Note for Strassen Algorithm

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1 Ideas of Basic Matrix Multiplication

The idea behind Strassen algorithm is in the formulation of matrix multiplication as a recursive problem. Assume that we have two matrices in $\mathbb{R}^{n\times n}$ A and B. We can write A and B as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{2}$$

Then we can write the matrix multiplication as follows:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \tag{3}$$

However, we still need to do 8 matrix multiplications to get the result. The time complexity is $O(n^3)$.

2 Strassen Algorithm

The idea of Strassen algorithm is to reduce the number of matrix multiplications. We can write the matrix multiplication as follows:

$$M_1 = (A_{11} + A_{22})(B_{11} + B_{22}) \tag{4}$$

$$M_2 = (A_{21} + A_{22})B_{11} \tag{5}$$

$$M_3 = A_{11}(B_{12} - B_{22}) \tag{6}$$

$$M_4 = A_{22}(B_{21} - B_{11}) \tag{7}$$

$$M_5 = (A_{11} + A_{12})B_{22} \tag{8}$$

$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12}) \tag{9}$$

$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22}) \tag{10}$$

Then we can write the matrix multiplication $AB = C$ as follows:

$$AB = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{11}$$

$$C_{11} = M_1 + M_4 - M_5 + M_7 \tag{12}$$

$$C_{12} = M_3 + M_5 \tag{13}$$

$$C_{21} = M_2 + M_4 \tag{14}$$

$$C_{22} = M_1 - M_2 + M_3 + M_6 \tag{15}$$

We can see that we only need to do 7 matrix multiplications to get the result. The time complexity is $O(n^{\log_2 7})$. In general, we will recursively do the matrix multiplication until the size of the matrix is 1×1. Then we can use the basic matrix multiplication to get the result.