Optimization - Exercises Christoph Müller

Exercise 6 :

Consider the linear optimization problem

$$(P') \quad \begin{array}{ll} \max & y^T x \\ \text{s.t.} & x^T Q x \le 1 \end{array} , \quad \rightsquigarrow (P) \quad \begin{array}{ll} \min & -y^T x \\ \text{s.t.} & -x^T Q x \ge -1 \end{array}$$

$y \in \mathbb{R}^n$ given, Q pos. def. (p.d.) and symm.

Show: (i) $\sqrt{y^T Q^{-1} y}$ is the optimal value

(ii) $(x^T y)^2 \le (x^T Q x)(y^T Q^{-1} y)$

Proof :

(i) Using the weak necessary optimality conditions (Lecture 4 / 28), we only need the continuity of $y^T x$ and $x^T Q x$ which is obvious.

Define the weak Lagrangian $\tilde{L}$ of (P):

$$\tilde{L}(x, \lambda', \lambda) = -\lambda' y^T x - \lambda(1 - x^T Q x)$$

Then, by above mentioned theorem there exist $\lambda, \lambda' \in \mathbb{R}$ s.t.

$$\frac{\partial L}{\partial x}(x, \lambda', \lambda) = -\lambda' y^T + 2\lambda x^T Q = 0 \qquad (1)$$

$$\lambda(1 - x^T Q x) = 0 \qquad (2)$$

$$\lambda, \lambda' \ge 0 \qquad (3)$$

$(1) \Rightarrow$ $\underbrace{\lambda x^T = \frac{1}{2}\lambda' y^T Q^{-1}}$, since Q p.d. & symm. $\rightsquigarrow$ Q invertible

$(2) \Rightarrow$ $\lambda \overset{(2)}{=} \lambda x^T Q x = \frac{1}{2}\lambda' y^T \underbrace{Q^{-1} Q}_{=I} x = \frac{1}{2}\lambda' y^T x$ $\quad (*)$

Assume now, $\lambda \neq 0$, then with (1) we have $\qquad (\lambda \neq 0 \Rightarrow \lambda' \neq 0)$

$$2\lambda x^T Q = \lambda' y^T \iff x^T = \frac{\lambda'}{2\lambda} y^T Q^{-1}$$

$$\Rightarrow x^T y = \frac{\lambda'}{2\lambda} y^T Q^{-1} y \overset{(*)}{=} \frac{\lambda'}{2 \cdot \frac{1}{2} \cdot \lambda' \cdot y^T x} y^T Q^{-1} y$$

$$\overset{x^T y = y^T x}{\iff} (x^T y)^2 = y^T Q^{-1} y$$

$\Rightarrow$ the optimal value is $x^T y = \sqrt{y^T Q^{-1} y}$

(ii) We have $(x^T y)^2 \le (x^T Q x)(y^T Q^{-1} y)$

$\iff \underbrace{(x^T y)^2 - (x^T Q x)(y^T Q^{-1} y)} \le 0$, since any feasible solution x satisfies $x^T Q x \le 1$:

$$\le \underbrace{(x^T Q x)}_{\le 1}(x^T y)^2 - (x^T Q x)(y^T Q^{-1} y) = \underbrace{(x^T Q x)}_{\substack{\ge 0 \\ Q \text{ p.d.}}} \underbrace{\left[\underbrace{(x^T y)^2}_{\overset{(i)}{\le} y^T Q^{-1} y} - (y^T Q^{-1} y)\right]}_{\le 0} \le 0$$

$\square$