Given a $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration $f : (X, B), A \to (Z, H)$, note that the base Z is bounded by assumption and the general fiber $(F, B_F), A_F$ is bounded by [Bir23a]. We study the induced rational map from Z to a "moduli space" of the general fibers. For this purpose, we use the strongly embedded fine moduli space $\mathcal{S}$ of polarized log Calabi–Yau pairs constructed in [Bir22, Bir23a]. Since $\mathcal{S}$ also parametrizes the polarizations, the universal family $(\mathcal{X}, \mathcal{B}) \to \mathcal{S}$ is not necessarily of maximal variation, and the rational map $Z \dashrightarrow \mathcal{S}$ is not necessarily bounded. Applying [Amb05], we obtain a new family $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ of maximal variation with **b**-nef and big moduli part $\boldsymbol{\mathcal{M}}^!$. Therefore, since the moduli part $\mathbf{M}_Z$ of $f : (X, B) \to Z$ is controlled by H, a volume argument shows that, up to a generically finite cover, the map $Z \dashrightarrow \mathcal{S}^!$ is bounded; see Theorem 3.5.

The traditional strategy for proving boundedness of polarized fibrations is to modify the vertical part of the polarization A so as to obtain a global ample divisor on X with bounded volume; see [Bir22, Jia23, Bir24, Zhu25, HH25]. However, this approach fails in our setting. Instead, we construct a new polarization L arising from the family $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ such that $L \equiv mA$ over the generic point of Z for some fixed $m \in \mathbb{N}$. More precisely, we define a polarization $\mathcal{L}$ on $(\mathcal{X}, \mathcal{B}) \to \mathcal{S}$ by pulling back $\mathcal{A}^!$ to a Galois cover of $\mathcal{X}$, taking the Galois sum, and then descending it to $\mathcal{X}$. For every $s \in \mathcal{S}$, we have $\mathcal{L}_s \equiv m\mathcal{A}_s$; see Theorem 3.4(6). Finally, we define L as the closure of the pullback of $\mathcal{L}$ via the moduli map $Z \dashrightarrow \mathcal{S}$. Since L arises from the fixed family $(\mathcal{X}^!, \mathcal{B}^!), \mathcal{A}^! \to \mathcal{S}^!$ and the map $Z \dashrightarrow \mathcal{S}^!$ is bounded up to a generically finite cover, by the invariance of plurigenera and an argument about the descent of volume from a generically finite cover, we can show that, after modifying the vertical part of $(X, B), L \to Z$, the volume of L can be controlled on a suitable birational model; see Lemma 3.6 and Theorem 3.8.

To proceed, we apply weak semistable reduction [AK00] and the minimal model program for lc pairs [HX13] in Theorem 3.7 to obtain a birational model $(X', \Delta'), L' \to Z$ of $(X, B), L \to Z$ satisfying that

- $(X', \Delta' + \alpha L')$ is lc for some fixed positive real number α,
- $K_{X'} + \Delta' + \alpha L'$ is big, and
- $\mathrm{vol}(K_{X'} + \Delta' + \alpha L')$ is bounded from above.

Then, we apply [HMX13, HMX14] to obtain the log birational boundedness of (X', Δ'). Moreover, we can deduce that $\mathrm{Supp}(\Delta')$ contains both the strict transform of $\mathrm{Supp}(B)$ on X' and the exceptional divisors over X. We can then apply the MMP in family [HMX18] to bound (X, B) in codimension one.

Regarding the proof of Theorem 1.4, we consider two cases. If the horizontal part B^h of B is nonzero, then K_X is not pseudo-effective over Z. Let t be the pseudo-effective threshold of A with respect to K_X over Z. By the argument as in [Bir23b, Theorem 11.1], we can run an MMP on $K_X + tA$ over Z to decompose the new fibration into a Fano type fibration and a lower-dimensional polarized Calabi–Yau fibration. We then proceed by applying the boundedness of Fano type fibrations [Bir24] and induction. If $B^h = 0$, then B is vertical over Z. After reducing to the case where Z is $\mathbb{Q}$-factorial and modifying B to be a very exceptional divisor over Z, we run an MMP on $K_X + B$ over Z by [Bir12] to contract all components of $\mathrm{Supp}(B)$. This case then follows directly from Theorem 1.3 with $\Phi = \{0\}$.

Now we turn to the sketch of the proof of Theorem 1.6. The polarization L constructed in Theorem 1.3 satisfies $L \equiv mA$ over the generic point of Z for some fixed $m \in \mathbb{N}$, and one might hope to modify its vertical part to extend this equivalence over all of Z. However, [Xie25, Example 5.1] shows that this is not possible in general. Nonetheless, under the assumption that $\mathrm{Supp}\, R^1 f_* \mathcal{O}_X \subsetneq Z$, we may assume that $L \sim_{\mathbb{Q}} mA$ over the generic point of Z. Then we construct a new polarization J such that $J \sim_{\mathbb{Q}} mA$ over Z and the components of $\mathrm{Supp}(J - L)$ can be controlled uniformly, see Lemma 5.2. The remaining difficulty is that the components of $J - L$ are vertical over Z, hence non-big over Z, and the coefficients appearing in $J - L$ are uncontrolled. To address this issue, we study the finiteness of log canonical models where the boundary divisors vary in a polytope whose boundary contains non-big divisors, see Lemma 5.3. Finally, by the relative ampleness of A over Z and a standard argument of running an MMP in a family [HMX18], we establish the pure boundedness of (X, B) from boundedness in codimension one.

Structure of the paper. This paper is organized as follows. In §2 we recall some definitions and preliminary results. In §3, we prove Theorem 1.3, which establishes boundedness for fibrations with finite coefficient sets. In §4, we extends the argument to arbitrary coefficients and proves Theorem 1.4. In §5, we focus on fibrations whose general fibers have vanishing irregularity and prove Theorem 1.6. Finally, in §6, we deduce Corollaries 1.7 and 1.8 as consequences of our main results.

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2. Preliminaries

2.1. **Notations and conventions.** We collect some notations and conventions used in this paper.

(1) A projective morphism $f : X \to Z$ between normal varieties is called a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Z$. In particular, f is surjective with connected fibers.
(2) A birational map $\phi : X \dashrightarrow Z$ of varieties is called a *birational contraction* if ϕ^{-1} does not contract any divisor.
(3) For a fibration $f : X \to Z$, we use X_η to denote the generic fiber of f and X_g to denote the general fiber of f. For an $\mathbb{R}$-divisor B on X, we write $B_\eta := B|_{X_\eta}$ and $B_g := B|_{X_g}$.
(4) Let $f : X \to Z$ be a morphism between normal varieties, and let M and L be $\mathbb{R}$-Cartier divisors on X. We say $M \sim L/Z$ (resp. $M \sim_{\mathbb{Q}} L/Z$, $M \sim_{\mathbb{R}} L/Z$) if there is a Cartier (resp. $\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier) divisor N on Z such that $M - L \sim f^*N$ (resp. $M - L \sim_{\mathbb{Q}} f^*N$, $M - L \sim_{\mathbb{R}} f^*N$).
(5) Let X be a normal variety, and let M be an $\mathbb{R}$-divisor on X. Writing $M = \sum m_i M_i$, where M_i are the distinct irreducible components, the notation $M_{\geq a}$ means $\sum_{m_i \geq a} m_i M_i$. One similarly defines $M_{\leq a}$, $M_{>a}$, and $M_{<a}$.
(6) Let $f : X \to Z$ be a morphism between normal varieties, and D be a $\mathbb{R}$-divisor on X. We say D is *horizontal* over Z if the induced map $\operatorname{Supp} D \to Z$ is dominant, otherwise we say D is *vertical* over Z. Given an $\mathbb{R}$-divisor D on X, there is a unique decomposition $D = D^h + D^v$ such that
    - $\operatorname{Supp} D^h$, $\operatorname{Supp} D^v$ have no common components,
    - every component of $\operatorname{Supp} D^h$ is horizontal over Z, and
    - D^v is vertical over Z.

    We call D^h the *horizontal part* of D and D^v the *vertical part* of D with respect to $f : X \to Z$.
(7) We say that a set $\Phi \subset \mathbb{R}$ satisfies the *descending chain condition* (DCC, for short) if Φ does not contain any strictly decreasing infinite sequence. Similarly, we say that a set $\Phi \subset \mathbb{R}$ satisfies the *ascending chain condition* (ACC, for short) if Φ does not contain any strictly increasing infinite sequence.
(8) Let X be a normal projective variety of dimension d, and let D be a $\mathbb{Q}$-divisor on X such that the Iitaka dimension $\kappa(D)$ is non-negative. The *Iitaka volume* of D, denoted by $\operatorname{Ivol}(D)$, is defined as
$$\operatorname{Ivol}(D) = \limsup_{m\to\infty} \frac{\kappa(D)!h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$
    When D is big, this is also called the *volume* of D, denoted by $\operatorname{vol}(D)$. If D is semi-ample and defines a contraction $f : X \to Z$ such that $D \sim_{\mathbb{Q}} f^*H$ for some ample $\mathbb{Q}$-divisor H on Z, then $\operatorname{Ivol}(D) = \operatorname{vol}(H) = H^{\dim Z}$.

2.2. **b-divisors.** Let X be a normal variety. A ***b**-divisor* $\mathbf{M}$ over X is a collection of $\mathbb{R}$-divisors $\mathbf{M}_Y$ on Y for each birational contraction $Y \to X$ from a normal variety that are compatible with respect to pushdown, that is, if $Y' \to X$ is another birational contraction and $\psi : Y' \dashrightarrow Y$ is a morphism, then $\psi_*\mathbf{M}_{Y'} = \mathbf{M}_Y$.

A $\mathbf{b}$-divisor $\mathbf{M}$ is ***b**-$\mathbb{R}$-Cartier* if there is a birational contraction $Y \to X$ such that $\mathbf{M}_Y$ is $\mathbb{R}$-Cartier and $\mathbf{M}_{Y'}$ is the pullback of $\mathbf{M}_Y$ on Y' for any birational contraction $Y' \to Y$. In this case, we say that $\mathbf{M}$ descends to Y, and it is represented by $\mathbf{M}_Y$, we write $\mathbf{M} = \overline{\mathbf{M}_Y}$.

A $\mathbf{b}$-$\mathbb{R}$-Cartier divisor $\mathbf{M}$ represented by $\mathbf{M}_Y$ for some birational model $Y \to X$ is ***b**-nef* if $\mathbf{M}_Y$ is nef. Similarly, $\mathbf{M}$ is ***b**-nef and big* if $\mathbf{M}_Y$ is nef and big.

Definition 2.1 (Discrepancy $\mathbf{b}$-divisors)**.** The discrepancy $\mathbf{b}$-divisor $\mathbf{A} = \mathbf{A}(X, B)$ of a sub-pair (X, B) is the $\mathbf{b}$-$\mathbb{R}$-divisor of X with the trace $\mathbf{A}_Y = \sum a_i A_i$ defined by the formula

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, B)$ by

$$\mathbf{A}^*_Y = \sum_{a_i > -1} a_i A_i.$$

Note that $\mathbf{A}(X, B) = \mathbf{A}^*(X, B)$ when (X, B) is sub-klt. By the definition, we have $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) \simeq \mathcal{O}_X$ if (X, B) is lc. We also have $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq \mathcal{O}_X$ when (X, B) is klt.

2.3. **(Generalized) pairs and singularities.** A *generalized sub-pair* $(X, B, \mathbf{M})/Z$ consists of:

- a normal variety X equipped with a projective morphism $X \to Z$,
- an $\mathbb{R}$-divisor B on X, and
- a $\mathbf{b}$-$\mathbb{R}$-Cartier $\mathbf{b}$-divisor $\mathbf{M}$ over X, represented by a projective birational morphism $f : X' \to X$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\mathbf{M}_{X'}$ on X' such that $\mathbf{M}_{X'}$ is nef over Z and $K_X + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier, where $\mathbf{M}_X := f_*\mathbf{M}_{X'}$.

When Z is a point, we omit it and say that the pair is projective, in which case we also say that $(X, B + \mathbf{M}_X)$ is a generalized sub-pair with nef part $\mathbf{M}_{X'}$. If, in addition, $B \geq 0$, then $(X, B + \mathbf{M}_X)$ is a *generalized pair*.

Let D be a prime divisor over X. Replace X' with a log resolution of (X, B) such that D is a prime divisor on X'. We can write

$$K_{X'} + B' + \mathbf{M}_{X'} = \pi^*(K_X + B + \mathbf{M}_X).$$

Then the *generalized log discrepancy* of D is defined as

$$a(D, X, B, \mathbf{M}_X) = 1 - \operatorname{mult}_D B'.$$

We say that $(X, B + \mathbf{M}_X)$ is *generalized sub-klt* (resp. *generalized sub-lc*, *generalized sub-ϵ-lc*) if $a(D, X, B, \mathbf{M}_X) > 0$ (resp. $a(D, X, B, \mathbf{M}_X) \geq 0$, $a(D, X, B, \mathbf{M}_X) \geq \epsilon$) for every prime divisor D over X. If $(X, B + \mathbf{M}_X)$ is a generalized pair, we remove the prefix "sub" and say the pair is *generalized klt* (resp. *generalized lc*, *generalized ϵ-lc*).

If $\mathbf{M}_X = 0$, then we say $(X, B)/Z$ is a *sub-pair*, and we define its singularities similarly.

2.4. **Minimal models.** Suppose that $f : X \to Z$ and $f^m : X^m \to Z$ are projective morphisms, $\phi : X \dashrightarrow X^m$ is a projective birational contraction over Z and (X, B) and (X^m, B^m) are lc pairs, where $B^m = \phi_* B$. If $a(E, X, B) > a(E, X^m, B^m)$ (resp. $a(E, X, B) \geq a(E, X^m, B^m)$) for all prime ϕ-exceptional divisors $E \subset X$, X^m is $\mathbb{Q}$-factorial and $K_{X^m} + B^m$ is nef over Z, then we say that $\phi : X \dashrightarrow X^m$ is a *minimal model* (resp. *weak log canonical model*) of (X, B) over Z.

A minimal model (resp. weak log canonical model) $\phi : X \dashrightarrow X^m$ of (X, B) over Z is called a *good minimal model* (resp. *semi-ample model*) if $K_{X^m} + B^m$ is semi-ample over Z. In this case,

$$R(X/Z, K_{X^m} + B^m) := \bigoplus_{l \geq 0} f^m_* \mathcal{O}_{X^m}(l(K_{X^m} + B^m))$$

is a finitely generated $\mathcal{O}_Z$-algebra, and let

$$X^c = \operatorname{Proj} R(X/Z, K_{X^m} + B^m).$$

If $K_{X^m} + B^m$ is semi-ample and big over Z, then X^c is called the *log canonical model* of (X, B) over Z.

Definition 2.2. ([Bir17, Definition 1.3]) Let $f : X \to Z$ be a contraction between two projective varieties and L be an $\mathbb{R}$-Cartier divisor on X. The *relative exceptional locus* of L (also called the *relative null locus* when L is nef over Z) is defined as

$$\mathbb{E}(L/Z) = \bigcup_{L|_V \text{ is not big over } f(V)} V,$$

where the union runs over the integral subvarieties $V \subseteq X$ with positive dimension.

Lemma 2.3. *Assume that*

- (X, B) *is a lc pair and* $f : X \to Z$ *is a contraction,*
- $\mu : Z' \to Z$ *is a finite cover,*
- X' *is the normalization of* $X \times_Z Z'$ *and denote the natural finite cover* $X' \to X$ *by* π*, and the contraction* $X' \to Z'$ *by* f'*,*
- (X', B') *is a lc pair such that* $K_{X'} + B' = \pi^*(K_X + B)$*, and*
- $\eta : X'' \dashrightarrow X'/Z'$ *is isomorphic in codimension one and* B'' *is the strict transform of* B' *on* X''*.*

$$\begin{tikzcd}
(X'', B'') \arrow[r, dashed, "\eta"] \arrow[dr] & (X', B') \arrow[r, "\pi"] \arrow[d, "f'"] & (X, B) \arrow[d, "f"'] \\
& Z' \arrow[r, "\mu"] & Z
\end{tikzcd}$$

Then we have the following statements:

(1) *If* $(X, B) \dashrightarrow (X^m, B^m)$ *is a good minimal model of* $K_X + B$ *over* Z *and* $(X'', B'') \dashrightarrow (X''^m, B''^m)$ *is a good minimal model of* $K_{X''} + B''$ *over* Z'*, then* (X''^m, B''^m) *is isomorphic in codimension one to the normalization of* $(X^m, B^m) \times_Z Z'$*,*

(2) *If furthermore* $K_X + B$ *is big over* Z*, assume that* $(X, B) \dashrightarrow (X^c, B^c)$ *is the log canonical model of* $K_X + B$ *over* Z*, and* $(X'', B'') \dashrightarrow (X''^c, B''^c)$ *is the*

log canonical model of $K_{X''}+B''$ over Z'. Then (X''^c, B''^c) is isomorphic to the normalization of $(X^c, B^c)\times_Z Z'$.

Proof. (1). By the proof of [HX13, Lemma 2.4], the set of exceptional divisors of $X \dashrightarrow X^m$ coincides the support of $N_\sigma(K_X+B/Z)$, and the set of exceptional divisors of $X'' \dashrightarrow X''^m$ coincides the support of $N_\sigma(K_{X''}+B''/Z)$. Thus it suffices to prove

$$\mathrm{Supp}(N_\sigma(K_{X''}+B''/Z)) = \eta^{-1}\pi^{-1}\,\mathrm{Supp}(N_\sigma(K_X+B/Z)).$$

Since $X'\to X$ is a finite cover, by [Nak04, §3, Theorem 5.16], we have

$$\pi^{-1}\,\mathrm{Supp}(N_\sigma(K_X+B/Z)) = \mathrm{Supp}(N_\sigma(K_{X'}+B'/Z')).$$

Since (X', B') is isomorphic in codimension one to (X'', B''), there is a one to one correspondence between $|m(K_{X'}+B')/Z'|$ and $|m(K_{X''}+B'')/Z'|$ for every $m\in\mathbb{N}$, hence

$$\eta^{-1}\,\mathrm{Supp}(N_\sigma(K_{X'}+B'/Z')) = \mathrm{Supp}(N_\sigma(K_{X''}+B''/Z')),$$

and we finish the proof.

(2). Since $X'\to X$ is a finite cover, we have

$$\pi^{-1}\mathbb{E}(K_X+B/Z) = \mathbb{E}(K_{X'}+B'/Z')$$

by [Gom22, Theorem 1.1]. Since (X'', B'') is isomorphic in codimension one to (X', B'), the divisorial part of $\mathbb{E}(K_{X''}+B''/Z')$ coincides with the strict transform of the divisorial part of $\mathbb{E}(K_{X'}+B'/Z')$. Let $(\tilde{X}^c, \tilde{B}^c)$ be the normalization of $(X^c, B^c)\times_Z Z'$. Since $X^m\to X^c$ contracts $\mathbb{E}(K_{X^m}+B^m/Z)$ and $X''^m\to X''^c$ contracts $\mathbb{E}(K_{X''^m}+B''^m/Z')$, we conclude that (X''^c, B''^c) is isomorphic in codimension one to $(\tilde{X}^c, \tilde{B}^c)$.

Note that $K_{\tilde{X}^c}+\tilde{B}^c$ is ample because $K_{X^c}+B^c$ is ample and $\tilde{X}^c\to X^c$ is a finite cover. Since $K_{\tilde{X}^c}+\tilde{B}^c$ and $K_{X''^c}+B''^c$ are both ample, and since (X''^c, B''^c) and $(\tilde{X}^c, \tilde{B}^c)$ are isomorphic in codimension one, we conclude that (X''^c, B''^c) is isomorphic to $(\tilde{X}^c, \tilde{B}^c)$. □

2.5. **Canonical bundle formula.**

Definition 2.4. An *lc-trivial fibration* (resp. *klt-trivial fibration*) $f:(X,B)\to Z$ consists of a projective surjective morphism $f: X\to Z$ with connected fibers between normal varieties and a pair (X,B) satisfying the following properties:

- (X, B) is sub-lc (resp. sub-klt) over the generic point of Z,
- $\mathrm{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X,B)\rceil) = 1$, and
- there exists a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L_Z on Z such that

$$K_X+B\sim_{\mathbb{R}} f^*L_Z.$$

In [Amb04, Amb05], klt-trivial fibrations as in Definition 2.4 are called lc-trivial fibrations.

