参考：https://www.cnblogs.com/slgkaifa/p/7265010.html

圆的方程，x_i 和 y_i 分别代表要拟合的数据（数量为 N），x_c 和 y_c 分别代表圆心的 x 和 y 坐标，R 为圆的半径:

$$(x_i - x_c)^2 + (y_i - y_c)^2 = R^2 \tag{1}$$

最小二乘法建模:

$$f(x_i, y_i) = \sum_{i=1}^{N}[(x_i - x_c)^2 + (y_i - y_c)^2 - R^2]^2 \tag{2}$$

令:

$$g(x_i, y_i) = (x_i - x_c)^2 + (y_i - y_c)^2 - R^2 \tag{3}$$

则(2)可以写为:

$$f(x_i, y_i) = \sum_{i=1}^{N} g(x_i, y_i)^2 \tag{4}$$

要求在数据$\{(x_i, y_i)\}_{i=1}^{N}$下最优的 x_c，y_c 和 R，则对其求偏导:

$$\frac{\partial f}{\partial x_c} = -4\sum_{i=1}^{N} g(x_i, y_i)(x_i\text{-}x_c) = 0 \tag{5}$$

$$\frac{\partial f}{\partial x_y} = -4\sum_{i=1}^{N} g(x_i, y_i)(y_i\text{-}y_c) = 0 \tag{6}$$

$$\frac{\partial f}{\partial R} = -4R\sum_{i=1}^{N} g(x_i, y_i) = 0 \tag{7}$$

在公式(7)中由于 $R \neq 0$，则有:

$$\sum_{i=1}^{N} g(x_i, y_i) = 0 \tag{8}$$

则(5)和(6)可以简化为:

$$\frac{\partial f}{\partial x_c} = \sum_{i=1}^{N} g(x_i, y_i) x_i = 0 \tag{9}$$

$$\frac{\partial f}{\partial y_c} = \sum_{i=1}^{N} g(x_i, y_i) y_i = 0 \tag{10}$$

为了便于后续步骤的求解，进行换元:

$$
\begin{aligned}
u_i &= x_i - \overline{x} \\
u_c &= x_c - \overline{x} \\
v_i &= y_i - \overline{y} \\
v_c &= y_c - \overline{y}
\end{aligned}
\tag{11}
$$

其中:

$$
\begin{aligned}
\overline{x} &= \frac{\sum_{i=1}^{N} x_i}{N} \\
\overline{y} &= \frac{\sum_{i=1}^{N} y_i}{N}
\end{aligned}
\tag{12}
$$

对上面公式(3),(9),(10)进行换元:

$$g(u_i, v_i) = (u_i - u_c)^2 + (v_i - v_c)^2 - R^2 \tag{13}$$

$$\sum_{i=1}^{N} u_i g(u_i, v_i) = 0 \tag{14}$$

$$\sum_{i=1}^{N} v_i g(u_i, v_i) = 0 \tag{15}$$

将公式(14)和(15)展开可得:

$$\sum_{i=1}^{N} (u_i^3 - 2u_i^2 u_c + u_i u_c^2 + u_i v_i^2 - 2u_i v_i v_c + u_i v_c^2 - u_i R^2) = 0 \tag{16}$$

$$\sum_{i=1}^{N} (v_i u_i^2 - 2v_i u_i u_c + v_i u_c^2 + v_i^3 - 2v_i^2 v_c + v_i {v_c}^2 - v_i R^2) = 0 \tag{17}$$

由于(由公式(11)易得此结论):

$$
\begin{aligned}
\sum_{i=1}^{N} u_i &= 0 \\
\sum_{i=1}^{N} v_i &= 0
\end{aligned}
\tag{18}
$$

因此,公式(16)和(17)可以被简化为:

$$\sum_{i=1}^{N} (u_i^3 - 2u_i^2 u_c + u_i v_i^2 - 2u_i v_i v_c) = 0 \tag{19}$$

$$\sum_{i=1}^{N} (v_i u_i^2 - 2v_i u_i u_c + v_i^3 - 2v_i^2 v_c) = 0 \tag{20}$$

公式(19)和(20)可以整理为:

$$u_c \sum_{i=1}^{N} u_i^2 + v_c \sum_{i=1}^{N} u_i v_i = \frac{\sum_{i=1}^{N} u_i^3 + \sum_{i=1}^{N} u_i v_i^2}{2} \tag{21}$$

$$u_c \sum_{i=1}^{N} u_i v_i + v_c \sum_{i=1}^{N} v_i^2 = \frac{\sum_{i=1}^{N} v_i^3 + \sum_{i=1}^{N} u_i^2 v_i}{2} \tag{22}$$

将公式(21)和(22)写成线性方程组的形式:

$$\begin{bmatrix} \sum_{i=1}^{N} u_i^2 & \sum_{i=1}^{N} u_i v_i \\ \sum_{i=1}^{N} u_i v_i & \sum_{i=1}^{N} v_i^2 \end{bmatrix} \begin{bmatrix} u_c \\ v_c \end{bmatrix} = \begin{bmatrix} \dfrac{\sum_{i=1}^{N} u_i^3 + \sum_{i=1}^{N} u_i v_i^2}{2} \\ \dfrac{\sum_{i=1}^{N} v_i^3 + \sum_{i=1}^{N} u_i^2 v_i}{2} \end{bmatrix} \tag{23}$$

由克莱姆法则:

运用**克莱姆法则**可以很有效地解决以下方程组。

已知:

$$ax + by = e$$
$$cx + dy = f$$

使用矩阵来表示时就是:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

当矩阵可逆时，x和y可以从克莱姆法则中得出:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc}$$

以及

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

可得 u_c 和 v_c 的解为:

$$u_c = \frac{\sum_{i=1}^{N} u_i^3 \sum_{i=1}^{N} v_i^2 + \sum_{i=1}^{N} u_i v_i^2 \sum_{i=1}^{N} v_i^2 - \sum_{i=1}^{N} v_i^3 \sum_{i=1}^{N} u_i v_i - \sum_{i=1}^{N} u_i^2 v_i \sum_{i=1}^{N} u_i v_i}{2[\sum_{i=1}^{N} u_i^2 \sum_{i=1}^{N} v_i^2 - (\sum_{i=1}^{N} u_i v_i)^2]} \tag{24}$$

$$v_c = \frac{\sum_{i=1}^{N} u_i^2 \sum_{i=1}^{N} v_i^3 + \sum_{i=1}^{N} u_i^2 v_i \sum_{i=1}^{N} u_i^2 - \sum_{i=1}^{N} u_i^3 \sum_{i=1}^{N} u_i v_i - \sum_{i=1}^{N} u_i v_i^2 \sum_{i=1}^{N} u_i v_i}{2[\sum_{i=1}^{N} u_i^2 \sum_{i=1}^{N} v_i^2 - (\sum_{i=1}^{N} u_i v_i)^2]} \tag{25}$$

易得 R 为（其实就是所有半径的均值）:

$$R=\frac{\sum_{i=1}^{N}\sqrt{(u_i-u_c)^2+(v_i-v_c)^2}}{N} \tag{26}$$

结果展示: