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Exercise 5:

Solve the two-dimensional problem

minimize $(x-a)^2 + (x-b)^2 + xy$, subject to: $0 \leq x \leq 1; 0 \leq y \leq 1$

for all possible values of the scalars a and b.

Answer:

We have 4 inequality constraints, we define the Lagrangian associated with problem as:

$$L(x, y, \mu) = ((x-a)^2 + (x-b)^2 + xy - u_1 x - u_2 y - u_3(1-x) - u_4(1-y) = 0$$

According to the KKT, there exist u_i's such that:

$$\begin{aligned} \frac{\delta L}{\delta x} = 2(x-a) + 2(x-b) + y - u_1 + u_3 &= 0 \\ \frac{\delta L}{\delta y} = x - u_2 + u_4 &= 0 \\ u_1 x &= 0 \\ u_2 y &= 0 \\ u_3(1-x) &= 0 \\ u_4(1-y) &= 0 \\ 0 \leq u_i, \forall i &\in \{1, 2, 3, 4\} \end{aligned}$$

Solving the above system using the online tool wolfram (https://www.wolframalpha.com/) we get eight solutions:

Input interpretation:

solve

$$2(x-a) + 2(x-b) + y - u1 + u3 = 0$$
$$x - u2 + u4 = 0$$
$$u3(1-x) = 0$$
$$u4(1-y) = 0$$
$$u1\, x = 0$$
$$u2\, y = 0$$

for $x, y, u1, u2, u3, u4$

Open code

Results:

$x = 0$ and $u1 = -2a - 2b + y$ and $u2 = 0$ and $u3 = 0$ and $u4 = 0$ (1)

$x = 0$ and $y = 0$ and $u1 = -2(a+b)$ and $u2 = 0$ and $u3 = 0$ and $u4 = 0$ (2)

$x = 0$ and $y = 1$ and $u1 = -2a - 2b + 1$ and $u2 = 0$ and $u3 = 0$ and $u4 = 0$ (3)

$x = 0$ and $y = 2(a+b)$ and $u1 = 0$ and $u2 = 0$ and $u3 = 0$ and $u4 = 0$ (4)

$x = 1$ and $y = 0$ and $u1 = 0$ and $u2 = 1$ and $u3 = 2(a+b-2)$ and $u4 = 0$ (5)

$x = 1$ and $y = 1$ and $u1 = 0$ and $u2 = 0$ and $u3 = 2a + 2b - 5$ and $u4 = -1$ (6)

$x = \frac{1}{4}(2a + 2b - 1)$ and $y = 1$ and $u1 = 0$ and $u2 = 0$ and $u3 = 0$ and $u4 = \frac{1}{4}(-2a - 2b + 1)$ (7)

$x = \frac{a+b}{2}$ and $y = 0$ and $u1 = 0$ and $u2 = \frac{a+b}{2}$ and $u3 = 0$ and $u4 = 0$ (8)

Analysing the solutions:

Solution (6) is not feasible because u_4 can not be negative. Solution (7) is not feasible because x and u_4 have different signs, but they both must be positive.

Solutions (2)(3)(4) can be consider as special cases of (1): $x = 0, y = 2a + 2b + u_1, u_2 = u_3 = u_4 = 0$ by vary the value of u_1

- Solution(2)($x = 0, y = 0$): $u_1 = -2(a+b)$, as $u_1 \geq 0$, we can conclude that $a + b \leq 0$
- Solution(3)($x = 0, y = 1$): $u_1 = -2(a+b) + 1$, as $u_1 \geq 0$, we can conclude that $-2(a+b) + 1 \geq 0 \Leftrightarrow -2(a+b) \geq -1 \Leftrightarrow a + b \leq \frac{1}{2}$
- Solution(4)($x = 0, y = 2(a+b)$): as $0 \leq y \leq 1$, we can conclude that $0 \leq a + b \leq \frac{1}{2}$, in this case $u_1 = 0$.