Let $f:(X,B)\to Z$ be an lc-trivial fibration such that $\dim Z>0$. Fix a prime divisor D on Z and let t_D be the lc threshold of f^*D with respect to (X,B) over the generic point of D. Now let $B_Z := \sum(1-t_D)D$, where the sum runs over all the prime divisors on Z. Let $M_Z := L_Z - (K_Z + B_Z)$, then we have the following

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).$$

We call B_Z the *discriminant divisor* and M_Z the *moduli divisor* of adjunction. Note that B_Z is uniquely determined but M_Z is determined only up to $\mathbb{R}$-linear equivalence.

Now let $\phi : X' \to X$ and $\psi : Z' \to Z$ be birational morphisms from normal projective varieties and assume the induced map $f' : X' \dashrightarrow Z'$ is a morphism. Let $K_{X'}+B'$ be the pullback of K_X+B on X' and similarly we can define a discriminant divisor $B_{Z'}$ and $L_{Z'} = \psi^* L_Z$ gives a moduli divisor $M_{Z'}$ so that

$$K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Z'} + B_{Z'} + M_{Z'}),$$

$B_Z = \psi_* B_{Z'}$ and $M_Z = \psi_* M_{Z'}$. In particular, the lc-trivial fibration $f : (X, B) \to Z$ induces $\mathbf{b}$-$\mathbb{R}$-divisors $\mathbf{B}$ and $\mathbf{M}$ on Z, called the *discriminant* and *moduli* $\mathbf{b}$-divisor respectively.

Theorem 2.5 ([Amb04, FG14, Hu20])**.** *With the above notation and assumptions, suppose that (X, B) is lc over the generic point of Z. If $Z' \to Z$ is a high resolution, then $\mathbf{M}_{Z'}$ is nef and for any birational morphism $Z'' \to Z'$ from a normal projective variety, $\mathbf{M}_{Z''}$ is the pullback of $\mathbf{M}_{Z'}$. In particular, we can view $(Z, B_Z + \mathbf{M}_Z)$ as a generalized pair with nef part $\mathbf{M}_{Z'}$.*

Proposition 2.6 ([Amb05, Proposition 3.1])**.** *Let $f : (X, B) \to Z$ be a klt-trivial fibration. Let $\tau : Z' \to Z$ be a surjective morphism from a proper normal variety Z', let X' be the normalization of the main component of $X \times_Z Z'$, and B' be the divisor on X' such that $K_{X'} + B' = \tau_X^*(K_X + B)$. Then we say that $f' : (X', B') \to Z'$ is the klt-trivial fibration induced by base change. Let $\mathbf{M}$ and $\mathbf{M}'$ be the corresponding moduli $\boldsymbol{b}$-divisors of f and f' respectively. Then we have*

$$\tau^*\mathbf{M} = \mathbf{M}'.$$

Theorem 2.7 ([Amb05])**.** *Let $f : (X, B) \to Z$ be a klt-trivial fibration over projective variety Z such that B is a $\mathbb{Q}$-divisor. Suppose that the geometric generic fiber $X_{\overline{\eta}} = X \times_Z \operatorname{Spec}(\overline{k(Z)})$ is projective and $B_{\overline{\eta}}$ is effective. Then there exist non-singular projective varieties $\bar{Z}$, T and V, and a commutative diagram*

$$\begin{tikzcd}
(X,B) \arrow[d, "f"'] & & (X_T,B_T) \arrow[d, "f_T"'] & \\
Z \arrow[rr, dashed, bend left, "i"] \arrow[rrr, dashed, bend right, "\gamma"'] & \bar{Z} \arrow[l, "\tau"'] \arrow[r, "\rho"] & T \arrow[r, "\pi"] & V
\end{tikzcd}$$

such that

(1) $f_T : (X_T, B_T) \to T$ is a klt-trivial fibration,
(2) τ is generically finite and surjective, and ρ is surjective,

(3) there exists a nonempty open subset $U \subset \bar{Z}$ and an isomorphism

$$\begin{tikzcd}
(X,B)\times_Z U \arrow[rr, "\cong"'] \arrow[dr] & & (X_T,B_T)\times_T U \arrow[dl] \\
& U, &
\end{tikzcd}$$

(4) let $\mathbf{M}$, $\mathbf{N}$ be the corresponding moduli $\mathbf{b}$-divisors of f and f_T, then $\mathbf{N}$ is $\mathbf{b}$-nef and big, and $\tau^*\mathbf{M} = \rho^*\mathbf{N}$,

(5) π is generically finite and surjective, $\Phi : Z \dashrightarrow V$ is bimeromorphic to the period map defined in [Amb05, Proposition 2.1], and

(6) $i : T \dashrightarrow Z$ is a rational map such that $f_T : (X_, B_T) \to T$ is equal to the pullback of $f : (X,B) \to Z$ via i.

Proof. The assertions (1)–(4) are stated in [Amb05, Theorem 3.3], while (5) and (6) are derived from the proof of [Amb05, Theorem 2.2]. Indeed, by algebraization theorem in [Kaw83, Theorem 11], the period map defined in [Amb05, Proposition 2.1] is bimeromorphic to a morphism $\gamma^o : Z^o \to V^o$ from a non-empty open subset of Z to a non-singular quasi-projective variety V^o. Let $T^o \to V^o$ be a generically finite surjective morphism from a non-singular quasi-projective variety T^o such that if $\bar{Z}^o$ is the main part of $Z^o \times_{V^o} T^o$, then the induced morphism $\rho^o : \bar{Z}^o \to T^o$ has a section α. By base change via the section $i^o : T^o \xrightarrow{\alpha} \bar{Z}^o \xrightarrow{\tau^o} Z^o$, we induce a family $f_{T^o} : (X_{T^o}, B_{T^o}) \to T^o$. After replacing $\bar{Z}^o$ and T^o by generically finite covers from non-singular quasi-projective varieties, we have an isomorphism of pairs over $\bar{Z}^o$

$$(X,B)\times_Z \bar{Z}^o \xrightarrow{\sim} (X_{T^o}, B_{T^o}) \times_{T^o} \bar{Z}^o.$$

Let $\bar{Z}$, T and V be non-singular projective compactifications of $\bar{Z}^o$, T^o and V^o respectively, and let (X_T, B_T) be a normal projective compactification of (X_{T^o}, B_{T^o}) so that f_{T^o} induces a klt-trivial fibration $f_T : (X_T, B_T) \to T$. Then (5) and (6) are satisfied. $\square$

The following lemma allows us to modify a generically finite cover into a finite cover.

Lemma 2.8. *Let $\pi : T \to V$ be a generically finite cover between projective varieties. Then there exists a generically finite cover $S^! \to T$ from a smooth projective variety $S^!$ and a birational map $S^* \dashrightarrow V$ from a projective variety S^* such that $S^! \to S^*$ is a finite cover.*

Proof. Let $S' \to T$ be a finite cover such that $S' \to V$ is Galois over an open subset of V with Galois group G. Let S'' be the closure of V in $K(S')$, then $S'' \dashrightarrow S'$ is birational and $V = S''/G$.

Let $S^! \to S''$ be a G-equivariant resolution such that $S^! \to S'$ is a morphism, and let S^* be the quotient of $S^!$ by G. Then, the map $S^* \dashrightarrow V$ is birational, $S^! \to T$ is a generically finite surjective morphism, and $S^! \to S^*$ is a finite cover.

$$\begin{tikzcd}
S' \arrow[d] & S'' \arrow[l, dashed] \arrow[d] & S^! \arrow[l] \arrow[ll, bend right] \arrow[d] \\
T \arrow[r, "\pi"] & V & S^* \arrow[l, dashed]
\end{tikzcd}$$

□

The following lemma shows that relative $\mathbb{Q}$-linear triviality can descend under finite covers.

Lemma 2.9. *Assume that*

- *$f : X \to Z$ is a contraction between two normal projective varieties,*
- *D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
- *$\mu : Z' \to Z$ is a finite cover,*
- *X' is the normalization of $X \times_Z Z'$, and*
- *denote the induced finite cover $X' \to X$ by π and the induced contraction $X' \to Z'$ by f'.*

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Z' \arrow[r, "\mu"] & Z
\end{tikzcd}$$

If $\pi^*D \sim_{\mathbb{Q}} 0/Z'$, then $D \sim_{\mathbb{Q}} 0/Z$.

Proof. Replacing Z' with a finite cover and replacing X' accordingly, we can assume that $\mu : Z' \to Z$ is a Galois cover with Galois group G. Then G acts on X' by base change, hence $\pi : X' \to X$ is also a Galois cover. Since $\pi^*D \sim_{\mathbb{Q}} 0/Z'$, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L' on Z' such that $\pi^*D \sim_{\mathbb{Q}} f'^*L'$. Since π^*D is G-invariant, replacing L' with $\frac{1}{|G|}\sum_{g\in G} g^*L'$, we can assume that L' is G-invariant. Therefore, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on Z such that $L' = \mu^*L$. Then $\pi^*D \sim_{\mathbb{Q}} f'^*\mu^*L$, hence $D \sim_{\mathbb{Q}} f^*L$ and we finish the proof. □

2.6. **Locally stable family.**

Definition 2.10 (Relative Mumford divisor)**.** Let $f : X \to Z$ be a flat finite type morphism with S_2 fibers of pure dimension d. A subscheme $D \subset X$ is a *relative Mumford divisor* if there is an open set $U \subset X$ such that

- $\operatorname{codim}_{X_z}(X_z \setminus U_z) \geq 2$ for each $z \in Z$,
- $D|_U$ is a relative Cartier divisor,
- D is the closure of $D|_U$, and
- X_z is smooth at the generic points of D_z for every $z \in Z$.

By $D|_U$ being relative Cartier we mean that $D|_U$ is a Cartier divisor on U and that its support does not contain any irreducible component of any fiber U_z.

If $D \subset X$ is a relative Mumford divisor for $f : X \to Z$ and $T \to Z$ is a morphism, then the *divisorial pullback* D_T on $X_T := X \times_Z T$ is the relative Mumford divisor defined to be the closure of the pullback of $D|_U$ to U_T. In particular, for each $z \in Z$, we define $D_z = D|_{X_z}$ to be the closure of $D|_{U_z}$ which is the divisorial pullback of D to X_z.

Definition 2.11 (Locally stable family)**.** A *locally stable family of slc pairs* $(X, B) \to Z$ over a reduced Noetherian scheme Z is a flat finite type morphism $X \to Z$ with S_2 fibers and a $\mathbb{Q}$-divisor B on X satisfying

- each prime component of B is a relative Mumford divisor,
- $K_{X/Z} + B$ is $\mathbb{Q}$-Cartier, and
- (X_z, B_z) is an slc pair for any point $z \in Z$.

Slc pairs naturally appear in the degeneration of lc pairs. For background on slc singularities, see [Kol13, §5].

Given a morphism $T \to Z$ of reduced schemes, we get the *induced locally stable family* $(X_T, B_T) \to T$ where $X_T = X \times_Z T$ and B_T is defined by divisorial pullback.

Definition 2.12 (Hodge line bundle)**.** If $f : (X, B) \to Z$ is a locally stable family of pairs such that $N(K_{X/Z} + B) \sim 0/Z$, we set

$$\lambda_{\mathrm{Hodge},f,N} := f_*(\mathcal{O}_X(N(K_{X/Z} + B)).$$

Proposition 2.13. *Let $f : (X, B) \to Z$ be a locally stable family of pairs such that $N(K_{X/Z} + B) \sim 0/Z$. Then we have the following statements:*

(1) $\lambda_{\mathrm{Hodge},f,N}$ is the unique line bundle (up to isomorphism) satisfying

$$\mathcal{O}_X(N(K_{X/Z} + B)) \cong f^*\lambda_{\mathrm{Hodge},f,N}.$$

(2) If $\varphi : Z' \to Z$ is a morphism and $f' : (X', B') \to Z'$ denotes the pullback of $(X, B) \to Z$ by φ, then there is a canonical isomorphism

$$\varphi^*\lambda_{\mathrm{Hodge},f,N} \xrightarrow{\sim} \lambda_{\mathrm{Hodge},f',N}.$$

*(3) If Z is smooth and the generic fiber of $X \to Z$ is normal, then $f : (X, B) \to Z$ is an lc-trivial fibration with $\mathcal{O}_Z(N\mathbf{M}_Z) \cong \lambda_{\mathrm{Hodge},f,N}$, and the moduli **b**-divisor $\mathbf{M}$ of f descends on Z.*

Proof. This is [ABB+23, Proposition 14.7]. While the proposition is stated only for families of boundary polarized Calabi–Yau pairs, their proof also applies to families of general Calabi–Yau pairs. □

We need the following lemma about numerically trivial divisors in a flat family.

Lemma 2.14. *Let $f : X \to S$ be a projective flat morphism with integral fibers and of relative dimension d, and let L be a flat family of divisors over S. If there exists a point $0 \in S$ such that $L_0 \equiv 0$, then $L_s \equiv 0$ for all $s \in S$.*

Proof. Let H be a relatively very ample line bundle on X. Take a closed point s on S. Choose $m \gg 0$ such that

$$\chi(X_s, n(mH_s + L_s)) = h^0(X_s, n(mH_s + L_s)) \quad \text{for } n \geq 1.$$

Since L is flat over S, it follows that

$$\chi(X_s, n(mH_s + L_s)) = \chi(X_0, n(mH_0 + L_0)).$$

Therefore, we have

$$h^0(X_s, n(mH_s + L_s)) = h^0(X_0, n(mH_0 + L_0)).$$

From the leading term in the polynomial expansion in n, we obtain

$$(mH_s + L_s)^d = (mH_0 + L_0)^d.$$

Similarly, we have

$$(mH_s)^d = (mH_0)^d.$$

Since $L_0 \equiv 0$, it follows that

$$(mH_s + L_s)^d = (mH_s)^d.$$

Expanding the left-hand side and canceling the dominant terms, we obtain

$$H_s^{d-1} \cdot L_s = H_s^{d-2} \cdot L_s^2 = 0.$$

Restricting to a surface by taking general hyperplane sections and applying the Hodge index theorem, we conclude that $L_s \equiv 0$. □

2.7. **Bounded families of pairs and morphisms.** We say that a collection of log pairs $\mathcal{P}$ is *log birationally bounded* (resp., *log bounded*, or *log bounded in codimension one*) if there is a quasi-projective scheme $\mathcal{X}$, a reduced divisor $\mathcal{E}$ on $\mathcal{X}$, and a projective morphism $h : \mathcal{X} \to T$, where T is of finite type and $\mathcal{E}$ does not contain any fiber, such that for every $(X, B) \in \mathcal{P}$, there is a closed point $t \in T$ and a birational map $f : \mathcal{X}_t \dashrightarrow X$ (resp. isomorphic, or isomorphic in codimension one) such that $\mathcal{E}_t$ contains the support of $f_*^{-1}B$ and any f-exceptional divisor (resp. $\mathcal{E}_t$ coincides with the support of $f_*^{-1}B$, or $\mathcal{E}_t$ coincides with the support of $f_*^{-1}B$).

We say that a collection of morphisms $\mathcal{F}$ is *bounded* if there exist quasi-projective schemes $\mathcal{X}, \mathcal{Z}$, and projective morphisms $\mathcal{X} \xrightarrow{\phi} \mathcal{Z} \to T$, where T is of finite type, such that for every morphism $X \to Z$ in $\mathcal{F}$, there is a closed point of $t \in T$ satisfying that $\mathcal{X}_t \to \mathcal{Z}_t$ is isomorphic to $X \to Z$.

3. Polarized log Calabi–Yau fibrations: finite coefficients

In this section, we consider the boundedness of weak polarized log Calabi–Yau fibration $f : (X, B), A \to (Z, H)$ such that the coefficients of B belong to a finite set Φ. We will prove the following more general form of Theorem 1.3.

Theorem 3.1. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer l and a bounded set of couples $\mathcal{P}$ depending only on d, Φ, v, r, ϵ satisfying the following.*

Assume that $f : (X, B), A \to (Z, H)$ is a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration, and $H_Z \geq 0$ is a general element of $|6dH|$. Then there exists a couple (V, Θ) and an effective integral divisor J on V such that

- *there is a contraction $h : V \to Z$ and V is $\mathbb{Q}$-factorial,*
- *$V \dashrightarrow X/Z$ is isomorphic in codimension one,*
- *$(V, \Theta + \mathrm{Supp}(J))$ belongs to $\mathcal{P}$,*
- *Θ contains h^*H_Z and the strict transform of B, and*
- *$J \equiv lA_V$ over the generic point of Z, where A_V is the strict transform of A on V.*

Lemma 3.2. *Assume that Theorem 3.1 holds when A is an effective integral divisor and $\mathrm{vol}(A|_F) = v$ for some fixed $v \in \mathbb{Q}^{>0}$, where F is the general fiber of $f : X \to Z$. Then the theorem holds in general.*

Proof. If (F, B_F) is the general fiber of $f : (X, B) \to Z$ and $A_F := A|_F$, then by [Bir23a, Theorem 1.3], there exists a positive integer m depending only on $\dim F$ and ϵ such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. Thus, we have $mA \sim G$ for some integral divisor G on X, whose horizontal part G^h is an effective integral divisor. Replacing A and v with G^h and $m^{\dim F}v$ respectively, we may assume that $A \geq 0$. Moreover, by [Bir23a, Corollary 1.6], the pair $(F, \mathrm{Supp}(B_F + A_F))$ belongs to a log bounded family. Hence, we can assume that $\mathrm{vol}(A_F)$ is fixed. □

From now until the end of this section, we will assume that A is an effective integral divisor and that $\mathrm{vol}(A|_F) = v$ for some fixed $v \in \mathbb{Q}^{>0}$.

3.1. Family of polarized log Calabi-Yau pairs.

Definition 3.3. ([Bir22, Bir23a]) Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$ and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. A (d, Φ, v)*-polarized log Calabi-Yau pair* $(X, B), A$ is defined by the data:

- (X, B) is projective slc pair of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- the coefficients of B are in Φ,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + tA)$ is slc for some $t \in \mathbb{Q}^{>0}$.

If (X, B) is klt, then $(X, B), A$ is called a *klt* (d, Φ, v)*-polarized log Calabi-Yau pair.*

Given a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration $f : (X, B), A \to (Z, H)$, it follows that the general fiber $(F, B_F), A_F$ of f is a klt $(\dim F, \Phi, v)$-polarized log Calabi–Yau pair, hence it is bounded by [Bir23a, Corollary 1.6].

In the following theorem, we use the moduli theory for polarized log Calabi–Yau pairs [Bir22] to construct a locally stable family of polarized log Calabi-Yau pairs $f_{\mathcal{S}} : (\mathcal{X}, \mathcal{B}), \mathcal{A} \to \mathcal{S}$ such that over an open subset of Z, the fibration $f : (X, B), A \to Z$ is the pullback of $f_{\mathcal{S}}$. We then apply [Amb05] to $f_{\mathcal{S}}$ to obtain a new family $f_{\mathcal{S}^!} : (\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ of maximal variation. Consequently, the moduli **b**-divisor $\boldsymbol{\mathcal{M}}^!$ of $f_{\mathcal{S}^!}$ descends to a nef and big divisor $\boldsymbol{\mathcal{M}}_{\mathcal{S}^!}$ on $\mathcal{S}^!$, which plays a crucial role in the boundedness of the moduli map in Theorem 3.5. A key step in this theorem is constructing a new polarization $\mathcal{L}$ on $\mathcal{X}$ coming from $\mathcal{X}^!$ such that $\mathcal{L}_s \equiv m\mathcal{A}_s$ for some bounded $m \in \mathbb{N}$ and all $s \in \mathcal{S}$, which allows us to prove the boundedness of the log canonical volume of a certain log general type pair in Theorem 3.8. It is also important for the proof of Theorem 1.6. We also prove some additional results that will be used in later subsections.

