ZK notes

Some keynotes made while learning about ZK, SNARK, STARK and ZK-VM.

Lectures: Youtube Playlist

1 SNARK

SNARK is not necessary ZK and can be quite different from STARK and ZK-VM. But it should be a good starting ground to understand some basic concept.

1.1 Polynomial Commitment Scheme (PCS)

A PCS is a functional commitment for the family $\mathcal{F} \in \mathbb{F}_p^{(\leq d)}[X]$. A prover commits to univariate polynomial f in $\mathbb{F}_p^{(\leq d)}[X]$ and can later prove to the verifier that $v = f(u)$ for public $u, v \in \mathbb{F}_p$.

Some examples PCS (here we focus on KZG'10):

1. Bulletproof (elliptic curves, but verification is $O(d)$)
2. KZG'10 (trusted setup, bilinear), Dory'20 (bilinear)
3. Dark'20 (groups of unknown order)
4. Hash (FRI)

1.2 KZG'10

Set cyclic group $\mathbb{G} = \{0, G, 2 \cdot G, 3 \cdot G, \ldots, (p-1) \cdot G\}$ of order p.

Setup algorithm

1. Sample random $\alpha \in \mathbb{F}_p$.
2. $pp = (H_0 = G, H_1 = \alpha \cdot G, \ldots, H_d = \alpha^d \cdot G) \in \mathbb{G}^{d+1}$.
3. **delete** α (i.e., A trusted setup)

Commitment

In short: $commit(pp, f) \rightarrow com_f$, where $com_f = f(\alpha) \cdot G \in \mathbb{G}$.[1]

Remark. *As a result, the committed message is extremely short (an element G) regardless how large our polynomial is.*

But α is deleted during trusted setup, how does prover compute $f(\alpha)$? Observe:

[1] This is not a hiding commitment

$\Rightarrow f(X) = f_0 + f_1 X + \ldots + f_d X^d$

$\Rightarrow f(\alpha) \cdot G = f_0 \cdot G + f_1 \cdot \alpha \cdot G + \ldots + f_d \cdot \alpha^d \cdot G$

$\Rightarrow f(\alpha) \cdot G = f_0 \cdot H_0 + f_1 \cdot H_1 + \ldots + f_d \cdot H_d$

Evaluation

How to prove $f(u) = v$?

First Observe:

1. If $f(u) = v \iff u$ is a root of polynomial $\hat{f}(X) = f(X) - v$.
2. If u is a root of $\hat{f}(X) \iff \hat{f}(X)$ is divisible by $(x - u)$.
3. $f(u) = v \iff \exists q \in \mathbb{F}_p^{(\leq d)}[X]$ s.t. $q(X)(X - u) = f(X) - v$

The prover then computes quotient polynomial $q(X) = (f(X) - v)/(X - u)$ and sends com_q to verifier.

The verifier accepts if $(\alpha - u) \cdot com_q = com_f - v \cdot G$.

LHS:

$\Rightarrow (\alpha - u) \cdot com_q$

$\Rightarrow (\alpha - u) \cdot (q_0 H_0 + q_1 H_1 + \ldots + q_d H_d)$

$\Rightarrow (\alpha - u) \cdot (q_0 G + q_1 \alpha G + \ldots + q_d \alpha^d G)$

$\Rightarrow commit(pp, (X - u)q(X))$

RHS is similar. [2]

Remark. *The verification work only take constant time, regardless of the degree of the polynomial.*

Extension

1. KZG for k-variant polynomial (PST'13)
2. Batch proofs: prove a batch of commitments in a single step.

[2]Important: verifier does not actually need to know about α. The *pairing* is used here to allow verifier to compute $(\alpha - u) * com_q$ with only G and H_1

1.3 A Useful Observation

A Useful and important observation.

For $0 \neq f \in \mathbb{F}_p^{(\leq d)}[X]$. Let r be a random point $r \leftarrow \mathbb{F}_p$, the probability $pr[f(r) = 0] = d/p$.[3]

For large enough p and reasonable d, e.g., $p \approx 2^{256}$ and $d \leq 2^{40}$, d/p is negligible.

Lemma 1. *for $r \leftarrow \mathbb{F}_p$, if $f(r) = 0$, we can conclude f is identically zero w.h.p.*[4]

Further more, with the same settings.

Lemma 2. *Let $f, g \in \mathbb{F}_p^{(\leq d)}[X]$. For $r \leftarrow \mathbb{F}_p$, if $f(r) = g(r)$ then $f = g$.*

$\Rightarrow f(r) - g(r) = 0$

$\Rightarrow$ Let $h = f - g$, from Lemma 1, h is identical zero w.h.p.

