BOUNDEDNESS OF KLT GOOD MINIMAL MODELS

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ABSTRACT. For good minimal models with klt singularities, polarized by Weil divisors that are relatively nef and big over the bases of the Iitaka fibration, we show that, after fixing appropriate numerical invariants, they form a bounded family. As an application, we construct separated coarse moduli spaces for klt good minimal models polarized by line bundles.

CONTENTS

1. INTRODUCTION

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of characteristic zero.

The central problem in birational geometry is the classification of algebraic varieties. According to the standard minimal model conjecture and the abundance conjecture, any

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variety Y with mild singularities is birational to a variety X such that either X admits a Mori–Fano fibration $X \to Z$ or X is a good minimal model, that is, K_X is semiample. Therefore, canonically polarized varieties, Fano varieties, Calabi–Yau varieties, and their iterated fibrations play a central role in birational geometry.

One of the main problems in the classification of algebraic varieties is whether there are only finitely many families of such objects after fixing certain numerical invariants; in other words, whether they form a bounded family. Establishing the boundedness of a given class of varieties is a natural first step toward constructing the corresponding moduli space.

For canonically polarized varieties, boundedness was established in [HMX13, HMX14, HMX18], while for Fano varieties it was proved in [Bir19, Bir21b]. However, for Calabi–Yau varieties there is no natural choice of polarization, and in general they are not bounded in the category of algebraic varieties. For example, projective K3 surfaces and abelian varieties of any fixed dimension are not bounded. Nevertheless, there has been recent progress toward the (birational) boundedness of fibered Calabi–Yau varieties and rationally connected Calabi–Yau varieties; see [Bir23b, BDCS24, FHS24, JJZ25, EFG+25].

When studying the moduli of Calabi–Yau varieties, one typically fixes a polarization despite its non-uniqueness. Recently, Birkar established the following boundedness results for polarized Calabi–Yau varieties, which are crucial for constructing moduli spaces of such polarized varieties.

Theorem 1.1 ([Bir23a])**.** *Let $d \in \mathbb{N}$, $u \in \mathbb{Q}^{>0}$, and $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Consider Calabi–Yau pairs (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X. Then the following hold:*

(1) (klt case) If
- *(X, B) is a klt pair of dimension d,*
- *the coefficients of B are contained in Φ,*
- *A is a nef and big divisor on X such that $\operatorname{vol}(A) = u$,*

then the set of such (X, B) forms a bounded family. If in addition $A \geq 0$, then the set of such $(X, B + A)$ also forms a bounded family.

(2) (slc case) If
- *(X, B) is an slc pair of dimension d,*
- *the coefficients of B are contained in Φ,*
- *A is an ample divisor on X such that $\operatorname{vol}(A) = u$,*
- *$A \geq 0$ does not contain any non-klt center of (X, B),*

then the set of such $(X, B + A)$ forms a bounded family.

Since the boundedness results for good minimal models of maximal and minimal Kodaira dimension have been established, it remains to investigate good minimal models with intermediate Kodaira dimension. Recently, Birkar proved the following boundedness result for slc good minimal models polarized by effective Weil divisors that are relatively ample over the bases of the Iitaka fibration, and he constructed their projective coarse moduli spaces.

Theorem 1.2 ([Bir22]). *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Consider pairs (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:*

- *(X, B) is an slc pair of dimension d,*
- *the coefficients of B are contained in Φ,*
- *$K_X + B$ is semiample, defining a contraction $f : X \to Z$,*
- *A is a divisor on X that is ample over Z,*
- *$\operatorname{vol}(A|_F) \in \Gamma$, where F is any general fiber of $f : X \to Z$ over any irreducible component of Z,*
- *$(K_X + B + tA)^d = \sigma(t)$, and*
- *$A \geq 0$ does not contain any non-klt center of (X, B),*

then the set of such $(X, B + A)$ forms a bounded family.

Theorem 1.2 can be regarded as a relative version of Theorem 1.1(2). One naturally wonders whether a relative version of Theorem 1.1(1) exists; that is, for klt pairs (X, B), the polarization A need not be an effective divisor. This paper addresses this question and uses it to construct the moduli space of klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of the Iitaka fibration (see Appendix A).

Theorem 1.3. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. Let $\mathcal{G}_{klt}(d, \Phi, \Gamma, \sigma)$ be the set of pairs (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:*

- *(X, B) is a klt pair of dimension d,*
- *the coefficients of B are contained in Φ,*
- *$K_X + B$ is semiample, defining a contraction $f : X \to Z$,*
- *A is a divisor on X that is nef and big over Z,*
- *$\operatorname{vol}(A|_F) \in \Gamma$, where F is the general fiber of $f : X \to Z$, and*
- *$(K_X + B + tA)^d = \sigma(t)$.*

Then the set of such (X, B) forms a bounded family. If in addition $A \geq 0$, then the set of such $(X, B + A)$ also forms a bounded family.

Recently, there have been some other related results on the (birational) boundedness of klt good minimal models, see [MST20, FS20, FHS24, Li24, HH25, Jia25, Zhu25, JJZ25].

Remark 1.4. In Theorems 1.2 and 1.3, conjecturally, the condition on $\operatorname{vol}(A|_F)$ can be removed; this is related to the effective b-semiampleness conjecture [PS09, Conjecture 7.13], and related discussions can be found in [Bir21a, Bir22]. Note that in Theorem 1.3 the condition that (X, B) is klt cannot be replaced by lc [HJ25, §4.2]. While in Theorem

1.2 the condition on fixing $\sigma(t)$ cannot be replaced by fixing only $\mathrm{Ivol}(K_X+B)$ [BH22, §1], one expects that in the klt case of Theorem 1.3 such a replacement may be possible [JJZ25]; however, in this case the polarization A cannot be controlled as Theorem C.

Description of the proof.

Theorem A (Boundedness of nef threshold)**.** *Under the same assumptions as Theorem 1.3, there exists a positive rational number τ, depending only on (d,Φ,Γ,σ), such that K_X+B+tA is nef and big for all $0<t<\tau$.*

Theorem B (Boundedness of pseudo-effective thereshold)**.** *Under the same assumptions as Theorem 1.3, there exists a positive rational number λ, depending only on (d,Φ,Γ,σ), such that K_X+B+tA is big for all $0<t<\lambda$.*

Theorem C. *Under the same assumptions as Theorem 1.3, there exist a natural number r depending only on (d,Φ,Γ,σ) and a very ample divisor H on X such that*

$$H^d\le r,\quad (K_X+B)\cdot H^{d-1}\le r,\quad \text{and}\quad H-A \text{ is pseudo-effective.}$$

In particular, by Lemma 2.10, the set of such (X,B) forms a bounded family. If in addition $A\ge 0$, then the set of such $(X,B+A)$ also forms a bounded family.

It is clear that Theorem A implies Theorem B, while Theorem C yields Theorem 1.3. The proof of Theorem A, B and C proceeds by induction on the dimension of X:

- Theorem B_d + Theorem C_{d-1} $\Longrightarrow$ Theorem A_d; cf. (3.5).
- Theorem A_{d-1} $\Longrightarrow$ Theorem B_d; cf. (3.6).
- Theorem A_d $\Longrightarrow$ Theorem C_d; cf. (3.9).

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2. Preliminaries

Notations and conventions. We collect some notations and conventions used in this paper.

(1) A projective morphism $f: X\to Z$ of schemes is called a *contraction* if $f_*\mathcal{O}_X=\mathcal{O}_Z$ (f is not necessarily birational). In particular, f is surjective with connected fibers.
(2) Suppose that X is a normal variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. We say that D_1 and D_2 are $\mathbb{Q}$-linear equivalent, denoted by $D_1\sim_{\mathbb{Q}} D_2$, if there exists $m\in\mathbb{Z}_{>0}$ such that mD_1 and mD_2 are Cartier and $mD_1\sim mD_2$. Moreover, fixed $l\in\mathbb{Z}_{>0}$, the notation $D_1\sim_l D_2$ means that $lD_1\sim lD_2$.

(3) Let $f : X \to Z$ be a morphism between normal varieties, and let M and L be $\mathbb{Q}$-Cartier divisors on X. We say $M \sim L/Z$ (resp. $M \sim_{\mathbb{Q}} L/Z$) if there is a Cartier (resp. $\mathbb{Q}$-Cartier) divisor N on Z such that $M - L \sim f^*N$ (resp. $M - L \sim_{\mathbb{Q}} f^*N$).

(4) We say that a set $\Phi \subset \mathbb{Q}$ satisfies the *descending chain condition* (DCC, for short) if Φ does not contain any strictly decreasing infinite sequence. Similarly, we say that a set $\Phi \subset \mathbb{Q}$ satisfies the *ascending chain condition* (ACC, for short) if Φ does not contain any strictly increasing infinite sequence.

(5) Let X be a normal variety. A *b-divisor* over X is a collection of $\mathbb{Q}$-divisors M_Y on Y for each birational contraction $Y \to X$ from a normal variety that are compatible with respect to pushdown, that is, if $Y' \to X$ is another birational contraction and $\psi : Y' \dashrightarrow Y$ is a morphism, then $\psi_* M_{Y'} = M_Y$.

A b-divisor is *b-$\mathbb{Q}$-Cartier* if there is a birational contraction $Y \to X$ such that M_Y is $\mathbb{Q}$-Cartier and $M_{Y'}$ is the pullback of M_Y on Y' for any birational contraction $Y' \to Y$. In this case, we say that the b-divisor descends to Y, and it is represented by M_Y.

2.1. **(Generalised) pairs and singularities.**

Definition 2.1 (Generalised pairs)**.** A *generalised pair* (X, B, M) consists of:

- a normal projective variety X,
- an effective $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a b-$\mathbb{Q}$-Cartier b-divisor M over X, represented by a projective birational morphism $f\colon X' \to X$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M' on X' such that:
  (1) M' is nef, and
  (2) $K_X + B + M$ is $\mathbb{Q}$-Cartier, where $M := f_* M'$.

We say that $(X, B + M)$ is a generalised pair with nef part M'.