Theorem 3.4. *Let $d \in \mathbb{N}$, $v \in \mathbb{Q}^{>0}$, and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Let $f : (X, B) \to Z$ be a klt-trivial fibration, and A be an effective integral divisor on X. Assume that the general fiber $(F, B_F), A_F$ of f is a klt (d, Φ, v)-polarized log*

Calabi-Yau pair. Then there exists a commutative diagram

$$\begin{tikzcd}
(X,B),A \arrow[d, "f"] & (X_U,B_U),A_U \arrow[l, hook'] \arrow[r] \arrow[d, "f_U"] & (\mathcal{X},\mathcal{B}),\mathcal{A},\mathcal{L} \arrow[d, "f_{\mathcal{S}}"] & \bar{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & (\mathcal{X}^!,\mathcal{B}^!),\mathcal{L}^! \arrow[d, "f_{\mathcal{S}^!}"] & \\
Z & U \arrow[l, hook'] \arrow[r, "\phi"] & \mathcal{S} \arrow[rrr, bend right, "\gamma"'] & \bar{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{S}^!,\mathcal{M}^! \arrow[r, "\pi"] & \mathcal{S}^*,\mathcal{H}
\end{tikzcd}$$

satisfying the following:

(1) *$\mathcal{S},\bar{\mathcal{S}},\mathcal{S}^!$ are smooth schemes,*
(2) *$\mathcal{S}^!,\mathcal{S}^*$ are projective schemes,*
(3) *$\tau:\bar{\mathcal{S}}\to\mathcal{S}$, $\pi:\mathcal{S}^!\to\mathcal{S}^*$ are finite covers, $\rho:\bar{\mathcal{S}}\to\mathcal{S}^!$ is a dominant morphism, and $\gamma:\mathcal{S}\to\mathcal{S}^*$ is a morphism,*
(4) *the generic fiber of the base change of $(\mathcal{X},\mathcal{B})\to\mathcal{S}$ to $\bar{\mathcal{S}}$ is isomorphic to the generic fiber of the base change of $(\mathcal{X}^!,\mathcal{B}^!)\to\mathcal{S}^!$ to $\bar{\mathcal{S}}$,*
(5) *$\bar{\mathcal{X}}$ is a common resolution of the main components of $\mathcal{X}\times_{\mathcal{S}}\bar{\mathcal{S}}$ and $\mathcal{X}^!\times_{\mathcal{S}^!}\bar{\mathcal{S}}$,*
(6) *there exist $\mathbb{Q}$-Cartier integral divisors $\mathcal{A}$ and $\mathcal{L}$ on $\mathcal{X}$, and $\mathcal{L}^!$ on $\mathcal{X}^!$, such that for some $m\in\mathbb{N}$ depending only on (d,Φ,v), the relation $\mathcal{L}_s\equiv m\mathcal{A}_s$ holds for all $s\in\mathcal{S}$, and the equality $\tau_{\mathcal{X}}^*\mathcal{L}=\rho_{\mathcal{X}}^*\mathcal{L}^!$ holds,*
(7) *$(\mathcal{X},\mathcal{B}+\alpha\mathcal{L})\to\mathcal{S},(\mathcal{X}^!,\mathcal{B}^!+\alpha\mathcal{L}^!)\to\mathcal{S}^!$ are locally stable morphisms for some $\alpha\in\mathbb{Q}^{>0}$ depending only on (d,Φ,v),*
(8) *there exist a very ample divisor $\mathcal{H}\geq 0$ on $\mathcal{S}^*$ such that*
   - *π is étale and Galois over $\mathcal{S}^*\setminus\mathcal{H}$, and*
   - *every fiber of $(\mathcal{X}^!,\mathcal{B}^!),\mathcal{L}^!\to\mathcal{S}^!$ over $\mathcal{S}^!\setminus\mathrm{Supp}(\pi^*\mathcal{H})$ is a klt (d,Φ,m^dv)-polarized log Calabi–Yau pair,*

(9) *the moduli $\mathbf{b}$-divisor $\boldsymbol{\mathcal{M}}^!$ of $(\mathcal{X}^!,\mathcal{B}^!)\to\mathcal{S}^!$ descends on $\mathcal{S}^!$, and there exists $0\leq\mathcal{M}^!\sim_{\mathbb{Q}}\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$ such that $l\mathcal{M}^!$ is Cartier and $l\mathcal{M}^!\geq\pi^*\mathcal{H}$ for some $l\in\mathbb{N}$ depending only on (d,Φ,v),*
(10) *there exists an open subset $U\subset Z$ and a morphism $\phi:U\to\mathcal{S}$ such that $(X_U,B_U),A_U\to U$ is isomorphic to the base change of $(\mathcal{X},\mathcal{B}),\mathcal{A}\to\mathcal{S}$ via ϕ,*
(11) *if $\gamma\circ\phi$ extends to a morphism $\psi:Z\to\mathcal{S}^*$, then $\psi(Z)\not\subset\pi(\mathrm{Supp}(\mathcal{M}^!))$.*

Proof. *Step* 1. In this step, we construct a universal family parametrizing the general fibers of $f:(X,B),A\to Z$.

By [Bir22, Lemma 10.2], there exist $\alpha\in\mathbb{Q}^{>0}$ and $r\in\mathbb{Z}^{>0}$ depending only on (d,Φ,v) such that:

- $(F,B_F+\alpha A_F)$ is klt for the general fiber $(F,B_F),A_F$ of f, and
- $r(K_F+B_F+\alpha A_F)$ is very ample without higher cohomology.

Let $n=h^0(r(K_F+B_F+\alpha A_F))-1$. Then, $r(K_F+B_F+\alpha A_F)$ defines an embedding $F\hookrightarrow\mathbb{P}^n$. Since $r(K_F+B_F+\alpha A_F)$ is very ample without higher cohomology, there exists an open subset $U\hookrightarrow Z$ such that $r(K_{X_U}+B_U+\alpha A_U)$ defines an embedding $X_U\hookrightarrow\mathbb{P}^n_U$.

By [Bir22, Proposition 9.5], there exists a finite type scheme $\mathcal{S}_{(1)}$ representing the functor of strongly embedded $(d,\Phi_{1/c},v,\alpha,r,\mathbb{P}^n)$-polarized log Calabi–Yau families (see [Bir22, Definition 9.3]) over reduced schemes, where $c\in\mathbb{N}^{>0}$ satisfies $c\Phi\subset\mathbb{N}$. Replacing $\mathcal{S}_{(1)}$ by its locally closed subset, we may assume that $\mathcal{S}_{(1)}$ parametrizes klt (d, Φ, v)-polarized log Calabi-Yau pairs. Let

$$(\mathcal{X}_{(1)} \subset \mathbb{P}^n_{\mathcal{X}_{(1)}}, \mathcal{B}_{(1)}), \mathcal{A}_{(1)} \to \mathcal{S}_{(1)}$$

be the corresponding universal family. Then, $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)} + \alpha\mathcal{A}_{(1)}) \to \mathcal{S}_{(1)}$ is locally stable and $K_{\mathcal{X}_{(1)}} + \mathcal{B}_{(1)} \sim_{\mathbb{Q},\mathcal{S}_{(1)}} 0$. Moreover, there exists a moduli morphism $\phi : U \to \mathcal{S}_{(1)}$ such that $(X_U, B_U), A_U \to U$ is isomorphic to the pullback of $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}), \mathcal{A}_{(1)} \to \mathcal{S}_{(1)}$ via ϕ.

Step 2. In this step, we apply Theorem 2.7 to the universal family obtained in Step 1.

By applying Theorem 2.7 to a projective compactification of $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \to \mathcal{S}_{(1)}$, we have a non-singular quasi-projective variety $\bar{\mathcal{S}}_{(1)}$, non-singular projective varieties $\mathcal{T}$ and $\mathcal{V}$, and a commutative diagram

$$\begin{tikzcd}
(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \arrow[d, "f_{\mathcal{S}_{(1)}}"'] & & (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \arrow[d, "f_{\mathcal{T}}"'] & \\
\mathcal{S}_{(1)} \arrow[rrr, dashed, bend right, "\gamma"'] & \bar{\mathcal{S}}_{(1)} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{T} \arrow[r, "\pi"] \arrow[ll, dashed, bend right, "i"'] & \mathcal{V},
\end{tikzcd}$$

such that

- $(\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \to \mathcal{T}$ is a klt-trivial fibration,
- $\tau : \bar{\mathcal{S}}_{(1)} \to \mathcal{S}_{(1)}$ and $\pi : \mathcal{T} \to \mathcal{V}$ are generically finite, surjective morphisms, $\rho : \bar{\mathcal{S}}_{(1)} \to \mathcal{T}$ is a dominant morphism,
- there exist a nonempty open subset $\mathcal{U} \subset \bar{\mathcal{S}}_{(1)}$ and an isomorphism

$$\begin{tikzcd}
(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}) \times_{\mathcal{S}_{(1)}} \mathcal{U} \arrow[rr, "\cong"] \arrow[dr] & & (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \times_{\mathcal{T}} \mathcal{U} \arrow[dl] \\
& \mathcal{U} &
\end{tikzcd}$$

- the moduli **b**-divisor of $f_{\mathcal{T}}$ is **b**-nef and big,
- $\gamma : \mathcal{S}_{(1)} \dashrightarrow \mathcal{V}$ is bimeromorphic to the period map defined in [Amb05, Proposition 2.1], and
- $i : \mathcal{T} \dashrightarrow \mathcal{S}_{(1)}$ is a generically finite rational map such that $f_{\mathcal{T}} : (\mathcal{X}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}) \to \mathcal{T}$ is equal to the pullback of $f_{\mathcal{S}_{(1)}}$ via i.

Step 3. In this step, we shrink $\mathcal{S}_{(1)}$ and construct a smooth projective variety $\mathcal{S}^!$ over which $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is a locally stable family of maximal variation. Then, we verify (1)–(4).

Let $\mathcal{S}_{(2)}$ be an open subset of $\mathcal{S}_{(1)}$ and $\bar{\mathcal{S}}_{(2)}$ be an open subset of $\mathcal{U}$ such that

- $\mathcal{S}_{(2)}$ is smooth,
- γ is a morphism on $\mathcal{S}_{(2)}$,
- $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is a finite cover, and
- $i|_{\mathcal{T}^o} : \mathcal{T}^o \to \mathcal{S}_{(2)}$ is a finite morphism for some open subset $\mathcal{T}^o$ of $\mathcal{T}$.

Let $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)}), \mathcal{A}_{(2)} \to \mathcal{S}_{(2)}$ be the corresponding base change. Then, the pullback of $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)} + \alpha\mathcal{A}_{(2)}) \to \mathcal{S}_{(2)}$ via i defines a locally stable morphism $(\mathcal{X}_{\mathcal{T}^o}, \mathcal{B}_{\mathcal{T}^o} + \alpha\mathcal{A}_{\mathcal{T}^o}) \to \mathcal{T}^o$.

By [KX20, Lemma 4], there exists a generically finite cover $\bar{\mathcal{T}}^o \to \mathcal{T}^o$ and a compactification $\bar{\mathcal{T}}^o \hookrightarrow \mathcal{S}^!$ such that the pullback of $(\mathcal{X}_{\mathcal{T}^o}, \mathcal{B}_{\mathcal{T}^o} + \alpha\mathcal{A}_{\mathcal{T}^o}) \to \mathcal{T}^o$ on $\bar{\mathcal{T}}^o$ extends to a locally stable morphism $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$.

By Lemma 2.8, after replacing $\mathcal{S}^!$ with a generically finite cover from a smooth projective variety and $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$ with the corresponding base change, we may assume that there exists a birational map $\mathcal{S}^* \dashrightarrow \mathcal{V}$ such that $\mathcal{S}^! \to \mathcal{S}^*$ is a finite cover. Replacing $\mathcal{S}_{(2)}$ by an open subset and shrinking $\bar{\mathcal{S}}_{(2)}$ accordingly, we may assume that $\gamma : \mathcal{S}_{(2)} \to \mathcal{S}^*$ is a morphism.

After replacing $\bar{\mathcal{S}}_{(2)}$ by a finite cover, we may assume that $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}^!$ is a dominant morphism. In this case, we have an isomorphism

$$(\mathcal{X}_{(2)}, \mathcal{B}_{(2)}) \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)} \cong (\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \bar{\mathcal{S}}_{(2)}.$$

Next, after replacing $\bar{\mathcal{S}}_{(2)}$ by another finite cover, we may assume that $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is a Galois cover with Galois group G. Replacing $\mathcal{S}_{(2)}$ by an open subset and shrinking $\bar{\mathcal{S}}_{(2)}$ accordingly, we may assume that $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is an étale Galois cover. Therefore, $\bar{\mathcal{S}}_{(2)}$ is smooth.

Step 4. In this step, we construct new polarizations $\mathcal{L}_{(2)}$ and $\mathcal{L}^!$ on $\mathcal{X}_{(2)}$ and $\mathcal{X}^!$ respectively that satisfy (6).

Consider the following diagram:

$$\mathcal{X}_{(2)} \xleftarrow{\tau_{\mathcal{X}}} \mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)} \cong \mathcal{X}^! \times_{\mathcal{S}^!} \bar{\mathcal{S}}_{(2)} \xrightarrow{\rho_{\mathcal{X}}} \mathcal{X}^!$$

Since $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is an étale Galois cover with Galois group G, the morphism

$$\mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)} \to \mathcal{X}_{(2)}$$

is also an étale Galois cover with Galois group G. Indeed, the action of G on $\mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)}$ is induced by base change, i.e., $g \cdot (x, \bar{s}) = (x, g \cdot \bar{s})$ for $g \in G$ and $(x, \bar{s}) \in \mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)}$, and hence it is G-equivariant with respect to the projection $\mathcal{X}_{(2)} \times_{\mathcal{S}_{(2)}} \bar{\mathcal{S}}_{(2)} \to \bar{\mathcal{S}}_{(2)}$. Let

$$\bar{\mathcal{L}}_{(2)} := \sum_{g \in G} g^* \rho_{\mathcal{X}}^* \mathcal{A}^!,$$

since $\bar{\mathcal{L}}_{(2)}$ is G-invariant, then there exist an effective $\mathbb{Q}$-Cartier integral divisor $\mathcal{L}_{(2)}$ on $\mathcal{X}_{(2)}$ such that $\bar{\mathcal{L}}_{(2)} = \tau_{\mathcal{X}}^* \mathcal{L}_{(2)}$.

Denote the image of $\bar{\mathcal{S}}_{(2)}$ in $\mathcal{S}^!$ by $\mathcal{S}^!_{(2)}$, and let $\mathcal{X}_{\mathcal{S}^!_{(2)}}$ be the base change of $\mathcal{X}^!$ over $\mathcal{S}^!_{(2)}$. Let $s \in \mathcal{S}^!_{(2)}$ be a closed point, $\bar{S}'$ the preimage of s in $\bar{\mathcal{S}}_{(2)}$, S an irreducible component of the image of $\bar{S}'$ in $\mathcal{S}_{(2)}$, and $\bar{S}$ the preimage of S on $\bar{\mathcal{S}}_{(2)}$. Then G acts on $\bar{S}$ by base change. Let $(\mathcal{X}_s, \mathcal{B}_s) \to s$, $(\mathcal{X}_S, \mathcal{B}_S) \to S$, and $(\mathcal{X}_{\bar{S}}, \mathcal{B}_{\bar{S}}) \to \bar{S}$ be the corresponding families by base change. Then we have the isomorphisms

$$(\mathcal{X}_s, \mathcal{B}_s) \times \bar{S} \cong (\mathcal{X}_{\bar{S}}, \mathcal{B}_{\bar{S}}) \cong (\mathcal{X}_S, \mathcal{B}_S) \times_S \bar{S}.$$

Now, the group G acts on $\mathcal{X}_{\bar{S}} \cong \mathcal{X}_S \times_S \bar{S}$ by base change, and the projection $\mathcal{X}_{\bar{S}} \cong \mathcal{X}_s \times \bar{S} \to \bar{S}$ is G-equivariant. Hence, the action of an element $g \in G$ on $\mathcal{X}_s \times \bar{S}$ is given by $g \cdot (x, \bar{s}) = (\phi_g(\bar{s}) \cdot x, g \cdot \bar{s})$ for $x \in \mathcal{X}_s$ and $\bar{s} \in \bar{S}$, where ϕ_g denotes the morphism

$$\bar{S} \to \mathrm{Aut}(\mathcal{X}_s, \mathcal{B}_s) = \{\sigma \in \mathrm{Aut}(\mathcal{X}_s) \mid \sigma^*\mathcal{B}_s = \mathcal{B}_s\}.$$

By [Amb05, Proposition 4.6], the connected component $\mathrm{Aut}^0(\mathcal{X}_s, \mathcal{B}_s)$ of $\mathrm{Aut}(\mathcal{X}_s, \mathcal{B}_s)$ containing the identity is an Abelian variety. Then by the same proof of [Kol15, Theorem 44], possibly after passing to a finite cover, the map $\phi_g(\bar{s})$ is independent of $\bar{s} \in \bar{S}$. Thus, G acts diagonally on $\mathcal{X}_{\bar{S}} \cong \mathcal{X}_s \times \bar{S}$.

Since s can be any closed point in $\mathcal{S}^!_{(2)}$, the action of G on $\mathcal{X}_s$ for all $s \in \mathcal{S}^!_{(2)}$ induces an action of G on $\mathcal{X}_{\mathcal{S}^!_{(2)}}$. Specifically, for each $g \in G$ and $x \in \mathcal{X}_{\mathcal{S}^!_{(2)}}$, we define the action $g \cdot x$ as the element in $\mathcal{X}_{\mathcal{S}^!_{(2)}}$ that lies in the same fiber as x but is mapped to $g \cdot x$ under the action of g within that fiber. Since G acts diagonally on $\mathcal{X}_{\bar{S}} \cong \mathcal{X}_s \times \bar{S}$, the projection $\mathcal{X}_{\bar{S}} \cong \mathcal{X}_s \times \bar{S} \to \mathcal{X}_s$ is G-equivariant, which implies that the map

$$\mathcal{X}_{\mathcal{S}^!_{(2)}} \times_{\mathcal{S}^!_{(2)}} \bar{\mathcal{S}}_{(2)} \xrightarrow{\rho_{\mathcal{X}}} \mathcal{X}_{\mathcal{S}^!_{(2)}}$$

is G-equivariant. Therefore, we obtain the equality

$$\sum_{g \in G} g^* \rho^*_{\mathcal{X}} \mathcal{A}^! = \sum_{g \in G} \rho^*_{\mathcal{X}} g^* \mathcal{A}^!.$$

Let

$$\mathcal{L}^! := \sum_{g \in G} g^* \mathcal{A}^!,$$

then we have $\bar{\mathcal{L}}_{(2)} = \rho^*_{\mathcal{X}} \mathcal{L}^!$.

Note that i is a morphism on $\mathcal{S}^!_{(2)}$. Let $\bar{\mathcal{T}}$ be the preimage of $i(\mathcal{S}^!_{(2)})$ on $\bar{\mathcal{S}}$. Because $\mathcal{A}^!$ is equal to the pullback of $\mathcal{A}_{(2)}$ via i, then

$$\rho^*_{\mathcal{X}} \mathcal{A}^!|_{\bar{\mathcal{T}}} = \rho^*_{\mathcal{X}} i^* \mathcal{A}_{(2)}|_{\bar{\mathcal{T}}} = \tau^*_{\mathcal{X}} \mathcal{A}_{(2)}|_{\bar{\mathcal{T}}}.$$

Then we have $(\rho^*_{\mathcal{X}} \mathcal{A}^!)_s \equiv (\tau^*_{\mathcal{X}} \mathcal{A}_{(2)})_s$ for all $s \in \bar{\mathcal{S}}_{(2)}$ by Lemma 2.14. Also because $\tau_{\mathcal{X}}$ is quotient by G, $\tau^*_{\mathcal{X}} \mathcal{A}_{(2)}$ is G-invariant, then $\tau^*_{\mathcal{X}} \mathcal{A}_{(2)} = \frac{1}{|G|} \sum_{g \in G} g^* \tau^*_{\mathcal{X}} \mathcal{A}_{(2)}$. Therefore, we have

$$(\tau^*_{\mathcal{X}} \mathcal{L}_{(2)})_s = (\bar{\mathcal{L}}_{(2)})_s = (\sum_{g \in G} g^* \rho^*_{\mathcal{X}} \mathcal{A}^!)_s \equiv (\sum_{g \in G} g^* \tau^*_{\mathcal{X}} \mathcal{A}_{(2)})_s = |G| (\tau^*_{\mathcal{X}} \mathcal{A}_{(2)})_s$$

for all $s \in \bar{\mathcal{S}}_{(2)}$. Since $\bar{\mathcal{S}}_{(2)} \to \mathcal{S}_{(2)}$ is surjective, we have $(\mathcal{L}_{(2)})_s \equiv |G| (\mathcal{A}_{(2)})_s$ for all $s \in \mathcal{S}_{(2)}$.

