

Review of Kinematics and Dynamics

Modeling and Simulation Outline

- I. **Review of Kinematics, Dynamics, and 6-DOF Equations of Motion**
 - A. Reference Frames / Notation
 - B. Useful kinematical equations
 - C. Translational and Rotational Dynamics
 - D. Euler Angles / Kinematics
- II. Overview of Rotorcraft Modeling and Simulation
 - A. Modular Model Structure
 - B. Simple Helicopter Model
 - C. Numerical Integration
 - D. Numerical Linearization
 - E. Trim
- III. Rotor Dynamic Model
 - A. Flapping Equations of Motion
 - B. Multi-blade Coordinates
 - C. Rotor Forces and Moments
 - D. Dynamic Inflow
- IV. Complete Modular Rotorcraft Simulation Model
 - A. Tail Rotor
 - B. Fuselage and Empennage Aerodynamics
 - C. Engine Dynamics
 - D. Flight Controls
 - E. Integrated Simulation Model
 - F. Other Configurations (tandem rotors, tilt-rotors, compounds, multi-copters)

Equations of Motion

Want to develop a set of differential equations that describe aircraft motion.

First order non-linear ODE's (ordinary differential equations) where the independent variable is time.

$f(x, \dot{x}, u, t) = 0$ Implicit State Equations

$f(x, \dot{x}, u) = 0$ Time invariant (usually a good assumption)

$\dot{x} = f(x, u)$ Explicit State Equations (sometimes a good approximation)

x = State Vector

u = Control Vector

t = Time

$\dot{x} = \frac{dx}{dt}$

Will apply Newton's Laws of motion:

\mathbf{F} = Force Vector

\mathbf{M} = Moment Vector

\mathbf{V} = Velocity Vector

\mathbf{H} = Angular Momentum Vector

$$\mathbf{F} = m \frac{^N d\mathbf{V}}{dt} \quad \mathbf{M} = \frac{^N d\mathbf{H}}{dt}$$

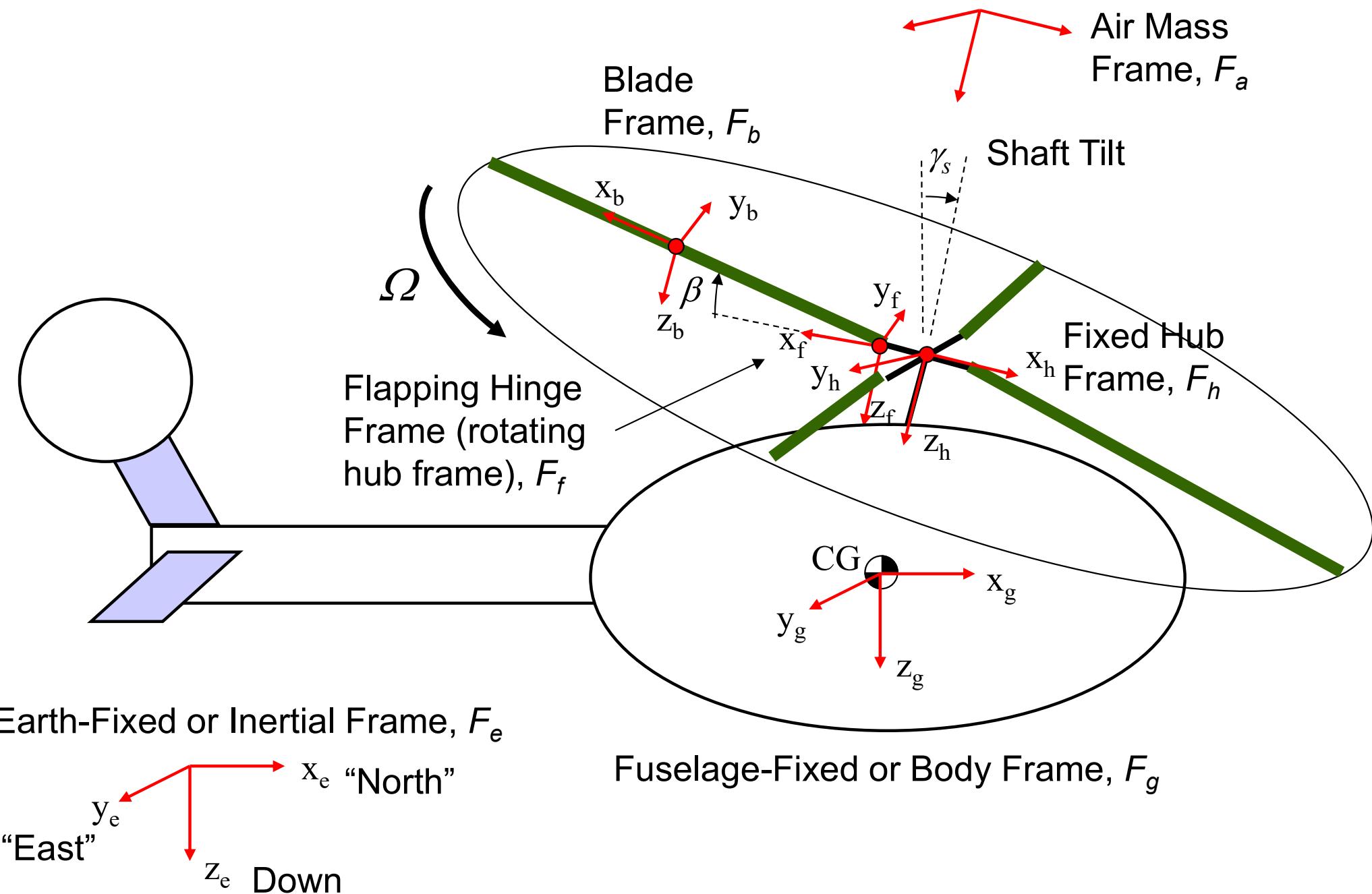
ASSUMPTIONS:

$\frac{^N d}{dt}$ = Derivative Taken in Newtonian Frame

1. Fuselage and rotor blades are rigid bodies, no elastic deformation.
2. "Flat Earth" assumption – surface of the Earth is defined by a flat plane and is a Newtonian reference frame. Ignore curvature of Earth or effect of Earth's rotation.
(Because helicopter is usually slow)
3. Constant gravitational field, and constant aircraft weight.