Now only left solution (5) and (8):

- Solution(8)($x = \frac{a+b}{2}, y = 0$): as $0 \leq x \leq 1$, we can conclude that $0 \leq a + b \leq 2$, which also satisfies the condition of $u_2 = \frac{a+b}{2} \geq 0$.
- Solution(5)($x = 1, y = 0$): as $u_3 \geq 0$, we can conclude that $a + b \geq 2$.

Below we present a plot where we played with values a and b to check if the solutions above were satisfied correctly. You can also play with these values.

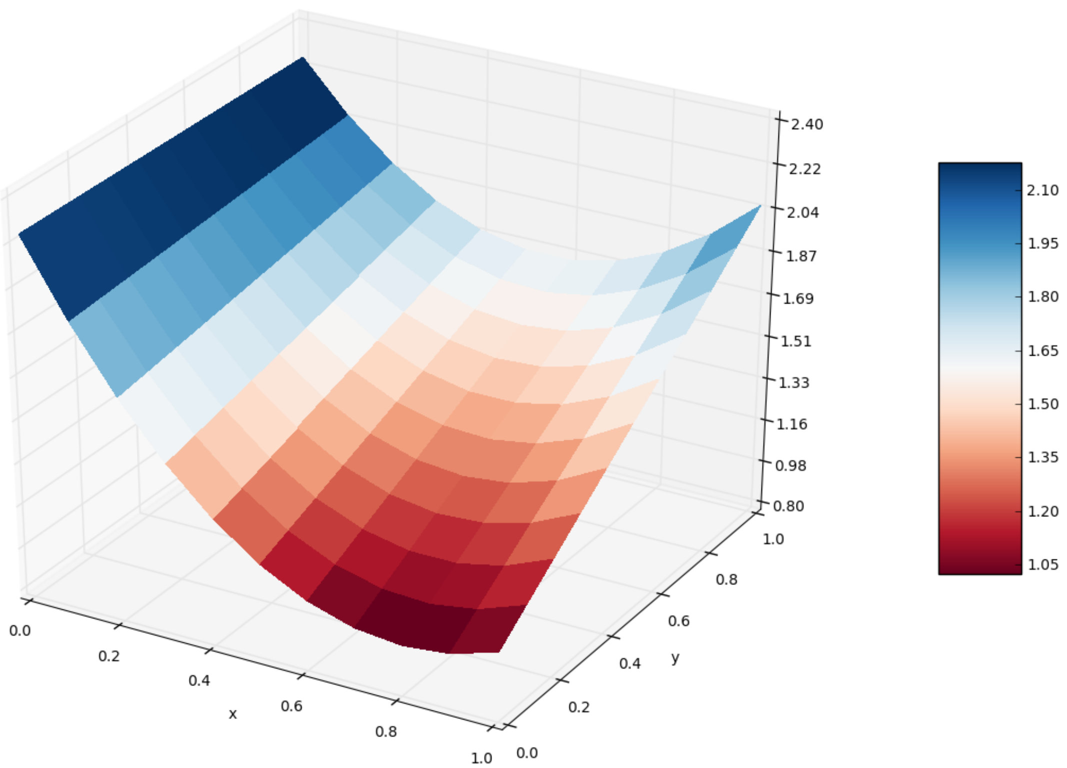
In [1]:

```
from matplotlib import pyplot as plt
import numpy as np
from __future__ import division

from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import matplotlib.pyplot as plt
%matplotlib inline

def f(x, y, a, b):
    return (x - a)**2 + (x - b)**2 + x*y
```