Step 5. In this step, we verify (7)–(9).

By the construction in the previous step, the general fiber of $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)}), \mathcal{L}_{(2)} \to \mathcal{S}_{(2)}$ is a $(d, \Phi, |G|^d v)$-polarized Calabi-Yau pair. After replacing $\mathcal{S}_{(2)}$ with an open subset and decreasing α, we may assume that $(\mathcal{X}_{(2)}, \mathcal{B}_{(2)} + \alpha \mathcal{L}_{(2)}) \to \mathcal{S}_{(2)}$ is locally stable. Applying [KX20, Lemma 4] to an open subset $(\mathcal{S}^!)^o$ of $\mathcal{S}^!$ over which $(\mathcal{X}^!, \mathcal{B}^! + \alpha \mathcal{L}^!) \to \mathcal{S}^!$ is locally stable, and then repeating the same arguments as in step 3, we may assume that $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ is locally stable. In the process, we may have lost the local stability of $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{A}^!) \to \mathcal{S}^!$, but this will not be used later. Therefore, (7) holds.

For (8), let $\mathcal{H} \geq 0$ be a very ample divisor on $\mathcal{S}^*$. Because $\mathcal{S}^! \to \mathcal{S}^*$ is a generically finite cover, $\pi^*\mathcal{H}$ is a big divisor on $\mathcal{S}^!$. Then we can choose $\mathcal{H}$ general such that

- π is étale and Galois over $\mathcal{S}^* \setminus \mathrm{Supp}(\mathcal{H})$, and
- every fiber of $(\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \to \mathcal{S}^!$ over $\mathcal{S}^! \setminus \mathrm{Supp}(\pi^*\mathcal{H})$ is a klt (d, Φ, v)-polarized log Calabi–Yau pair.

Now, we address (9). Since $\mathcal{S}^!$ is smooth and $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is locally stable of maximal variation, by Proposition 2.13, the moduli $\mathbf{b}$-divisor $\boldsymbol{\mathcal{M}}^!$ descends to a nef and big divisor $\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$ on $\mathcal{S}^!$. We can choose a general member $0 \leq \mathcal{M}^! \in |\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}|_{\mathbb{Q}}$ such that $l\mathcal{M}^!$ is Cartier and $\pi^*\mathcal{H} \leq l\mathcal{M}^!$ for some $l \in \mathbb{N}$ depending only on (d, Φ, v).

Step 6. In this step, we construct $\mathcal{S}$ and verify (10) and (11). Let

$$\mathcal{S}_{(3)} := \gamma^{-1}(\mathcal{S}^* \setminus \pi(\mathrm{Supp}(\mathcal{M}^!))) \cap \mathcal{S}_{(2)},$$

and $\bar{\mathcal{S}}_{(3)}$ be the preimage of $\mathcal{S}_{(3)}$. Let $(\mathcal{X}_{(3)}, \mathcal{B}_{(3)}), \mathcal{A}_{(3)}, \mathcal{L}_{(3)} \to \mathcal{S}_{(3)}$ and $\bar{\mathcal{X}}_{(3)} \to \bar{\mathcal{S}}_{(3)}$ be the corresponding base change.

Note that $\mathcal{S}_{(3)}$ is an open subset of $\mathcal{S}_{(1)}$, and the moduli map $\phi : U \to \mathcal{S}_{(1)}$ obtained in Step 1 may map onto $\mathcal{S}_{(1)} \setminus \mathcal{S}_{(3)}$. Thus, we repeat the same arguments on $\mathcal{S}_{(1)} \setminus \mathcal{S}_{(3)}$, obtaining a stratification of $\mathcal{S}_{(1)}$, denoted by $\mathcal{S}$. Let $\bar{\mathcal{S}}$ be the preimage of $\mathcal{S}$, and replace $\mathcal{S}^!$ and $\mathcal{S}^*$ accordingly. Let $\bar{\mathcal{X}}$ be a common resolution of the main components of $\mathcal{X} \times_{\mathcal{S}} \bar{\mathcal{S}}$ and $\mathcal{X}^! \times_{\mathcal{S}^!} \bar{\mathcal{S}}$. Then, we have the following diagram

$$\begin{tikzcd}
(\mathcal{X}, \mathcal{B}), \mathcal{A}, \mathcal{L} \arrow[d] & \bar{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & (\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \arrow[d] & \\
\mathcal{S} \arrow[rrr, bend right, "\gamma"'] & \bar{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{S}^! \arrow[r, "\pi"] & \mathcal{S}^*,
\end{tikzcd}$$

that satisfies the requirements (1)–(9).

Recall that $(X_U, B_U), A_U \to U$ is isomorphic to the pullback of $(\mathcal{X}_{(1)}, \mathcal{B}_{(1)}), \mathcal{A}_{(1)} \to \mathcal{S}_{(1)}$ via the moduli morphism $\phi : U \to \mathcal{S}_{(1)}$. After replacing U by an open subset, we may assume ϕ induces an morphism $\phi : U \to \mathcal{S}$, then $(X_U, B_U), A_U \to U$ is isomorphic to the pullback of $(\mathcal{X}, \mathcal{B}), \mathcal{A} \to \mathcal{S}$ via $U \to \mathcal{S}$. Therefore, (10) follows.

Finally, we deal with (11). Suppose that $\gamma \circ \phi$ extends to a morphism $\psi : Z \to \mathcal{S}^*$. By the construction of $\mathcal{S}_{(3)}$, we have $\gamma^{-1}(\pi(\mathrm{Supp}(\mathcal{M}^!))) = \emptyset$. Since $\psi|_U$ factor through $\mathcal{S}$, we have $\psi(Z) \not\subset \pi(\mathrm{Supp}(\mathcal{M}^!))$.

$\square$

3.2. **Boundedness of moduli map.** In this subsection, we construct a birational model (W, D) of Z such that (W, D) is log bounded and the map $W \dashrightarrow \mathcal{S}^*$ induced by the moduli map $Z \dashrightarrow \mathcal{S}^*$ is a bounded morphism.

Theorem 3.5. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. Let $f : (X, B), A \to (Z, H)$ be a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration.*

Let

$$\begin{tikzcd}
(X,B),A \arrow[r, dashed] \arrow[d] & (\mathcal{X},\mathcal{B}),\mathcal{A},\mathcal{L} \arrow[d] & \bar{\mathcal{X}} \arrow[l, "\tau_{\mathcal{X}}"'] \arrow[r, "\rho_{\mathcal{X}}"] \arrow[d] & (\mathcal{X}^!,\mathcal{B}^!),\mathcal{L}^! \arrow[d] & \\
(Z,H) \arrow[r, dashed, "\phi"'] & \mathcal{S} \arrow[rrr, bend right, "\gamma"'] & \bar{\mathcal{S}} \arrow[l, "\tau"'] \arrow[r, "\rho"] & \mathcal{S}^!,\mathcal{M}^! \arrow[r, "\pi"] & \mathcal{S}^*,\mathcal{H}
\end{tikzcd}$$

be the commutative diagram obtained in Theorem 3.4. Then there exists a birational morphism $h: W \to Z$ from a normal projective variety W and a reduced divisor D on W such that

(1) the induced rational map $\psi_W: W \dashrightarrow \mathcal{S}^*$ is a morphism,
(2) $D \supset \mathrm{Supp}(h_*^{-1}B_Z + E + \psi_W^*\mathcal{H})$, where E is the sum of reduced exceptional divisors of h, and
(3) $K_W + D - h^*H$ is big.

Moreover, the set of (W, D) forms a log bounded family, and the morphism $\psi_W: W \to \mathcal{S}^*$ is bounded.

Proof. *Step* 1. In this step, we construct a birational model W of Z such that $W \dashrightarrow Z$ and $W \dashrightarrow \mathcal{S}^*$ are morphisms.

Since $\mathrm{coeff}(B)$ belongs to a finite set, by [BH22, Lemma 6.7], there exist $q \in \mathbb{Z}^{>0}$ and $\delta \in \mathbb{Q}^{>0}$ depending only on d, Φ, v, ϵ such that we can write the adjunction formula

$$q(K_X + B) \sim qf^*(K_Z + B_Z + \mathbf{M}_Z)$$

with $q\mathbf{M}_{Z'}$ Cartier and qB_Z integral, where $\mathbf{M}_{Z'}$ is the moduli divisor on any sufficiently high resolution $Z' \to Z$. Moreover, $(Z, B_Z + \mathbf{M}_Z)$ is generalized δ-lc. In particular, $\mathrm{coeff}(B_Z)$ belongs to a finite set $\mathcal{I}$. Replacing $\mathcal{I}$ by $\mathcal{I} \cup \{1 - \frac{\delta}{2}\}$, we may assume that $1 - \frac{\delta}{2} \in \mathcal{I}$.

Let $g: Z' \to Z$ be a log resolution of (Z, B_Z) such that the moduli $\mathbf{b}$-divisor $\mathbf{M}$ of f descends to Z', and the rational map $\gamma \circ \phi: Z \dashrightarrow \mathcal{S}^*$ extends to a morphism $\psi': Z' \to \mathcal{S}^*$. In particular, $\mathbf{M}_{Z'}$ is nef. Define

$$B_{Z'} := g_*^{-1}B_Z + (1 - \frac{\delta}{2})E_{Z'},$$

where $E_{Z'}$ is the sum of all reduced g-exceptional divisors. Since $(Z, B_Z + \mathbf{M}_Z)$ is generalized δ-lc, it follows that

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective and has the same support as $E_{Z'}$. Moreover, $\mathrm{coeff}(B_{Z'})$ belongs to the finite set $\mathcal{I}$.

By the boundedness of the length of extremal rays, $K_Z + B_Z + \mathbf{M}_Z + 3dH$ is ample. Since

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective, it follows that

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$$

is big. Consider $(Z', B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H})$ as a generalized $\frac{\delta}{2}$-lc pair with nef part $\mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$. By [BZ16, Lemma 4.4], the divisor

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$$

admits a generalized log canonical model $Z' \dashrightarrow W$. In particular, $Z' \dashrightarrow W$ is a birational contraction. Since $d \geq \dim Z'$, the boundedness of the length of extremal rays ensures that the birational contraction $Z' \dashrightarrow W$ is automatically over both Z and $\mathcal{S}^*$, inducing morphisms $h : W \to Z$ and $\psi_W : W \to \mathcal{S}^*$. Let B_W and $\mathbf{M}_W$ be the pushforwards of $B_{Z'}$ and $\mathbf{M}_{Z'}$, respectively. Then,

$$\mathrm{Supp}(B_W) \supset \mathrm{Supp}(h_*^{-1}B_Z + E),$$

where E is the sum of reduced exceptional divisors of h.

$$\begin{tikzcd}
\bar{Z} \arrow[r, "\pi_{Z'}"'] \arrow[dr, "\bar{\psi}"'] & Z' \arrow[r, "g"'] \arrow[rr, dashed, bend left] \arrow[dr, "\psi'"'] & Z \arrow[d, dashed, "\psi"] & W \arrow[l, "h"'] \arrow[dl, "\psi_W"] \\
& \mathcal{S}^! \arrow[r, "\pi"'] & \mathcal{S}^* &
\end{tikzcd}$$

Step 2. In this step, we show that $l\mathbf{M}_{Z'} - \psi'^*\mathcal{H}$ is pseudo-effective for some $l \in \mathbb{Z}^{>0}$ depending only on (d, Φ, v).

Let $\pi_{Z'} : \bar{Z} \to Z'$ be a generically finite cover from a smooth varitiey $\bar{Z}$ such that

- $\psi' : Z' \to \mathcal{S}^*$ lifts to a morphism $\bar{\psi} : \bar{Z} \to \mathcal{S}^!$, and
- the generic fiber of $(X, B) \times_Z \bar{Z} \to \bar{Z}$ is isomorphic to the generic fiber of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \bar{Z} \to \bar{Z}$.

Since $(\mathcal{X}^!, \mathcal{B}^!) \to \mathcal{S}^!$ is locally stable over the smooth base $\mathcal{S}^!$, the morphism

$$(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \bar{Z} \to \bar{Z}$$

is also locally stable over the smooth base $\bar{Z}$. By parts (2) and (3) of Proposition 2.13, the moduli **b**-divisor $\overline{\mathbf{M}}$ of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \bar{Z} \to \bar{Z}$ descends to $\bar{Z}$ and satisfies

$$\bar{\psi}^*\mathcal{M}^! \sim_{\mathbb{Q}} \overline{\mathbf{M}}_{\bar{Z}}.$$

Since the moduli **b**-divisor depends only on the generic fiber and the base $\bar{Z}$ by [Bir19, Lemma 3.5], the moduli **b**-divisor of $(X, B) \times_Z \bar{Z} \to \bar{Z}$ is the same as that of $(\mathcal{X}^!, \mathcal{B}^!) \times_{\mathcal{S}^!} \bar{Z} \to \bar{Z}$. We may still denote the moduli **b**-divisor of $(X, B) \times_Z \bar{Z} \to \bar{Z}$ by $\overline{\mathbf{M}}$ without confusion, and it descends to $\bar{Z}$.

Since $\pi_{Z'} : \bar{Z} \to Z'$ is a generic finite cover and $\mathbf{M}$ descends to Z', it follows from Proposition 2.6 that

$$\overline{\mathbf{M}}_{\bar{Z}} \sim_{\mathbb{Q}} \pi_{Z'}^*\mathbf{M}_{Z'}.$$

By parts (9) and (11) of Theorem 3.4, there exists $l \in \mathbb{Z}^{>0}$ depending only on (d, Φ, v) such that

$$\pi^*\mathcal{H} \leq l\mathcal{M}^!,$$

and $\psi'(Z') \not\subset \pi(\mathrm{Supp}(\mathcal{M}^!))$. Then, we obtain

$$\pi_{Z'}^* l\mathbf{M}_{Z'} \sim_{\mathbb{Q}} l\overline{\mathbf{M}}_{\bar{Z}} \sim_{\mathbb{Q}} \bar{\psi}^* l\mathcal{M}^! \geq \bar{\psi}^*\pi^*\mathcal{H} \sim_{\mathbb{Q}} \pi_{Z'}^*\psi'^*\mathcal{H}.$$

Therefore, $l\mathbf{M}_{Z'} - \psi'^*\mathcal{H}$ is pseudo-effective.

Step 3. In this step, we show that $\mathrm{vol}(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H})$ is bounded from above.

Since $Z' \dashrightarrow W$ is the generalized log canonical model of

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H},$$

and $l\mathbf{M}_{Z'} - \psi'^*\mathcal{H}$ is pseudo-effective by Step 2, we have

$$\begin{aligned} &\mathrm{vol}(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}) \\ \leq &\mathrm{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}) \\ \leq &\mathrm{vol}(K_{Z'} + B_{Z'} + (3dl+1)\mathbf{M}_{Z'} + 3dg^*H). \end{aligned} \tag{3.1}$$

By Step 1, $q\mathbf{M}_{Z'}$ is Cartier. Hence, replacing l with ql, we may assume that

$$l(\mathbf{M}_{Z'} + 3dg^*H)$$

is Cartier.

Since the coefficients of $B_{Z'}$ belong to a finite set $\mathcal{I}$, by [BZ16, Theorem 8.1], there exists $e \in (0,1)$ depending only on $d, \mathcal{I}, l$ such that

$$K_{Z'} + B_{Z'} + e\mathbf{M}_{Z'} + 3dg^*H$$

is big. Choose $\lambda \in (0,1)$ such that

$$\lambda e + (1-\lambda)(3dl+1) = 1.$$

Then, we have

$$\begin{aligned} &\lambda(K_{Z'} + B_{Z'} + e\mathbf{M}_{Z'} + 3dg^*H) \\ +&(1-\lambda)(K_{Z'} + B_{Z'} + (3dl+1)\mathbf{M}_{Z'} + 3dg^*H) \\ =&K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\mathrm{vol}(K_{Z'} + B_{Z'} + (3dl+1)\mathbf{M}_{Z'} + 3dg^*H) \\ \leq &\frac{1}{(1-\lambda)^d}\,\mathrm{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H). \end{aligned} \tag{3.2}$$

By the definition of the weak polarized log Calabi–Yau fibration, $H - (K_Z + B_Z + \mathbf{M}_Z)$ is pseudo-effective. Since

$$K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} - g^*(K_Z + B_Z + \mathbf{M}_Z)$$

is effective and exceptional over Z, and $\dim(Z) \leq d$, we have

$$\begin{aligned} &\mathrm{vol}(K_{Z'} + B_{Z'} + \mathbf{M}_{Z'} + 3dg^*H) \\ = &\mathrm{vol}(K_Z + B_Z + \mathbf{M}_Z + 3dH) \\ \leq &(3d+1)^d H^{\dim Z} \\ \leq &(3d+1)^d r. \end{aligned} \tag{3.3}$$

Combining equations (3.1)–(3.3), we conclude that

$$\mathrm{vol}(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}) \leq \frac{(3d+1)^d}{(1-\lambda)^d} r.$$

Step 4. In this step, we show that W belongs to a bounded family. Moreover, the morphism $\psi_W : W \to \mathcal{S}^*$ is also bounded.

Since Z' is smooth and $\mathbf{M}$ descends to Z', after replacing Z' with a higher model so that $Z' \to W$ is a morphism, $(W, B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H})$ is a generalized $\frac{\delta}{2}$-lc pair with nef part $\mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H}$, satisfying the following conditions:

- The coefficients of B_W belong to a finite set $\mathcal{I}$,
- $l(\mathbf{M}_{Z'} + 3dg^*H + 3d\psi'^*\mathcal{H})$ is Cartier, and
- $K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H}$ is ample with bounded volume,

it follows from [BH22, Lemma 6.6] that $(W, B_W+\mathbf{M}_W+3dh^*H+3d\psi_W^*\mathcal{H})$ is bounded. In particular, there exists $m \in \mathbb{Z}^{>0}$ depending only on d, Φ, v, ϵ, r such that

$$H_W := m(K_W + B_W + \mathbf{M}_W + 3dh^*H + 3d\psi_W^*\mathcal{H})$$

is very ample and $\mathrm{vol}(H_W)$ is bounded from above.

Let $\Gamma_{\psi_W} \subset W \times \mathcal{S}^*$ be the graph of the morphism $\psi_W : W \to \mathcal{S}^*$. Since H_W and $\mathcal{H}$ are very ample, the product $W \times \mathcal{S}^*$ can be embedded into a projective space via the Segre embedding $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \subset \mathbb{P}^N$. Moreover, the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\Gamma_{\psi_W} \cong W$ is given by $H_W + \psi_W^*\mathcal{H}$. By a similar argument as before, we conclude that $\mathrm{vol}(H_W + \psi_W^*\mathcal{H})$ is bounded from above, which implies that Γ_{ψ_W} is bounded. Since every morphism $\psi_W : W \to \mathcal{S}^*$ is determined by its graph Γ_{ψ_W}, it follows that the morphism $\psi_W : W \to \mathcal{S}^*$ is bounded.

Step 5. In this step, we define a reduced divisor D on W and conclude the proof.

Since h^*H and $\psi_W^*\mathcal{H}$ are base point free, it follows that $3dH_W + h^*H + \psi_W^*\mathcal{H}$ is very ample. We can find a positive integer $p \in \mathbb{Z}^{>0}$ and a general reduced divisor

$$0 \le D \in |p(3dH_W + h^*H + \psi_W^*\mathcal{H})|$$

such that D contains the support of $h_*^{-1}B_Z + E + \psi_W^*\mathcal{H}$, where E is the sum of the reduced exceptional divisors of h. Moreover, by the boundedness of the length of extremal rays, the divisor $K_W + D - h^*H$ is big.