70, flat earths

Reference Frames / Coordinate Systems



Notation

Vector Notation, Velocity and Accelerations:

$\mathbf{V}_{P/f}$ = Velocity vector of point P in frame F_f

$\mathbf{r}_{P/O}$ = Position vector from point O to point P

If O is fixed in reference frame F_f , then

$\mathbf{V}_{P/f} = \frac{^f d\mathbf{r}_{P/O}}{dt}$, left superscript indicates the derivative is taken in the frame F_f

$\mathbf{a}_{P/f} = \frac{^f d\mathbf{V}_{P/f}}{dt} =$ acceleration vector of point P in frame F_f

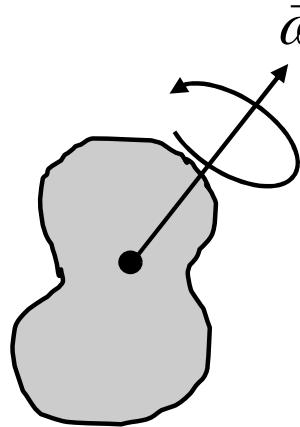
Right superscript denotes coordinate system that defines components:

$$\mathbf{V}_{P/f}^f = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

where v_x, v_y, v_z are components of $\mathbf{V}_{P/f}^f$ in coordinate system f

Notation

Angular Velocity Vector:



$\vec{\omega}_{g/e}$ = Angular velocity of body frame g in reference frame e

- Note that a rigid body defines a reference frame
- Angular velocity vector defines axis of rotation at an instant in time
- Magnitude defines rate of rotation in right hand sense

$$\vec{\omega}_{g/e}^g = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \text{ Components in of body frame coordinate system } g$$

Aircraft Attitude:

Use standard 3-2-1 Euler angles

ϕ = Roll Attitude (bank angle)

θ = Pitch Attitude

ψ = Yaw Attitude (heading)

Notation

Force and Moment Vectors:

$$\mathbf{F}^g = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + m\mathbf{g}^g$$

to $\hat{x}\hat{y}\hat{z}$ is B -Frame

Aerodynamic forces

$\mathbf{M}^g = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$ Roll Moment
Pitch Moment about CG
Yaw Moment

A subscript on force and moment terms denotes contribution of a specific rotorcraft component, e.g. L_R is roll moment due to Main Rotor

Body Axis Coordinate System: Forces, Moments, Velocities, Angular Rates, and Mass Properties

Mass = m

Weight = $W = mg$

Products of Inertia = I_{xz} , I_{xy} , I_{yz}

Typically I_{xy} , $I_{yz} \approx 0$

Principal axes are a special set of body axes chosen such that:

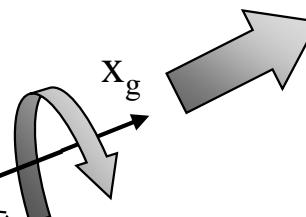
$$I_{xz} = I_{xy} = I_{yz} = 0$$

Roll Moment = L

Roll Rate = p

Positive right wing down

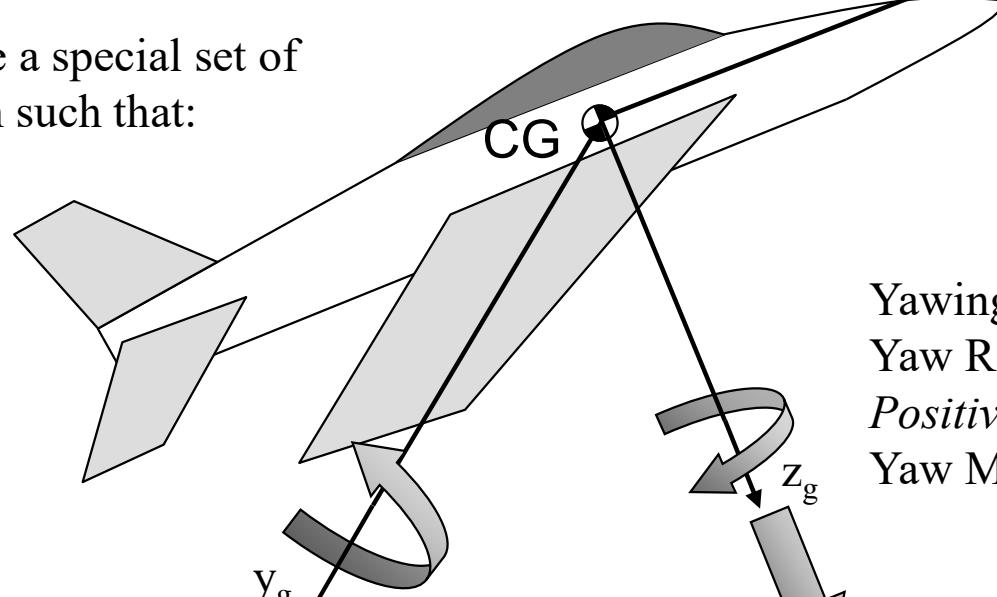
Roll Moment of Inertia = I_{xx}



Longitudinal (or Axial) Force = X

Longitudinal Velocity = u

Positive Forward



Lateral (or Side) Force = Y

Lateral Velocity = v

Positive Right

Pitch Moment = M

Pitch Rate = q

Positive nose up

Pitch Moment of Inertia = I_{yy}

Yawing Moment = N

Yaw Rate = r

Positive nose right

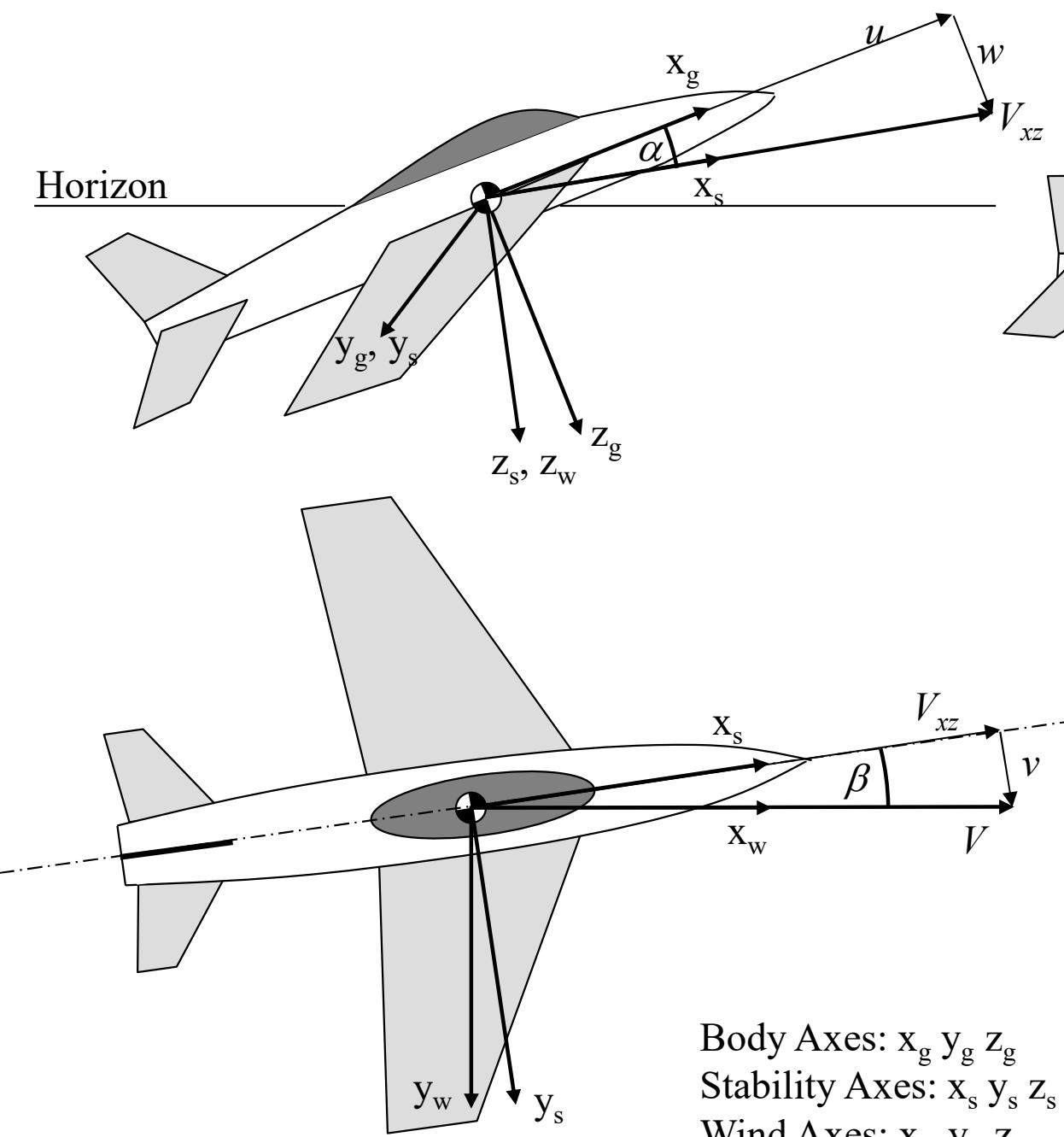
Yaw Moment of Inertia = I_{zz}

Vertical (or Normal) Force = Z

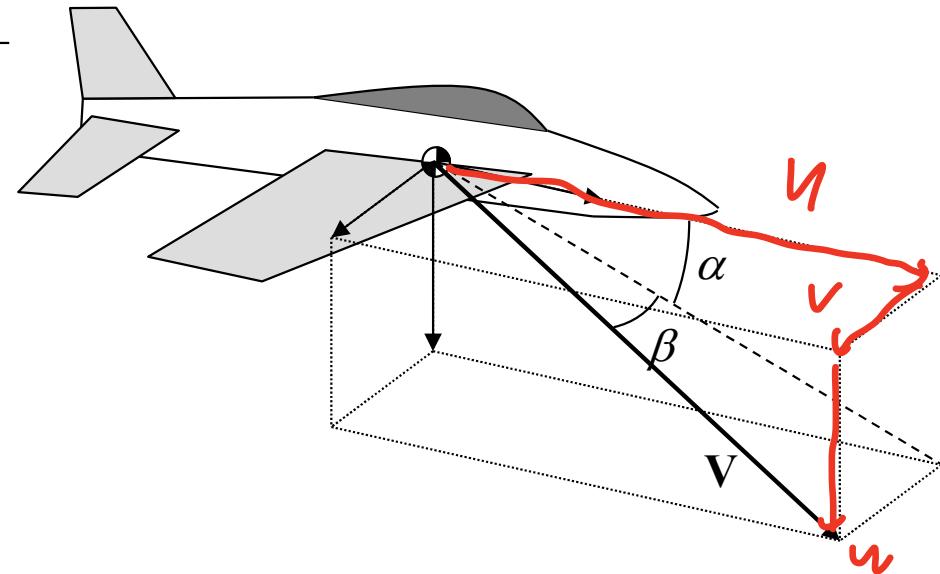
Vertical Velocity = w

Positive down

Stability and Wind Axes



α, β = Angle of Attack and Sideslip Angle (note they are ill-defined in hover, $V = 0$)



$$\alpha = \tan^{-1}\left(\frac{w}{u}\right), \quad \beta = \sin^{-1}\left(\frac{v}{V}\right)$$

$$u = V \cos \alpha \cos \beta$$

$$v = V \sin \beta$$

$$w = V \sin \alpha \cos \beta$$

$$V = \sqrt{u^2 + v^2 + w^2}$$

$$V_{xz} = \sqrt{u^2 + w^2}$$

Body Axes: $x_g \ y_g \ z_g$
 Stability Axes: $x_s \ y_s \ z_s$
 Wind Axes: $x_w \ y_w \ z_w$

Vector Derivative Transport Theorem

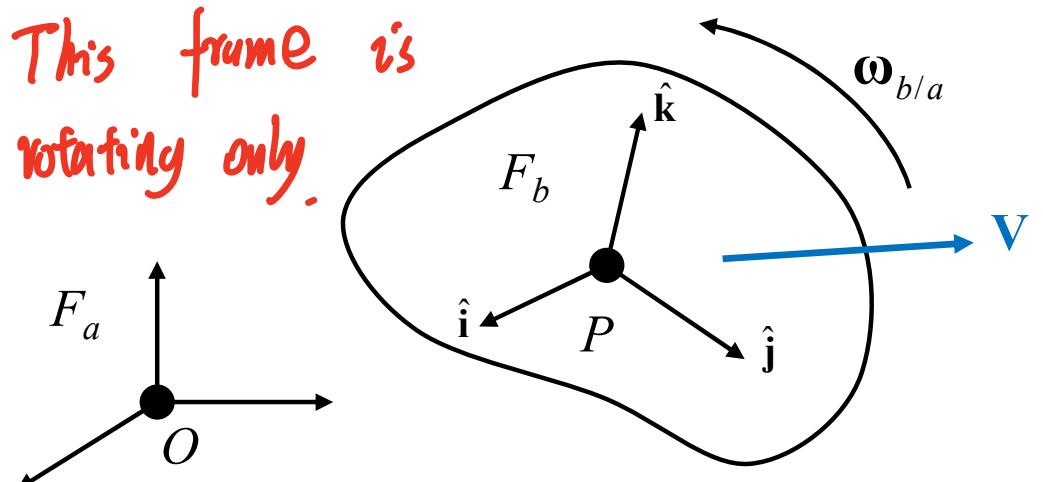
- Suppose we have two reference frames F_a and F_b that are rotating relative to one another
- For clarity we can consider F_a a “fixed frame” and a F_b “rotating frame”
- Consider a vector \mathbf{V} defined in terms of unit vectors fixed in F_b

$$\mathbf{V} = V_x \hat{\mathbf{i}} + V_y \hat{\mathbf{j}} + V_z \hat{\mathbf{k}}$$

- We can take derivative \mathbf{V} in F_a or in F_b . The results will be different

$$\frac{^a d\mathbf{V}}{dt} \neq \frac{^b d\mathbf{V}}{dt}$$

- But we can use transport theorem to relate to the two derivatives



$$\frac{^a d\mathbf{V}}{dt} = \frac{dV_x}{dt} \hat{\mathbf{i}} + V_x \frac{^a d\hat{\mathbf{i}}}{dt} + \frac{dV_y}{dt} \hat{\mathbf{j}} + V_y \frac{^a d\hat{\mathbf{j}}}{dt} + \frac{dV_z}{dt} \hat{\mathbf{k}} + V_z \frac{^a d\hat{\mathbf{k}}}{dt}$$

$$\frac{^a d\mathbf{V}}{dt} = \left(\frac{dV_x}{dt} \hat{\mathbf{i}} + \frac{dV_y}{dt} \hat{\mathbf{j}} + \frac{dV_z}{dt} \hat{\mathbf{k}} \right) + V_x \frac{^a d\hat{\mathbf{i}}}{dt} + V_y \frac{^a d\hat{\mathbf{j}}}{dt} + V_z \frac{^a d\hat{\mathbf{k}}}{dt}$$

Note that:

$$\frac{^b d\mathbf{V}}{dt} = \frac{dV_x}{dt} \hat{\mathbf{i}} + \frac{dV_y}{dt} \hat{\mathbf{j}} + \frac{dV_z}{dt} \hat{\mathbf{k}} \text{ since } \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \text{ are fixed in } F_b$$

$$\text{and } \frac{^a d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{b/a} \times \hat{\mathbf{i}}, \quad \frac{^a d\hat{\mathbf{j}}}{dt} = \boldsymbol{\omega}_{b/a} \times \hat{\mathbf{j}}, \quad \frac{^a d\hat{\mathbf{k}}}{dt} = \boldsymbol{\omega}_{b/a} \times \hat{\mathbf{k}}$$

$$\frac{^a d\mathbf{V}}{dt} = \frac{^b d\mathbf{V}}{dt} + V_x (\boldsymbol{\omega}_{b/a} \times \hat{\mathbf{i}}) + V_y (\boldsymbol{\omega}_{b/a} \times \hat{\mathbf{j}}) + V_z (\boldsymbol{\omega}_{b/a} \times \hat{\mathbf{k}})$$

$$\frac{^a d\mathbf{V}}{dt} = \frac{^b d\mathbf{V}}{dt} + \boldsymbol{\omega}_{b/a} \times (V_x \hat{\mathbf{i}} + V_y \hat{\mathbf{j}} + V_z \hat{\mathbf{k}})$$

$$\frac{^a d\mathbf{V}}{dt} = \frac{^b d\mathbf{V}}{dt} + \boldsymbol{\omega}_{b/a} \times \mathbf{V}$$

Useful Kinematical Relationships

Vector Derivative Transport Theorem (Equation of Coriolis)

$$\frac{^a d\mathbf{V}}{dt} = \frac{^b d\mathbf{V}}{dt} + \boldsymbol{\omega}_{b/a} \times \mathbf{V}$$

Properties of Angular Velocity and Angular Acceleration

$$\begin{aligned}\boldsymbol{\omega}_{c/a} &= \boldsymbol{\omega}_{c/b} + \boldsymbol{\omega}_{b/a} \\ \boldsymbol{\omega}_{b/a} &= -\boldsymbol{\omega}_{a/b}\end{aligned}\quad \text{这两个性质还是很重要的}$$

$$\boldsymbol{\alpha}_{b/a} = \frac{^a d\boldsymbol{\omega}_{b/a}}{dt}$$

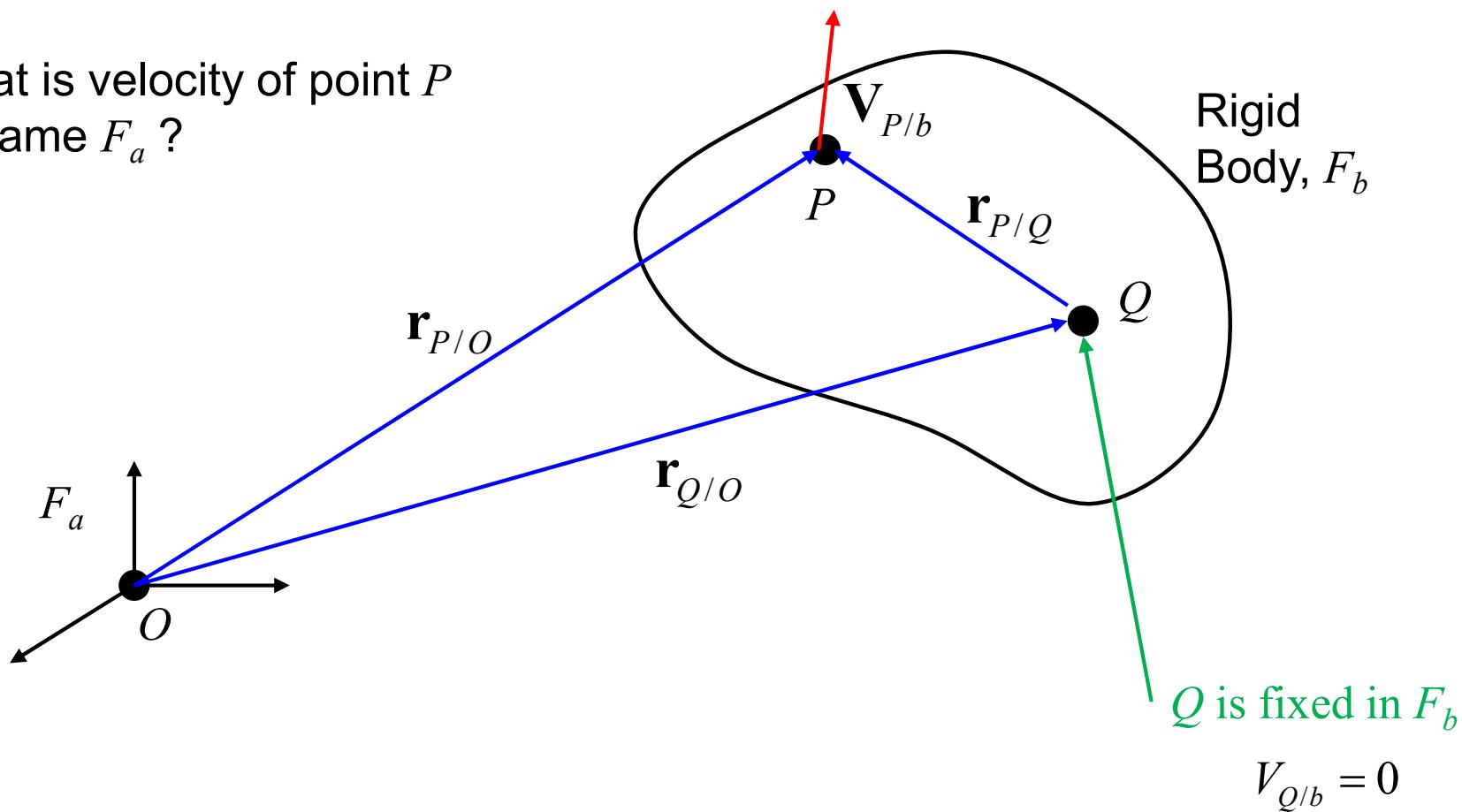
$$\text{Using transport theorem: } \boldsymbol{\alpha}_{b/a} = \frac{^a d\boldsymbol{\omega}_{b/a}}{dt} = \frac{^b d\boldsymbol{\omega}_{b/a}}{dt} + \boldsymbol{\omega}_{b/a} \times \boldsymbol{\omega}_{b/a}$$