$\Rightarrow f = g$, w.h.p

1.4 Zero Test On H

One of the (and the simplest) poly-IOP tasks that the verifier would like the prover to do.

Let $\omega \in \mathbb{F}_p$ be a primitive k-th root of unity (such that $\omega^k = 1$ and $\omega^n \neq 1$ for $n < k$).

Set $H = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\} \in \mathbb{F}_p$.

Let polynomial $f \in \mathbb{F}_p^{(\leq d)}[X]$.

A zero test is a test from verifier to prover to prove that: f **is identically zero on set** H**.**

Lemma 3. *f is zero on H iff $f(X)$ is divisible by $X^k - 1$.*

1. The prover can compute the quotient polynomial $q(X) = f(X)/(X^k - 1)$. and send the commitment of q to the verifier.
2. The verifier then choose random $r \in \mathbb{F}_p$ and ask prover to open $f(X)$ and $q(X)$ at r.
3. The verifier then accepts the test if $f(r) = q(r) \cdot (r^k - 1)$

As mentioned in Lemma 2, two polynomials that agree on a random point r has a high probability that the two polynomials are identical. Therefore, the above implies $f(X) = q(X)(X^k - 1)$. This proves $f(X)$ is indeed divisible by $X^k - 1$, hence from Lemma 3, f is identical on H.

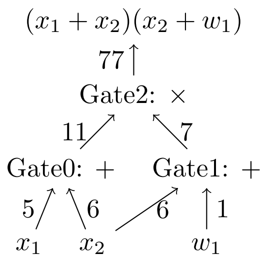
[3] Given f has at most d roots and p elements

[4] Also holds for multivariate polynomial, see SZDL lemma.

1.5 Interpolate Polynomial

Plonk.

1.5.1 Compile a circuit into a computation trace



inputs:	5	6	1
	left	right	out
Gate0	5	6	11
Gate1	6	1	7
Gate2	11	7	77

1.5.2 Encoding the trace as polynomial

$C \leftarrow$: total # of gates

$I \leftarrow |I_x| + |I_w|$: # inputs to circuit

$d \leftarrow 3|C| + |I| = 12$ for our example. (3 since each gate has 3 inputs).

$H \leftarrow \{1, \omega, \ldots, \omega^{(d-1)}\}$

The goal here is to interpolate a polynomial P that encodes the computation trace. To achieve that, we want to

1. let P encodes all inputs, such that $P(\omega^{-j}) =$ inputs # j for all $j = 1, \ldots, |I|$.

2. let P encodes all wires, such that $\forall l = 0, \ldots, |C| - 1$:

   (a) $P(\omega^{3l}) =$ left input of gate # l.

   (b) $P(\omega^{3l+1}) =$ right input of gate # l.

   (c) $P(\omega^{3l+2}) =$ output of gate # l.

This results in 12 constraints for P, which means there exists a P with degree at most 11 that satisfies all the constraints. Prover can then constructs P using Fast Fourier Transform in time $O(d \log d)$, which I don't know how yet.

1.5.3 Prove that encoding is correct

There are four things to prove.

Inputs are correctly encoded.
Both prover and verifier takes input x and interpolate a polynomial $v(X) \in \mathbb{F}_p^{(\leq d)}[X]$ that satisfies $\forall j = 1, \ldots, |I_x| : v(\omega^{-j}) =$input #j.

From the slides, it says construction takes time linear to the size of x, shouldn't it still be using FFT and the time is actually $O(n \log n)$?.

Then prover just proves that $P(y) - v(y) = 0 \;\; \forall y \in H_{inp}$ where H_{inp} is all the input points, i.e., $\{\omega^{-1}, \ldots, \omega^{-|I_x|}\}$. This can be done using zero-test.

Gates evaluations are correctly encoded.
Interpolate selector polynomial $S(X) \in \mathbb{F}_p^{(\leq d)}[X]$ such that $\forall l = 0, \ldots, |C| - 1$:

1. $S(\omega^{3l}) = 1$ if gate l is addition
2. $S(\omega^{3l}) = 0$ if gate l is multiplication

Observe $\forall y \in H_{gates} = \{1, \omega^3, \omega^6, \ldots, \omega^{3(|C|-1)}\}$:

$$S(y) \cdot [P(y) + P(\omega y)] + (1 - S(y)) \cdot P(y) \cdot P(\omega y) = P(\omega^2 y)$$

When $S(y) = 1$, which means a gate is addition gate, and $[P(y)+P(\omega y)]$ encodes the two inputs of that gate, which equals to $P(\omega^2 y)$ (where $\omega^2 y$ encodes the output of the circuit). At the same time, since the gate is addition, the right operand $((1 - S(y)) \cdot \ldots)$ must evaluated to zero. The same goes for when $S(y) = 0$, i.e., multiplication gate.