Let D be a prime divisor over X. Replace X' with a log resolution of (X, B) such that D is a prime divisor on X'. We can write

$$K_{X'} + B' + M' = \pi^*(K_X + B + M).$$

Then the *generalised log discrepancy* of D is defined as

$$a(D, X, B, M) = 1 - \operatorname{mult}_D B'.$$

We say that $(X, B + M)$ is *generalised klt* (resp. *generalised lc*, *generalised ϵ-lc*) if $a(D, X, B, M) > 0$ (resp. $a(D, X, B, M) \geq 0$, $a(D, X, B, M) \geq \epsilon$) for every prime divisor D over X.

If $M = 0$, then we say (X, B) is a *pair*, and we define its singularities similarly.

Definition 2.2 (Lc threshold of $\mathbb{Q}$-linear systems)**.** Let (X, B) be an lc pair. The *lc threshold* of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L \geq 0$ with respect to (X, B) is defined as

$$\operatorname{lct}(X, B, L) := \sup\{\, t \in \mathbb{R} \mid (X, B + tL) \text{ is lc}\,\}.$$

Now let H be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The $\mathbb{Q}$-linear system of H is

$$|H|_{\mathbb{Q}} := \{\, L \geq 0 \mid L \sim_{\mathbb{Q}} H \,\}.$$

We then define the *lc threshold* of $|H|_{\mathbb{Q}}$ with respect to (X, B) (also called the *global lc threshold* or *α-invariant*) as

$$\operatorname{lct}(X, B, |H|_{\mathbb{Q}}) := \inf\{\, \operatorname{lct}(X, B, L) \mid L \in |H|_{\mathbb{Q}} \,\},$$

which is equivalent to

$$\sup\{\, t \in \mathbb{R} \mid (X, B + tL) \text{ is lc for every } L \in |H|_{\mathbb{Q}} \,\}.$$

Definition 2.3 (Good minimal models)**.** Let $\phi : X \dashrightarrow X^m$ be a projective birational contraction between normal projective varieties. Suppose that (X, B) and (X^m, B^m) are lc pairs, where $B^m = \phi_* B$. If

$$a(E, X, B) > a(E, X^m, B^m)$$

for all prime ϕ-exceptional divisors $E \subset X$, X^m is $\mathbb{Q}$-factorial, and $K_{X^m} + B^m$ is nef, then we say that $\phi : X \dashrightarrow X^m$ is a *minimal model* of (X, B). If, further, $K_{X^m} + B^m$ is semiample, then the minimal model $\phi : X \dashrightarrow X^m$ is called a *good minimal model.*

2.2. **Canonical bundle formula.** An *lc-trivial fibration* $f\colon (X, B) \to Z$ consists of a projective surjective morphism $f\colon X \to Z$ with connected fibers between normal varieties such that

- (X, B) is an lc pair, and
- there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L_Z on Z such that

$$K_X + B \sim_{\mathbb{Q}} f^* L_Z.$$

Let $f : (X, B) \to Z$ be an lc-trivial fibration with $\dim Z > 0$. Fix a prime divisor D on Z, and let t_D be the lc threshold of f^*D with respect to (X, B) over the generic point of D. Define

$$B_Z := \sum (1 - t_D) D,$$

where the sum runs over all prime divisors on Z. Set

$$M_Z := L_Z - (K_Z + B_Z),$$

so that

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z).$$

We call B_Z the *discriminant divisor* and M_Z the *moduli divisor* of adjunction. Note that B_Z is uniquely determined, whereas M_Z is determined only up to $\mathbb{Q}$-linear equivalence.

Now let $\phi : X' \to X$ and $\psi : Z' \to Z$ be birational morphisms from normal projective varieties, and assume that the induced map $f' : X' \dashrightarrow Z'$ is a morphism. Let $K_{X'} + B'$ be the pullback of $K_X + B$ to X'. Similarly, we can define a discriminant divisor $B_{Z'}$ on

Z' and, setting $L_{Z'} := \psi^* L_Z$, obtain a moduli divisor $M_{Z'}$ such that

$$K_{X'} + B' \sim_{\mathbb{Q}} f'^*(K_{Z'} + B_{Z'} + M_{Z'}),$$

with $B_Z = \psi_* B_{Z'}$ and $M_Z = \psi_* M_{Z'}$.

In particular, the lc-trivial fibration $f : (X, B) \to Z$ induces b-$\mathbb{Q}$-divisors B and M on Z, called the *discriminant* and *moduli* b-divisors, respectively.

Theorem 2.4 ([Amb04, FG14]). *With the above notation and assumptions. If $Z' \to Z$ is a high resolution, then $M_{Z'}$ is nef and for any birational morphism $Z'' \to Z'$ from a normal projective variety, $M_{Z''}$ is the pullback of $M_{Z'}$. In particular, we can view $(Z, B_Z + M_Z)$ as a generalized pair with nef part $M_{Z'}$.*

2.3. **Volume of divisors.** We recall the definition of various types of volumes for divisors. In this paper, we mainly consider $\mathbb{Q}$-divisors. However, for the proof of Proposition 3.6, we need to deal with $\mathbb{R}$-divisors.

Definition 2.5 (Volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. The *volume* of D is

$$\operatorname{vol}(X, D) = \limsup_{m\to\infty} \frac{d!h^0(X, \mathcal{O}_X(\lfloor mD\rfloor))}{m^d}.$$

Definition 2.6 (Iitaka volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{R}$-divisor on X. The *Iitaka volume* of D, denoted by $\operatorname{Ivol}(D)$, is defined as

$$\operatorname{Ivol}(D) := \begin{cases} \limsup\limits_{m\to\infty} \dfrac{\kappa(D)!\, h^0(X, \mathcal{O}_X(\lfloor mD\rfloor))}{m^{\kappa(D)}} & \text{if } \kappa(D) \geq 0, \\ 0 & \text{if } \kappa(D) = -\infty, \end{cases}$$

where $\kappa(D)$ denotes the Iitaka dimension of D. By convention, when $\kappa(D) = 0$ we set $\kappa(D)! = 1$, so in this case $\operatorname{Ivol}(D) = 1$.

If $f\colon X \to Z$ is a contraction and $D \sim_{\mathbb{Q}} f^*D_Z$ for some big $\mathbb{Q}$-divisor D_Z on Z, then $\operatorname{Ivol}(D) = \operatorname{vol}(D_Z)$.

Definition 2.7 (Restricted volumes)**.** Let X be a normal irreducible projective variety of dimension d, and let D be an $\mathbb{Q}$-divisor on X. Let $S \subset X$ be a normal irreducible subvariety of dimension n. Suppose that S is not contained in the augmented base locus $\mathbf{B}_+(D)$. Then the *restricted volume of D along S* is

$$\operatorname{vol}_{X|S}(D) = \limsup_{m\to\infty} \frac{n!(\dim \operatorname{Im}(H^0(X, \mathcal{O}_X(\lfloor mD\rfloor)) \to H^0(S, \mathcal{O}_S(\lfloor mD|_S\rfloor))))}{m^n}.$$

For the precise definition of the *augmented base locus* $\mathbf{B}_+(D)$, see [ELM+06]. In this paper, we only use the fact that $\mathbf{B}_+(D)$ is a Zariski-closed subset of X such that $\mathbf{B}_+(D) \subsetneq X$ if and only if D is big. The restricted volume $\operatorname{vol}_{X|S}(D)$ measures asymptotically the number of sections of the restriction $\mathcal{O}_S(\lfloor mD|_S\rfloor)$ that can be lifted to X. If D is ample, then the restriction maps are eventually surjective, and hence

$$\mathrm{vol}_{X|S}(D) = \mathrm{vol}(D|_S).$$

In general, it can happen that $\mathrm{vol}_{X|S}(D) < \mathrm{vol}(D|_S)$.

Theorem 2.8 ([LM09, Corollary 4.27]). *Let X be an irreducible projective variety of dimension d, and let $S \subset X$ be an irreducible (and reduced) Cartier divisor on X. Suppose that D is a big $\mathbb{R}$-divisor such that $S \not\subseteq \mathbf{B}_+(D)$. Then the function $t \mapsto \mathrm{vol}(D+tS)$ is differentiable at $t=0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{vol}(D+tS))\Big|_{t=0} = d\,\mathrm{vol}_{X|S}(D).$$

By [LM09, Remark 4.29], volume function has continuous partials in all directions at any point $D \in \mathrm{Big}(X)$, i.e., the function $\mathrm{vol} : \mathrm{Big}(X) \to \mathbb{R}$ is $\mathcal{C}^1$.

2.4. **Bounded family of pairs.**

Definition 2.9 (Bounded families of couples and pairs). A *couple* consists of a projective normal variety X and a reduced divisor D on X. We call (X, D) a couple rather than a pair because $K_X + D$ is not assumed to be $\mathbb{Q}$-Cartier and (X, D) is not assumed to have good singularities.

Two couples (X, D) and (X', D') are said to be *isomorphic* if there exists an isomorphism $X \to X'$ that maps D onto D'.

Let $\mathcal{P}$ be a set of couples. We say that $\mathcal{P}$ is *bounded* if the following conditions hold:

- There exist finitely many projective morphisms $V^i \to T^i$ of varieties,
- C^i is a reduced divisor on V^i, and
- for each $(X, D) \in \mathcal{P}$, there exist an index i, a closed point $t \in T^i$, and an isomorphism $\phi : V^i_t \to X$ such that (V^i_t, C^i_t) is a couple and $\phi_* C^i_t \geq D$.

A set of projective pairs (X, B) is said to be *bounded* if the set of couples $(X, \mathrm{Supp}B)$ forms a bounded family.

Boundedness for couples is equivalent to the following criterion.

Lemma 2.10 ([Bir19, Lemma 2.20]). *Let $d, r \in \mathbb{N}$. Assume $\mathcal{P}$ is a set of couples (X, D) where X is of dimension d and there is a very ample divisor H on X with $H^d \leq r$ and $H^{d-1} \cdot D \leq r$. Then $\mathcal{P}$ is bounded.*

Lemma 2.11 ([Bir22, Lemma 4.6]). *Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set. Then there exists $l \in \mathbb{N}$ satisfying the following. Assume*

- *X is a normal projective variety of dimension d,*
- *H is a very ample divisor,*
- *B is a divisor with coefficients in Φ, and*
- *$H^d \leq r$ and $B \cdot H^{d-1} \leq r$.*

Then $lH - B$ is pseudo-effective.