By a similar argument as in Step 4, we conclude that $H_W^{\dim W - 1} \cdot D$ is bounded from above. Hence, (W, D) is log bounded, completing the proof.

□

Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. Let α be the positive rational number defined in Theorem 3.4. Let $f : (X, B), A \to (Z, H)$ be a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi–Yau fibration. By Theorem 3.5, there exists a family of pairs $(\mathcal{W}, \mathcal{D}) \to T$ over a finite type scheme T, and a projective morphism $\Theta : \mathcal{W} \to \mathcal{S}^*$ such that $(W, D) \cong (\mathcal{W}_t, \mathcal{D}_t)$, and $\psi_W : W \to \mathcal{S}^*$ is equivalent to $\Theta_t : \mathcal{W}_t \to \mathcal{S}^*$ for some closed point $t \in T$.

Let $\bar{\mathcal{W}}$ be the normalization of the main component of $\mathcal{W} \times_{\mathcal{S}^*} \mathcal{S}^!$, and let $\bar{\mathcal{D}}_{\bar{\mathcal{W}}}$ denote the preimage of $\mathcal{D}$ via the map $\bar{\mathcal{W}} \to \mathcal{W}$. After replacing $(\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\bar{\mathcal{W}}})$ with its log resolution and passing to a stratification of T, we may assume that the pair $(\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\bar{\mathcal{W}}})$ is log smooth over T. Let $\bar{\Theta} : (\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\mathcal{W}}) \to \mathcal{S}^!$ be the induced morphism, and let $\bar{\mathcal{F}} : (\mathcal{X}_{\bar{\mathcal{W}}}, \mathcal{B}_{\bar{\mathcal{W}}}), \mathcal{L}_{\bar{\mathcal{W}}} \to \bar{\mathcal{W}}$ be the pullback of $(\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \to \mathcal{S}^!$ via $\bar{\Theta}$. We have the following commutative diagram.

$$\begin{tikzcd}
(\bar{\mathcal{X}}_{\bar{\mathcal{W}}}, \bar{\mathcal{B}}_{\bar{\mathcal{W}}}), \bar{\mathcal{L}}_{\bar{\mathcal{W}}} \arrow[r, "\bar{\mathcal{F}}"] \arrow[d] & (\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\bar{\mathcal{W}}}) \arrow[r] \arrow[d, "\bar{\Theta}"] & (\mathcal{W}, \mathcal{D}) \arrow[r] \arrow[d, "\Theta"] & T \\
(\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \arrow[r] & \mathcal{S}^! \arrow[r, "\pi"] & \mathcal{S}^* &
\end{tikzcd}$$

Lemma 3.6. *There exists $w \in \mathbb{N}$ depending only on d, Φ, v, r, ϵ such that*

$$\mathrm{vol}(K_{\bar{\mathcal{X}}_{\bar{\mathcal{W}}_t}} + \bar{\mathcal{B}}_{\bar{\mathcal{W}}_t} + \alpha\bar{\mathcal{L}}_{\bar{\mathcal{W}}_t} + \bar{\mathcal{F}}_t^*\bar{\mathcal{D}}_{\bar{\mathcal{W}}_t}) \leq w$$

for every closed point $t \in T$.

Proof. Since $(\mathcal{X}^!, \mathcal{B}^! + \alpha\mathcal{L}^!) \to \mathcal{S}^!$ is locally stable, it follows that

$$(\bar{\mathcal{X}}_{\bar{\mathcal{W}}}, \bar{\mathcal{B}}_{\bar{\mathcal{W}}} + \alpha\bar{\mathcal{L}}_{\bar{\mathcal{W}}}) \to \bar{\mathcal{W}}$$

is also locally stable. Since $(\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\bar{\mathcal{W}}})$ is log smooth, it follows from [Kol23, Corollary 4.55] that

$$(\bar{\mathcal{X}}_{\bar{\mathcal{W}}}, \bar{\mathcal{B}}_{\bar{\mathcal{W}}} + \alpha\bar{\mathcal{L}}_{\bar{\mathcal{W}}} + \bar{\mathcal{F}}^*\bar{\mathcal{D}}_{\bar{\mathcal{W}}})$$

is lc. After taking a locally closed decomposition of T, we may assume that

$$(\bar{\mathcal{X}}_{\bar{\mathcal{W}}}, \bar{\mathcal{B}}_{\bar{\mathcal{W}}} + \alpha\bar{\mathcal{L}}_{\bar{\mathcal{W}}} + \bar{\mathcal{F}}^*\bar{\mathcal{D}}_{\bar{\mathcal{W}}}) \to T$$

admits a fiberwise log resolution $(\bar{\mathcal{Y}}_{\bar{\mathcal{W}}}, \bar{\mathcal{R}}_{\bar{\mathcal{W}}}) \to T$. Then, by [HMX13, Theorem 1.8 (3)],

$$\mathrm{vol}(K_{\bar{\mathcal{X}}_{\bar{\mathcal{W}}_t}} + \bar{\mathcal{B}}_{\bar{\mathcal{W}}_t} + \alpha\bar{\mathcal{L}}_{\bar{\mathcal{W}}_t} + \bar{\mathcal{F}}_t^*\bar{\mathcal{D}}_{\bar{\mathcal{W}}_t}) = \mathrm{vol}(K_{\bar{\mathcal{Y}}_{\bar{\mathcal{W}}_t}} + \bar{\mathcal{R}}_{\bar{\mathcal{W}}_t, >0})$$

is independent of $t \in T$. □

3.3. **Log birational boundedness.** In this subsection, we do some preparation for the proof of log birational boundedness of weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration $f : (X, B), A \to (Z, H)$.

In the following theorem, we construct a special birational model $(X', \Delta'), A' \to (Z', D')$ of $(X, B), A \to Z$, where $(Z', D') \to Z$ factors through the log bounded birational model (W, D) of Z constructed in Theorem 3.5.

Theorem 3.7. *Let $d \in \mathbb{N}$, $v \in \mathbb{Q}_{>0}$, and let $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Let α be the rational number defined in Theorem 3.4. Assume that*

- *$f : (X, B) \to Z$ is a log Calabi–Yau fibration such that (X, B) is klt, and A is an effective integral divisor on X,*
- *the general fiber $(X_g, B_g), A_g$ is a (d, Φ, v)-polarized log Calabi–Yau pair,*
- *there is an adjunction formula $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z)$,*
- *$W \to Z$ is a birational morphism, and*
- *D is a reduced divisor on W containing the strict transform of $\mathrm{Supp}(B_Z)$ together with the exceptional divisors over Z.*

Then we can construct a diagram

$$\begin{tikzcd}
(\bar{X},\bar{B}),\bar{A} \arrow[r] \arrow[d, "\bar{f}"] & (X',B'),A' \arrow[rr, dashed] \arrow[d, "f'"] & & (X,B),A \arrow[d, "f"] \\
(\bar{Z},\bar{D}) \arrow[r, "\pi'"] & (Z',D') \arrow[r] & (W,D) \arrow[r] & Z
\end{tikzcd}$$

satisfying the following properties:

- $Z' \to W$ *is a birational morphism,*
- $f' : X' \to Z'$ *is a contraction, and* B', A' *are horizontal* $\mathbb{Q}$*-divisors on* X'*,*
- *the generic fiber of* $(X', B' + \alpha A') \to Z'$ *is isomorphic to the generic fiber of* $(X, B + \alpha A) \to Z$*,*
- $\pi' : \bar{Z} \to Z'$ *is a finite cover,*
- (Z', D') *and* $(\bar{Z}, \bar{D})$ *are log smooth, where* D' *is the sum of the strict transform of* D *and all exceptional divisors over* W*, and* $\bar{D}$ *is the preimage of* D' *under* π'*,*
- $\bar{X}$ *is the normalization of* $X' \times_{Z'} \bar{Z}$*, and* $\bar{B}, \bar{A}$ *are horizontal* $\mathbb{Q}$*-divisors equal to the pullback of* B', A' *on* $\bar{X}$ *over the generic point of* Z'*, and*
- $\bar{f} : (\bar{X}, \bar{B}), \bar{A} \to \bar{Z}$ *is a family of* (d, Φ, v)*-polarized log Calabi–Yau pairs.*

Furthermore, if $\bar{f} : (\bar{X}, \bar{B}) \to \bar{Z}$ *has klt fibers over codimension one points of* $\bar{Z} \setminus \bar{D}$*, then, setting* $\Delta' := B' + \mathrm{red}(f'^*D')$*, we have:*

(1) f' *has integral fibers over codimension one points of* $Z' \setminus D'$*,*
(2) $(X', \Delta' + \alpha A')$ *is lc,*
(3) $K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + D' + \mathbf{M}_{Z'})$*, and*
(4) $\mathrm{Supp}(\Delta')$ *contains the strict transform of* $\mathrm{Supp}(B)$ *together with all exceptional divisors over* X*.*

Proof. *Step* 1. In this step we construct a birational morphism $Z' \to W$ and a finite cover $\bar{Z} \to Z'$.

Let Y be a log resolution of $(X, B + \alpha A)$. Let B_Y be the strict transform of B plus the reduced horizontal exceptional divisors over Z, and let A_Y be the strict transform of A. Let $Z^o \subset Z$ be an open subset such that

- $W \to Z$ is an isomorphism over Z^o,
- $f : (X, B), A \to Z$ is a family of (d, Φ, v)-polarized log Calabi-Yau pair over Z^o, and
- $f_Y : (Y, B_Y + A_Y) \to Z$ is log smooth over Z^o.

Then B_Y and A_Y are effective $\mathbb{Q}$-divisors which are horizontal over Z^o. By [AK00, Theorem 2.1 and Proposition 4.4], there is an extension $Z^o \hookrightarrow Z'$ such that

- Z' is a log resolution of (W, D),
- there is an equidimensional toroidal morphism $f'_Y : Y' \to Z'$,
- if B'_Y, A'_Y are the closures of $B_Y|_{Z^o}$ and $A_Y|_{Z^o}$, respectively, then they are contained in the toroidal boundary of Y', and
- $(Y', B'_Y), A'_Y \to Z'$ is an extension of $(Y, B_Y), A_Y \times_Z Z^o \to Z^o$.

Let D' be the strict transform of D plus the reduced exceptional divisors over W. By [AK00, Proposition 5.1], there exists a finite cover $\pi' : \bar{Z} \to Z'$ so that $\bar{f}_Y : \bar{Y} \to \bar{Z}$ is an equidimensional toroidal morphism with reduced fibers, where $\bar{Y}$

is the normalization of $Y' \times_{Z'} \bar{Z}$. Note that the finite cover $\bar{Z} \to Z'$ is a Kawamata covering, to ensure the smoothness of $\bar{Z}$ in the construction, we add extra branch loci artificially. Let R' be the divisor on Z' whose support contains the union of the support of D' and the branch divisors of π'. Define $\bar{R} := \mathrm{red}(\pi'^*R')$, then $(\bar{Z}, \bar{R})$ is log smooth by [AK00, Lemma 5.9]. Let $\bar{D} := \mathrm{red}(\pi'^*D')$ be the reduced divisor on $\bar{Z}$, which is contained in $\bar{R}$. Let $\bar{B}_Y, \bar{A}_Y$ on $\bar{Y}$ be the pullback of B'_Y, A'_Y, then they are contained in the toroidal boundary of $\bar{Y}$. By [ACSS21, Proposition 2.16], $(\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y + \bar{f}_Y^*\Sigma)$ is lc for any reduced simple normal crossing divisor Σ on $\bar{Z}$, where $\mu \in (0,1)$ is small enough. Then $\bar{f}_Y : (\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y) \to \bar{Z}$ is a locally stable morphism by [Kol23, Corollary 4.55].

Step 2. In this step we construct a family of (d, Φ, v)-polarized log Calabi–Yau pairs $(\bar{X}, \bar{B}), \bar{A} \to \bar{Z}$.

Since $(\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y) \to \bar{Z}$ is locally stable and $\bar{Z}$ is smooth, every lc center of $(\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y)$ dominates $\bar{Z}$ by [Kol23, Corollary 4.56]. As a general fiber $(Y'_g, B'_g + \mu A'_g)$ is klt, we conclude that $(\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y)$ is klt. Since $(\bar{Y}_g, \bar{B}_{Y_g})$ has a semi-ample model (X_g, B_g), it admits a good minimal model by [HMX18, Lemma 2.9.1]. Thus, by [HX13, Theorem 1.1], running an MMP on $K_{\bar{Y}} + \bar{B}_Y$ over $\bar{Z}$ yields a good minimal model $(\bar{X}', \bar{B}')$ over $\bar{Z}$, and let $\bar{A}'$ be the pushforward of $\bar{A}_Y$.

By [Kol23, Corollary 4.57.1], $(\bar{X}', \bar{B}') \to \bar{Z}$ is also locally stable. Since $K_{\bar{X}'} + \bar{B}'$ is semi-ample over $\bar{Z}$ with Kodaira dimension 0 on the generic fiber, and $\bar{X}' \to \bar{Z}$ is equidimensional, upper semi-continuity of fiber dimensions gives

$$K_{\bar{X}'} + \bar{B}' \sim_{\mathbb{Q},\bar{Z}} 0.$$

Define $(\bar{X}', \bar{B}' + \mu\bar{A}') \dashrightarrow (\bar{X}, \bar{B} + \mu\bar{A})$ to be the log canonical model of $K_{\bar{X}'} + \bar{B}' + \mu\bar{A}'$ over $\bar{Z}$. As $\bar{X}' \dashrightarrow \bar{X}$ is a birational contraction,

$$K_{\bar{X}} + \bar{B} \sim_{\mathbb{Q},\bar{Z}} 0. \tag{3.4}$$

Finally, by [Kol23, Corollary 4.57.2], $(\bar{X}, \bar{B}), \bar{A} \to \bar{Z}$ is a stable family of polarized log Calabi–Yau pairs. Since the general fiber is (d, Φ, v)-polarized, by definition of α the family $(\bar{X}, \bar{B} + \alpha\bar{A}) \to \bar{Z}$ is locally stable.

Step 3. In this step we construct a contraction $f' : X' \to Z'$ together with horizontal $\mathbb{Q}$-divisors B', A' on X', and show that the generic fiber of $(X', B' + \alpha A') \to Z'$ is isomorphic to that of $(X, B + \alpha A) \to Z$.

By the Hurwitz formula [Kol13, §2.41.4] we have

$$K_{\bar{Z}} + \bar{R} = \pi^*(K_{Z'} + R'),$$

where both $(\bar{Z}, \bar{R})$ and (Z', R') are log smooth by construction. By [Kol23, Corollary 4.55], $(\bar{Y}, \bar{B}_Y + \mu\bar{A}_Y + \bar{f}_Y^*\bar{R})$ is lc. Let $\pi_Y : \bar{Y} \to Y'$ be the natural finite cover. Since étale morphisms are stable under base change, the ramification divisor of π_Y is contained in $\mathrm{Supp}(\bar{f}_Y^*\bar{R})$. Thus, by [Kol13, §2.41.4],

$$K_{\bar{Y}} + \bar{B}_Y + \mu\bar{A}_Y + \bar{f}_Y^*\bar{R} = \pi_Y^*\big(K_{Y'} + B'_Y + \mu A'_Y + \mathrm{red}(f'^*_Y R')\big),$$

and $(Y', B'_Y + \mu A'_Y + \mathrm{red}(f'^*_Y R'))$ is lc.

Since the general fiber (Y'_g, B'_{Y_g}) admits a semi-ample model (X_g, B_g), by [HX13, Theorem 1.1], we can run an MMP on $K_{Y'} + B'_Y + \mathrm{red}(f'^*_Y R')$ over Z' (equivalently on $K_{Y'} + B'_Y + \mathrm{red}(f'^*_Y R') - af'^*_Y R'$ for $a \ll 1$). This yields a good minimal model $(X'', B'' + \mathrm{red}(f''^* R'))$ over Z', with $f'' : X'' \to Z'$. Let A'' be the pushforward of A'_Y. By Lemma 2.3(1), $\bar{X}'$ is isomorphic in codimension one to the normalization of $X'' \times_{Z'} \bar{Z}$. Now let $(X', B' + \mathrm{red}(f'^* R') + \mu A')$ be the log canonical model of $K_{X''} + B'' + \mathrm{red}(f''^* R') + \mu A''$ over Z', where $f' : X' \to Z'$. Then the generic fiber of $(X', B' + \mathrm{red}(f'^* R') + \alpha A') \to Z'$ coincides with that of $(X, B + \alpha A) \to Z$.

Since both B'_Y and A'_Y are horizontal over Z', so are B' and A'. By Lemma 2.3(2), $\bar{X}$ is isomorphic to the normalization of $X' \times_{Z'} \bar{Z}$ and

$$K_{\bar{X}} + \bar{B} + \alpha \bar{A} + \bar{f}^* \bar{R} = \pi_X^* \big(K_{X'} + B' + \alpha A' + \mathrm{red}(f'^* R')\big),$$

where $\pi_X : \bar{X} \to X'$. Since $\bar{A} = \pi_X^* A'$, it follows that

$$K_{\bar{X}} + \bar{B} + \bar{f}^* \bar{R} = \pi_X^* (K_{X'} + B' + \mathrm{red}(f'^* R')).$$

By (3.4) and Lemma 2.9, we conclude

$$K_{X'} + B' + \mathrm{red}(f'^* R') \sim_{\mathbb{Q}} 0/Z'.$$

Finally, since $(\bar{X}, \bar{B} + \alpha \bar{A} + \bar{f}^* \bar{R})$ is lc, [Kol13, Corollary 2.43] implies that $(X', B' + \mathrm{red}(f'^* R') + \alpha A')$ is also lc.

Step 4. In this step we prove the furthermore part. From now on we assume that $(\bar{X}, \bar{B}) \to \bar{Z}$ has klt fibers over codimension one points in $\bar{Z} \setminus \bar{D}$, and denote $\Delta' := B' + \mathrm{red}(f'^* D')$.

Let P be a prime divisor on Z' not contained in $\mathrm{Supp}(D')$, and let $\widetilde{B_Z}$ be the strict transform of B_Z on W. Since $\mathrm{Supp}(\widetilde{B_Z}) \subseteq \mathrm{Supp}(D)$ and $\mathrm{Supp}(D')$ contains both the strict transform of D and all exceptional divisors over Z, by the definition of the discriminant part in the canonical bundle formula we conclude that $f' : X' \to Z'$ has a reduced fiber over the generic point of P.

Let $\bar{P}$ be an irreducible component of the preimage of P on $\bar{Z}$. By assumption, $(\bar{X}, \bar{B})$ has a klt fiber over the generic point of $\bar{P}$. By inversion of adjunction, $(\bar{X}, \bar{B} + \bar{f}^* \bar{P})$ is plt near the fiber over the generic point of $\bar{P}$. By [Kol13, §2.41.4], over the generic point of $\bar{P}$ the divisor $K_{\bar{X}} + \bar{B} + \bar{f}^* \bar{P}$ is equivalent to the pullback of $K_{X'} + B' + f'^* P$. Hence, near the fiber of the generic point of P, the pair $(X', B' + f'^* P)$ is plt by [Kol13, Corollary 2.43]. Therefore, $f'^* P$ is irreducible over the generic point of P. This proves (1).

Because f' is equidimensional and has reduced fibers over codimension one points of $Z' \setminus D'$, we obtain

$$\mathrm{red}(f'^* R') = \mathrm{red}(f'^* D) + f'^* (R' - D').$$

Since

$$K_{X'} + B' + \mathrm{red}(f'^* R') \sim_{\mathbb{Q}} 0/Z'$$

and $(X', B' + \mathrm{red}(f'^* R') + \alpha A')$ is lc, it follows that

$$K_{X'} + \Delta' = K_{X'} + B' + \mathrm{red}(f'^* D) \sim_{\mathbb{Q}} 0/Z'$$

and $(X', \Delta' + \alpha A')$ is also lc. This proves (2).

