Report

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Proofs

Thm1. Angular acceleration derivation for rigid body: $\beta = I^{-1}\tau - I^{-1}[\omega]I\omega$, where I is momentum of inertia according to centroid under ground reference frame, τ is total moment of force according to centroid, ω is angular velocity, $[\omega]$ is the skew matrix of ω which is defined as

$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{1}$$

(In this case: $\omega \times v = [\omega]v$)

Proof: Firstly prove some lemmas.

Lemma 1. $I_{\text{centroidsystem}} = RI_{\text{rigidbodysystem}}R^\top$, where R is the rotation matrix that transfer position vector from rigid body system to centroid system (Notice: this is a **translational system**), i.e. $r^{(\text{centroidsystem})} = Rr^{(\text{rigidbodysystem})}$.

Proof: From definition of I:

$$\begin{aligned}
I_{\text{centroidsystem}} &= \int \left[\|r^{(\text{centroidsystem})}\|^2 I_3 - r^{(\text{centroidsystem})} r^{(\text{centroidsystem})^\top} \right] \mathrm{d}m \\
&= \int \left[\|Rr^{(\text{rigidbodysystem})}\|^2 I_3 - Rr^{(\text{rigidbodysystem})} (Rr^{(\text{rigidbodysystem})})^\top \right] \mathrm{d}m \\
&= \int \left[\|Rr^{(\text{rigidbodysystem})}\|^2 I_3 - Rr^{(\text{rigidbodysystem})} r^{(\text{rigidbodysystem})^\top} R^\top \right] \mathrm{d}m \\
&= \int \left[\|r^{(\text{rigidbodysystem})}\|^2 I_3 - Rr^{(\text{rigidbodysystem})} r^{(\text{rigidbodysystem})^\top} R^\top \right] \mathrm{d}m \\
&= \int \left[R\|r^{(\text{rigidbodysystem})}\|^2 I_3 R^\top - Rr^{(\text{rigidbodysystem})} r^{(\text{rigidbodysystem})^\top} R^\top \right] \mathrm{d}m \\
&= R \left\{ \int \left[\|r^{(\text{rigidbodysystem})}\|^2 I_3 - r^{(\text{rigidbodysystem})} r^{(\text{rigidbodysystem})^\top} \right] \mathrm{d}m \right\} R^\top \\
&= RI_{\text{rigidbodysystem}}R^\top
\end{aligned} \tag{2}$$

Notice that I is an (positive) symmetric matrix, so carefully choosing reference frame of rigid body system results $I_{\text{rigidbodysystem}}$ to be a **diagonal** matrix.

Lemma 2: Angular momentum theorem according to the reference point on the rigid body: $\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I_{\text{centroidsystem}}\beta + \frac{\mathrm{d}I_{\text{centroidsystem}}}{\mathrm{d}t}\omega$.

Proof: From definition of angular momentum,

$$L = \int r \times v \mathrm{d}m \tag{3}$$

Thus

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int v \times v + r \times a \mathrm{d}m = \int r \times a \mathrm{d}m \tag{4}$$

Applying Newton's second law,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int r \times \mathrm{d}F = \tau \tag{5}$$

where τ is total moment of force according to this reference point.

Calculating L:

$$L = \int r \times (\omega \times r) \mathrm{d}m = \int (r \cdot r)\omega - r(r \cdot \omega) \mathrm{d}m = \int \|r\|^2\omega - rr^\top\omega \mathrm{d}m = \left[\int \|r\|^2 - rr^\top \mathrm{d}m\right]\omega = I\omega \tag{6}$$

where I is the momentum of inertia under centroid (translational) reference, i.e. $I = I_{\text{centroidsystem}}$. (Also is the momentum of inertia calculated under ground system.) Omit subscripts where there is no ambiguity in the following text.

Without proof, using a conclusion in derivative of exponential matrix, we introduce that $\frac{\mathrm{d}R}{\mathrm{d}t} = [\omega]R$.

Lemma 3: $\frac{\mathrm{d}R^\top}{\mathrm{d}t} = -R^\top[\omega]$.

Proof: From $RR^\top \equiv I_3$, take derivative of both side:

$$\frac{\mathrm{d}R}{\mathrm{d}t}R^\top + R\frac{\mathrm{d}R^\top}{\mathrm{d}t} = 0 \tag{7}$$

Where

$$L.H.S. = [\omega]RR^\top + R\frac{\mathrm{d}R^\top}{\mathrm{d}t} = [\omega] + R\frac{\mathrm{d}R^\top}{\mathrm{d}t} \tag{8}$$

Which draws the result.

Lemma 4: $\frac{\mathrm{d}I}{\mathrm{d}t} = [\omega]I - I[\omega]$.

This follows from Lemma 1 and 3 immediately.

Then the proof of Thm. 1 begins:

From Angular momentum theorem according to centroid on the rigid body, combining previous lemmas,

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I\beta + \frac{\mathrm{d}I}{\mathrm{d}t}\omega = I\beta + ([\omega]I - I[\omega])\omega = I\beta + [\omega]I\omega \tag{9}$$

Last step follows from $[\omega]\omega = \mathbf{0}$. Thus

$$\beta = I^{-1}\tau - I^{-1}[\omega]I\omega \tag{10}$$

Proof 2. Collision of rigid body

TODO.

