OPTIMIZATION

Xiaoxia Lin, Christoph Mueller

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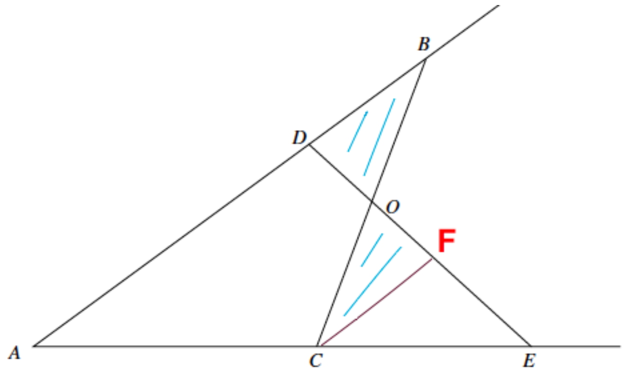
1 Exercise 1: Smallest area problem

Given an angle with vertex A and a point O in its interior. Pass a line BC through the point O that cuts off from the angle a triangle of minimal area.

Hint: proof that for the triangle of minimal area the segments $\overline{OB}$ and $\overline{OC}$ should be equal.



Solution:
Suppose line BC is such that $\overline{OB} = \overline{OC}$, and line DE is any other line passing through O.
We will see that area $\Delta ABC <$ area ΔADE.

As we can see from the figure:

$$\text{area } \Delta ABC = \text{area } ADOC + \text{area } \Delta DOB$$
$$\text{area } \Delta ADE = \text{area } ADOC + \text{area } \Delta COE$$

Prove $\Delta ABC <$ area ΔADE is equivalent to prove that area $\Delta DOB <$ area ΔCOE.

Let F be a point such that $\overline{BD}$ is parallel to $\overline{CF}$. Since $\overline{OB} = \overline{OC}$ and $\angle DOB = \angle COF$, area $\Delta DOB =$ area ΔCOF.

Therefore $\Delta DOB <$ (area ΔCOF + area ΔCEF) = area ΔCOE.

Note: if the line DE is on the other side of BC, the prove is proceed exactly the same way.

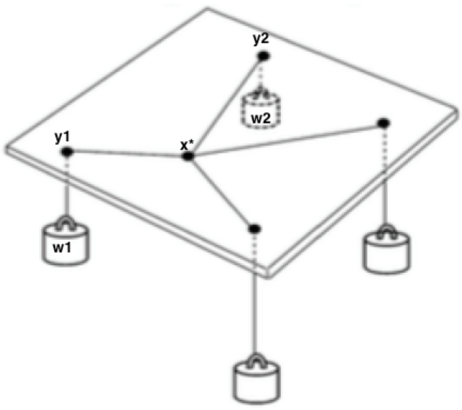
2 Exercise 2: The Fermat-Weber point of a set of points

We want to find a point x^* in the plane whose sum of weighted distances from a given set of points y_1,...,y_m is minimized. Mathematically, the problem is

$$\text{minimize } \sum_{i=1}^{m} \omega_i \|x^* - y_i\|, \text{ subject to } x^* \in \mathbf{R}^n$$

where $w_1, \ldots w_m$ are given positive real numbers.

2.1. Show that there exists a global minimum for this problem and that it can be realized by means of the mechanical model shown in the figure.



Solution:
Define $f : \mathbf{R}^n \to \mathbf{R}$:

$$f(x) = \sum_{i=1}^{m} \omega_i \|x - y_i\|. \tag{1}$$

First of all, note that if the given points are collinear, i.e. $y_1, \cdots y_m$ lie in a straight line, the minimum of f(x) is achieved at any one-dimensional median.

Let us study the case that the points y_i are not collinear.

PROPOSITION*: If the points are not collinear, then f is strictly convex.*

Proof: For $x_1, x_2, y \in \mathbf{R}^n$ with $x_1 \neq x_2$ and $0 < \lambda < 1$ the Cauchy-Schwarz inequality implies

$$\begin{aligned}
&\big\|\lambda(x_1 - y) + (1-\lambda)(x_2 - y)\big\|^2 \\
=&\big\|\lambda(x_1 - y)\big\|^2 + 2(\lambda(x_1 - y))^T((1-\lambda)(x_2 - y)) + \big\|(1-\lambda)(x_2 - y)\big\|^2 \\
\leq&\big\|\lambda(x_1 - y)\big\|^2 + 2\big\|(\lambda x_1 - y)\big\| \cdot \big\|(1-\lambda)(x_2 - y)\big\| + \big\|(1-\lambda)(x_2 - y)\big\|^2 \\
=&(\lambda\big\|(x_1 - y)\big\| + (1-\lambda)\big\|(x_2 - y)\big\|)^2
\end{aligned}$$

with strict inequality if and only if x_1, x_2 and y are not collinear.