Next, observe that if P is a prime divisor on Z' not contained in $\mathrm{Supp}(D')$, then $(X', \Delta' + f'^*P)$ is plt over the generic point of P, which implies that the discriminant divisor of $f' : (X', \Delta') \to Z'$ is contained in $\mathrm{Supp}(D')$. If P is a prime divisor contained in $\mathrm{Supp}(D')$, then $\mathrm{lct}(X', \Delta'; P) = 0$. Thus,

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + D' + \mathbf{M}_{Z'}),$$

where $\mathbf{M}$ is the moduli $\mathbf{b}$-divisor corresponding to $f : (X, B) \to Z$. This proves (3).

Finally, we prove (4). First, we show that $\mathrm{Supp}(f'^*D')$ contains all exceptional divisors over X. Suppose E' is a prime divisor on X' exceptional over X but not contained in $\mathrm{Supp}(f'^*D')$. Since $(X, B) \to Z$ and $(X', B') \to Z'$ have the same generic fiber, E' is vertical over Z'. As $f' : X' \to Z'$ is equidimensional, $P' := f'(E')$ is a prime divisor on Z' not contained in $\mathrm{Supp}(D')$. Because $\mathrm{Supp}(D')$ contains all exceptional divisors over Z, the image of P' on Z is also a prime divisor P. Let F be a component of $f^{-1}P$ dominating P. Then F is a non-klt center of $(X', B' + f'^*P')$ over the generic point of P', distinct from E', contradicting the fact that $(X', B' + f'^*P')$ is plt near the fiber over the generic point of P'.

By construction, D' contains the strict transform of $\mathrm{Supp}(B_Z)$. By [Jia25, Lemma 2.6.(b)], every f-vertical log center of (X, B) dominates a generalized log center of $(Z, B_Z + \mathbf{M}_Z)$. Hence, $\mathrm{Supp}(f'^*D')$ contains the strict transform of $\mathrm{Supp}(B^v)$. Since $(X, B) \to Z$ and $(X', B') \to Z'$ share the same generic fiber, $\mathrm{Supp}(B')$ contains the strict transform of $\mathrm{Supp}(B^h)$. Therefore, $\mathrm{Supp}(\Delta')$ contains the strict transform of $\mathrm{Supp}(B)$ and all exceptional divisors over X. This proves (4).

□

In the following theorem, we aim to bound the log canonical volume of the special birational model constructed in Theorem 3.7.

Theorem 3.8. *Let $d \in \mathbb{N}$, $v, r, \epsilon \in \mathbb{Q}^{>0}$ and $\Phi \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists a rational number $\alpha \in (0, 1)$ and positive numbers m, w depending only on d, Φ, v, r, ϵ satisfying the following:*

If $f : (X, B), A \to (Z, H)$ is a weak $(d, \Phi, v, r, \epsilon)$-polarized log Calabi-Yau fibration, then there exists a polarized log Calabi-Yau fibration $f' : (X', \Delta'), L' \to Z'$ such that

(1) *$X' \dashrightarrow X$ is a birational map, and $Z' \to Z$ is a birational morphism,*
(2) *the generic fiber of $f : (X, B) \to Z$ is isomorphic to the generic fiber of $f' : (X', \Delta') \to Z'$,*
(3) *$L'_g := L'|_{X'_g}$ is numerically equivalent to $mA'_g := mA'|_{X_g}$ on X'_g, where A' is the strict transform of A on X', and*
(4) *The coefficients of Δ' are in $\Phi \cup \{1\}$.*

Moreover, we have

(5) *Δ' contains the strict transform of $\mathrm{Supp}(B)$ on X' and all exceptional divisors over X,*
(6) *$(X', \Delta' + \alpha L')$ is lc,*
(7) *$K_{X'} + \Delta' + \alpha L' - h'^*H$ is big, where $h' : X' \to Z$, and*
(8) *$\mathrm{vol}(K_{X'} + \Delta' + \alpha L') \leq w$.*

Proof. *Step* 1. In this step we construct a polarized log Calabi–Yau fibration

$$f' : (X', \Delta'), L' \to Z'$$

by Theorem 3.7.

By Theorem 3.5, there exists a birational morphism $h : W \to Z$ and a reduced divisor D on W such that

- (W, D) is log bounded,
- the induced rational map $\psi_W : W \dashrightarrow \mathcal{S}^*$ is a bounded morphism,
- $D \supset \mathrm{Supp}(h_*^{-1} B_Z + E + \psi_W^* \mathcal{H})$, where E is the sum of the reduced exceptional divisors of h, and $\mathcal{H}$ is a very ample divisor on $\mathcal{S}^*$,
- $K_W + D - h^* H$ is big.

Let $\bar{W}$ be the normalization of the main component of $W \times_{\mathcal{S}^*} \mathcal{S}^!$, and let $D_{\bar{W}}$ denote the preimage of D via $\bar{W} \to W$. After replacing $(\bar{W}, D_{\bar{W}})$ with its log resolution, we may assume that $(\bar{W}, D_{\bar{W}})$ is log smooth. Then $\bar{W} \to W$ is generically finite. Let $(X_{\bar{W}}, B_{\bar{W}}), L_{\bar{W}} \to \bar{W}$ be the pullback of $(\mathcal{X}^!, \mathcal{B}^!), \mathcal{L}^! \to \mathcal{S}^!$ via $\bar{W} \to \mathcal{S}^!$.

Let L on X be the closure of the pullback of $\mathcal{L}$ via the moduli map $U \to \mathcal{S}$ for some open subset $U \subset Z$. By Theorem 3.4, the general fiber $(X_g, B_g), L_g$ is a $(\dim X_g, \Phi, v')$-polarized log Calabi–Yau pair, where v' depends only on d, Φ, v. Applying Theorem 3.7, we obtain a family of $(\dim X_g, \Phi, v')$-polarized log Calabi–Yau pairs $\bar{f} : (\bar{X}, \bar{B}), \bar{L} \to \bar{Z}$, and a polarized log Calabi–Yau fibration $f' : (X', \Delta'), L' \to Z'$ satisfying (1)–(4). We may assume that Z' is the log resolution of (W, D) extracting all exceptional divisors of $\bar{W} \to W$.

Step 2. In this step we prove (5)–(7).

By Theorem 3.7 (2)(4), to show that Δ' contains the strict transform of $\mathrm{Supp}(B)$ on X' and all exceptional divisors over X, and that $(X', \Delta' + \alpha L')$ is lc, it suffices to prove that $(\bar{X}, \bar{B}) \to \bar{Z}$ has klt fibers in $\bar{Z} \setminus \bar{D}$.

Let $\tilde{Z} \to \bar{Z}$ be a generically finite morphism such that

- $\tilde{Z} \to Z \dashrightarrow \mathcal{S}$ is a morphism and factors through $\bar{\mathcal{S}} \to \mathcal{S}$, and
- $\tilde{Z} \to Z' \to W$ factors through $\bar{W} \to W$.

We have the following commutative diagram.

$$\begin{tikzcd}
 & & (Z', D') \arrow[d, "g"] & (\bar{Z}, \bar{D}) \arrow[l] & (\tilde{Z}, \tilde{D}) \arrow[l] \arrow[dl] \arrow[dd] \\
U \arrow[r, hook] \arrow[dr] & Z & (W, D) \arrow[l, "h"'] \arrow[d] & (\bar{W}, D_{\bar{W}}) \arrow[l] \arrow[d] & \\
 & \mathcal{S} \arrow[r, hook] & (\mathcal{S}^*, \mathcal{H}) & \mathcal{S}^! \arrow[l] & \bar{\mathcal{S}} \arrow[l] \arrow[lll, bend left]
\end{tikzcd}$$

Let $(\tilde{X}, \tilde{B}), \tilde{L}$ be the normalization of the main component of the base change of $(\bar{X}, \bar{B}), \bar{L}$ by $\tilde{Z} \to \bar{Z}$. Let $(\tilde{X}_W, \tilde{B}_W), \tilde{L}_W$ be the normalization of the main component of the base change of $(X_{\bar{W}}, B_{\bar{W}}), L_{\bar{W}}$ by $\tilde{Z} \to \bar{W}$.

By Theorem 3.4 (4) and (6), the generic fiber of $(\tilde{X}, \tilde{B}), \tilde{L} \to \tilde{Z}$ is isomorphic to the generic fiber of $(\tilde{X}_W, \tilde{B}_W), \tilde{L}_W \to \tilde{Z}$. Moreover, since both $(\tilde{X}, \tilde{B}), \tilde{L} \to \tilde{Z}$

and $(\tilde{X}_W, \tilde{B}_W), \tilde{L}_W \to \tilde{Z}$ are families of polarized log Calabi–Yau pairs, by the separatedness of the moduli of polarized log Calabi–Yau pairs, we obtain

$$(\tilde{X}, \tilde{B}), \tilde{L} \cong (\tilde{X}_W, \tilde{B}_W), \tilde{L}_W.$$

By Theorem 3.4 (8), Theorem 3.5 (2), and the fact that $D_{\bar{W}}$ is the preimage of D, we conclude that $(X_{\bar{W}}, B_{\bar{W}}) \to \bar{W}$ has klt fibers over $\bar{W} \setminus D_{\bar{W}}$. Therefore, $(\tilde{X}, \tilde{B}) \to \tilde{Z}$ has klt fibers over $\tilde{Z} \setminus \tilde{D}'$, where $\tilde{D}'$ is the preimage of $D_{\bar{W}}$. Since $\bar{D}$ contains the preimage of D on $\bar{Z}$, it follows that $\mathrm{Supp}(\tilde{D}') \subseteq \mathrm{Supp}(\tilde{D})$, where $\tilde{D}$ is the preimage of $\bar{D}$. Hence $(\bar{X}, \bar{B})$ has klt fibers over $\bar{Z} \setminus \bar{D}$.

We now show that $K_{X'} + \Delta' + \alpha L' - h'^* H$ is big. By Theorem 3.5 (3), $K_W + D - h^* H$ is big. Since D' contains the strict transform of D plus the reduced exceptional divisors over W, it follows that $K_{Z'} + D' - (K_W + D)$ is effective. Hence $K_{Z'} + D' - g^* h^* H$ is big. Let $0 < a \ll 1$. By Theorem 3.7 (3), we have

$$K_{X'} + \Delta' + \alpha L' - h'^* H = f'^*(K_{Z'} + D' + \mathbf{M}_{Z'} - g^* h^* H) + aL' + (\alpha - a)L'.$$

Since L' is big over Z' and $L' \geq 0$, it follows that $K_{X'} + \Delta' + \alpha L' - h'^* H$ is the sum of a big $\mathbb{Q}$-divisor and an effective $\mathbb{Q}$-divisor, and hence big.

Step 3. In this step we prove that $\mathrm{vol}(K_{X'} + \Delta' + \alpha L')$ is bounded from above.
Consider the following commutative diagram:

$$\begin{tikzcd}
(X', B' + \alpha L') \arrow[d, "f'"'] & (\tilde{X}, \tilde{B} + \alpha \tilde{L}) \arrow[l, "\mu"'] \arrow[r, "\eta"] \arrow[d, "\tilde{f}"'] & \tilde{X}' \arrow[r, "\nu"] & (X_{\bar{W}}, B_{\bar{W}} + \alpha L_{\bar{W}}) \arrow[d, "f_{\bar{W}}"] \\
(Z', D') & (\tilde{Z}, \tilde{D}) \arrow[l, "\pi"'] \arrow[rr, "\tau"] & & (\bar{W}, D_{\bar{W}})
\end{tikzcd}$$

Here $\tilde{X} \xrightarrow{\eta} \tilde{X}' \xrightarrow{\nu} X_{\bar{W}}$ is the Stein factorization of $\tilde{X} \to X_{\bar{W}}$, hence ν is a finite morphism and η is a birational morphism. Now we claim that

$$\eta_* \mu^*(K_{X'} + \Delta' + \alpha L') = \nu^*(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}}).$$

Since the generic fiber of

$$(X', \Delta' + \alpha L') \times_{Z'} \tilde{Z} \to \tilde{Z}$$

is equal to the generic fiber of

$$(X_{\bar{W}}, B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_W) \times_{\bar{W}} \tilde{Z} \to \tilde{Z},$$

it suffices to compare vertical divisors. Let $\tilde{P}$ be a prime divisor on $\tilde{X}$ vertical over $\tilde{Z}$, and set $\tilde{P}' := \eta(\tilde{P})$, which is also a prime divisor on $\tilde{X}'$. Let $P_{\bar{W}}$ denote the image of $\tilde{P}$ on $X_{\bar{W}}$. We claim that the image of $\tilde{P}$ on X' is also a prime divisor. Indeed, since $P_{\bar{W}}$ is prime and $f_{\bar{W}}$ is equidimensional, $f_{\bar{W}}(P_{\bar{W}})$ is a prime divisor.

As Z' is a log resolution of (W, D) extracting all exceptional divisors of $\bar{W} \to W$, it follows that $\pi \circ \tilde{f}(\tilde{P})$ is a prime divisor on Z'. Since f' and $\tilde{f}$ are equidimensional with fibers of the same dimension, $\mu(\tilde{P})$ is a prime divisor on X', which we denote by P'.

Now we consider two cases:

Case (1): $\mathrm{coeff}_{P'}\Delta' = 1$. Then $f'(P')$ is a prime divisor contained in $\mathrm{Supp}(D')$. By construction, $\pi^{-1}(D')$ is the union of $\tau^{-1}(D_{\bar{W}})$ and some τ-exceptional divisors. Therefore, $f_{\bar{W}}(P_{\bar{W}})$ is a prime divisor contained in $\mathrm{Supp}(D_{\bar{W}})$. Hence, by [Kol13, §2.41.4], over the generic point of $\tilde{P}'$, we have

$$\nu^*(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}}) = K_{\tilde{X}'} + \tilde{P}' = \eta_* \mu^*(K_{X'} + \Delta' + \alpha L').$$

Case (2): $\mathrm{coeff}_{P'}\Delta' = 0$. Then $f'(P')$ is not contained in $\mathrm{Supp}(D')$ and $f_{\bar{W}}(P_{\bar{W}})$ is not contained in $\mathrm{Supp}(D_{\bar{W}})$. Hence, by Theorem 3.7 (1), f' has reduced fibers over the generic point of $f'(P')$. Since the ramification locus of $\bar{W} \to W$ is contained in $\mathrm{Supp}(D_{\bar{W}})$ by Theorem 3.4 (8), the map $\bar{W} \dashrightarrow Z'$ is étale over the generic point of $f'(P')$. Therefore, the ramification index of μ along $\tilde{P}$ equals that of ν along $\tilde{P}$. By the Hurwitz formula, we have

$$\nu^*(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}}) = \eta_* \mu^*(K_{X'} + \Delta' + \alpha L')$$

over the generic point of $\tilde{P}$. This proves the claim.

By the claim, we obtain

$$\mathrm{vol}(\mu^*(K_{X'} + \Delta' + \alpha L')) \leq \mathrm{vol}(\nu^*(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}})).$$

By [Hol12, Lemma 4.3], it follows that

$$\mathrm{vol}(\mu^*(K_{X'} + \Delta' + \alpha L')) = \deg(\mu)\ \mathrm{vol}(K_{X'} + \Delta' + \alpha L'),$$
$$\mathrm{vol}(\nu^*(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}})) = \deg(\nu)\ \mathrm{vol}(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}}).$$

Since $\deg(\nu) \cdot \deg(\bar{W}/Z) = \deg(\mu)$, we conclude

$$\mathrm{vol}(K_{X'} + \Delta' + \alpha L') \leq \frac{1}{\deg(\bar{W}/Z)} \mathrm{vol}(K_{X_{\bar{W}}} + B_{\bar{W}} + \alpha L_{\bar{W}} + f_{\bar{W}}^* D_{\bar{W}}) \leq w$$

by Lemma 3.6, where w is a positive integer depending only on d, Φ, v, r, ϵ.

□

3.4. **Log boundedness in codimension one.** We now proceed to establish the main theorem of this section.

Proof of Theorem 3.1. *Step* 1. Let $h' : (X', \Delta'), L' \to Z' \to Z$ be the fibration constructed in Theorem 3.8. By [HMX14, Theorem 1.3], there exists a fixed positive integer n such that the linear system

$$|n(K_{X'} + \Delta' + \alpha L')|$$

defines a birational map. Let $\pi : Y' \to X'$ be a log resolution of $(X', \Delta' + L')$ such that

$$|n\pi^*(K_{X'} + \Delta' + \alpha L')| = |M| + F,$$

where $|M|$ is the free part and F is the fixed part.

Set $G = M + \pi^* h'^* H$. Then $|G|$ is base point free and defines a birational morphism $\mu : Y' \to Y$ such that $\mu_* G$ is very ample on Y. Since every curve contracted by μ intersects the pullback of H trivially, it follows that the induced map $g : Y \dashrightarrow Z$ is in fact a morphism.

By construction, we have

$$G + F \sim_{\mathbb{Q}} n\pi^*(K_{X'} + \Delta' + \alpha L') + \pi^* h'^* H.$$

Let η_Z denote the generic point of Z. Since $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0/\eta_Z$, we obtain
$$G + F \sim_{\mathbb{Q}} n\alpha\pi^* L'/\eta_Z.$$

Step 2. Let
$$\Sigma' = \operatorname{red}(\pi^{-1}\Delta') + G + F + \pi^* h'^* H_Z + E',$$
where E' is the reduced exceptional divisor of $\pi : Y' \to X'$. Set $\Sigma = \mu_*\Sigma'$. In this step, we prove that (Y, Σ) belongs to a log bounded family and that $g : Y \to Z$ is bounded.

Since $K_{X'} + \Delta' + \alpha L' - h'^* H$ is big by Theorem 3.8 (7), it follows that
$$\operatorname{vol}(G) \le \operatorname{vol}\big((n+1)(K_{X'} + \Delta' + \alpha L')\big) \le (n+1)^d w.$$
By [HMX14, Lemma 7.3], there exists a fixed positive number $\beta < 1$ such that $K_{X'} + \beta(\Delta' + \alpha L')$ is big.

Define
$$c := \frac{1}{\min\{\, c_i \in \Phi \cup \{1\} \mid c_i \neq 0 \,\}},$$
and choose a fixed positive number t satisfying
$$\frac{c + t\beta}{1+t} \le 1 \quad \Longleftrightarrow \quad t \ge \frac{c-1}{1-\beta}.$$
Then we conclude that
$$\begin{aligned}
&\operatorname{vol}(K_{Y'} + \Sigma' + (4d+2)G) \\
\le &\operatorname{vol}(K_{X'} + \pi_*\Sigma' + (4d+2)\pi_* G) \\
\le &\operatorname{vol}(K_{X'} + c\Delta' + (10d+3)(n+1)(K_{X'} + \Delta' + \alpha L')) \\
\le &\operatorname{vol}(K_{X'} + c\Delta' + t(K_{X'} + \beta(\Delta' + \alpha L')) + (10d+3)(n+1)(K_{X'} + \Delta' + \alpha L')) \\
\le &\operatorname{vol}((1 + t + (10d+3)(n+1))(K_{X'} + \Delta' + \alpha L')) \\
\le &(1 + t + (10d+3)(n+1))^d w,
\end{aligned}$$
where the second inequality holds since $K_{X'} + \Delta' + \alpha L' - h'^* H$ is big. Therefore, by [HMX13, Lemma 3.2],
$$\begin{aligned}
\Sigma \cdot ((4d+2)\mu_* G)^{d-1} &= \Sigma' \cdot ((4d+2)G)^{d-1} \\
&\le 2^d \operatorname{vol}(Y', K_{Y'} + \Sigma' + (4d+2)G) \\
&\le 2^d (1 + t + (10d+3)(n+1))^d w.
\end{aligned}$$
Thus by [HMX13, Lemma 2.4.2 (4)], (Y, Σ) forms a log bounded family. By [HJ22, Lemma 2.8], $g : Y \to Z$ is a bounded morphism.