Proof 3. Updating error of velocity, angular velocity ($O(\delta t)^3$); position, rotation matrix ($O(\delta t)^4$)

Updating rules:

$$v_{t+\delta t} = v_t + \frac{3a_t - a_{t-\delta t}}{2}\delta t \tag{11}$$

$$\omega_{t+\delta t} = \omega_t + \frac{3\beta_t - \beta_{t-\delta t}}{2}\delta t \tag{12}$$

$$x_{t+\delta t} = x_t + v_t\delta t + \frac{4a_t - a_{t-\delta t}}{6}(\delta t)^2 \tag{13}$$

$$R_{t+\delta t} = \left\{ e^{[\omega(t) + \frac{4\beta(t) - \beta(t-\delta t)}{6}\delta t]\delta t} - \frac{1}{12}([\beta(t)][\omega(t)] + [\omega(t)][\beta(t)])(\delta t)^3 \right\} R(t) \tag{14}$$

Proof: For the first 2 terms,

$$f(t+\delta t) = f(t) + f'(t)\delta t + \frac{1}{2}f''(t)(\delta t)^2 + O((\delta t)^3) \tag{15}$$

$$f'(t-\delta t) = f'(t) - f''(t)\delta t + O((\delta t)^2) \tag{16}$$

Combining those two we get

$$\begin{aligned} f(t+\delta t) &= f(t) + f'(t)\delta t + \frac{1}{2}(f'(t) - f'(t-\delta t) + O((\delta t)^2))\delta t + O((\delta t)^3) \\ &= f(t) + \frac{1}{2}(3f'(t) - f'(t-\delta t))\delta t + O((\delta t)^3) \end{aligned} \tag{17}$$

For position x:

$$f(t+\delta t) = f(t) + f'(t)\delta t + \frac{1}{2}f''(t)(\delta t)^2 + \frac{1}{6}f'''(t)(\delta t)^3 + O((\delta t)^4) \tag{18}$$

$$f''(t-\delta t) = f''(t) - f'''(t)\delta t + O((\delta t)^2) \tag{19}$$

Combining those two

$$\begin{aligned} f(t+\delta t) &= f(t) + f'(t)\delta t + \frac{1}{2}f''(t)(\delta t)^2 + \frac{1}{6}(f''(t) - f''(t-\delta t) + O((\delta t)^2))(\delta t)^2 + O((\delta t)^4) \\ &= f(t) + f'(t)\delta t + \frac{1}{6}(4f''(t) - f''(t-\delta t))(\delta t)^2 + O((\delta t)^4) \end{aligned} \tag{20}$$

For the last term (Here big-O means F-norm):

$$R(t+\delta t) = R(t) + \dot{R}(t)\delta t + \frac{1}{2}\ddot{R}(t)(\delta t)^2 + \frac{1}{6}\dddot{R}(t)(\delta t)^3 + O((\delta t)^4) \tag{21}$$

$$\dot{R}(t) = [\omega]R \tag{22}$$

$$\ddot{R}(t) = [\beta]R + [\omega]^2 R \tag{23}$$

$$\dddot{R}(t) = [\dot{\beta}]R + 2[\beta][\omega]R + [\omega][\beta]R + [\omega]^3 R \tag{24}$$

Thus

$$\begin{aligned} R(t+\delta t) = \Big\{ & I + [\omega(t)]\delta t + \frac{1}{2}([\beta(t)] + [\omega(t)]^2)(\delta t)^2 \\ & + \frac{1}{6}([\dot{\beta}(t)] + 2[\beta(t)][\omega(t)] + [\omega(t)][\beta(t)] + [\omega(t)]^3)(\delta t)^3 \Big\} R(t) + O((\delta t)^4) \end{aligned} \tag{25}$$

From the fact that $[a][b] = ba^\top - \langle a, b\rangle I_3$, $[a \times b] = ba^\top - ab^\top$ (actually this is from $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, $(a \times b) \times c = b(a \cdot c) - a(b \cdot c)$)

$$([\beta][\omega] - [\omega][\beta]) = \omega\beta^\top - \beta\omega^\top = [\beta \times \omega] \tag{26}$$

Combine with $\dot{\beta}(t) = \frac{\beta(t) - \beta(t-\delta t)}{\delta t} + O(\delta t)$:

$$\begin{aligned} R(t+\delta t) &= \Big\{ I + [\omega(t)]\delta t + \left(\frac{1}{2}[\beta(t)] + \frac{1}{2}[\omega(t)]^2 + \frac{1}{6}[\beta(t) - \beta(t-\delta t)]\right)(\delta t)^2 \\ &\quad + \frac{1}{6}(2[\beta(t)][\omega(t)] + [\omega(t)][\beta(t)] + [\omega(t)]^3)(\delta t)^3 \Big\} R(t) + O((\delta t)^4) \\ &= \Big\{ e^{[\omega(t) + \frac{4\beta(t) - \beta(t-\delta t)}{6}\delta t]\delta t} + \frac{1}{12}([\beta(t)][\omega(t)] - [\omega(t)][\beta(t)])(\delta t)^3 \Big\} R(t) + O((\delta t)^4) \\ &= \Big\{ e^{[\omega(t) + \frac{4\beta(t) - \beta(t-\delta t)}{6}\delta t + \frac{1}{12}\beta(t) \times \omega(t)(\delta t)^2]\delta t} \Big\} R(t) + O((\delta t)^4) \end{aligned} \tag{27}$$

ACK: huang-wj22@mails.tsinghua.edu.cn, idea on applying Taylor's expansion of rotation matrix and checking proof details.