Overall, another zero-test on H_{gates}.

Wirings are encoded correctly.
For example, the input 6 flows to right input of Gate0 and left input of Gate1, we need to prove that does data flows (wiring) are encoded correctly. For our examples, the equivalent constraints are:

$$\begin{cases} P(\omega^{-2}) = P(\omega^1) = P(\omega^3) \\ P(\omega^{-1}) = P(\omega^0) \\ P(\omega^2) = P(\omega^6) \\ P(\omega^3) = P(\omega^4) \end{cases}$$

To do so, define a rotation polynomial $W : H \to H$ such that:

$$\begin{cases} W(\omega^{-2}, \omega^1, \omega^3) = (\omega^3, \omega^{-2}, \omega^1) \\ W(\omega^{-1}, \omega^0) = (\omega^0, \omega^1) \\ \ldots \end{cases}$$

Lemma 4. $\forall y \in H : P(y) = P(W(y)) \Rightarrow$ *wiring constraints are satisfied.*

Since W has degree of d and P has degree of d, the verification can takes quadratic time. The trick here is to use prod-check (another IOP check) to reduce it to linear complexity. Not sure how to yet, another time. :P

Outputs are encoded correctly (is zero).
Just let prover to open P at the output of the final gate.

2 STARK

This follows the tutorial at here

2.1 Extended Euclidean Algorithm

Refresh myself with the Extended Euclidean algorithm...

Extended Euclidean algorithm is an extension of the Euclidean algorithm, in addition to computing the greatest common divisor of two integer a and b, it also gives x and y such that:

$$ax + by = gcd(a, b)$$

Recall the standard Euclidean algorithm in recursive form: $gcd(a, b) = gcd(b, a \mod b)$, stops at $gcd(r, 0)$ and returns r.

The Extended Euclidean Algorithm works the same, but keeps the quotient at each iterations.

$$\begin{aligned}
r_0 = a \quad s_0 &= 1 & t_0 &= 0 \\
r_1 = b \quad s_1 &= 0 & t_1 &= 1 \\
&\cdots \\
r_i &= r_{i-2} - q_{i-1} r_{i-1} \\
s_i &= s_{i-2} - q_{i-1} s_{i-1} \\
t_i &= t_{i-2} - q_{i-1} t_{i-1}
\end{aligned}$$

The EEA is useful as it defines the inverse of an element under $\mathbb{F}_p$. Give an element $x \in \mathbb{F}_p$, the inverse of x is therefore a:

$\Rightarrow gcd(x, p) = 1$

$\Rightarrow ax + bp \equiv 1 \mod p$

$\Rightarrow ax \equiv 1 \mod p$

2.2 Lagrange Interpolation

Refresh myself with Lagrange Interpolation.

The Lagrange Interpolation returns a polynomial of lowest degree that pass through a set of points D.

As an example, consider three points $(3, 1), (4, 2), (7, -3)$. The algorithm first construct three polynomial such that each polynomial f_i go through $(x_i, 1)$ for the i^{th} point in D and $(x_j, 0)\ \forall j \neq i$. This is extremely easy:

$$\begin{cases}
f_1(x) = {}^1\!/_4 (x-4)(x-7) \\
f_2(x) = {}^1\!/_3 (x-3)(x-7) \\
f_3(x) = {}^1\!/_{12} (x-3)(x-4)
\end{cases}$$

Finally, just scale each f_i so when $x = x_i \implies y = y_i$, and let $p = \sum_i f_i$

$$p(X) = f_1(X) + 2 * f_2(X) + -3 * f_3(X)$$

We can also prove the resulting polynomial $p(X)$ is the unique polynomial of degree $(n-1)$ that go through those points, by contridiciton.

Assuming the opposite, let $q(X)$ be another polynomial of degree $n-1$ that statistifes points $d \in D$. Since $q(X) \neq p(X)$, $r(X) = q(X) - p(X)$ is not a zero polynomial and $degree(r) \leq (n-1)$. However, we know for points $d \in D$, $r(X) = q(X) - p(X) = 0$, which means $degree(r) = n$, therefore contridiciton.

2.3 Generator

Not sure yet how generator is applied in SNARK or how it is produced — How to produce a generator of multiplicative group of order prime?

2.4 Polynomial Implementation

Just some notes on the notations here.

leading coefficient is the coefficient of the highest degree. It is confusing as I interpreted it as the first coefficient (it is the last one).

The zerofier takes a set of domain D where $\forall x \in D$, the resulting polynomial $f(x) = 0$.