$$\boldsymbol{\omega}_{b/a} \times \boldsymbol{\omega}_{b/a} = 0$$

$$\Rightarrow \boldsymbol{\alpha}_{b/a} = \frac{^a d\boldsymbol{\omega}_{b/a}}{dt} = \frac{^b d\boldsymbol{\omega}_{b/a}}{dt}$$

Velocity in Moving Body (or Moving Reference Frame)

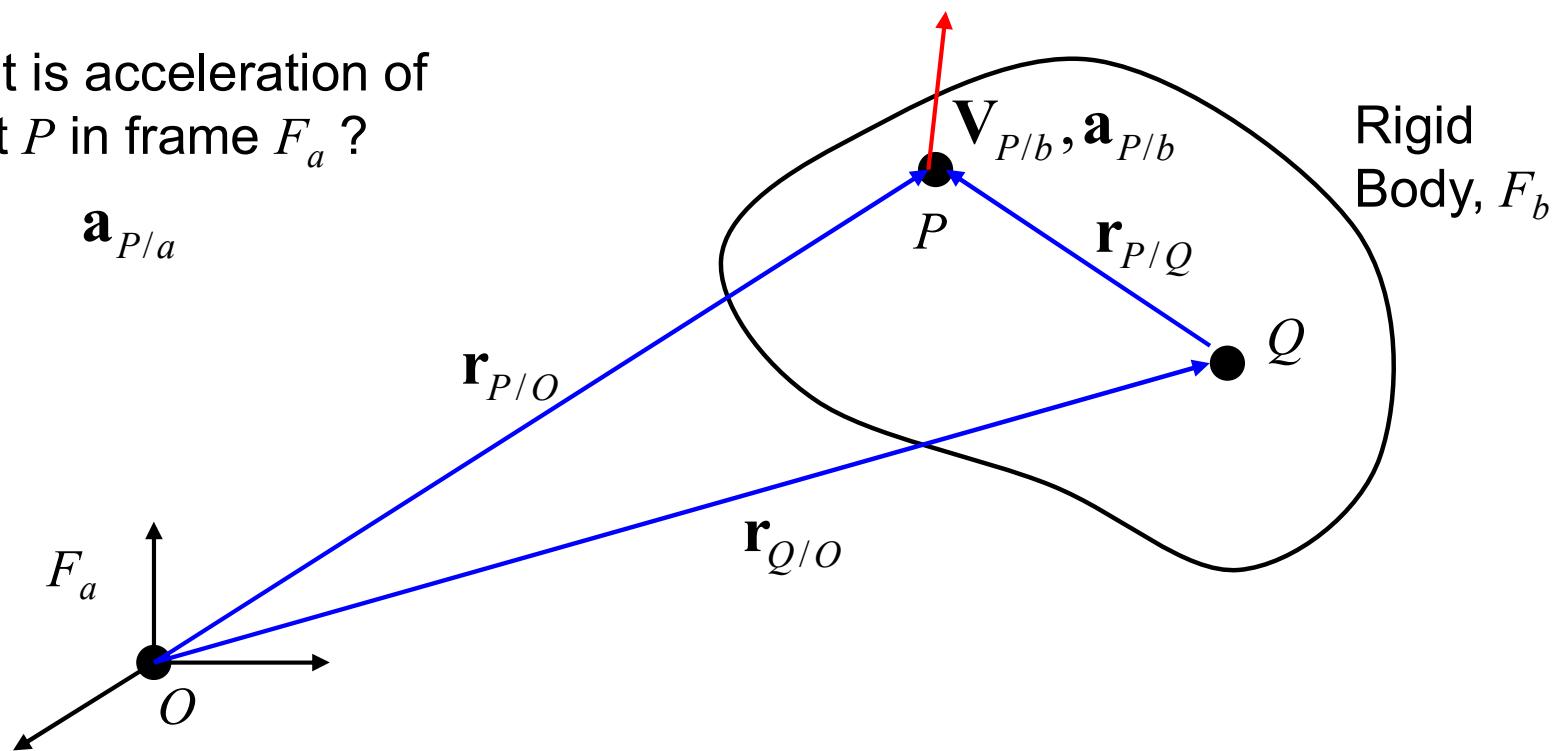
What is velocity of point P in frame F_a ?



$$\mathbf{V}_{P/a} = \mathbf{V}_{P/b} + \mathbf{V}_{Q/a} + \boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q}$$

Moving Body (or Moving Reference Frame)

What is acceleration of point P in frame F_a ?



$$\mathbf{a}_{P/a} = \frac{d\mathbf{V}_{P/a}}{dt}$$

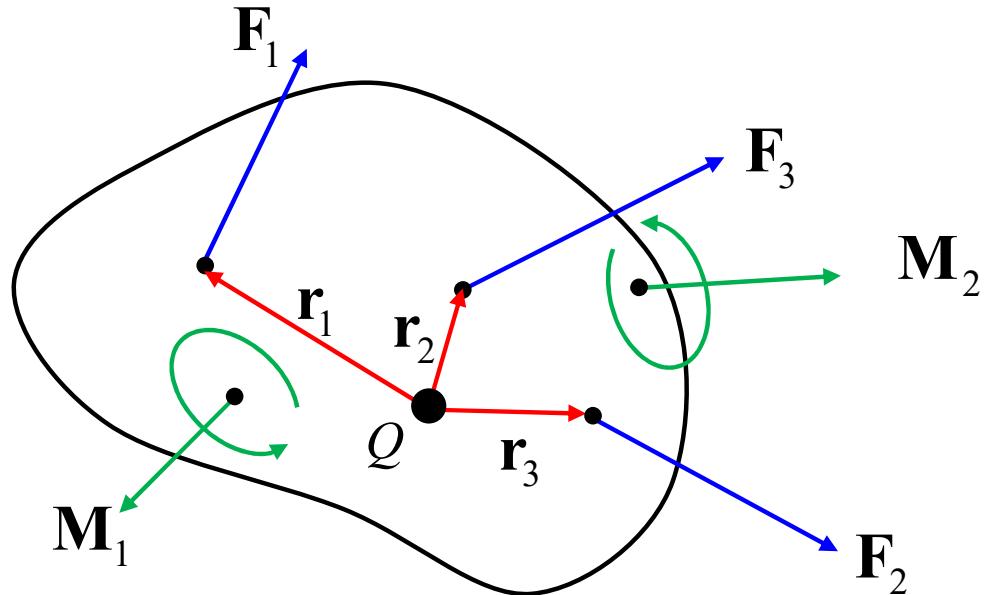
$$\mathbf{a}_{P/a} = \mathbf{a}_{P/b} + \mathbf{a}_{Q/a} + \mathbf{a}_{b/a} \times \mathbf{r}_{P/Q} + \boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q}) + 2\boldsymbol{\omega}_{b/a} \times \mathbf{V}_{P/b}$$

Centripetal Acceleration

Coriolis Acceleration

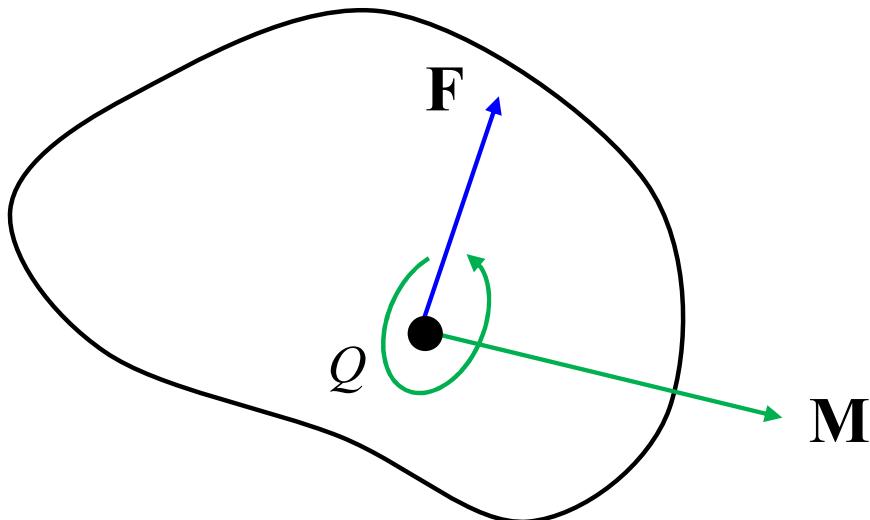
Resultant Force and Moment on a Rigid Body

What if we have multiple forces and moments acting on a rigid body?



For forces need the force vector and a position vector to its line of action

For applied moments (a.k.a. torques), point of action is irrelevant for rigid body dynamics



Resultant force and moment

$$\mathbf{F} = \sum_i \mathbf{F}_i$$

$$\mathbf{M} = \sum_i (\mathbf{M}_i + \mathbf{r}_i \times \mathbf{F}_i)$$

Rigid Body Equations of Motion

Applying Newton's Laws We Can Derive Equations of Motion of an unconstrained Rigid Body.

Translational Dynamics in Inertial Coordinates:

$\mathbf{V} = \mathbf{V}_{G/e}$, where G = origin of the body frame coordinate system (typically the rotorcraft CG)

$$\mathbf{F} = m \frac{^e d\mathbf{V}}{dt}$$

$$\begin{bmatrix} F_N \\ F_E \\ F_D \end{bmatrix} = m \begin{bmatrix} \dot{V}_N \\ \dot{V}_E \\ \dot{V}_D \end{bmatrix}$$

Translational Dynamics in Body Coordinates:

Apply equation of Coriolis (derivative transport theorem)

$$\mathbf{F}^g = m \left(\frac{^g d\mathbf{V}^g}{dt} + \boldsymbol{\omega}_{g/e}^g \times \mathbf{V}^g \right)$$

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}$$



We analyze
force and
moments in B .



Rigid Body Equations of Motion

Rotational Dynamics in Inertial Coordinates:

$$\mathbf{M} = \frac{^e d\mathbf{H}}{dt}$$

\mathbf{H} = Angular momentum vector

Rotational Dynamics in Body Coordinates:

$$\mathbf{H}^g = \mathbf{I}^g \boldsymbol{\omega}^g$$

$\boldsymbol{\omega}^g = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

Because $\mathbf{I} \equiv$ does not change
in B

$$\mathbf{I}^g = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = \text{Inertia Matrix in Body Coordinate System}$$

$$\mathbf{M}^g = \mathbf{I}^g \frac{^g d\boldsymbol{\omega}_{g/e}^g}{dt} + \boldsymbol{\omega}_{g/e}^g \times \mathbf{I}^g \boldsymbol{\omega}_{g/e}^g$$

Rigid Body Equations of Motion

Rotational Dynamics in Body Coordinates:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \mathbf{I}_B \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} (I_z - I_y)qr + I_{yz}(r^2 - q^2) + I_{xy}rp - I_{xz}pq \\ (I_x - I_z)rp + I_{xz}(p^2 - r^2) + I_{yz}pq - I_{xy}qr \\ (I_y - I_x)pq + I_{xy}(q^2 - p^2) + I_{xz}qr - I_{yz}rp \end{bmatrix}$$

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_x \dot{p} \\ I_y \dot{q} \\ I_z \dot{r} \end{bmatrix} + \begin{bmatrix} -I_{xy}\dot{q} - I_{xz}\dot{r} \\ -I_{xy}\dot{p} - I_{yz}\dot{r} \\ -I_{xz}\dot{p} - I_{yz}\dot{q} \end{bmatrix} + \begin{bmatrix} (I_z - I_y)qr + I_{yz}(r^2 - q^2) + I_{xy}rp - I_{xz}pq \\ (I_x - I_z)rp + I_{xz}(p^2 - r^2) + I_{yz}pq - I_{xy}qr \\ (I_y - I_x)pq + I_{xy}(q^2 - p^2) + I_{xz}qr - I_{yz}rp \end{bmatrix}$$

(Can usually assume the xz plane of the body frame is a plane of symmetry. This implies that $I_{xy} = I_{yz} = 0$.)

↓
Inertia coupling

Euler Angles

Euler angles define the aircraft attitude or the orientation of the aircraft and its body-fixed coordinate system relative to the Earth-fixed coordinate system.

ϕ = Roll Attitude or Bank Angle

θ = Pitch Attitude

ψ = Yaw Attitude or Heading

Defined using a sequence of three rotations that transform a set of axes that are initially aligned with the Earth-fixed system so that they are aligned with the body-fixed coordinate system of the aircraft.

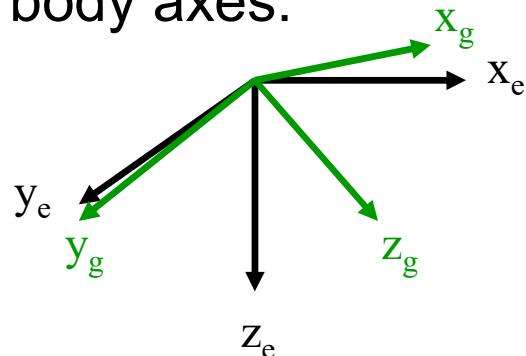
The order of the sequence of rotations is very important. Using a different order of rotations will result in different values for the Euler angles.

For aircraft, the standard order of the sequence of rotations is always:
Yaw (about z-axis), then pitch (about y-axis), then roll (about x-axis).

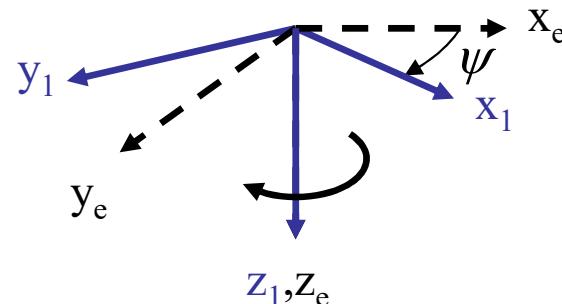
Euler angles are used to do coordinate system transformations between Earth-fixed and body-fixed systems.

Euler Angles – Standard Sequence of Rotations

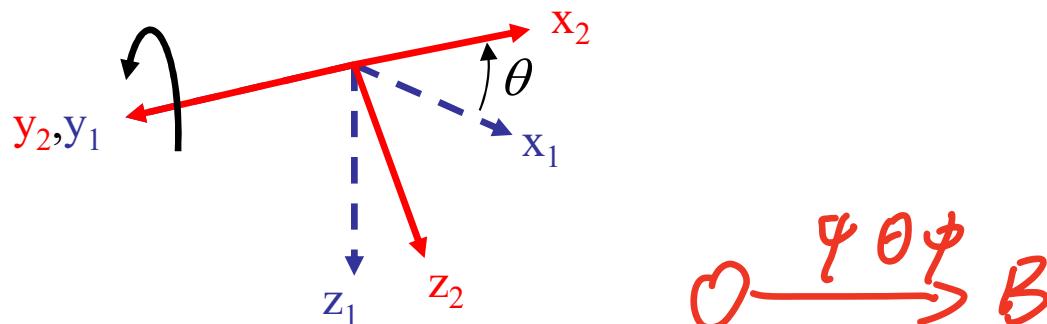
Starting with earth-fixed axes, find sequence of rotations that align it with body axes:



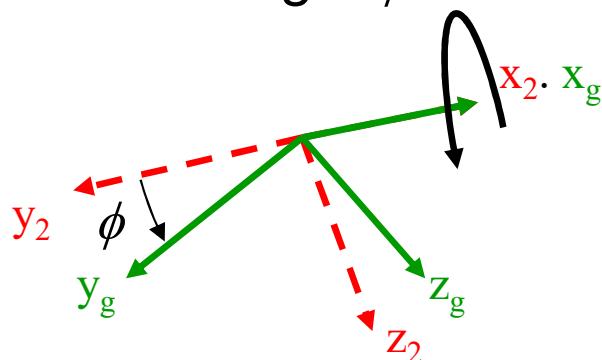
1. Rotate about z_E axis by heading angle ψ



2. Rotate about y_1 axis by pitch attitude angle θ

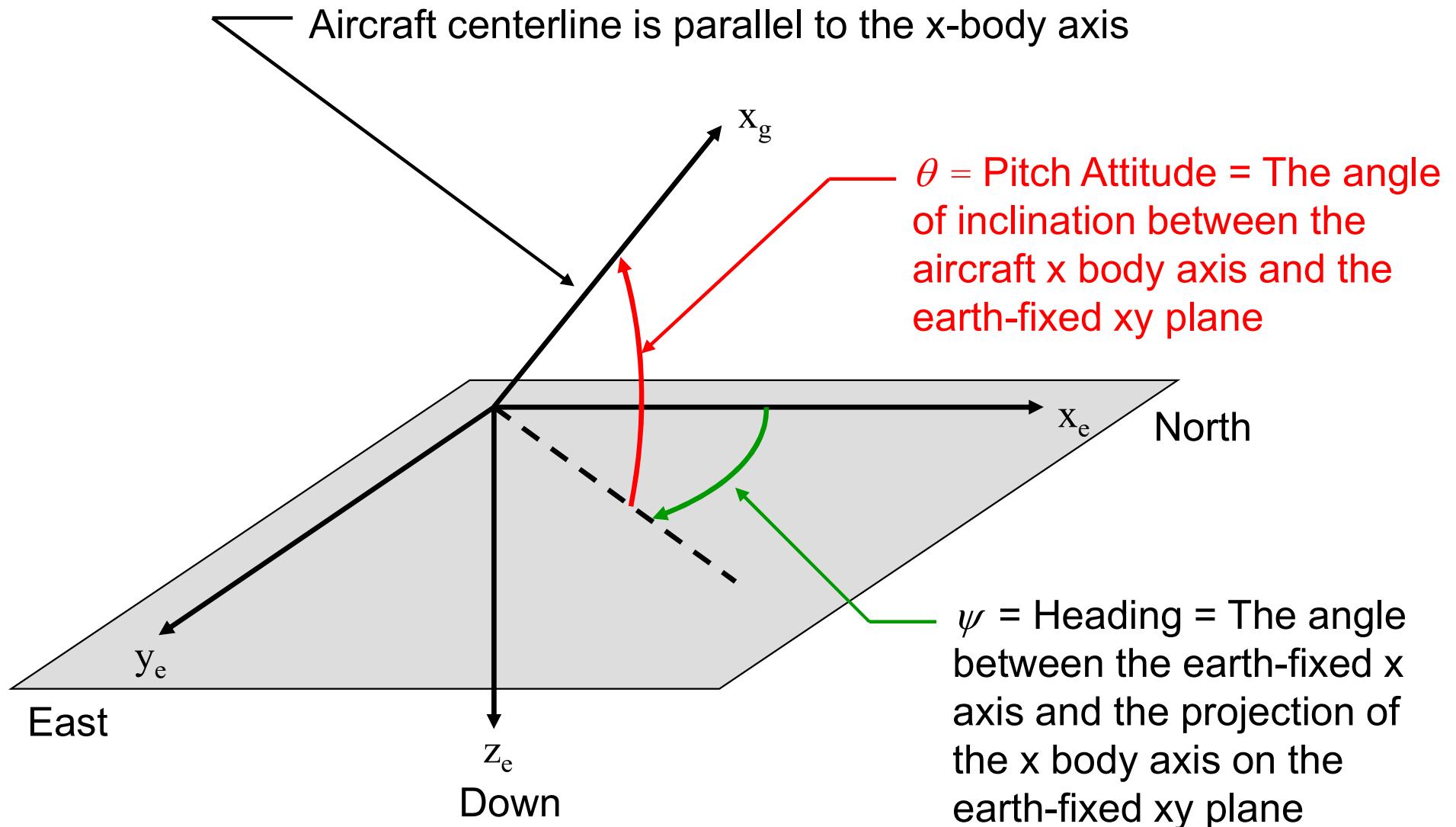


3. Rotate about x_2 axis by roll attitude angle ϕ



推导一下 $P g r$ 与 ψ, θ, ϕ 的关系。

Euler Angles – interpretation of pitch and heading



Euler Angles

To eliminate some of the ambiguity we normally we constrain the Euler angles as follows:



$$\begin{aligned} -180^\circ &\leq \phi \leq 180^\circ \\ -90^\circ &\leq \theta \leq 90^\circ \\ -180^\circ &\leq \psi \leq 180^\circ \text{ or } 0^\circ \leq \psi \leq 360^\circ \end{aligned}$$

Euler Angle Kinematics: Time rate of change of Euler angles are related to the body-axis angular rates by the following equations:

$$\begin{aligned} \dot{\phi} &= p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\ \dot{\theta} &= q \cos \phi - r \sin \phi \\ \dot{\psi} &= q \frac{\sin \phi}{\cos \theta} + r \frac{\cos \phi}{\cos \theta} \end{aligned}$$

这有了，但我还是自己
推导一下。

Coordinate System Transformations

Can derive transformation matrices between Earth-fixed coordinate system and body coordinate system using Euler angles:

$T_{g/e}$ = Coordinate system transform from Earth coordinate system to Body coordinate system

$$\mathbf{V}^g = T_{g/e} \mathbf{V}^e$$

$$T_{g/e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$T_\phi \qquad \qquad \qquad T_\theta \qquad \qquad \qquad T_\psi$$

$$T_{g/e} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Coordinate System Transformations (cont'd)

Can use to write gravity vector in body coordinate system:

$$\mathbf{g}^g = T_{g/e} \mathbf{g}^e = T_{g/e} \begin{bmatrix} 0 \\ 0 \\ g_D \end{bmatrix}, \quad g_D \approx 9.81 \text{ m/s}^2 \approx 32.2 \text{ ft/s}^2, \quad \mathbf{g}^g = g_D \begin{bmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{bmatrix}$$

Use inverse transform to determine North, East, Down velocities from body velocities. Direction Cosine Matrices are orthogonal – can get inverse by taking transpose of matrix:

$$T_{e/g} = T_{g/e}^{-1} = T_{g/e}^T$$

$$\mathbf{V}^e = \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix} = T_{e/g} \mathbf{V}^g = T_{e/g} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Aircraft 6-DOF Rigid Body Equations of Motion

$$\dot{u} = \frac{X}{m} - g \sin \theta - qw + rv$$

$$\dot{v} = \frac{Y}{m} + g \cos \theta \sin \phi - ru + pw$$

$$\dot{w} = \frac{Z}{m} + g \cos \theta \cos \phi - pv + qu$$

$$\dot{p} = \frac{1}{I_x I_z - I_{xz}^2} \left(I_z \underline{L} + I_{xz} \underline{N} + I_{xz} (I_x - I_y + I_z) pq - (I_z^2 - I_z I_y + I_{xz}^2) qr \right)$$

$$\dot{q} = \frac{1}{I_y} \left(\underline{M} - (I_x - I_z) rp - I_{xz} (p^2 - r^2) \right)$$

$$\dot{r} = \frac{1}{I_x I_z - I_{xz}^2} \left(I_x N + I_{xz} L - I_{xz} (I_x - I_y + I_z) qr + (I_x^2 - I_x I_y + I_{xz}^2) pq \right)$$

$$\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta$$

$$\dot{\theta} = q \cos \phi - r \sin \phi$$

$$\dot{\psi} = q \sin \phi / \cos \theta + r \cos \phi / \cos \theta$$

$$\dot{x} = u \cos \theta \cos \psi + v (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) + w (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)$$

$$\dot{y} = u \cos \theta \sin \psi + v (\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi) + w (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)$$

$$\dot{z} = -u \sin \theta + v \sin \phi \cos \theta + w \cos \phi \cos \theta$$

Flight Simulation

- Note that the equations of motion include 6 parameters: X, Y, Z, L, M, N that represent the total aerodynamic forces and moments resolved at the aircraft CG
- Flight simulation involves the calculation of these forces and moments and the integration of the equations of motion based on prescribed control inputs and external disturbances)
- For fixed-wing aircraft, the forces and moments can be represented with look-up tables based on wind-tunnel data. *helicopter cannot use*
- For helicopters the problem is more difficult: *it's because too much if, we cannot capture all information using wind-tunnel*
 - Additional differential equations need to be solved to account for rotor dynamics and the engine dynamics
 - There are complex aerodynamic interactions between the rotor and the fuselage and tail
- Helicopter simulation is usually formulated using a modular approach, in which the forces and moments due to sub-components (fuselage, main rotor, tail surfaces, tail rotor) are calculated individually and summed to get the total resultant forces and moments