We tailor it for our problem, multiplying both side by $\omega_i > 0$ and summing up for all the $y_i's$:

$$\sum_{i=1}^{m} \omega_i \big\| \lambda(x_1 - y_i) + (1-\lambda)(x_2 - y_i) \big\| \le \lambda \sum_{i=1}^{m} \omega_i \big\| (x_1 - y_i) \big\| + (1-\lambda) \sum_{i=1}^{m} \omega_i \big\| (x_2 - y_i) \big\|$$
$$\sum_{i=1}^{m} \omega_i \big\| (\lambda x_1 + (1-\lambda) x_2) - y_i \big\| \le \lambda \sum_{i=1}^{m} \omega_i \big\| (x_1 - y_i) \big\| + (1-\lambda) \sum_{i=1}^{m} \omega_i \big\| (x_2 - y_i) \big\|$$
$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

with strict inequality if and only if x_1, x_2 and y_i are not collinear. Hence f is strictly convex.

Thus f is minimized at a unique point $x^* \in \mathbf{R}^n \backslash \{y_1, \cdots, y_m\}$, such that $\nabla f(x^*) = 0$.

$$\nabla f(x^*) = \sum_{i=1}^{m} \omega_i \frac{x^* - y_i}{\|x^* - y_i\|} = 0 \tag{2}$$

$$\sum_{i=1}^{m} \frac{\omega_i x^*}{\|x^* - y_i\|} = \sum_{i=1}^{m} \frac{\omega_i y_i}{\|x^* - y_i\|} \tag{3}$$

"Extract" x^*, disregarding the dependency in $\|x^* - a_i\|$ from $x^*, i = 1, 2, \cdots, m$, and to obtain the relation

$$x^* = \frac{1}{\sum_{i=1}^{m} \frac{\omega_i}{\|x^* - y_i\|}} \sum_{i=1}^{m} \frac{\omega_i y_i}{\|x^* - y_i\|} \tag{4}$$

Weiszfeld's method is just a fixed point method for solving the relation (4).

$$x^* = T(x^*)$$

where the operator $T : \mathbf{R}^m \backslash \{y_1, \cdots, y_m\} \to \mathbf{R}$ is defined by

$$T(x) = \frac{1}{\sum_{i=1}^{m} \frac{\omega_i}{\|x - y_i\|}} \sum_{i=1}^{m} \frac{\omega_i y_i}{\|x - y_i\|} \tag{5}$$

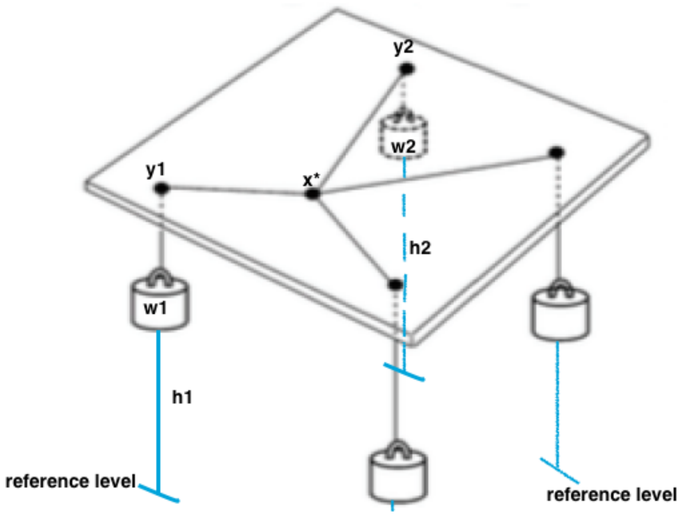
2.2. Is the optimal solution always unique?

We have seen in the previous part that the optimal solution is unique when y_i are not collinear because f is strictly convex in this case.

When $y_{i,i=1,\cdots,m}$ are collinear, the optimal solution may not be unique.

2.3. Show that an optimal solution minimizes the potential energy of the mechanical model defined as $\sum_{i=1}^{m} w_i h_i$, where h_i is the height of the ith weight measured from some reference level.

We can think the figure as a dinner table, and take the ground as the reference level. The length of ropes are constant, the optimal solution minimizes the sum of length of the rope "on the table", therefore maximize the length of rope "under the table". Consequently minimize the sum of distances from the ground to the weights ($\sum_{i=1}^{m} h_i$). This shows that the optimal solution minimize the potential energy of the mechanical model $\sum_{i=1}^{m} w_i h_i$ as required.


y2
w2
y1
x*
h2
w1
h1
reference level
reference level