Step 3. There exists a family of contractions $\mathcal{Y} \to \mathcal{Z} \to T$ and three effective divisors Ω, $\mathcal{G}$, and $\mathcal{F}$ on $\mathcal{Y}$ such that for some closed point $t \in T$, the fiber $\mathcal{Y}_t \to \mathcal{Z}_t$ is isomorphic to $g : Y \to Z$, $\Omega_t \simeq \Sigma$, $\mathcal{G}_t \simeq \mu_* G$, and $\mathcal{F}_t \simeq \mu_* F$. Since $\mu_* G$ is a very ample divisor on Y and ampleness is an open condition, after passing to a stratification we may assume that $\mathcal{G}$ is ample over the generic point of $\mathcal{Z}$. If we write $\mathcal{J}_{\mathcal{Y}} = \mathcal{G} + \mathcal{F}$, then $\mathcal{J}_{\mathcal{Y}}$ is big over $\mathcal{Z}$. Setting $J_Y = \mu_* G + \mu_* F$, we have

$$J_Y \sim_{\mathbb{Q}} n\alpha\mu_*\pi^* L'/\eta_Z.$$

Taking a log resolution and passing to a stratification of T, we may assume that T is smooth and $(\mathcal{Y}, \Omega)$ is log smooth over T. Since we replace $\mathcal{J}_{\mathcal{Y}}$ by its pullback, it remains big over $\mathcal{Z}$. After passing to a finite étale cover of a stratification of T (see [Kol13, Claim 4.38.1]), we may assume that every prime component of Ω restricts to a prime divisor fiberwise. Moreover, after replacing $(\mathcal{Y}, \Omega)$ by a sequence of blowups of strata, we extract all divisors whose log discrepancies with respect to $(\mathcal{Y}, (1-\epsilon)\Omega)$ are at most one. Up to a further stratification of T, this process can be assumed to be fiberwise. Therefore, the induced birational map $\mathcal{Y}_t \dashrightarrow X' \dashrightarrow X$ is a birational contraction.

Since H is very ample and $H_Z \in |6dH|$ is general, we may assume that $(X, B + \frac{1}{2}f^*H_Z)$ is ϵ-lc. By the canonical bundle formula,

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z).$$

By the boundedness of the length of extremal rays, $K_Z + B_Z + \mathbf{M}_Z + 3dH$ is ample, and hence

$$K_X + B + \tfrac{1}{2}f^*H_Z \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \mathbf{M}_Z + \tfrac{1}{2}H_Z)$$

is semi-ample.

Let $\Gamma_{\mathcal{Y}_t}$ be the strict transform of $B + \frac{1}{2}f^*H_Z$ on $\mathcal{Y}_t$, together with $(1 - \frac{1}{2}\epsilon)E$, where E is the reduced exceptional divisor of $\mathcal{Y}_t \dashrightarrow X$. Define $\Gamma_{\mathcal{Y}}$ to be the divisor supported on Ω whose restriction to $\mathcal{Y}_t$ is $\Gamma_{\mathcal{Y}_t}$. Since the coefficients of $B + \frac{1}{2}f^*H_Z$ lie in a finite set, the possible coefficients appearing in $\Gamma_{\mathcal{Y}_t}$ also belong to a finite set $\Phi \cup \{\frac{1}{2}, 1 - \frac{1}{2}\epsilon\}$. Therefore, without loss of generality, we may assume that $\Gamma_{\mathcal{Y}}$ is fixed on $\mathcal{Y}$.

By construction, $(X, B + \frac{1}{2}f^*H_Z)$ is a good minimal model of $(\mathcal{Y}_t, \Gamma_{\mathcal{Y}_t})$. By [HMX18, Theorem 1.2], the pair $(\mathcal{Y}, \Gamma_{\mathcal{Y}})$ admits a relative good minimal model $(\mathcal{V}, \Gamma)$ over T, which, up to a stratification of T, induces good minimal models fiberwise. By the boundedness of the length of extremal rays, the induced map $\mathcal{V} \dashrightarrow \mathcal{Z}$ is a morphism. If we denote the pushforward of $\mathcal{J}_{\mathcal{Y}}$ by $\mathcal{J}$, then $\mathcal{J}$ is big over $\mathcal{Z}$. By [HX13, Lemma 2.4], the pair $(\mathcal{V}_t, \Gamma_t)$ is isomorphic in codimension one to $(X, B + \frac{1}{2}f^*H_Z)$. Since L' is numerically equivalent to the strict transform of mA on the generic fiber of $X' \to Z$, and since $J_Y \sim_{\mathbb{Q}} n\alpha\mu_*\pi^*L'/\eta_Z$, we conclude that $\mathcal{J}_t$ is numerically equivalent to the strict transform of $mn\alpha A$ on the generic fiber of $\mathcal{V}_t \to Z$. Replacing n by a bounded multiple, we may assume that $l := mn\alpha$ is an integer. Thus, the pair $(\mathcal{V}_t, \Gamma_t)$ and the integral divisor $\mathcal{J}_t$ are what we need.

$\square$

Remark 3.9. we remark that the relative bigness of $\mathcal{J}$ over $\mathcal{Z}$ will be used in the proof of Theorem 1.6.

4. Polarized log Calabi–Yau fibrations: arbitrary coefficients

In this section, we consider the boundedness of polarized log Calabi–Yau fibration $f : (X, B), A \to (Z, H)$ where the coefficients of B are arbitrary.

We first recall the boundedness result for Fano type fibrations.

Theorem 4.1 ([Bir24, Theorem 1.3])**.** *Let $d \in \mathbb{N}$ and $r, \epsilon, \delta \in \mathbb{R}^{>0}$. Consider the set of all (d, r, ϵ)-Fano type fibrations $(X, B) \to (Z, H)$ and $\mathbb{R}$-divisors $0 \leq \Delta \leq B$ where the non-zero coefficients of Δ are larger than δ. Then the set of such $(X, \Delta + f^*H)$ is log bounded.*

Proof. By [Bir24, Theorem 1.4], there exists a positive number $t < 1$ depending only on d, r, ϵ such that $(X, B + tf^*H)$ is $\frac{\epsilon}{2}$-lc. Then $(X, B + tf^*H) \to Z$ is a $(d, 2^d r, \frac{\epsilon}{2})$-Fano type fibration, hence by [Bir24, Theorem 1.3], $(X, \Delta + f^*H)$ is log bounded. □

Lemma 4.2. *Let $d \in \mathbb{N}$ and $r, \epsilon, \delta \in \mathbb{R}^{>0}$. Assume that*

- *(X, B) is an ϵ-lc pair of dimension d,*
- *$f : X \to Z$ is a contraction to a projective normal variety,*
- *$K_X + B \sim_{\mathbb{R}} f^*N$ for some $\mathbb{R}$-divisor N on Z,*
- *H is a very ample divisor on Z such that $H^{\dim Z} \leq r$ and $H - N$ is ample,*
- *$0 \leq \Delta \leq B$ is an $\mathbb{R}$-divisor on X such that the non-zero coefficients of Δ are larger than δ,*
- *$f : X \to Z$ factors through a contraction $h : X \to Y$, and denote the morphism $Y \to Z$ by g,*
- *$-K_X$ is big over Y,*
- *$\mu : Y \dashrightarrow Y'/Z$ is a birational contraction, and denote the morphism $Y' \to Z$ by g', and*
- *(Y', g'^*H) is log bounded.*

Then there exists a $\mathbb{Q}$-factorial projective variety X' and a contraction $f' : X' \to Z$ satisfying that

- *$\nu : X \dashrightarrow X'/Z$ is isomorphic in codimension one,*
- *(X', B') is ϵ-lc, where $B' = \nu_* B$,*
- *$f' : X' \to Z$ factors through $h' : X' \to Y'$, where $-K_{X'}$ is big over Y', and*
- *$(X', \Delta' + f'^*H)$ is log bounded, where $\Delta' = \nu_*\Delta$.*

$$\begin{tikzcd}
X \arrow[r, dashed, "\nu"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\
Y \arrow[r, dashed, "\mu"] \arrow[dr, "g"'] & Y' \arrow[d, "g'"] \\
& Z
\end{tikzcd}$$

Proof. Since $K_X + B \sim_{\mathbb{R}} 0/Z$, it follows that $K_X + B \sim_{\mathbb{R}} 0/Y$. By [BDCS24, Proposition 3.6], we may assume that Y is $\mathbb{Q}$-factorial. Then, by the relative version of [BDCS24, Proposition 3.7], which holds by running a relative MMP instead of a global MMP in its proof, there exists a birational map

$$\nu : X \dashrightarrow X'/Z$$

that is isomorphic in codimension one, and a contraction $h' : X' \to Y'$. Let $f' : X' \to Z$ denote the induced morphism $X' \to Y' \to Z$. Set $K_{X'} + B' := \nu_*(K_X + B)$.

Since $K_X + B \sim_{\mathbb{R}} f^*N$, we have

$$K_{X'} + B' \sim_{\mathbb{R}} f'^*N,$$

and (X', B') is also ϵ-lc. As (Y', g'^*H) is log bounded, there exist $r' \in \mathbb{R}^{>0}$ and a very ample divisor $H_{Y'}$ on Y' such that

$$H_{Y'}^{\dim Y'} \leq r', \quad \text{and} \quad H_{Y'} - g'^*H \text{ is ample},$$

which implies that $H_{Y'} - g'^*N$ is ample. Note that $-K_{X'}$ is big over Y' because $-K_X$ is big over Y and $\nu : X \dashrightarrow X'$ is isomorphic in codimension one. Therefore,

$$h' : (X', B') \to (Y', H_{Y'})$$

is a (d, r', ϵ)-Fano type fibration. Hence, by Theorem 4.1, $(X', \Delta' + f'^*H)$ is log bounded. □

Remark 4.3. Let $X = Y$ in Lemma 4.2. We then conclude that if

$$X \dashrightarrow X''/Z$$

is a birational contraction and (X'', f''^*H) is log bounded, where $f'' : X'' \to Z$, then there exists a $\mathbb{Q}$-factorial variety X' that is isomorphic to X in codimension one over Z, and

$$(X', \Delta' + f'^*H)$$

is log bounded, where $f' : X' \to Z$.

For the polarized log Calabi-Yau fibration $(X, B), A \to (Z, H)$, if the horizontal part $B^h \neq 0$, we can decompose it into a Fano type fibration and a lower-dimensional polarized log Calabi-Yau fibration.

Proposition 4.4. *Assume that Theorem 1.4 holds in dimension $\leq d-1$. Moreover, assume it also holds when X is of dimension d and B is vertical over Z. Then Theorem 1.4 holds in dimension d.*

Proof. The proof is similar to that of [Bir23b, Theorem 11.1].

Step 1. By assumption we only need to consider the case where the horizontal part B^h of B is non-zero. Then K_X is not pseudo-effective over Z because $K_X + B \sim_{\mathbb{R}} 0/Z$. Let t be the smallest number such that $K_X + tA$ is pseudo-effective over Z. By the proof of [Bir23a, Lemma 4.11], t is a rational number bounded from above.

Step 2. In this step we reduce to the case when X is $\mathbb{Q}$-factorial and $t \geq 1$.

Let l be the largest integer such that $\tilde{A} = lK_X + A$ is big over Z. Then

$$\mathrm{vol}(\tilde{A}_F) = \mathrm{vol}(-lB_F + A_F) \leq \mathrm{vol}(A_F) \leq v,$$

where F is the general fiber of $f\colon X \to Z$. Let $f_1\colon X_1 \to Z$ be the ample model of $\tilde{A}$ over Z, and let B_1, A_1 be the pushdowns of $B, \tilde{A}$ on X_1. If $B_1^h = 0$, then (X_1, f_1^*H) is log bounded in codimension one by assumption. By Remark 4.3, $(X, \Delta + f^*H)$ is log bounded in codimension one. Therefore, we may assume $B_1^h \neq 0$.

Repeating the process, we obtain a chain of birational contractions over Z:

$$(X, B), A \dashrightarrow (X_1, B_1), A_1 \dashrightarrow \cdots \dashrightarrow (X_k, B_k), A_k \dashrightarrow \cdots$$

satisfying $B_i^h \neq 0$. Since the Picard number $\rho(X)$ is finite, there exists $k \in \mathbb{N}$ such that $X_i \dashrightarrow X_{i+1}$ is isomorphic in codimension one for all $i \geq k$. By the definition of $\tilde{A}_k$, we know $K_{X_k} + \tilde{A}_k$ is not big over Z, hence $K_{X_{k+1}} + A_{k+1}$ is not big over Z. Denote by t_{k+1} the smallest number such that $K_{X_{k+1}} + t_{k+1}A_{k+1}$ is pseudo-effective over Z. Then $t_{k+1} \geq 1$. By Remark 4.3, to prove that $(X, \Delta + f^*H)$ is log bounded in codimension one, it suffices to show that (X_{k+1}, f_{k+1}^*H) is log bounded in codimension one, where $f_{k+1}\colon X_{k+1} \to Z$. Thus we may replace $(X, B), A$ with $(X_{k+1}, B_{k+1}), A_{k+1}$ and assume that X is $\mathbb{Q}$-factorial and $t \geq 1$.

Step 3. By the proof of [Bir23a, Lemma 4.11], t belongs to a fixed set of rational numbers which is discrete away from zero. Since $t \geq 1$ and t is bounded above, there are only finitely many possibilities for t. In the following we assume that t is fixed.

View (X, tA) as a generalized pair over Z with nef part tA. Then (X, tA) is generalized ϵ-lc. By [BZ16, Lemma 4.4], there exists a good minimal model $f'\colon X' \to Z$ of $K_X + tA$ over Z. Let B', Δ', A' be the pushdowns of B, Δ, A on X'. By Remark 4.3, it suffices to prove that (X', f'^*H) is log bounded in codimension one.

Let $h\colon X' \to Y/Z$ be the non-birational contraction induced by $K_{X'} + tA'$, and denote the morphism $Y \to Z$ by g. By [Fil20], there is a generalized adjunction formula

$$K_{X'} + tA' \sim_{\mathbb{Q}} h^*(K_Y + C_Y + R_Y).$$

Since A' is big over Z, it follows that $-K_{X'}$ is big over Y. Hence $(Y, C_Y + R_Y)$ is generalized τ-lc for some fixed $\tau > 0$ by [Bir23b, Theorem 9.3].

Since t is fixed, there exists $p \in \mathbb{N}$ such that $p(K_{X'} + tA')$ is integral. Let G be the general fiber of $h\colon X' \to Y$. Then G is ϵ-lc and belongs to a bounded family by [Bir21b]. Replacing p by a bounded multiple, we may assume $p(K_G + A_G)$ is Cartier by [HLQ23, Theorem 1.10]. Since G is of Fano type, $\mathrm{Pic}^0(G) = 0$, hence $p(K_G + tA_G) \sim 0$. Thus there exists a rational function α on X' such that $p(K_{X'} + tA') + \mathrm{Div}(\alpha)$ is vertical over Y. Since

$$p(K_{X'} + tA') + \mathrm{Div}(\alpha) \sim_{\mathbb{Q}} 0/Y,$$

we deduce that $p(K_{X'} + tA') + \mathrm{Div}(\alpha)$ is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y by [CHL24, Lemma 2.5]. Hence we obtain the adjunction formula

$$p(K_{X'} + tA') \sim ph^*(K_Y + C_Y + R_Y).$$

Since $p(K_{X'}+tA')$ is integral and the multiplicities of the fibers of h over codimension one points are bounded, after replacing p by a bounded multiple we may assume that

$$J := p(K_Y + C_Y + R_Y)$$

is an integral divisor.

Step 4. In this step we prove that the volume of the restriction of J on the general fiber of $g : Y \to Z$ is bounded from above.

Let $\phi : W \to X$ and $\psi : W \to X'$ be common resolutions. Pick a general point of Z and let $F_W, F_X, F_{X'}, F_Y$ be the corresponding fiber over this point. By

[Bir23a, Theorem 1.1], there exists a fixed positive integer m such that $|mA|_{F_X}|$ defines a birational map. Let $c = \dim F_W$ and $e = \dim F_Y$, then

$$\begin{aligned}\operatorname{vol}(J|_{F_Y}) &\leq (\phi^*(mA)|_{F_W})^{c-e} \cdot (\psi^*(p(K_{X'} + tA'))|_{F_W})^e \\ &\leq m^{c-e}p^e \operatorname{vol}(\phi^*A|_{F_W} + \psi^*(K_{X'} + tA')|_{F_W}) \\ &\leq m^{c-e}p^e \operatorname{vol}(\phi^*(K_X + (1+t)A)|_{F_W}) \\ &\leq m^{c-e}p^e \operatorname{vol}((1+t)A|_{F_X}) \leq (1+t)m^{c-e}p^e v.\end{aligned}$$

Step 5. Applying [Bir23a, Theorem 1.1] to a $\mathbb{Q}$-factorialization of F_Y and J, we conclude that there exists a fixed positive integer n such that $|nJ|_{F_Y}|$ defines a birational map. Therefore, we can find an effective integral divisor J' such that $J' \sim nJ/Z$, and then

$$\operatorname{vol}(J'|_{F_Y}) \leq v',$$

where $v' = (1+t)m^{c-e}n^e p^e v$.

By [Zhu25, Lemma 2.11] (see also [Bir23b, Theorem 1.8]), there exists a fixed $\tau \in \mathbb{R}_{>0}$ such that we can write an adjunction formula

$$K_{X'} + B' \sim_{\mathbb{R}} h^*(K_Y + D_Y + S_Y),$$

where $(Y, D_Y + S_Y)$ is generalised τ-lc. By the proof of [Amb05, Theorem 4.1], we can find a boundary $\tilde{D}_Y$ such that

$$K_Y + \tilde{D}_Y \sim_{\mathbb{R}} K_Y + D_Y + S_Y$$

and $(Y, \tilde{D}_Y)$ is $\frac{\tau}{2}$-lc. Then

$$g : (Y, \tilde{D}_Y), J' \to (Z, H)$$

is a $(\dim Y, v', r, \frac{\tau}{2})$-polarized log Calabi–Yau fibration. Therefore, (Y, g^*H) is log bounded in codimension one by induction. Hence (X', f'^*H) is log bounded in codimension one by Lemma 4.2. It follows that $(X, \Delta + f^*H)$ is log bounded in codimension one by Remark 4.3.

□

If the horizontal part B^h vanishes, we can run an MMP for very exceptional divisors to reduce the problem to Theorem 3.1 with $\Phi = \{0\}$.

Proof of Theorem 1.4. By Proposition 4.4, it suffices to consider the case where B is vertical over Z.

By [Zhu25, Lemma 2.11] (see also [Bir23b, Theorem 1.8]), there exists a fixed $\delta \in \mathbb{R}^{>0}$ such that we can write an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where $(Z, B_Z + M_Z)$ is generalized δ-lc. By [Bir24, Theorem 2.3], there exists a $\mathbb{Q}$-factorialization $\mu\colon Z' \to Z$ such that Z' belongs to a bounded family. Let H' be a very ample divisor on Z' such that $H'^{\dim Z'} \leq r'$ for some fixed $r' \in \mathbb{R}_{>0}$ and $H' - \mu^*H$ is ample. By [BDCS24, Proposition 3.6], there exists a $\mathbb{Q}$-factorial ϵ-lc pair (X', B') which is isomorphic in codimension one to (X, B), together with a contraction $f'\colon X' \to Z'$. Let A' be the strict transform of A in X'. Then

$$\mathrm{vol}(A'|_{F'}) = \mathrm{vol}(A|_F) \leq v,$$

where F, F' denote the general fibers of $f\colon X \to Z$ and $f'\colon X' \to Z'$, respectively.

If $f'\colon X' \to Z'$ has a very exceptional divisor E, run an MMP on $(X', B' + \lambda E)$ over Z', where λ is a sufficiently small positive number. By [Bir12, Theorem 1.8], this MMP terminates with a model X'' that contracts E. If $X'' \to Z'$ also has a very exceptional divisor, we repeat this process. Since $\rho(X'/Z')$ strictly decreases each time, after finitely many steps we reach a contraction $g\colon Y \to Z'$ with no very exceptional divisor. Let B_Y, A_Y be the pushdowns of B', A' to Y. Then $K_Y + B_Y \sim_{\mathbb{R}} 0/Z'$ and (Y, B_Y) is ϵ-lc. Moreover, since $X' \dashrightarrow Y$ is an isomorphism over an open subset of Z', we have

$$\mathrm{vol}(A_Y|_{F_Y}) = \mathrm{vol}(A'|_{F'}) \leq v,$$

where F_Y is the general fiber of $g\colon Y \to Z'$.