The following theorem is one of the main ingredients in the proof of Theorem A. We emphasise that it imposes no restriction on the coefficients of B and M.

Theorem 2.12 ([Bir21b, Theorem 1.8]). *Let $d, r \in \mathbb{N}$ and $\epsilon \in \mathbb{Q}^{>0}$. Then there is a positive rational number t depending only on d, r, ϵ, satisfying the following. Assume*

- *(X, B) is projective ϵ-lc of dimension d,*
- *H is a very ample divisor on X with $H^d \leq r$,*
- *$H - B$ is pseudo-effective, and*
- *$M \geq 0$ is an $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $H - M$ pseudo-effective.*

Then

$$\operatorname{lct}(X, B, |M|_{\mathbb{Q}}) \geq \operatorname{lct}(X, B, |H|_{\mathbb{Q}}) \geq t.$$

We will use the following boundedness result for polarized nef pairs to deduce Theorem C from Theorem A.

Theorem 2.13 ([Bir23a, Theorem 1.5]). *Let $d \in \mathbb{N}$, $\delta, v \in \mathbb{Q}^{>0}$. Consider pairs (X, B) and nef and big Weil divisors N on X such that*

- *(X, B) is projective ϵ-lc of dimension d,*
- *the coefficients of B are in $\{0\} \cup [\delta, \infty)$,*
- *$K_X + B$ is nef,*
- *$\operatorname{vol}(K_X + B + N) \leq v$.*

Then the set of such (X, B) forms a bounded family. If in addition $N \geq 0$, then the set of such $(X, B + N)$ forms a bounded family.

3. Boundedness

In this section, we prove Theorem 1.3.

3.1. **Boundedness of generalised pairs on bases of fibrations.**

In this subsection, we consider the set of good minimal models whose general fibers of the Iitaka fibration belong to a bounded family and whose Iitaka volume is fixed.

Definition 3.1. Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Let $\mathcal{G}_{klt}(d, \Phi, u, v)$ be the set of (X, B) and $\mathbb{Q}$-Cartier Weil divisors A on X satisfying the following conditions:

- (X, B) is a klt pair of dimension d,
- the coefficients of B are contained in Φ,
- $K_X + B$ is semiample, defining a contraction $f : X \to Z$,
- A is a divisor on X that is nef and big over Z,
- $\operatorname{vol}(A|_F) = u$, where F is the general fiber of $f : X \to Z$, and
- $\operatorname{Ivol}(K_X + B) = v$.

Since $K_X + B$ is semiample, there exists a contraction $f : X \to Z$ onto a normal variety Z. By the canonical bundle formula in §2.2, we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),$$

and we may then regard $(Z, B_Z + M_Z)$ as a generalised pair with ample $K_Z + B_Z + M_Z$, that is, a *generalised log canonical (lc) model.*

Lemma 3.2. *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exist $p, q \in \mathbb{N}$ depending only on (d, Φ, u), and $l \in \mathbb{N}$, $\epsilon \in \mathbb{Q}^{>0}$ depending only on (d, Φ, u, v), such that for any*

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, u, v),$$

the following hold:

(1) We have an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where $pM_{Z'}$ is Cartier on some high resolution $Z' \to Z$.

(2) The pair (X, B) is ϵ-lc, and lB is a Weil divisor.

Proof. Replacing X with the ample model of A over Z, we may assume that A is ample over Z. Applying [Bir23a, Corollary 1.4] to $(F, B|_F)$ and $A|_F$, there exists $m \in \mathbb{N}$, depending only on d and Φ, such that $H^0(F, \mathcal{O}_X(mA|_F)) \neq 0$. Hence $mA \sim G$ for some Weil divisor G. Replacing A with the horizontal part of G, we may assume that A is effective.

Applying [Bir21a, Lemma 7.4] yields integers p, q satisfying (1). Moreover, by [Bir21a, Lemma 8.2], the set of log discrepancies

$$\{a(D, X, B) \leq 1 \mid D \text{ a prime divisor over } X\}$$

is finite, and hence (2) holds. Note that the proof of [Bir21a, Lemma 8.2] uses A only in the relative sense over Z. □

Definition 3.3 ([Bir21a, Definition 1.1])**.** Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $v \in \mathbb{Q}^{>0}$. Let $\mathcal{F}_{gklt}(d, \Phi, v)$ be the set of projective generalised pairs $(X, B + M)$ with nef part M' such that

- $(X, B + M)$ is generalised klt of dimension d,
- the coefficients of B are in Φ,
- $M' = \sum \mu_i M'_i$ where $\mu_i \in \Phi$ and M'_i are nef Cartier, and
- $K_X + B + M$ is ample with volume $\mathrm{vol}(K_X + B + M) = v$.

Now we can prove the boundedness of bases of Iitaka fibrations with their induced generalised pair structure under natural assumptions.

Theorem 3.4 ([Bir21a]). *Let $d \in \mathbb{N}$, $\Phi \subset \mathbb{Q}^{\geq 0}$ be a DCC set, and $u, v \in \mathbb{Q}^{>0}$. Then there exists $l \in \mathbb{N}$ depending only on d, Φ, u, v such that for any*

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, u, v),$$

we can write an adjunction formula

$$K_X + B \sim_l f^*(K_Z + B_Z + M_Z)$$

such that the corresponding set of generalized pairs $(Z, B_Z + M_Z)$ forms a bounded family. Moreover, $l(K_Z + B_Z + M_Z)$ is very ample.

Proof. By Lemma 3.2 (1), there exist $p, q \in \mathbb{N}$ depending only on d, Φ, u such that we can write an adjunction formula

$$K_X + B \sim_q f^*(K_Z + B_Z + M_Z),$$

where $pM_{Z'}$ is Cartier on some higher resolution $Z' \to Z$.

By definition of the discriminant part of the canonical bundle formula and the ACC for lc thresholds [HMX14, Theorem 1.1], we see that the coefficients of B_Z belong to a DCC subset of $\mathbb{Q}^{>0}$ depending only on d and Φ, which we denote by Ψ. Moreover, $(Z, B_Z + M_Z)$ is generalised klt pair and

$$\text{Ivol}(K_X + B) = \text{vol}(K_Z + B_Z + M_Z) = v.$$

Adding $\frac{1}{p}$, we can assume $\frac{1}{p} \in \Psi$, we see that

$$(Z, B_Z + M_Z) \in \mathcal{F}_{gklt}(\dim Z, \Psi, v).$$

In the proof of [Bir21a, Theorem 1.4], a divisor Θ is constructed such that

$$l(K_X + \Theta) \sim l(1 + t)(K_X + B + M)$$

is ample, (X, Θ) is ϵ-lc, and the coefficients of Θ belong to a fixed DCC set Ψ'. Here $l \in \mathbb{N}$, $t, \epsilon \in \mathbb{Q}^{>0}$, and $\Psi' \subset \mathbb{Q}^{>0}$ depend only on (d, Φ, u, v). Moreover, (X, Θ) is log birationally bounded. By [HMX14, Theorem 1.6], (X, Θ) belongs to a bounded family. Thus, we may replace l so that both $l(K_X + \Theta)$ and $l(K_X + B + M)$ are very ample. Hence, the set of generalised pairs $(Z, B_Z + M_Z)$ forms a bounded family. Replacing q, l with ql, we conclude the proof. □

3.2. **Boundedness of nef threshold.** In this subsection, we show that the nef threshold of $K_X + B$ with respect to A is bounded for all

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \Gamma, \sigma).$$

We follow the argument of [Bir22, Theorem 4.1] with some modifications. The main difference is that, since A may not be an effective divisor in our situation, we cannot directly apply the cone theorem to bound the nef threshold.

Therefore, we first assume that $K_X + B + \lambda A$ is big for some natural mumber α and rational number $\lambda \in [0,1]$. We can then replace $K_X + B + \lambda A$ by an effective $\mathbb{Q}$-divisor E, but this loses control of the coefficients of E. For this reason, we require a stronger boundedness result on singularities in Theorem 2.12 compared to [Bir22, Lemma 4.7]. To make the induction argument go through, we also need to show that $H - A$ is pseudo-effective, as in Theorem C.

Proposition 3.5. *Theorem B_d and Theorem C_{d-1} imply Theorem A_d.*

Proof. We proceed by induction on the dimension of X.

Step 1. For each

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \Gamma, \sigma),$$

we have

$$\sigma(t) = (K_X + B + tA)^d = \sum_{i=0}^{d} \binom{d}{i} (K_X + B)^{d-i} \cdot A^i t^i,$$

so the intersection numbers $(K_X+B)^{d-i} \cdot A^i$ are determined by d and σ for each $0 \le i \le d$. In particular, for a general fiber F of $X \to Z$,

$$\mathrm{Ivol}(K_X + B) \cdot \mathrm{vol}(A|_F) = (K_X + B)^{\dim Z} \cdot A^{d-\dim Z}$$

is a fixed number depending only on d and σ. Since $\mathrm{vol}(A|_F)$ belongs to the finite set Γ, there are only finitely many possibilities for $\mathrm{Ivol}(K_X + B)$. Therefore, we may fix both

$$u := \mathrm{vol}(A|_F) \quad \text{and} \quad v := \mathrm{Ivol}(K_X + B).$$

Step 2. By Theorem B and Lemma 3.2 (2), we may choose $\alpha \in \mathbb{N}$ depending only on d, Φ, u, v, λ such that $\alpha(K_X + B + \frac{\lambda}{2}A)$ is a big Weil divisor. Moreover,

$$\begin{aligned} &\mathrm{vol}\left(K_X + B + t\alpha(K_X + B + \tfrac{\lambda}{2}A)\right) \\ =&\mathrm{vol}\left((1+t\alpha)(K_X + B) + \tfrac{t\alpha\lambda}{2}A\right) \\ =&\left((1+t\alpha)(K_X + B) + \tfrac{t\alpha\lambda}{2}A\right)^d \end{aligned}$$

is a polynomial γ in t whose coefficients are uniquely determined by the intersection numbers $(K_X + B)^{d-i} \cdot A^i$, α and λ. Therefore, γ is determined by $d, \Phi, \Gamma, \sigma, \lambda$.

