Authors: Xiaoxia Lin, Samuel Noriega

Exercise 3

Suppose that we evaluate f at three points $x_1 < x_2 < x_3$. Show that the inequalities $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$ imply that the coeffcient c of the quadratic approximation $\phi(x) = a + bx + cx^2$ is positive and that the predicted stationary point of ϕ is indeed a minimum.

Solution:

Let $\phi(x) = a + bx + cx^2$ be the cuadratic function that approximate f at x_1, x_2, x_3, where $x_1 < x_2 < x_3$. Hence $f(x_1) = \phi(x_1)$, $f(x_2) = \phi(x_2)$, and $f(x_3) = \phi(x_1)$, using the hypotesis $x_1 < x_2 < x_3$ then:

$$\left.\begin{array}{r} f(x_1) > f(x_2) \implies \phi(x_1) > \phi(x_2) \\ x_1 < x_2 \end{array}\right\} \implies \phi \text{ decreases at } x_1, \text{ i.e. } \frac{d\phi}{dx}(x_1) < 0$$

$$\left.\begin{array}{r} f(x_3) > f(x_2) \implies \phi(x_3) > \phi(x_2) \\ x_2 < x_3 \end{array}\right\} \implies \phi \text{ increases at } x_3, \text{ i.e. } \frac{d\phi}{dx}(x_3) > 0$$

As $\frac{d\phi}{dx}(x_1) < 0$ and $\frac{d\phi}{dx}(x_3) > 0$, there exists a minimum x^* between x_1 and x_3, such that $\frac{d\phi}{dx}(x^*) = 0$, i.e. the coeffcient c of the quadratic approximation $\phi(x) = a + bx + cx^2$ is positive.

Exercise 4

Let f be a real function on R^n. Also, let $x_0 \in R^n$, $z \in R^n$, and $\theta \in R$. Define

$$F(\theta) = f(x_0 + \theta z)$$

and suppose that we are looking for the minimum of F (that is, for the minimum of f in the direction z through the point x_0). Let $x_0 + \theta_1 z$, $x_0 + \theta_2 z$, and $x_0 + \theta_3 z$ be three points, where f is evaluated. Show that the minimum predicetd by applying the quadratic approximation method is $x_0 + \theta^* z$, where

$$\theta^* = \frac{[\theta_2^2 - \theta_3^2]F(\theta_1) + [\theta_3^2 - \theta_1^2]F(\theta_2) + [\theta_1^2 - \theta_2^2]F(\theta_3)}{2[(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)]}$$

and it is indeed the minimum of the parabola passing through the above three points if

$$\frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} < 0$$

Solution:

Let $\phi(x) = a + bx + cx^2$ be the quadratic approximation of F evaluated at three points θ_1, θ_2 and θ_3, where $F(\theta) = f(x_0 + \theta z)$, we have:

$$\begin{cases} F(\theta_1) = f(x_0 + \theta_1 z) = \phi(\theta_1) \implies F(\theta_1) = c\theta_1^2 + b\theta_1 + a \\ F(\theta_2) = f(x_0 + \theta_2 z) = \phi(\theta_2) \implies F(\theta_2) = c\theta_2^2 + b\theta_2 + a \\ F(\theta_3) = f(x_0 + \theta_3 z) = \phi(\theta_3) \implies F(\theta_3) = c\theta_3^2 + b\theta_3 + a \end{cases}$$

Solving the obtained system of equations gives us the following expressions for a, b and c coefficients of the quadratic approximation:

$$a = \frac{F(\theta_1)(\theta_2^2\theta_3 - \theta_2\theta_3^2) + \theta_1(F(\theta_2)\theta_3^2 - F(\theta_3)\theta_2^2) + \theta_1^2(F(\theta_3)\theta_2 - F(\theta_2)\,\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)}$$

$$b = \frac{(\theta_2^2 - \theta_3^2)F(\theta_1) + (\theta_3^2 - \theta_1^2)F(\theta_2) + (\theta_1^2 - \theta_2^2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)}$$

$$c = -\frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)}$$

For the quadratic function to have a minimum, the c coefficient has to be positive. Therefore, in our case

$$c = \frac{(\theta_3 - \theta_2)F(\theta_1) + (\theta_1 - \theta_3)F(\theta_2) + (\theta_2 - \theta_1)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} > 0$$

This is equivalent to

$$-c = \frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} < 0$$

And the minimum of the quadratic funtion that approximate F is determined by $\theta^* = -\frac{b}{2c}$. Using the previously obtained expressions for the b and c coefficients, we obtain:

$$\theta^* = \frac{(\theta_2^2 - \theta_3^2)F(\theta_1) + (\theta_3^2 - \theta_1^2)F(\theta_2) + (\theta_1^2 - \theta_2^2)F(\theta_3)}{2[(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)]}$$

Finally, $F(\theta) = f(x_0 + \theta z)$ implies that the minimum of the function f, predicetd by applying the quadratic approximation method passing through the points $x_0 + \theta_1 z$, $x_0 + \theta_2 z$, and $x_0 + \theta_3 z$, is indeed $x_0 + \theta^* z$, where

$$\theta^* = \frac{(\theta_2^2 - \theta_3^2)F(\theta_1) + (\theta_3^2 - \theta_1^2)F(\theta_2) + (\theta_1^2 - \theta_2^2)F(\theta_3)}{2[(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)]},$$

and it is in deed the minimum of the parabola passing through the above three points if

$$-c = \frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} < 0$$