CEE 451

Coding Problem Number 3: Euler Step Method and Predictor-Corrector Method for Solving First-order Differential Equations.

We wish to solve the following equation and boundary condition

$$\frac{dy}{dx} = f(x,y) \qquad \left. y\right|_{x=x_A} = y_A \tag{1a,b}$$

for the function y(x). Here x_A and y_A are specified, and the function f(x,y) is known.

We discretize x as

$$x_i = x_A + (i-1)\Delta x \quad ,i = 1, 2, 3\ldots \tag{2}$$

Now (1a) integrates exactly to give

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x,y)dx \tag{3}$$

We cannot implement (3) directly, because y, which is the parameter we want to solve for, is needed in order to compute y.

Euler Step Method Now suppose we know y_i. If Δx is not too large, we can approximate

$$f(x,y) \cong f(x_i,y_i) \quad , \quad x_i \le x \le x_{i+1} \tag{4}$$

In this approximation, (3) reduces to

$$y_{i+1} = y_i + f(x_i,y_i)\Delta x \tag{5}$$

Thus if we know y_i we can compute y_{i+1}, out to as far a value of x_i as we want. We start with $y_1 = y_A$, a value we know from the boundary condition. The Euler Step method is an extension of the Rectangular Rule.

Predictor-Corrector Method (2nd Order Runge-Kutta Scheme) A better approximation of (3) uses the trapezoidal rule:

$$y_{i+1} = y_i + \frac{1}{2}\left[f(x_i,y_i) + f(x_{i+1},y_{i+1})\right]\Delta x \tag{6}$$

But again, in order to compute y_i on the left-hand side of the equation, we need to know y_i on the right-hand side. In principle, we can iterate using (e.g. Newton-Raphson) in order to do this. But a simpler way involves estimating y_{i+l} on the right-hand side using the Euler Step Method. Let y_p be the value of y_{i+1} predicted by (5);

$$y_p = y_i + f(x_i,y_i)\Delta x \tag{7}$$

We can now estimate y_{i+1} from (6) as

$$y_{i+1} = y_i + \frac{1}{2}\left[f(x_i,y_i) + f(x_{i+1},y_p)\right]\Delta x \tag{8}$$

Grid Invariance You should choose your value of Δx to be large enough for accuracy, but no larger. Suppose we want to compute y(x) from x_A to x_B. As in Coding Problem 2, we can divide the interval into N subintervals, so that

$$\Delta x = \frac{x_B - x_A}{N} \qquad (9)$$

We then discretize this interval as

$$x_i = x_A + (i-1)\Delta x \quad , \quad i = 1..N+1 \qquad (10)$$

We can now compute y from x_A to x_B using various values of N, e.g. N = 2, 4, 6, 8… By choosing even numbers, we know that the point $x_{N/2+1}$ always denotes a center point x_C.

Let $y_{C,N}$ and $y_{B,N}$ denote the values of y_C and y_B computed using N number of intervals

Now the relative errors ε_C and ε_B between the result of the computation using a value N = N_1 and a larger value $N_2 = N_1 + 2$ at $x_C = x_{N/2+1}$ (center point) and x_B, respectively, are given as

$$\varepsilon_C = \left| \frac{\left(y_{C,N_2} - y_{C,N_1}\right)}{\frac{1}{2}\left(y_{C,N_2} + y_{C,N_1}\right)} \right|$$

$$\varepsilon_B = \left| \frac{\left(y_{B,N_2} - y_{B,N_1}\right)}{\frac{1}{2}\left(y_{B,N_2} + y_{B,N_1}\right)} \right| \qquad (11a,b)$$

You can define grid invariance as where the maximum of ε_B and ε_C falls below some tolerance δ which you choose (e.g. 0.001 for 0.1% accuracy, or 0.0001 for 0.01% accuracy.

Problem:

Write a program to compute y(x) from x_A to x_B, where the function f(x,y) in (1a) is

$$f(x,y) = -k\frac{\sqrt{y}}{\ell n(x+1)} \qquad (12)$$

and k is a specified constant. Your program should use both the Euler Step Method and the Predictor-Corrector Method. Implement it for values k = 0.25 and 0.5, with x_A = 1, x_B = 5, and y_A = 4. Plot your results for some representative (even) values of N (4 or 5 values?). How large does N have to be to achieve grid-invariance, using a tolerance that you have chosen?

Your assignment should consist of a) a printout of your code + GUI, b) your choice for δ, and c) Plots of y versus x for N = 2, 4, 6 up to the value corresponding to grid-invariance within your specified tolerance.

NOTE As in Coding Problem 2, it will be easier if you use arrays in this problem. The first step is to write a) a working code that computes y(i) for all value x(i) =

xA + (i – 1)*Dx with specified values of xA, xB, yA and N with the Euler method and b) the same for the predictor-corrector method. Don't try to automate the code for grid invariance (so that it runs through N = 2, 4, 6… to within a tolerance) until you have completed steps a) and b).