Replacing A, u, σ with $\alpha(K_X + B + \frac{\lambda}{2}A), (\frac{\alpha\lambda}{2})^{\dim F}u, \gamma$, we may assume that A is a big Weil divisor.

Step 3. Since when $\dim X = 1$, $K_X + B + tA$ is always ample, and when $\dim Z = 0$, $K_X + B + tA$ is nef and big for all $0 < t < 1$, we may assume that $\dim X \ge 2$ and $\dim Z \ge 1$.

We claim that it suffices to find $\tau \in (0,1]$, depending only on d, Φ, Γ, σ, such that $K_X + B + \tau A$ is nef. Indeed, once such a τ is found, $K_X + B + tA$ is nef and big for any $t \in (0, \tau)$. Since A is nef and big over Z, by the base point free theorem it is semiample over Z, so we may pick $0 < t' \ll t$ such that $K_X+B+t'A$ is nef and big. Then K_X+B+tA

is a positive linear combination of $K_X+B+t'A$ and $K_X+B+\tau A$, and hence is nef and big.

We aim to find such a τ in the subsequent steps.

Step 4. By Theorem 3.4, there exists $l\in\mathbb{N}$ depending only on d,Φ,u,v such that we can write an adjunction formula

$$K_X+B\sim_l f^*(K_Z+B_Z+M_Z)$$

and the generalised klt pair (Z,B_Z+M_Z) belongs to a bounded family. Moreover,

$$L:=l(K_Z+B_Z+M_Z)$$

is very ample.

Let T be a general member of $|L|$, and let S be its pullback to X. Define

$$K_S+B_S:=(K_X+B+S)|_S$$

and set $A_S:=A|_S$. Then

$$(S,B_S),\ A_S\to T\in\mathcal{G}_{klt}\big(d-1,\Phi,\Gamma,\psi\big)$$

for some polynomial $\psi(t)$ depending only on (d,Φ,Γ,σ).

Indeed, we may choose a general $T\in|L|$ such that $A|_S$ is nef and big over T and $(X,B+S)$ is plt. Hence (S,B_S) is a projective klt pair, and K_S+B_S is semi-ample, defining the contraction $g\colon S\to T$. If G is a general fibre of $S\to T$, then

$$\operatorname{vol}(A_S|_G)=\operatorname{vol}(A|_G)=u,$$

since G is among the general fibres of $X\to Z$. Moreover,

$$\begin{aligned}
\psi(t)&=(K_S+B_S+tA_S)^{d-1}\\
&=\big((K_X+B+S+tA)|_S\big)^{d-1}\\
&=(K_X+B+S+tA)^{d-1}\cdot S\\
&=\big((l+1)(K_X+B)+tA\big)^{d-1}\cdot S\\
&=\big((l+1)(K_X+B)+tA\big)^{d-1}\cdot l(K_X+B),
\end{aligned}$$

which is a polynomial in t whose coefficients are uniquely determined by the intersection numbers $(K_X+B)^{d-i}\cdot A^i$ and by l, and hence depend only on d, σ, and l.

Step 5. By Theorem C in lower dimension, there exists a fixed $r\in\mathbb{N}$ such that for any $(S,B_S),A_S$, we can find a very ample divisor H_S on S satisfying

$$H_S^{d-1}\le r,\quad (K_S+B_S)\cdot H_S^{d-2}\le r,\quad \text{and}\quad H_S-A_S \text{ is pseudo-effective.}$$

By Lemma 2.11, we may further assume that H_S-B_S is pseudo-effective.

Since A is big, there exists an effective $\mathbb{Q}$-divisor E such that $A\sim_{\mathbb{Q}}E$. As S is the pullback of a general element of a very ample linear system, we have $E_S:=E|_S$ effective and $A_S \sim_{\mathbb{Q}} E_S$. Moreover,

$$H_S - E_S \ \sim_{\mathbb{Q}} \ H_S - A_S$$

is also pseudo-effective.

By the same argument as in Step 1, $v' := \mathrm{Ivol}(K_S + B_S)$ is fixed. Therefore, (S, B_S) is ϵ-lc for some $\epsilon \in \mathbb{Q}^{>0}$ depending only on $(d-1, \Phi, u, v')$ by Lemma 3.2 (2).

Thus by Theorem 2.12, there is a fixed $\tau \in \mathbb{Q}^{>0}$ depending only on $d-1, \epsilon, r$ such that

$$\mathrm{lct}(S, B_S, |E_S|_{\mathbb{Q}}) > \tau,$$

hence $(S, B_S + \tau E_S)$ is klt. Then by inversion of adjunction [KM98, Theorem 5.50], $(X, B + S + \tau E)$ is plt near S. Therefore, $(X, B + \tau E)$ is lc over the complement of a finite set of closed points of Z: otherwise, the non-lc locus of $(X, B + \tau E)$ maps onto a closed subset of Z positive dimension which intersects T, hence S intersects the non-lc locus of $(X, B + \tau E)$, a contradiction.

Step 6. In this step, we assume that $K_X + B + \tau E$ is not nef. Otherwise, $K_X + B + \tau A \sim_{\mathbb{Q}} K_X + B + \tau E$ is nef, and we are done by Step 3.

Let R be a $(K_X + B + \tau E)$–negative extremal ray, since $K_X + B + \tau E$ is nef and big over Z, R is not contained in the fibers of $X \to Z$. By Step 5, the non-lc locus of $(X, B + \tau E)$ maps to finitely many points of Z, so R is not contained in the image

$$\mathrm{Im}(\overline{\mathrm{NE}}(\Pi) \to \overline{\mathrm{NE}}(X)),$$

where Π is the non-lc locus of $(X, B + \tau E)$.

Then by the length of extremal ray [Amb03] [Fuj11, Theorem 1.1], R is generated by a curve C with

$$(K_X + B + \tau E) \cdot C \geq -2d.$$

Since $L \in |l(K_Z + B_Z + M_Z)|$ is very ample, $f^*L \cdot C = L \cdot f_*C \geq 1$, we see that

$$(K_X + B + 2df^*L + \tau E) \cdot C \geq 0.$$

It follows that

$$K_X + B + 2df^*L + \tau E$$

is nef. Since $f^*L \sim l(K_X + B)$, we see that

$$K_X + B + \frac{\tau}{1 + 2dl} E \sim_{\mathbb{Q}} \frac{1}{1 + 2dl} \Big(K_X + B + 2df^*L + \tau E \Big)$$

is nef. Hence after replacing τ with $\frac{\tau}{1+2dl}$, we can assume that $K_X + B + \tau E$ is nef. $\square$

3.3. **Boundedness of pseudo-effective thereshold.** In this subsection, we show that the pseudo-effective threshold of $K_X + B$ with respect to A is bounded for all

$$(X, B), \ A \to Z \ \in \ \mathcal{G}_{klt}(d, \Phi, \Gamma, \sigma).$$

Proposition 3.6. *Theorem A_{d-1} implies Theorem B_d.*

Proof. Step 0. In this step, we introduce the top self-intersection function $\varsigma(t)$ and the volume function $\vartheta(t)$, and then outline the main idea of the proof using these functions.

Let

$$\varsigma(t) \in \mathbb{Q}[t], \quad \varsigma(t) := \big(A + t(K_X + B)\big)^d = \sum_{i=0}^{d} \binom{d}{i} A^{d-i} \cdot (K_X + B)^i t^i,$$

be the top self-intersection function. It is easy to see that fixing ς is equivalent to fixing σ. Let

$$\vartheta(t) := \operatorname{vol}\big(A + t(K_X + B)\big)$$

be the volume function. Then $\vartheta(t)$ is a non-negative, non-decreasing real function of t, and $\vartheta(t) = \varsigma(t)$ for $t \gg 0$.

It is enough to show that there exists a positive rational number τ, depending only on $(d, \Phi, \Gamma, \sigma)$, such that

$$A + t(K_X + B) \text{ is big for all } t > \tau.$$

In other words, it suffices to show that $\vartheta(t) > 0$ for all $t > \tau$.

We will prove the proposition by showing:

- There exists a positive rational number τ, such that $\varsigma(t) > 0$ and strictly increasing for all $t \geq \tau$.
- Since $\varsigma(t) = \vartheta(t)$ for $t \gg 0$, a comparison of their derivatives shows that $\vartheta(t)$ decreases no faster than $\varsigma(t)$ as t decreases. Hence, $\vartheta(t) \geq \varsigma(t) > 0$ for all $t \geq \tau$.

Step 1. We prove this proposition by induction on the dimension of Z. Since $A^{d-i} \cdot (K_X + B)^i = 0$ for $i > \dim Z$, the dimension of Z is determined by $\varsigma(t)$. Thus, we may assume $\dim Z = m$ is fixed. If $\dim Z = 0$, then clearly $A + t(K_X + B)$ is big. Hence, we may assume $\dim Z > 0$. By Step 1 of the proof of Proposition 3.5, we may fix both

$$u := \operatorname{vol}(A|_F) \quad \text{and} \quad v := \operatorname{Ivol}(K_X + B),$$

where F is a general fiber of $X \to Z$.