In [43]:

```
# Try to change the value of a and b

a = 0.1
b = 1.5

x = np.arange(0, 1.1,0.1)
y = np.arange(0, 1.1,0.1)
X, Y = np.meshgrid(x, y)
Z = f(X, Y, a, b)

fig = plt.figure(figsize=[15,10])
ax = fig.gca(projection='3d')
surf = ax.plot_surface(X, Y, Z, rstride=1, cstride=1,
                       cmap=cm.RdBu, linewidth=0, antialiased=False)

ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))
plt.xlabel('x')
plt.ylabel('y')
fig.colorbar(surf, shrink=0.5, aspect=5);
posX = np.argmin(Z)
posY = np.argmin(Z[:,posX])
print 'min_f: ', np.min(Z)
print 'posX: ', posX, 'posY: ', posY
print 'x_min: ', x[posX], 'Y_min: ', y[posY], 'f: ', f(x[posX],  y[posY], a, b)
```

```
min_f:  0.98
posX:  8 posY:  0
x_min:  0.8 Y_min:  0.0 f:  0.98
```



Exercise 6:

Given a vector y, consider the problem

$$\begin{aligned} maximize \ y^T x \\ subject \ to: \ \ x^T Q x \leq 1 \end{aligned}$$

where Q is a positive definite symmetric matrix. Show that the optimal value is $\sqrt{y^T Q^{-1} y}$ and use this fact to establish the inequality

$$(x^T y)^2 \leq (x^T Q x)(y^T Q^{-1} y)$$

Answer:

We have 1 inequality constraint, we define the Lagrangian associated with problem as:

$$L(x, y, \mu) = y^T x + \mu(1 - x^T Q x) = 0$$

According to the KKT, there exist μ such that:

$$\begin{aligned} \frac{\delta L}{\delta x} = y^T - 2\mu x^T Q = 0 \quad (1) \\ \mu(1 - x^T Q x) = 0 \quad (2) \\ 0 \leq \mu \end{aligned}$$

From (1) and the fact that Q is a positive definite symmetric matrix, $x^T = \frac{1}{2\mu} y^T Q^{-1}$ for $\mu \neq 0$ (Note that, from (1), $\mu = 0$ only if $y = 0$). Substitute it into (2),

$$\begin{aligned} \mu(1 - x^T Q x) = 0 \\ 1 - x^T Q x = 0 \\ 1 - \frac{1}{2\mu} y^T Q^{-1} Q \frac{1}{2\mu} (Q^{-1})^T y = 0 \\ 1 - \frac{1}{(2\mu)^2} y^T (Q^{-1})^T y = 0 \\ 1 = \frac{1}{(2\mu)^2} y^T (Q^{-1})^T y \\ \mu^2 = \frac{1}{2^2} y^T Q^{-1} y \\ \mu = +\frac{1}{2} \sqrt{y^T Q^{-1} y} \quad (3) \end{aligned}$$

Then, using (3), the optimal value is

$$y^T x = x^T y = \frac{1}{2\mu} y^T Q^{-1} y = \frac{1}{2\frac{1}{2}\sqrt{y^T Q^{-1} y}} y^T Q^{-1} y = \frac{1}{\sqrt{y^T Q^{-1} y}} y^T Q^{-1} y = \sqrt{y^T Q^{-1} y}$$

Now let us see the inequality. From (2), the optimal point is reached at the boundary of our constraint, i.e, $x^T Q x = 1$. Hence, for this point, the inequality becomes the trivial equality. Note that this solution is unique, thus, it remains to proof that when we are not in the boundary, i.e, $x^T Q x < 1$, the square of our objective function is strictly upper bounded by $(x^T Q x)(y^T Q^{-1} y)$. Formally, we want to proof:

$$\texttt{if } x \in \{x^T Q x < 1\} \implies (x^T y)^2 < (x^T Q x)(y^T Q^{-1} y)$$

The proof follows directly by an absurd discussing. Suppose that $(x^T y)^2 \geq (x^T Q x)(y^T Q^{-1} y)$, we know that the square of the maximum in our feasible space is $y^T Q^{-1} y$, and it is achieved by x s.t $x^T Q x = 1$, which is a contradiction $\square$