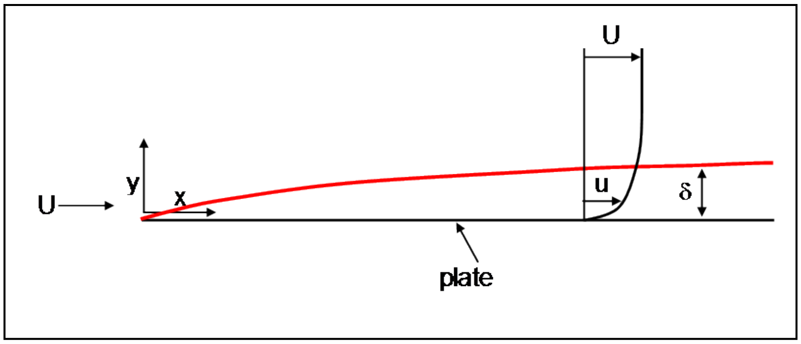
CEE 451G

Class Project

Laminar Boundary Layer on a Flat Plate



The equations governing a steady, laminar boundary layer on a flat plate are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

The boundary conditions are

$$u\big|_{y=0} = 0 \quad , \quad v\big|_{y=0} = 0 \quad , \quad u\big|_{y=\infty} = U$$

Continuity is identically satisfied with the use of a streamfunction ψ;

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x} \qquad (3)$$

The following similarity assumption is used to solve the problem:

$$\psi = \sqrt{U\nu x}\,F(\eta) \quad , \quad \eta = y\sqrt{\frac{U}{\nu x}} \qquad (4)$$

Substituting (4) into (1) yields the ordinary differential equation

$$F''' = -\frac{1}{2}FF'' \qquad (5)$$

The boundary conditions on F are

$$F(0) = 0 \quad , \quad F'(0) = 0 \quad , \quad F'(\infty) = 1 \qquad (6a,b,c)$$

Now (5) and (6) can be rewritten as

$$\begin{aligned} F' &= G \\ G' &= H \\ H' &= -\frac{1}{2}FH \\ F(0) &= 0 \\ G(0) &= 0 \\ G(\infty) &= 1 \end{aligned} \qquad (7a\text{-}f)$$

In (7f) replace ∞ with η_{large}, where η_{large} is some appropriately large value of η;

$$G(\eta_{large}) = 1 \tag{8}$$

In order to solve this problem by the shooting method, we in turn replace (8) with the boundary condition

$$H(0) = s \tag{9}$$

where s is a guessed value.

Now (7a,b,c) can be solved subject to (7d,e) and (9) using e.g. the Euler step method. Each value of s yields a different solution $F(\eta, s)$, $G(\eta, s)$ and $H(\eta, s)$. The correct value of s is the one that satisfies (8), i.e.

$$\Phi(s) = G(\eta_{large}, s) - 1 = 0 \tag{10}$$

To solve (10) it is necessary to solve the associated variational problem. Variational parameters are defined as:

$$F_V = \frac{\partial F}{\partial s} \quad , \quad G_V = \frac{\partial G}{\partial s} \quad , \quad H_V = \frac{\partial H}{\partial s} \tag{11a,b,c}$$

Now between (11a,b,c), (7a – e) and (9), the associated variational problem is found to be

$$\begin{aligned} F_V' &= G_V \\ G_V' &= H_V \\ H_V' &= -\frac{1}{2} F H_V - \frac{1}{2} H F_V \\ F_V(0) &= 0 \\ G_V(0) &= 0 \\ H_V(0) &= 1 \end{aligned} \tag{12a-f}$$

Condition (10) reduces to the Newton-Raphson problem

$$s^{new} = s - \frac{G(\eta_{large}, s) - 1}{G_V(\eta_{large,}\ s)} \tag{13}$$

Problem

Solve the problem using the shooting method. A reasonable guess for η_{large} should be about 10; your solution should not be too sensitive to the choice as long as it is in the right range. Solve for the correct value of s, and the functions $F(\eta)$, $G(\eta)$ and $H(\eta)$ at that value of s.

Use the solution to plot u/U as a function of η. The nominal boundary layer thickness $y = \delta(x)$ is attained where u = 0.99U. Find the value η_{99} where G = 0.99, and use this to find a relation for $\delta(x)$.

The boundary shear stress on the body is given as

$$\tau_o = \rho \nu \left. \frac{\partial u}{\partial y} \right|_{x=0} \tag{14}$$

Find the relation for τ_o. What is the dimensionless number $\tau_o/(\rho U^2)$ a function of?

Integrate τ_o from $x = 0$ to $x = L$ and determine the drag force F_D on the upper half of the plate (assuming that the plate has width b). What is the dimensionless combination $F_D/(\rho b L U^2)$ a function of?

A plate has a length $L = 0.2$ m and a width $b = 0.1$ m. The flowing fluid is air with a density ρ of 1.24 kg/m^3 and a kinematic viscosity ν of 1.5×10^{-5} m^2/s. What is the drag force F_D on one side of the plate for approach velocities U of 0.01 m/s, 0.1 m/s and 0.5 m/s?

Turn in a) a copy of your code and b) a writeup of your solution.