If $\dim Z = 1$, then

$$\varsigma(t) = \big(A + t(K_X + B)\big)^d = A^d + dA^{d-1} \cdot (K_X + B)t = A^d + duvt.$$

Let $\varsigma'(t)$ be the derivative of $\varsigma(t)$ with respect to t, it follows that $\varsigma'(t) = duv$. Since

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z) \sim_{\mathbb{Q}} vF,$$

we have

$$\vartheta(t) = \operatorname{vol}\big(A + t(K_X + B)\big) = v^d \operatorname{vol}(\frac{1}{v}A + tF).$$

For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose a sufficiently general fiber F_t of $X \to Z$ such that $F_t \not\subseteq \mathbf{B}_+(\frac{1}{v}A + tF_t)$. Then by Theorem 2.8, the function $s \mapsto \operatorname{vol}(\frac{1}{v}A + tF_t + sF_t)$ is differentiable at $s = 0$. Let $\vartheta'(t)$ denote the derivative of $\vartheta(t)$ with respect to t. This derivative is well-defined for all t such that $\vartheta(t) > 0$. By

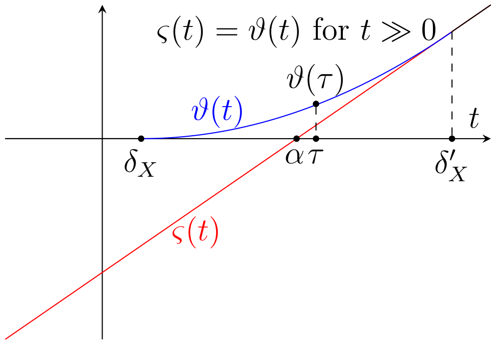
Theorem 2.8, we have

$$\frac{1}{v^d}\vartheta'(t) = \frac{1}{v^d}\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{vol}(\frac{1}{v}A + tF_t + sF_t)\Big)\Big|_{s=0} = d\,\mathrm{vol}_{X|F_t}(\frac{1}{v}A + tF_t).$$

It follows that for all t such that $\vartheta(t) > 0$,

$$\vartheta'(t) = dv^d\,\mathrm{vol}_{X|F_t}(\frac{1}{v}A + tF_t) \le dv^d\,\mathrm{vol}\big((\frac{1}{v}A + tF_t)|_{F_t}\big) = dv^d\frac{1}{v^{d-1}}u = duv = \varsigma'(t).$$

FIGURE 1. The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z = 1$



Let α be the root of $\varsigma(t)$ and set $\tau := \max\{\lceil\alpha\rceil + 1, 1\}$, so that τ is a positive rational number with $\varsigma(t) > 0$ for all $t \ge \tau$. Let δ_X be the largest real number such that $\vartheta(\delta_X) = 0$, where δ_X may *a priori* depend on X. We claim that $\vartheta(\tau) > 0$. Suppose, for a contradiction, that $\vartheta(\tau) = 0$. Then $\tau \le \delta_X$, hence $\varsigma(\delta_X) \ge \varsigma(\tau) > 0$. Since $\vartheta(t) = \varsigma(t)$ for all $t \gg 0$, there exists $\delta'_X \gg 0$ (possibly depending on X) such that $\vartheta(\delta'_X) = \varsigma(\delta'_X)$. By [Laz04, Corollary 2.2.45], the function $\vartheta(t)$ is continuous on $[\delta_X, \delta'_X]$, and since both $\varsigma(t)$ and $\vartheta(t)$ are differentiable on (δ_X, δ'_X), Lemma 3.7 yields some $\gamma_X \in (\delta_X, \delta'_X)$ such that

$$\big(\vartheta(\delta'_X) - \vartheta(\delta_X)\big)\,\varsigma'(\gamma_X) = \big(\varsigma(\delta'_X) - \varsigma(\delta_X)\big)\,\vartheta'(\gamma_X).$$

Since $\vartheta(\delta'_X) = \varsigma(\delta'_X)$, $\vartheta(\delta_X) = 0$, and $\varsigma(\delta_X) > 0$, it follows that $\vartheta'(\gamma_X) > \varsigma'(\gamma_X)$, contradicting the inequality $\vartheta'(t) \le \varsigma'(t)$ for all $t > \delta_X$ from the previous paragraph. Therefore $\vartheta(\tau) > 0$, and hence $\vartheta(t) \ge \vartheta(\tau) > 0$ for all $t \ge \tau$.

Step 2. From now on we assume that $\dim Z = m > 1$. Recall that in Step 4 of the proof of Proposition 3.5, we pick a general element T in the very ample linear system $|l(K_Z + B_Z + M_Z)|$ and let S be its pullback to X, so that

$$S \sim_{\mathbb{Q}} l(K_X + B).$$

Define

$$K_S + B_S := (K_X + B + S)|_S \quad \text{and} \quad A_S := A|_S,$$

so that

$$K_S + B_S \sim_{\mathbb{Q}} \Big(\frac{1}{l} + 1\Big)S|_S.$$

Moreover,

$$(S, B_S), A_S \to T \in \mathcal{G}_{klt}(d-1, \Phi, \Gamma, \psi)$$

for some fixed polynomial $\psi(t) \in \mathbb{Q}[t]$ depending only on d, Φ, Γ, σ, with $\dim T = m-1$. By Theorem A in lower dimension, there exists a positive rational number β, depending only on d, Φ, Γ, σ, such that $A_S + t(K_S + B_S)$ is nef and big for all $t > \beta$.

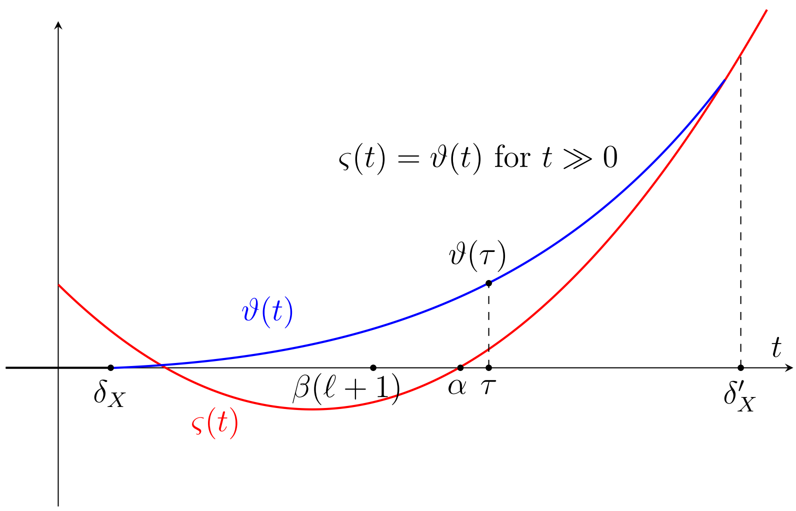
Step . Recall that $\varsigma(t) = (A + t(K_X + B))^d$. If $t > \beta(l+1)$, then $A_S + \frac{t}{l+1}(K_S + B_S)$ is nef and big by Step 2. We have

$$\begin{aligned}\varsigma'(t) &= d\big(A + t(K_X + B)\big)^{d-1} \cdot (K_X + B) \\ &= \frac{d}{l}\big(A + t(K_X + B)\big)^{d-1} \cdot S \\ &= \frac{d}{l}\left(A_S + \frac{t}{l+1}(K_S + B_S)\right)^{d-1} \\ &> 0.\end{aligned}$$

Hence $\varsigma(t)$ is an increasing function on $\big(\beta(l+1), +\infty\big)$.

If $\varsigma(t)$ has no roots (which occurs only when $\dim Z$ is even), set $\tau = \beta(l+1) + 1$. If $\varsigma(t)$ has roots, let α be the largest root of $\varsigma(t)$ and set $\tau = \max\{\beta(l+1), \lceil \alpha \rceil\} + 1$. Note that τ is a positive rational number. Moreover, on $[\tau, +\infty)$, $\varsigma(t)$ is a positive, increasing real function, and $\vartheta(t)$ is a non-negative, non-decreasing real function.

FIGURE 2. The graph of $\varsigma(t)$ and $\vartheta(t)$ when $\dim Z > 1$



Step 4. In this step, we conclude the proof. We see that

$$\vartheta(t) = \operatorname{vol}(A + t(K_X + B)) = \frac{1}{l^d}\operatorname{vol}(lA + tS),$$

for any $S \sim_{\mathbb{Q}} l(K_X + B)$. For each t such that $A + t(K_X + B)$ is big, i.e., $\vartheta(t) > 0$, we may choose S_t as the pullback of a sufficiently general element $T_t \in |l(K_Z + B_Z + M_Z)|$ such that $S_t \not\subseteq \mathbf{B}_+(lA + tS_t)$. Then by Theorem 2.8, the function $s \mapsto \operatorname{vol}(lA + tS_t + sS_t)$ is differentiable at $s = 0$. Let $\vartheta'(t)$ be the derivative of $\vartheta(t)$ with respect to t. This derivative is well-defined for all t such that $\vartheta(t) > 0$. By Theorem 2.8, we have

$$l^d\vartheta'(t) = l^d\frac{\mathrm{d}}{\mathrm{d}s}\vartheta(t+s)\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{vol}\,\big(lA+tS_t+sS_t\big)\big)\bigg|_{s=0} = d\,\mathrm{vol}_{X|S_t}(lA+tS_t).$$

It follows that for all $t \geq \tau$ such that $\vartheta(t) > 0$, we have

$$\begin{aligned}\vartheta'(t) =&\frac{d}{l^d}\,\mathrm{vol}_{X|S_t}(lA+tS_t)\\ \leq&\frac{d}{l}\,\mathrm{vol}\,\big((A+\frac{t}{l}S_t)|_{S_t}\big)\\ =&\frac{d}{l}\,\mathrm{vol}\,\big(A_{S_t}+\frac{t}{l+1}(K_{S_t}+B_{S_t})\big)\\ =&\frac{d}{l}\big(A_{S_t}+\frac{t}{l+1}(K_{S_t}+B_{S_t})\big)^{d-1}\\ =&\varsigma'(t),\end{aligned}$$

where the second-to-last equality follows from the fact that $A_{S_t} + \frac{t}{l+1}(K_{S_t} + B_{S_t})$ is nef on $[\tau, +\infty)$.

By the same argument as in the last paragraph of Step 1, we conclude that $\vartheta(t) \geq \vartheta(\tau) > 0$ for all $t > \tau$. □

We use the following elementary result in the proof of Proposition 3.6. Note that differentiability at the endpoints is not required.

Lemma 3.7 ([Rud76, Theorem 5.9])**.** *Let f and g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Then there exists a point $x \in (a, b)$ such that*

$$\big(f(b) - f(a)\big)g'(x) = \big(g(b) - g(a)\big)f'(x).$$

Remark 3.8. In the case $\dim X = 2$, by the Zariski decomposition for normal surfaces [Sak84, Corollary 7.5], the volume of a big divisor is greater than or equal to its self-intersection. Thus, when $\dim X = 2$, Proposition 3.6 follows immediately from this fact. However, this property does not necessarily hold in higher dimensions. For example, let Y be a smooth 3-fold with K_Y ample, and let $\pi : X = \mathrm{Bl}_P Y \to Y$ be the blow-up of Y at a closed point P. Then $K_X = \pi^*K_Y + 2E$, where E is the exceptional divisor over $P \in Y$, and K_X is big. It follows that $\mathrm{vol}(K_X) = \mathrm{vol}(K_Y) = (K_Y)^3$, while $(K_X)^3 = (K_Y)^3 + 8E^3 = (K_Y)^3 + 8 > \mathrm{vol}(K_X)$.

3.4. **Boundedness of klt good minimal models.** In this subsection, we prove the boundedness of klt good minimal models.

Proposition 3.9. *Theorem A_d implies Theorem C_d.*

Proof. For each

$$(X, B), A \to Z \in \mathcal{G}_{klt}(d, \Phi, \Gamma, \sigma),$$

by Step 1 of the proof of Proposition 3.5, we may fix both

$$u := \mathrm{vol}(A|_F) \quad \text{and} \quad v := \mathrm{Ivol}(K_X + B),$$

where F is a general fiber of $X \to Z$. By Lemma 3.2 (2), (X, B) is ϵ-lc and lB is a Weil divisor for some $\epsilon > 0$ and $l \in \mathbb{N}$ depending only on (d, Φ, u, v). Replacing l by a bounded multiple, Theorem A implies that

$$L := l\Big(K_X + B + \frac{\tau}{2}A\Big)$$

is a nef and big $\mathbb{Q}$-Cartier Weil divisor. Let

$$L' := l(K_X + B) + L,$$

which is also a nef and big Weil divisor. Then $L' - K_X = (l-1)(K_X + B) + B + L$ is pseudo-effective. By [Bir23a, Theorem 1.1], there exists $m \in \mathbb{N}$, depending only on d and ϵ, such that the linear system $|mL'|$ defines a birational map. Picking a general member $N \in |mL'|$, we have that $N \geq 0$ is a nef and big Weil divisor. It then follows that

$$\operatorname{vol}(K_X+B+N) = (2ml+1)^d \operatorname{vol}\Big(K_X+B+\frac{l\tau}{2(2ml+1)}A\Big) = (2ml+1)^d\,\sigma\Big(\frac{l\tau}{2(2ml+1)}\Big),$$

which is fixed. Consequently, by Theorem 2.13, the set of $(X, B+N)$ forms a bounded family.

Therefore, there exist a fixed $r \in \mathbb{N}$ and a very ample divisor H on X such that

$$H^d \leq r \quad \text{and} \quad H^{d-1} \cdot (K_X + B + N) \leq r.$$

By Lemma 2.11, $H - N$ is pseudo-effective. Since

$$N - \frac{\tau}{2}A = (ml+1)(K_X + B) + (ml-1)\Big(K_X + B + \frac{\tau}{2}A\Big)$$

is also pseudo-effective, it follows that $\frac{2}{\tau}H - A$ is pseudo-effective. Replacing H by a bounded multiple, we may assume that $H - A$ is pseudo-effective. □

***Proof of Theorem** 1.3.* This directly follows from Theorem C. □

Appendix A. Moduli space

In this appendix, we apply the boundedness results obtained in this paper to construct the moduli space of klt good minimal models of arbitrary Kodaira dimension, polarized by line bundles that are relatively ample over the bases of their respective Iitaka fibrations.

We refer readers to [Alp25] for the notions of stacks, algebraic stacks, Deligne-Mumford stacks and algebraic spaces.

Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{\geq 0}$ be a finite set, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. In this appendix, we will fix these data.

A.1. **Moduli functor of traditional stable minimal models.** Let $\Bbbk$ be an algebraically closed field of characteristic zero. We define the main object studied in this appendix, as introduced in Birkar's survey note [Bir24, §10].

Definition A.1 (Traditional stable minimal models)**.** A *traditional stable minimal model* $(X, B), A$ over $\mathbb{k}$ consists of a projective connected pair (X, B) and a Cartier divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f : X \to Z$, and
- $K_X + B + tA$ is ample for some $t > 0$.

A $(d, \Phi, \Gamma, \sigma)$-*traditional stable minimal model* is a traditional stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- the coefficients of B are in Φ,
- $\mathrm{vol}(A|_F) \in \Gamma$, where F is any general fiber of $f : X \to Z$, and
- $(K_X + B + tA)^d = \sigma(t)$.

We recall the notion of relative Mumford divisor from [Kol23, Definition 4.68].

Definition A.2 (Relative Mumford divisor)**.** Let $f : X \to S$ be a flat finite type morphism with S_2 fibers of pure dimension d. A subscheme $D \subset X$ is a *relative Mumford divisor* if there is an open set $U \subset X$ such that

- $\mathrm{codim}_{X_s}(X_s \setminus U_s) \geq 2$ for each $s \in S$,
- $D|_U$ is a relative Cartier divisor,
- D is the closure of $D|_U$, and
- X_s is smooth at the generic points of D_s for every $s \in S$.

By $D|_U$ being relative Cartier we mean that $D|_U$ is a Cartier divisor on U and that its support does not contain any irreducible component of any fiber U_s.

If $D \subset X$ is a relative Mumford divisor for $f : X \to S$ and $T \to S$ is a morphism, then the *divisorial pullback* D_T on $X_T := X \times_S T$ is the relative Mumford divisor defined to be the closure of the pullback of $D|_U$ to U_T. In particular, for each $s \in S$, we define $D_s = D|_{X_s}$ to be the closure of $D|_{U_s}$ which is the divisorial pullback of D to X_s.

Definition A.3 (Locally stable family)**.** A *locally stable family of klt pairs* $(X, B) \to S$ over a reduced Noetherian scheme S is a flat finite type morphism $X \to S$ with S_2 fibers and a $\mathbb{Q}$-divisor B on X satisfying

- each prime component of B is a relative Mumford divisor,
- $K_{X/S} + B$ is $\mathbb{Q}$-Cartier, and
- (X_s, B_s) is a klt pair for any point $s \in S$.

We define families of traditional minimal models and the corresponding moduli functor.

Definition A.4. Let S be a reduced scheme over $\mathbb{k}$.

(1) When $S = \mathrm{Spec}\, \mathbb{K}$ for a field $\mathbb{K}$, we define a traditional stable minimal model over $\mathbb{K}$ as in Definition A.1 by replacing $\mathbb{k}$ with $\mathbb{K}$ and replacing connected with

geometrically connected. Similarly we can define $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over $\mathbb{K}$.

(2) For general S, a *family of traditional stable minimal models* over S consists of a projective morphism $X \to S$ of schemes, a $\mathbb{Q}$-divisor B and a line bundle A on X such that
- $(X, B) \to S$ is a locally stable family,
- $(X_s, B_s), A_s$ is a traditional stable minimal model over $k(s)$ for every $s \in S$.

Here X_s is the fiber of $X \to S$ over s and B_s is the divisorial pullback of B to X_s. Moreover, $K_{X_s} + B_s$ is semi-ample which defines a contration $X_s \to Z_s$, and A_s is a line bundle on X_s which is ample over Z_s. We will denote this family by $(X, B), A \to S$.

(3) Let $d \in \mathbb{N}$, $\Phi = \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \mathbb{Q}^{\geq 0}$, $\Gamma \subset \mathbb{Q}^{>0}$ be a finite set, $\sigma \in \mathbb{Q}[t]$ be a polynomial. A *family of $(d, \Phi, \Gamma, \sigma)$-marked traditional stable minimal models* over S is a family of traditional stable minimal models $(X, B), A \to S$ such that
- $B = \sum_{i=1}^{m} a_i D_i$, where $D_i \geq 0$ are relative Mumford divisors, and
- $(X_s, B_s), A_s$ is a $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal model over $k(s)$ for every $s \in S$, where $B_s = \sum_{i=1}^{m} a_i D_{i,s}$.

(4) We define the moduli functor $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)$ of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models from the category of reduced $\mathbb{k}$-schemes to the category of groupoids by choosing:
- On objects: for a reduced $\mathbb{k}$-scheme S, one take
$$\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(S) = \{\text{family of } (d, \Phi, \Gamma, \sigma)\text{-traditional stable minimal models over } S\}.$$
We define an isomorphism $(f' : (X', B'), A' \to S) \to (f : (X, B), A \to S)$ of any two objects in $\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(S)$ to be an isomorphism $\alpha_X : (X', B') \to (X, B)$ over S such that $A' \sim_S \alpha_X^* A$.
- On morphisms: $(f_T : (X_T, B_T), A_T \to T) \to (f : (X, B), A \to S)$ consists of morphisms of reduced $\mathbb{k}$-schemes $\alpha : T \to S$ such that the natural map $g : X_T \to X \times_S T$ is an isomorphism, B_T is the divisorial pullback of B and $A_T \sim_T g^* \alpha_X^* A$. Here $\alpha_X : X \times_S T \to X$ is the base change of α.

Now we can state our main result on moduli.

Theorem A.5. *$\mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)$ is a separated Deligne-Mumford stack of finite type, which admits a coarse moduli space $TS_{klt}(d, \Phi, \Gamma, \sigma)$ as a separated algebraic space.*

A.2. **Moduli stack of traditional stable minimal models.**

Lemma A.6. *Let $\mathbb{K}$ be a field of characteristic zero. Then there exist natural number τ and I depending only on $(d, \Phi, \Gamma, \sigma)$ such that $\tau\Phi \subset \mathbb{N}$ and they satisfy the following. For any $(X, B), A \in \mathfrak{T}\mathfrak{S}_{klt}(d, \Phi, \Gamma, \sigma)(K)$ and nef Cartier divisor M on X, we have*

- *$\tau(K_X+B)$ is a base point free divisor, $A+\tau(K_X+B)$ is an ample Cartier divisor,*
- *Let $L_M := I(A+\tau(K_X+B))+M$, then L_M is strongly ample, i.e. L_M is very ample and $H^q(X, kL_M)=0$ for any $k, q>0$,*

Proof. By the same argument as [Bir22, Proof of Lemma 10.2], it is enough to find τ and I when $\mathbb{K}=\mathbb{C}$. Note that A is a line bundle in our setting. Hence, by the proof of Theorem 1.3, there exists $\tau\in\mathbb{N}$ such that $\tau(K_X+B)$ is base point free, and both $A+(\tau-1)(K_X+B)$ and $A+\tau(K_X+B)$ are ample Cartier divisors. Applying the effective base point free theorem [Kol93, Theorem 1.1] and the very ampleness lemma [Fuj17, Lemma 7.1] to $A+\tau(K_X+B)$, we obtain $I_0\in\mathbb{N}$ such that $L_0 := I_0(A+\tau(K_X+B))$ is very ample.

After replacing I_0 with a bounded multiple, we may assume that $L_0-(K_X+B)$ is nef and big. Let $I=(d+2)I_0$ and $\mathcal{F} := L_M - I_0(A+\tau(K_X+B))$, then

$$H^i(X, \mathcal{F}\otimes L_0^{\otimes(-i)})=0$$

for all $i>0$ by Kawamata-Viehweg vanishing theorem. Thus $\mathcal{F}$ is 0-regular with respect to L_0 ([Laz04, Definition 1.8.4]), and hence $\mathcal{F}$ is base point free by [Laz04, Theorem 1.8.5]. Therefore,

$$L_M = L_0+\mathcal{F}$$

is very ample by [Har77, Exercise II 7.5(d)]. Again we have $L_M-(K_X+B)$ is nef and big, hence $H^q(X, kL_M)=0$ for any $k, q>0$. $\square$

Notation A.7. From now on, we will fix the positive natural numbers I and τ obtained in Lemma A.6. Let S be a reduced scheme, for any $(f:(X,B),A\to S)\in\mathfrak{TS}_{klt}(d,\Phi,\Gamma,\sigma)(S)$, we define

$$L_{1,S} := I(A+\tau(K_{X/S}+B))+I(A+\tau(K_{X/S}+B)) = 2IA+2I\tau(K_{X/S}+B),$$
$$\begin{aligned} L_{2,S} :=& I(A+\tau(K_{X/S}+B))+(I-1)(A+\tau(K_{X/S}+B))+\tau(K_{X/S}+B)\\ =&(2I-1)A+2I\tau(K_{X/S}+B)\end{aligned}$$

and $L_{3,S} := L_{1,S}+L_{2,S}$ to be the divisorial sheaves on X. Then $L_{1,S}-L_{2,S}=A$, and $L_{j,S}$ are strongly ample line bundles over S for $j=1,2,3$ by Lemma A.6 and the proof of Lemma A.8.

Lemma A.8. *Let $(X, B=\sum_{i=1}^m a_iD_i), A\to S$ be a family of (d,Φ,Γ,σ)-marked traditional stable minimal models over reduced Noetherian scheme S. For $j=1,2,3$, let $L_{j,S}$ be the divisorial sheaves on X as Notation A.7. Then for every $k\in\mathbb{Z}_{>0}$, the functions $S\to\mathbb{Z}$ by sending*

(1) $s\mapsto h^0(X_s, kL_{j,s})$ for $j=1,2,3$ and
(2) $s\mapsto \deg_{L_{3,s}}(D_{i,s})$ for $i=1,2,\dots,m$

are locally constant on S, where $L_{j,s}=L_{j,S}|_{X_s}$ and $D_{i,s}=D_i|_{X_s}$ are the divisorial pullbacks to X_s, and $\deg_{L_{3,s}}(D_{i,s}) := D_{i,s}\cdot L_{3,s}^{d-1}$.

Proof. (1). For $j = 1, 2, 3$, it is enough to show that $L_{j,S}$ are flat over S: since then $\chi(X_s, kL_{j,s})$ are locally constant, and $L_{j,S}$ are strongly ample over S by Lemma A.6, hence $h^0(X_s, kL_{j,s})$ are locally constant. Since $X \to S$ is flat, it suffices to show that $\mathcal{O}_X(L_{j,S})$ are line bundles by [Har77, Proposition III 9.2(c)(e)].

Since $(X, B) \to S$ is a locally stable family, B is a relative Mumford divisor over S, we see that $\tau(K_{X/S} + B)$ is $\mathbb{Q}$-Cartier, and it is mostly flat ([Kol23, Definition 3.26]) over S. Moreover, since $\mathcal{O}_{X_s}(\tau(K_{X_s} + B_s))$ is a base point free line bundle for any $s \in S$ by Lemma A.6, $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a mostly flat family of line bundles. Therefore, by [Kol23, Corollary 4.34 and Proposition 5.29], $\mathcal{O}_X(\tau(K_{X/S} + B))$ is a line bundle on X. Furthermore, since A is a line bundle on X, $\mathcal{O}_X(L_{j,S})$ are line bundles for $j = 1, 2, 3$.

(2). It follows from [Kol23, Theorem 4.3.5]. □

Let $n, l \in \mathbb{Z}_{>0}$, $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$, and $h \in \mathbb{Q}[k]$ be a polynomial. Let S be a reduced scheme, for any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$ and $j = 1, 2, 3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation A.7. We define $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ to be a full subcategory of $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$ such that $\mathfrak{TS}_{h,n,l,\mathbf{c}}(S)$ is a groupoid whose objects consist of families of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models over S satisfying:

- the Hilbert polynomial of X_s with respect to $L_{3,s}$ is h,
- $h^0(X_s, L_{1,s}) - 1 = n$,
- $h^0(X_s, L_{2,s}) - 1 = l$, and
- $(\deg_{L_{3,s}}(D_{1,s}), \deg_{L_{3,s}}(D_{2,s}), \ldots, \deg_{L_{3,s}}(D_{m,s})) = \mathbf{c}$

for every $s \in S$.

Lemma A.9. *We can write*

$$\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma) = \bigsqcup_{h,n,l,\mathbf{c}} \mathfrak{TS}_{h,n,l,\mathbf{c}}$$

as disjoint union, and each $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is a union of connected components of $\mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)$. Moreover, there are only finitely many $n, l \in \mathbb{Z}_{>0}$, $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$ and $h \in \mathbb{Q}[k]$ such that $\mathfrak{TS}_{h,n,l,\mathbf{c}}$ is not empty.

Proof. Given any $(f : (X, B = \sum_{i=1}^{m} a_i D_i), A \to S) \in \mathfrak{TS}_{klt}(d, \Phi, \Gamma, \sigma)(S)$. By Lemma A.8, the Hilbert functions

$$h_s(k) = \chi(X_s, kL_{3,s}) = h^0(X_s, kL_{3,s})$$

of X_s with respect to $L_{3,s}$, and the numbers

$$n_s = h^0(X_s, L_{1,s}) - 1, \quad l_s = h^0(X_s, L_{2,s}) - 1 \quad \text{and} \quad c_{i,s} = \deg_{L_{3,s}}(D_{i,s})$$

are locally constant on $s \in S$ for all $1 \leq i \leq m$. The first assertion follows from this fact.

The second assertion follows from the fact that n_s, l_s, $c_{i,s}$ and h_s belong to a finite set for all $1 \leq i \leq m$ by Theorem 1.3 (these finiteness results can be reduced to the case when $s = \operatorname{Spec}\mathbb{C}$ by the same argument as [Bir22, Proof of Lemma 10.2]). □

Lemma A.10. *$\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is a stack.*

Proof. Since our argument follows the same strategy as in [Alp25, Proposition 2.5.14 and Example 2.5.9], we only sketch the proof here.

Axiom (1) of [Alp25, Definition 2.5.1] follows from descent [Alp25, Proposition 2.1.7, Proposition 2.1.19, Proposition 2.1.4(1) and Proposition 2.1.16(2)].

To verify Axiom (2) of [Alp25, Definition 2.5.1], i.e., given any descent datum (f', ξ) with respect to a covering $S' \to S$ (see [HH25, Remark 2.10] for notions of covering and descent datum), where $(f' : (X', B'), A' \to S') \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S')$, we need to show that f' descends to a family $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$. We use the strongly f'-ample line bundles $\mathcal{O}_{X'}(L'_{1,S'})$ and $\mathcal{O}_{X'}(L'_{2,S'})$ as Notation A.7 instead of $\omega_{C'/S'}^{\otimes 3}$ in [Alp25, Proposition 2.5.14], then the same argument as in *loc.cit.* implies that $(X', B') \to S'$ descends to $(X, B) \to S$. Moreover, by applying [Alp25, Proposition 2.1.4(2) and Proposition 2.1.16(2)] to the covering $X' \to X$, we see that A' descends to a line bundle A on X. Since every geometric fiber of $f : (X, B), A \to S$ is identified with a geometric fiber of $f' : (X', B'), A' \to S'$, $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$. □

For any scheme S and positive integer n, l, Let $\mathbb{P}^n_S \times_S \mathbb{P}^l_S \cong \mathbb{P}^n \times \mathbb{P}^l \times S$ be the natural isomorphism, and

$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times S \xrightarrow{p_2} \mathbb{P}^l$$

be the projections. Then for any $a, b \in \mathbb{Z}$, we denote $p_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^l}(b)$ by $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S}(a, b)$.

Theorem A.11. *$\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is an algebraic stack of finite type.*

Proof. Step 1. In this step, we consider a suitable Hilbert scheme parametrizing the total spaces of interest.

For any $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$ and for $j = 1, 2, 3$, let $L_{j,S}$ be the strongly ample line bundles over S as Notation A.7. We get an embedding

$$X \hookrightarrow \mathbb{P}(f_*\mathcal{O}_X(L_{1,S})) \times_S \mathbb{P}(f_*\mathcal{O}_X(L_{2,S})).$$

We proceed to parametrize such embedding.

Let $H = \operatorname{Hilb}_h(\mathbb{P}^n \times \mathbb{P}^l)$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n \times \mathbb{P}^l$ with Hilbert polynomial h. Let $X_H = \operatorname{Univ}_h(\mathbb{P}^n \times \mathbb{P}^l) \overset{i}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^l \times H$ be the universal family over H, and

$$\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^l \times H \xrightarrow{p_2} \mathbb{P}^l.$$

be the natural projections. Note that the $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on $\mathbb{P}^n \times \mathbb{P}^l$ induces a $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$ action on H. Let $M_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, 1)$ and $N_H := i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H}(1, -1)$ be the universal line bundles on X_H.

Step 2. In this step, we parametrize the boundary divisors in the moduli problem.

By [Gro66, Theorem 12.2.1 and Theorem 12.2.4], the locus $s \in H$ such that X_s is geometrically connected and reduced, equidimensional, and geometrically normal is an open subscheme H_1 of H.

Since $f_1 : X_{H_1} \to H_1$ is equidimensional, and over reduced bases relative Mumford divisors are the same as K-flat divisors [Kol23, Definition 7.1 and comment 7.4.2], there is a separated H_1-scheme $\mathrm{MDiv}_c(X_{H_1}/H_1)$ of finite type which parametrizes relative Mumford divisors of degree c with respect to M_{H_1} by [Kol23, Theorem 7.3]. Fixing $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{N}^m$, let

$$H_2 := \mathrm{MDiv}_{c_1}(X_{H_1}/H_1) \times_{H_1} \mathrm{MDiv}_{c_2}(X_{H_1}/H_1) \times_{H_1} \cdots \times_{H_1} \mathrm{MDiv}_{c_m}(X_{H_1}/H_1)$$

be the m-fold fiber product, we denote the universal family by

$$(X_{H_2}, B_{H_2} = \sum_{i=1}^{m} a_i D_{i,H_2}), N_{H_2} \to H_2,$$

where D_{i,H_2} are the universal families of relative Mumford divisors on X_{H_2} of degree c_i with respect to M_{H_2} for $1 \le i \le m$.

Step 3. By [Kol23, Theorem 4.8], there is a locally closed partial decomposition $H_3 \to H_2$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_2$, then the family obtained by base change $f_W : (X_W, B_W) \to W$ is locally stable iff q factors as $q : W \to H_3 \to H_2$.

Since $f_3 : (X_{H_3}, B_{H_3}) \to H_3$ is locally stable, By [Kol23, Theorem 4.28], there is a locally closed partial decomposition $H_4 \to H_3$ satisfying the following: for any reduced scheme W and morphism $q : W \to H_3$, the divisorial pullback of $\tau(K_{X_{H_3}/H_3} + B_{H_3})$ to $W \times_{H_3} X_{H_3}$ is Cartier iff q factors as $q : W \to H_4 \to H_3$.

Step 4. Since the fibers X_s of $f_4 : X_{H_4} \to H_4$ are reduced and connected by Step 2, we have $h^0(X_s, \mathcal{O}_{X_s}) = 1$. Since $\tau(K_{X_{H_4}/H_4} + B_{H_4})$ is Cartier by Step 3, by [Vie95, Lemma 1.19], there is a locally closed subscheme $H_5 \subset H_4$ with the following property: for any scheme W and morphism $q : W \to H_4$,

$$\mathcal{O}_{X_W}(1,0) \sim_W N_W^{2I} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W) \text{ and}$$

$$\mathcal{O}_{X_W}(0,1) \sim_W N_W^{2I-1} \otimes \omega_{X_W/W}^{[2I\tau]}(2I\tau B_W)$$

iff q factors as $q : W \to H_5 \to H_4$, where $\mathcal{O}_{X_W}(1,0)$ and $\mathcal{O}_{X_W}(0,1)$ are the pullbacks of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(1,0)$ and $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times H_4}(0,1)$ to X_W, respectively.

Step 5. In this step, we cut the locus parametrizing $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models.

(1). By [Bir22, Lemma 8.5], there is a locally closed subscheme $H_6 \subset H_5$ such that for any $s \in H_6$, $K_{X_s} + B_s$ is semi-ample defining a contraction $X_s \to Z_s$.

(2). Since ampleness and klt are open conditions, there is an open subscheme $H_7 \subset H_6$ such that $N_s + \tau(K_{X_s} + B_s)$ is ample and (X_s, B_s) is klt for any $s \in H_7$.

(3). By [Bir22, Lemma 8.7] (the condition of N_s being effective is not required in the proof), there is a locally closed subscheme $H_8 \subset H_7$ such that for any $s \in H_8$, $\mathrm{vol}(N_s|_F) \in \Gamma$ for the general fibres F of $X_s \to Z_s$.

(4). For each $s \in H_8$, since $K_{X_s} + B_s$ is semi-ample and $N_s + \tau(K_{X_s} + B_s)$ is ample, $K_{X_s} + B_s + tN_s$ is ample for each $t \in (0, \frac{1}{\tau}]$, then

$$\theta_s(t) = \mathrm{vol}(K_{X_s} + B_s + tN_s) = (K_{X_s} + B_s + tN_s)^d$$

is a polynomial in t of degree $\leq d$ on the interval $(0, \frac{1}{\tau}]$. By Step 3(iv) of [Bir22, Proof of Proposition 9.5], there is an open and closed subscheme $H_9 \subset H_8$ such that $\theta_s(t) = \sigma(t)$ on the interval $(0, \frac{1}{\tau}]$.

Therefore, $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is a family of $(d, \Phi, \Gamma, \sigma)$-traditional stable minimal models. For $j = 1, 2$, let L_{j,H_9} be the strongly ample line bundles over H_9 as Notation A.7. Then $f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})$ and $f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})$ are locally free sheaves of rank $n+1$ and $l+1$, respectively. Shrinking H_9, we may assume that they are free sheaves, and hence

$$\mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{1,H_9})) \cong \mathbb{P}^n_{H_9} \text{ and } \mathbb{P}(f_{9*}\mathcal{O}_{X_{H_9}}(L_{2,H_9})) \cong \mathbb{P}^l_{H_9}.$$

Step 6. In this step, we will prove that

$$\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}} \cong [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}].$$

Then since H_9 is a finite type scheme and $[H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$ is an algebraic stack, $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is a finite type algebraic stack.

We follow the arguments of [Alp25, Theorem 3.1.17] and [ABB+23, Proposition 3.9]. By our construction, the universal family $f_9 : (X_{H_9} \subset \mathbb{P}^n \times \mathbb{P}^l \times H_9, B_{H_9}), N_{H_9} \to H_9$ is an object in $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(H_9)$, which induces a morphism $H_9 \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$, where this morphism just forgets the projective embeddings. Moreover, this morphism is $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}$-invariant, hence descends to a morphism $\Psi^{\mathrm{pre}} : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]^{\mathrm{pre}} \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ of prestacks. Since $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$ is a stack by Lemma A.10, the universal property of stackification [Alp25, Theorem 2.5.18] yields a morphism $\Psi : [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}] \to \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}$.

To construct the inverse, consider $(f : (X, B), A \to S) \in \mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}}(S)$, since $f_*\mathcal{O}_X(L_{1,S})$ and $f_*\mathcal{O}_X(L_{2,S})$ are locally free by Step 1, there exists an open cover $S = \cup_i S_i$ over which their restrictions are free. Choosing trivializations induce embeddings $g_i : (X_{S_i}, B_{S_i}) \hookrightarrow \mathbb{P}^n \times \mathbb{P}^l \times S_i$. Moreover, we have $A_{S_i} \sim_{S_i} N_{S_i} := g_i^*\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^l \times S_i}(1, -1)$. Hence by our construction of H_9, we have morphisms $\Phi_i : S_i \to H_9$. Over the intersections $S_i \cap S_j$, the trivializations differ by a section $s_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1})$. Therefore the Φ_i glue to a morphism $\Phi : S \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, which induces a morphism $\mathfrak{T}\mathfrak{S}_{h,n,l,\mathbf{c}} \to [H_9/\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{l+1}]$, that is the inverse of Ψ. $\square$

A.3. **Moduli space of traditional stable minimal models.** We need the following separatedness result to obtain the coarse moduli space of traditional stable minimal models.

Theorem A.12. *Let $f:(X,B),A\to C$ and $f':(X',B'),A'\to C$ be two families of (d,Φ,Γ,σ)-traditional stable minimal models over a smooth curve C. Let $0\in C$ be a closed point and $C^o:=C\setminus\{0\}$ the punctured curve. Assume there exists an isomorphism*

$$g^o:((X,B),A)\times_C C^o\to((X',B'),A')\times_C C^o$$

over C^o, then g^o can be extended to an isomorphism $g:(X,B),A\to(X',B'),A'$ over C.

Proof. Consider $L:=A'+\tau(K_{X/C}+B)$ and $L':=A'+\tau(K_{X'/C}+B')$, where τ is the positive natural number as Lemma A.6. By the proof of Lemma A.8, L is an f-ample Cartier divisor on X (resp. L' is an f'-ample Cartier divisor on X'). Let $g:X\dashrightarrow X'$ be the birational map induced by g^o, then by the same argument as in [HH25, Proof of Proposition 4.4], g is an isomorphism over C. □

Corollary A.13. *For any $(X,B),A\in\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)(\Bbbk)$, $\mathrm{Aut}((X,B),A)$ is finite.*

Proof. It follows from Theorem A.12 and the argument of [BX19, Proof of Corollary 3.5]. □

***Proof of Theorem** A.5.* By Theorem A.11 and Lemma A.9, $\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)$ is an algebraic stack of finite type. By Corollary A.13 and [Alp25, Theorem 3.6.4], $\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)$ is a Deligne-Mumford stack. Moreover, Theorem A.12 and [Alp25, Theorem 3.8.2(3)] imply that $\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)$ is a separated Deligne-Mumford stack of finite type. Therefore, we may apply the Keel–Mori's theorem [KM97][Alp25, Theorem 4.3.12] to see that $\mathfrak{T}\mathfrak{S}_{klt}(d,\Phi,\Gamma,\sigma)$ has a coarse moduli space $TS_{klt}(d,\Phi,\Gamma,\sigma)$, which is a separated algebraic space. □

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