Let Y' be the ample model of A_Y over Z', and let B'_Y, A'_Y be the pushdowns of B_Y, A_Y to Y'. Then $K_{Y'} + B'_Y \sim_{\mathbb{R}} 0/Z'$ and (Y', B'_Y) is ϵ-lc. Furthermore,

$$\mathrm{vol}(A'_Y|_{F'_Y}) = \mathrm{vol}(A_Y|_{F_Y}) \leq v,$$

where F'_Y is the general fiber of $g'\colon Y' \to Z'$. Therefore, $g'\colon (Y', B'_Y), A'_Y \to Z'$ is a (d, v, r', ϵ)-polarized log Calabi–Yau fibration.

Note that $g'\colon Y' \to Z'$ has no very exceptional divisor. Since B'_Y is vertical over Z' and Z' is $\mathbb{Q}$-factorial, B'_Y is of fiber type over Z'. Hence, there exists an effective $\mathbb{R}$-divisor C' on Z' such that $B'_Y = g'^*C'$. We then have

$$K_{Y'} \sim_{\mathbb{R}} g'^*(\mu^*N - C').$$

Note that $H' - (\mu^*N - C')$ may not be ample, but only pseudo-effective. Therefore, $g'\colon Y', A'_Y \to Z'$ is only a weak $(d, 0, v, r', \epsilon)$-polarized log Calabi–Yau fibration. By Theorem 1.3, (Y', g'^*H') is log bounded in codimension one. Since $H' - \mu^*H$ is ample, it follows that $(Y', \tilde{g}'^*H)$ is log bounded in codimension one, where $\tilde{g}'\colon Y' \to Z$. Finally, since $X \dashrightarrow Y'$ is a birational contraction, Remark 4.3 implies that $(X, \Delta + f^*H)$ is log bounded in codimension one. □

5. Fibrations whose general fibers have vanishing irregularity

In this section, we consider the boundedness of polarized log Calabi–Yau fibration $f : (X, B), A \to (Z, H)$ such that $\operatorname{Supp} R^1 f_*\mathcal{O}_X \subsetneq Z$.

The following lemma addresses the issue when the base of a polarized log Calabi–Yau fibration is not $\mathbb{Q}$-factorial.

Lemma 5.1. *Let $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Assume that*

- *$(X, B + M)$ is a generalized ϵ-lc projective pair of dimension d,*
- *H is a very ample divisor such that $H^d \leq r$, and*
- *$H - (K_X + B + M)$ is ample.*

Then there exists a positive integer r' depending only on d, r, ϵ such that

- *there exists a couple (X', Σ') such that $\pi : X' \to X$ is a $\mathbb{Q}$-factorialization,*

- *the irreducible components of Σ' generate $N^1(X'/X)$,*
- *H' is a very ample divisor on X' such that $H'^d \leq r'$, and*
- *$H' - \Sigma'$ and $H' - \pi^* H$ are ample.*

Proof. By [Bir24, Theorem 2.3], there exists a $\mathbb{Q}$-factorialization $\pi : X' \to X$ such that X' is in a bounded family. Therefore, there exists a bounded resolution W of X' such that $\rho(X') \leq \rho(W)$ is bounded from above, hence $\rho(X'/X)$ is also bounded from above. In the following we apply induction on $\rho(X'/X)$ to find Σ' and H' on X' which satisfy the conditions.

By the cone theorem [KM98, Theorem 3.7], we can decompose $\pi : X' \to X$ into a sequence of extremal contractions

$$X' = X_1 \to X_2 \to \cdots \to X_{l-1} \to X_l = X.$$

By [Bir24, Theorem 2.3], X_{l-1} is also in a bounded family. Hence there exists a positive integer r_{l-1} depending only on d, r, ϵ and a very ample divisor H_{l-1} on X_{l-1} such that $H_{l-1}^d \leq r_{l-1}$ and $H_{l-1} - \mu^* H$ is ample, where $\mu : X_{l-1} \to X$. If we write $K_{X_{l-1}} + B_{l-1} + M_{l-1} = \mu^*(K_X + B + M)$, then $H_{l-1} - (K_{X_{l-1}} + B_{l-1} + M_{l-1})$ is ample. Since $\rho(X'/X_{l-1}) < \rho(X'/X)$, by induction we conclude that

- there exists a couple (X', Σ') such that the irreducible components of Σ' generate $N^1(X'/X_{l-1})$,
- there exists a fixed $r' \in \mathbb{N}$ and a very ample divisor H' on X' such that $H'^d \leq r'$, and
- $H' - \Sigma'$ and $H' - \nu^* H_{l-1}$ are ample, where $\nu : X' \to X_{l-1}$. Therefore, $H' - \pi^* H$ is also ample.

Since H_{l-1} is ample over X and $\mu : X_{l-1} \to X$ is an extremal contraction, replacing Σ' with $\Sigma' \cup \mathrm{Supp}(\nu^* H_{l-1})$, H' with $2H'$, and r' with $2^d r'$, we conclude that the irreducible components of Σ' generate $N^1(X'/X)$. □

The following lemma bounds certain vertical divisors in a log bounded family.

Lemma 5.2. *Let $\epsilon, \delta \in \mathbb{R}^{>0}$ and $\Phi \subset [0,1] \cap \mathbb{Q}$ be a finite set. Assume that*

- *(X, B) is a projective $\mathbb{Q}$-factorial ϵ-lc pair which belongs to a bounded family,*
- *the coefficients of B are in Φ,*
- *$K_X + B$ is semi-ample and defines a contraction $f : X \to Z$,*
- *there is an adjunction formula $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$ such that $(Z, B_Z + M_Z)$ is a generalized δ-lc pair, and*
- *N is an integral divisor on X such that $N \sim_{\mathbb{Q}} 0/\eta_Z$, where η_Z is the generic point of Z.*

Then there exists an effective $\mathbb{Q}$-divisor D on X such that

- *D is vertical over Z,*
- *$N \sim_{\mathbb{Q}} D/Z$, and*
- *$(X, \mathrm{Supp}(B) \cup \mathrm{Supp}(D))$ is log bounded.*

Proof. *Step* 1. Since (X, B) belongs to a log bounded family of ϵ-lc pairs and the coefficients of B are in a finite set, by [BDCS24, Lemma 2.17] and its proof, the set of morphisms $f:(X,B)\to Z$ is bounded. Moreover, there exists a bounded $m\in\mathbb{N}$ such that $m(K_X+B)$ is base point free and satisfies

$$m(K_X+B)\sim mf^*(K_Z+B_Z+M_Z).$$

Let $H=m(K_Z+B_Z+M_Z)$, then H is a very ample divisor on Z and $H^{\dim Z}$ is bounded from above. By Lemma 5.1, we conclude that

- there exists a couple (Z',Σ') such that $\pi:Z'\to Z$ is a $\mathbb{Q}$-factorialization,
- the irreducible components of Σ' generate $N^1(Z'/Z)$,
- H' is a very ample divisor on Z' such that $H'^{\dim Z'}$ is bounded from above, and
- $H'-\Sigma'$ and $H'-\pi^*H$ are ample.

Therefore, replacing m with a bounded multiple, we can assume that $H-\Sigma$ is pseudo-effective, where $\Sigma=\pi_*\Sigma'$.

By [BDCS24, Proposition 3.6], there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "\mu"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "\pi"] & Z \end{tikzcd}$$

such that $\mu:X'\dashrightarrow X$ is isomorphic in codimension one.

Step 2. Let $N'=\mu^*N$. Since $N\sim_{\mathbb{Q}}0/\eta_Z$, we have

$$N'\sim_{\mathbb{Q}}0/\eta_{Z'},$$

where $\eta_{Z'}$ is the generic point of Z'. Because Z' is $\mathbb{Q}$-factorial, there exists a very exceptional$/Z'$ $\mathbb{Q}$-divisor L' on X' such that

$$N'\sim_{\mathbb{Q}}L'/Z'.$$

Since the irreducible components of Σ' generate $N^1(Z'/Z)$, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C' on Z' with

$$\operatorname{Supp}(C')\subseteq\operatorname{Supp}(\Sigma')\quad\text{and}\quad N'\sim_{\mathbb{Q}}L'+f'^*C'/Z.$$

Let $L=\mu_*L'$. Then L is very exceptional over Z, because L' is very exceptional over Z' and π,μ are isomorphic in codimension one. Hence we have

$$N\sim_{\mathbb{Q}}L+\mu_*f'^*C'/Z.$$

Since $f:X\to Z$ is a bounded morphism, by [BDCS24, Lemma 2.20], we conclude that $\operatorname{Supp}(L)$ is bounded.

Possibly enlarging Σ and replacing H',H with bounded multiples, we may assume that there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor T on Z such that

$$\operatorname{Supp}(T)\subset\Sigma\quad\text{and}\quad C'+\pi^*T,\ L+f^*T\text{ are effective.}$$

Replacing C' and L by $C'+\pi^*T$ and $L+f^*T$, we can assume that C' and L are effective $\mathbb{Q}$-divisors.

Since $H-\Sigma$ is pseudo-effective, we have that $m(K_X+B)-\mu_*f'^*\Sigma'$ is pseudo-effective. Therefore,

$$(X,\operatorname{Supp}(B)\cup\operatorname{Supp}(L)\cup\operatorname{Supp}(\mu_*f'^*C'))$$

is log bounded. Take

$$D = L + \mu_* f'^* C'$$

and we finish the proof.

□

We also need the following result on the finiteness of log canonical models when the boundary divisors vary in a polytope.

Lemma 5.3. *Assume that*

- *(X, B) is a projective $\mathbb{Q}$-factorial klt pair,*
- *$X \to Z$ is a contraction to a projective normal variety,*
- *$K_X + B \sim_{\mathbb{Q}} 0/Z$,*
- *L is an effective $\mathbb{Q}$-divisor on X which is big over Z,*
- *D_i's are effective $\mathbb{Q}$-divisors on X which are vertical over Z for $1 \leq i \leq k$, and*
- *V is the affine subspace generated by L and all D_i in the real vector space of divisors, and P is the polytope in V generated by L and all D_i.*

Then there exist finitely many rational maps $\pi_j : X \dashrightarrow Y_j/Z$ for $1 \leq j \leq l$ satisfying the following.

For each point $C \in P$, there exists $1 \leq j \leq l$ such that π_j gives the ample model of C over Z.

Proof. Let δ be a positive rational number such that $(X, B + \delta(L + \sum_{i=1}^k D_i))$ is klt. Let V' be the affine subspace generated by $B + \delta L$ and $B + \delta D_i$ in the real vector space of divisors, and let P' be the polytope in V' generated by $B + \delta L$ and $B + \delta D_i$. Since $K_X + B \sim_{\mathbb{Q}} 0/Z$, it suffices to prove the finiteness of log canonical models of $K_X + C'$ over Z for all $C' \in P'$. By [MZ23, Theorem 5.1] (see also [Kaw24, Theorem 2.10.3]), it is enough to show that (X, C') has a good minimal model over Z for every $C' \in P'$.

Let $C' \in P'$ and write

$$C' = B + a_0 L + \sum_{i=1}^{k} a_i D_i, \quad \text{where } \sum_{i=0}^{k} a_i = \delta.$$

If $a_0 > 0$, then $K_X + C'$ is big over Z, and by [BCHM10], (X, C') has a good minimal model over Z. If $a_0 = 0$, then since each D_i is vertical over Z, we have $K_X + C' \sim_{\mathbb{Q}} 0/\eta_Z$, where η_Z is the generic point of Z. By [Bir12, Theorem 1.4] or [HX13, Theorem 1.1], (X, C') has a good minimal model over Z. □

With the necessary preparations complete, we can now prove the main theorem of this section.

Proof of Theorem 1.6. By Theorem 3.1, there exists a couple (V, Θ), an effective integral divisor J on V and a positive integer l depending only on d, Φ, v, r, ϵ such that

- V is $\mathbb{Q}$-factorial,
- there is a contraction $h : V \to Z$,
- $V \dashrightarrow X/Z$ is isomorphic in codimension one,

- $(V, \Theta + \mathrm{Supp}(J))$ is bounded,
- Θ contains B_V and h^*H_Z, where B_V is the strict transform of B, and H_Z is a general element of $|6dH|$, and
- $J_X \equiv lA$ over the generic point of Z, where J_X is the strict transform of J on X.

Since $\mathrm{Supp}\, R^1 f_*\mathcal{O}_X \subsetneq Z$, by Grauert's theorem, we conclude that $h^1(X_g, \mathcal{O}_{X_g}) = 0$, where X_g is the general fiber of $f : X \to Z$. Then $J_X \sim_{\mathbb{Q}} lA/\eta_Z$, where η_Z is the generic point of Z. Since $V \dashrightarrow X/Z$ is isomorphic in codimension one, we have $J \sim_{\mathbb{Q}} lA_V/\eta_Z$, where A_V is the strict transform of A on V. By Lemma 5.2, there exists a log bounded pair $(V, B_V + J + \sum D_i)$ and rational numbers $a_i \geq 0$ for $1 \leq i \leq k$ such that

$$J + \sum a_i D_i \sim_{\mathbb{Q}} lA_V/Z.$$

By log boundedness, we may assume there exists a family $(\mathcal{V}, \mathcal{B}_{\mathcal{V}}) \to \mathcal{S}$ and divisors $\mathcal{J}$ and $\mathcal{D}_i$ on $\mathcal{V}$ such that there exists a point $s \in \mathcal{S}$ with

$$(V, B_V) \simeq (\mathcal{V}_s, \mathcal{B}_{\mathcal{V}_s}), \quad \mathcal{J}_s \simeq J, \quad \text{and} \quad \mathcal{D}_{i,s} \simeq D_i.$$

By [HX15, Proposition 2.4], after passing to a stratification of $\mathcal{S}$, we can assume that $K_{\mathcal{V}} + \mathcal{B}_{\mathcal{V}}$ is $\mathbb{Q}$-Cartier and klt. By [HJ22, Lemma 2.8], there is a fibration $g : \mathcal{V} \to \mathcal{Z}$ over $\mathcal{S}$ such that $\mathcal{V}_s \to \mathcal{Z}_s$ is isomorphic to $V \to Z$. By Remark 3.9, we may assume that $\mathcal{J}$ is big over $\mathcal{Z}$. Since each D_i is vertical over Z, $\mathcal{D}_i$ is vertical over $\mathcal{Z}$ as well.

Let $\mathcal{H}$ be a Cartier divisor on $\mathcal{Z}$ which is ample over $\mathcal{S}$, and let $\mathcal{G} \in |6n\mathcal{H}|$ be a general member, where $n = \dim \mathcal{X}$. By the boundedness of the length of extremal rays, the log canonical model of

$$(\mathcal{V}, \mathcal{B}_{\mathcal{V}} + \mu(\mathcal{J} + \sum_{i=1}^{k} a_i \mathcal{D}_i))$$

over $\mathcal{Z}$ is also the log canonical model of

$$(\mathcal{V}, \mathcal{B}_{\mathcal{V}} + \mu(\mathcal{J} + \sum_{i=1}^{k} a_i \mathcal{D}_i) + \tfrac{1}{2}\mathcal{G})$$

over $\mathcal{S}$, where μ is sufficiently small. After passing to a stratification and applying [HMX18, Theorem 1.2] to a fiberwise log resolution of

$$(\mathcal{V}, \mathcal{B}_{\mathcal{V}} + \mu(\mathcal{J} + \sum_{i=1}^{k} \mathcal{D}_i) + \tfrac{1}{2}\mathcal{G})$$

over $\mathcal{S}$, we may assume that it admits a relative log canonical model over $\mathcal{S}$ which induces log canonical models fiberwise.

By Lemma 5.3, there exist finitely many rational maps

$$\mathcal{V} \dashrightarrow \mathcal{Y}_j/\mathcal{Z}$$

such that for every $(a_1, a_2, \ldots, a_k)$, there exists j for which $\mathcal{Y}_j$ is the log canonical model of

$$(\mathcal{V}, \mathcal{B}_{\mathcal{V}} + \mu(\mathcal{J} + \sum_{i=1}^{k} a_i \mathcal{D}_i))$$

over $\mathcal{Z}$. Then $\mathcal{Y}_{j,s}$ is the log canonical model of

$$(V, B_V + \mu(J + \sum_{i=1}^{k} a_i D_i))$$

over Z, hence also of $(V, B_V + \mu l A_V)$ over Z. Since $(X, B + \mu l A)$ is the log canonical model of $(V, B_V + \mu l A_V)$ over Z, it follows that $\mathcal{Y}_{j,s} \simeq X$. Therefore, we conclude that $(X, B + f^*H)$ is log bounded. $\square$

6. Stable minimal models and fibered Calabi-Yau varieties

In this section, we apply our boundedness results on polarized log Calabi-Yau fibrations to stable minimal models and fibered Calabi-Yau varieties.

Definition 6.1 ([Bir21a, Definition 1.1])**.** Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Let $\mathcal{F}_{gklt}(d, \Phi, u)$ be the set of projective generalized pairs $(X, B + M)$ with data $X' \to X$ and M' such that

- $(X, B + M)$ is generalized klt of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M'_i$ where M'_i is nef Cartier and $\mu_i \in \Phi$ for any i,
- $K_X + B + M$ is ample, and
- $\mathrm{vol}(K_X + B + M) = u$.

Proof of Corollary 1.7. By the proof of Lemma 3.2, we may assume that A is an effective integral divisor and $\mathrm{vol}(A|_F) = v$ is fixed. By [Bir21a, Lemma 8.2], there exists a positive number ϵ depending only on d, u, v, Φ such that (X, B) is ϵ-lc. By [Bir21a, Lemma 7.4], there exists a positive integer p depending only on d, u, Φ such that we can write an adjunction formula

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

where $pM_{Z'}$ is Cartier on some high resolution $Z' \to Z$.

By [HMX14, Theorem 1.1], the coefficients of B_Z are in a fixed DCC set Ψ depending only on d, Φ. Replacing Ψ with $\Psi \cup \{\frac{1}{p}\}$, we conclude that $(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(d', \Psi, u)$, where $d' = \dim Z$. By [Bir21a, Theorem 1.4], $(Z, B_Z + M_Z)$ belongs to a bounded family. Furthermore, by the remark of [Bir22, Theorem 4.3], there exists a fixed positive integer l such that $H := l(K_Z + B_Z + M_Z)$ is very ample. Then $f : (X, B), A \to (Z, H)$ is a $(d, v, l^{d'}u, \epsilon)$-polarized log Calabi-Yau fibration. Therefore, the corollary follows from Theorem 1.4 and Theorem 1.6. $\square$

Proof of Corollary 1.8. By [Zhu25, Lemma 2.11] (see also [Bir23b, Theorem 1.8]), there exists an adjunction formula

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

such that $(Z, B_Z + M_Z)$ is generalized δ-lc for some $\delta \in \mathbb{R}^{>0}$ depending only on d, ϵ, v. Since Z is rationally connected, by [Bir23b, Theorem 1.7], there exists a projective variety Z' satisfying that

- $Z' \dashrightarrow Z$ is isomorphic in codimension one, and

- there is a fixed positive integer r and a very ample divisor H' on Z' such that ${H'}^{\dim Z'} \leq r$.

By [BDCS24, Proposition 3.6, Proposition 3.7], there exists an ϵ-lc pair (X', B') which is isomorphic in codimension one to (X, B) and a contraction $f' : X' \to Z'$. Let A' be the strict transform of A on X', then $\mathrm{vol}(A'|_{F'}) = \mathrm{vol}(A|_F) \leq v$, where F' is the general fiber of $f' : X' \to Z'$. Let X'' be the ample model of A' over Z' and B'', A'' be the pushdown of B', A' on X''. Then we conclude that $f'' : (X'', B''), A'' \to Z'$ is a (d, v, r, ϵ)-polarized log Calabi-Yau fibration. Therefore, X'' is bounded in codimension one by Theorem 1.4. Since $X \dashrightarrow X''$ is a birational contraction, by [BDCS24, Corollary 2.13], X is bounded in codimension one.

□

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