Supplementary Material for Convergence Analysis of DRRA

Xuyang Wu, Sindri Magnússon, and Mikael Johansson

This document provides the proofs of Lemma 5 and Lemma 8 in [1].

Lemma 5. Suppose $X, Y \subseteq \mathbb{R}^d$ are convex, $f : X \to \mathbb{R}$ is convex and continuous, and X, Y satisfy the following:

(i) X is open and when x goes to the boundary of X from its interior, $f(x)$ goes to $+\infty$.
(ii) Y is closed.
(iii) $X \cap Y$ is non-empty and bounded.

Then, the optimal solution set of the following problem is non-empty and compact:

$$\underset{x \in X \cap Y}{\text{minimize}} \ f(x). \tag{1}$$

Proof: Suppose $\tilde{x} \in X \cap Y$. Then, the optimal objective value of problem (1) is smaller than or equal to $f(\tilde{x})$ and therefore, the optimal solution set of problem (1) is identical to that of the following problem:

$$\begin{aligned} &\underset{x \in X \cap Y}{\text{minimize}} \quad f(x) \\ &\text{subject to } f(x) \le f(\tilde{x}). \end{aligned} \tag{2}$$

To prove the optimal solution set of problem (2) is non-empty and compact, we first show its feasible solution set $X \cap Y \cap \{x \in \mathbb{R}^d : f(x) \le f(\tilde{x})\}$ is non-empty and compact. Since f is convex and continuous, it is closed. As a result, $\{x \in \mathbb{R}^d : f(x) \le f(\tilde{x})\}$ is closed, which, together with condition (i), implies the closeness of $X \cap \{x \in \mathbb{R}^d : f(x) \le f(\tilde{x})\}$. In addition, $X \cap Y$ is bounded and Y is closed. Therefore, $X \cap Y \cap \{x \in \mathbb{R}^d : f(x) \le f(\tilde{x})\}$ is compact. The set $X \cap Y \cap \{x : f(x) \le f(\tilde{x})\}$ is non-empty because $\tilde{x} \in X \cap Y \cap \{x \in \mathbb{R}^d : f(x) \le f(\tilde{x})\}$.

Since the feasible solution set of (2) is compact and non-empty, its optimal solution set is compact and non-empty [2, Lemma A.8], so is the optimal solution set of (1). ■

The proof of Lemma 8 makes use of the following proposition.

Proposition A. Suppose Assumption 1 holds and y_i, $i \in \mathcal{V}$ is feasible to problem (8). For any $i \in \mathcal{V}$ and $u_i \in \mathbb{R}^m$, $-u_i \in \partial\phi_i(y_i)$ if and only if u_i is the unique geometric multiplier of (7).

Proof: A proof for problem (7) with only inequality constraints is given in [2, Section 6.4.5], which can be straightforwardly extended to prove the result. ■

With Proposition A, we are ready to prove Lemma 8.

Lemma 8. Suppose Assumption 1 holds and y_i, $i \in \mathcal{V}$ is feasible to problem (8). Then, for any $i \in \mathcal{V}$, ϕ_i is differentiable at y_i and

$$\nabla\phi_i(y_i) = -u_i^\star(y_i), \tag{3}$$

where $u_i^\star(y_i)$ is the unique geometric multiplier of problem (7).

X. Wu and M. Johansson are with the Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden. Email: {xuyangw,mikaelj}@kth.se.

S. Magnússon is with the Department of Computer and System Science, Stockholm University, SE-164 07 Stockholm, Sweden. Email: sindri.magnusson@dsv.su.se.

This work was supported in part by the funding from Digital Futures and in part by the Swedish Research Council (Vetenskapsrådet) under grant 2020-03607.

Proof: Since y_i, $i \in \mathcal{V}$ is feasible to problem (8), by Lemma 7 in [1], the optimal solution set of (7) is non-empty. In addition, each $\tilde{\mathcal{X}}_i$, $i \in \mathcal{V}$ is convex and open. Then, strong duality holds between problem (7) and its Lagrange dual and therefore, there exists at least one geometric multiplier to problem (7).

Below, we show that the geometric multiplier of problem (7) is unique. Suppose u_i' and u_i'' are both geometric multipliers of problem (7). Then, there exist optimal solutions $x_i', x_i'' \in \tilde{\mathcal{X}}_i$ of problem (7) such that $\nabla F_i(x_i') + A_i^T u_i' = \mathbf{0}$ and $\nabla F_i(x_i'') + A_i^T u_i'' = \mathbf{0}$, which leads to

$$\begin{aligned} \|\nabla F_i(x_i') - \nabla F_i(x_i'')\|^2 &= \|A_i^T(u_i' - u_i'')\|^2 \\ &\geq \lambda_{\min}(A_i A_i^T)\|u_i' - u_i''\|^2. \end{aligned} \tag{4}$$

Here, $\lambda_{\min}(A_i A_i^T)$ is the minimal eigenvalue of $A_i A_i^T$, which is positive due to the full row rank property of A_i. By the optimality of x_i' and x_i'', for any feasible solution x_i of problem (7),

$$F_i(x_i) - F_i(x_i') \geq \langle \nabla F_i(x_i'), x_i - x_i' \rangle \geq 0, \tag{5}$$

$$F_i(x_i) - F_i(x_i'') \geq \langle \nabla F_i(x_i''), x_i - x_i'' \rangle \geq 0. \tag{6}$$

Letting $x_i = x_i''$ in (5) and $x_i = x_i'$ in (6) and due to $F_i(x_i') = F_i(x_i'')$, we have $\langle \nabla F_i(x_i'), x_i'' - x_i' \rangle = \langle \nabla F_i(x_i''), x_i' - x_i'' \rangle = 0$, which indicates

$$\langle \nabla F_i(x_i') - \nabla F_i(x_i''), x_i' - x_i'' \rangle = 0. \tag{7}$$

In addition, by the continuity of $\nabla^2 F_i$, there exists $L > 0$ such that $\nabla^2 F_i(x_i) \preceq L I_{d_i}$ for any x_i in the compact set $\{x \in \mathbb{R}^{d_i} : x = \alpha x_i' + (1-\alpha)x_i'', \alpha \in [0,1]\}$. Then,

$$\langle \nabla F_i(x_i') - \nabla F_i(x_i''), x_i' - x_i'' \rangle \geq \frac{1}{L}\|\nabla F_i(x_i') - \nabla F_i(x_i'')\|^2. \tag{8}$$

Combining (7), (8), and (4), we have $\nabla F_i(x_i') = \nabla F_i(x_i'')$ and $u_i' = u_i''$. As a result, the geometric multiplier of problem (7) is unique. Then, according to Proposition A, ϕ_i is differentiable at y_i and (3) holds. ■

References

[1] X. Wu, S. Magnússon, and M. Johansson, "A new family of feasible methods for distributed resource allocation," submitted to *IEEE Conference on Decision and Control*, 2021.

[2] D. P. Bertsekas, *Nonlinear Programming (3rd edition)*. Belmont, MA: Athena Scientific, 1999.