CSC165H1 Problem Set 3

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1. Induction and sequences

   (a) *Proof:* Let $n \in \mathbb{Z}^+$.
   Base Case: Let $n = 1$

$$\begin{aligned} d_{2n-1} = d_1 &= 1 \leq \sqrt{1} \\ &= \sqrt{2-1} \\ &= \sqrt{2n-1} \end{aligned}$$

   Induction Step: Let $n \in \mathbb{Z}^+$ and assume that $d_{2n+1} \leq \sqrt{2n-1}$. We want to prove that $d_{2(n+1)-1} \leq \sqrt{2(n+1)-1}$, which is $d_{2n+1} \leq \sqrt{2n+1}$

$$\begin{aligned} d_{2n+1} &= \frac{n+1}{d_{2n}} \\ &= \frac{2n+1}{\frac{2n}{d_{2n_1}}} \\ &= \frac{(2n+1)d_{2n-1}}{2n} \\ &\leq \frac{(2n+1)\sqrt{2n-1}}{2n} (since d2n-1 \leq \sqrt{2n-1}) \\ &= \frac{\sqrt{2n+1} * \sqrt{2n-1} * \sqrt{2n+1}}{2n} \\ &= \frac{\sqrt{4n^2-1} * \sqrt{2n+1}}{2n} \\ &\leq \frac{\sqrt{4n^2} * \sqrt{2n+1}}{2n} \\ &= \frac{2n * \sqrt{2n+1}}{2n} \\ &= \sqrt{2n+1} \end{aligned}$$

(b) *Proof:* Let $n \in \mathbb{N}$.
Since $n \in \mathbb{N}$ which means that $n \geq 1$.
So $2n \geq 2$, which means that d_{2n-1} always exist.
From part (a), we can know that $d_{2n-1} \leq \sqrt{2n-1}$.

$$\begin{aligned} d_{2n} &= \frac{2n}{d_{2n-1}} \\ &\geq \frac{2n}{\sqrt{2n-1}} \\ &> \frac{2n}{\sqrt{2n}} \\ &= \sqrt{2n} \end{aligned}$$

Thus, $d_2 n > \sqrt{2n}$

2. Number Representations

(a) i.

$$\begin{aligned} (T011T)_{bt} &= (-1) \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + (-1) \times 3^0 \\ &= -81 + 0 + 9 + 3 - 1 \\ &= -70 \end{aligned}$$

ii.

$$\begin{aligned} 210 &= 1 \times 3^5 + 0 \times 3^4 + (-1) \times 3^3 + (-1) \times 3^2 + 1 \times 3^1 + 0 \times 3^0 \\ &= (10TT10)_{bt} \end{aligned}$$

(b) *Proof:*Let $n \in \mathbb{Z}^+$
Base Case: Let $n = 1$
$3^n - 3 = 3^1 - 3 = 0$
6 can be divided by 0, so $6|0$, which is $6|3^1 - 3$.
Induction Step: Let $n \in \mathbb{Z}^+$, and assume that $6|3^n - 3$. We want to prove $6|3^{n+1} - 3$.
Since $6|3^n - 3$, there exist $k_1 \in \mathbb{Z}$ that $6k_1 = 3^n - 3$.
Let $k_2 = 3k_1 + 1$

$$\begin{aligned} 3^{n+1} - 3 &= (3^n - 3)3 + 6 \\ &= 6k_1 \times 3 + 6 \\ &= 18k_1 + 6 \\ &= 6(3k_1 + 1) \\ &= 6k_2 \end{aligned}$$

so $6|3^{n+1} - 3$

(c) Let $P(n) : \forall x \in \mathbb{N}$, (x is n-digit positively balanced),$\Rightarrow 6 \nmid x - 2 \wedge 6 \nmid x - 5$.
We want to prove that $\forall n \in \mathbb{Z}^+, P(n)$.
Proof. Let $n \in \mathbb{Z}^+$'
Base Case: Let $n = 1$.
$x = d_0 * 3^0$, where d_0 can be 1 or 0. Let us divide the proof in two cases.
Case 1: When d_0 is 0, so x = 0.
$x - 2 = 0 - 2 = -2$
$6 \nmid -2$, which means that $6 \nmid x - 2$
$x - 5 = 0 - 5 = -5$
$6 \nmid -5$, which means that $6 \nmid x - 5$
Case 2: When d_0 is 1, so x = 1.
$x - 2 = 1 - 2 = -1$
$6 \nmid -1$, which means that $6 \nmid x - 2$
$x - 5 = 1 - 5 = -4$
$6 \nmid -4$, which means that $6 \nmid x - 5$
Inductive Step: Let $n \in \mathbb{Z}^+$ and assume that $P(n)$ is true (i.e.$\forall x \in \mathbb{N}$, (x is n-digit positively balanced),$\Rightarrow 6 \nmid x-2 \wedge 6 \nmid x-5$.) We want to prove that $P(n+1)$ is true. Let $x_2 \in \mathbb{N}$ and assume that x_2 is n+1-digit positively balanced, we want to prove that $6 \nmid x_2 - 2 \wedge 6 \nmid x - 5$.
By the definition of balance ternary, we can know that $x_1 = (d_{n-1}d_{n-2}...d_1d_0)_{bt}, x_2 = (d_nd_{n-1}d_{n-2}...d_1d_0)_{bt}$.
By the definition of balance ternary and n-digit positively balanced, we can know that $x_2 = x_1 + c * 3^n$,where $c = 0$ or $c = 1$.
So we will divide the proof in two cases:
Case 1: When $c = 0$, which is $x_2 = x_1$.
Form the assumption, we can know that $6 \nmid x_1 - 2$ and $6 \nmid x_1 - 5$.
Thus $6 \nmid x_2 - 2$ and $6 \nmid x_2 - 5$
Case 2: When $c = 1$, which is $x_2 = x_1 + 3^n$
Let's divide the proof into two parts:
Part 1: We want to proof that $6 \nmid x_2 - 2$.
$x_2 - 2 = x_1 + 3^n - 2 = (x_1 - 5) + (3^n + 3) = (x_1 - 5) + (3^n - 3) + 6$
Form part (b), we already know that $6|3^n - 3$, also $6|6$.
However, from the assumption we can know that $6 \nmid x_1 - 5$
By the Quotient-Remainder Theorem, we can conclude that:
$6 \nmid x_1 - 5 + 3^n - 3 + 6$, which is $6 \nmid x_2 - 2$
Part 2: We want to prove that $6 \nmid x_2 - 5$
$x_2 - 5 = x_1 + 3^n - 5 = (x_1 - 2) + (3^n - 3)$
From part (b), we already know that $6|3^n - 3$. From the assumption, we can know that $6 \nmid x_1 - 2$
Thus, by the Quotient-Remainder Theorem, we can conclude that
$6 \nmid x_1 - 2 + 3^n - 3$, which is $6 \nmid x_2 - 5$

3. Properties of Asymptotic Notation

(a) **Translation:** $\exists k \in \mathbb{N}$, $\exists c, n_0 \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow n^n \leq c * n^k$
Negation: $\forall k \in \mathbb{N}, \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^n > c * n^k$

Proof: Let $k \in \mathbb{N}$, let $c, n_0 \in \mathbb{R}^+$, let $n = k+c+n_0+1$. We want to prove $n \geq n_0$ and $n^n > c * n^k$. We will divide the proof in two parts.
Part 1: Since $k \geq 0, c, n_0 > 0$

$$\begin{aligned} k + c + n_0 + 1 &\geq n_0 \\ So, n &\geq n_0 \end{aligned}$$

Part 2: Since $k \geq 0, c, n_0 > 0$ and $n = k+c+n_0+1$, $n > k+1, n > c$ and $n > n_0$.

$$\begin{aligned} c * n^k &< n * n^k \\ &= n^{k+1} \\ &< n^n (\text{since } n^x \text{ is a non-decreasing function and } n > k+1 \end{aligned}$$

So the original statement is false

(b) **Translation:**$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 165n^5 + n^2 \leq c(n^5 - n^3)$
Proof: Let $c = 166$, Let $n_0 = 13$, Let $n \in \mathbb{N}$, assume $n \geq n_0$, we want to prove that $165n^5 + n^2 \leq c(n^5 - n^3)$.

$$\begin{aligned} c(n^5 - n^3) &= 166(n^5 - n^3) \\ &= 165n^5 + (n^5 - 166n^3) \\ &= 165n^5 + n^2 * n * (n^2 - 166) \\ &\geq 165n^5 + n^2 * 13 * (13^2 - 166)\ (\text{since } n \geq n_0 = 13) \\ &= 165n^5 + 39n^2 \\ &> 165n^5 + n^2 \end{aligned}$$

So $c(n^5 - n^3) \geq 165n^5 + n^2$ has been proven

(c) **Translation:** $\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 * 4^{n^2+n} \leq 4^{n^2} \leq c_2 * 4^{n^2+n}$
Negation: $\forall c_1, c_2, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, (n \geq n_0 \wedge c_1 * 4^{n^2+n} > 4^{n^2}) \vee (n \geq n_0 \wedge 4^{n^2} > c_2 * 4^{n^2+n})$
Proof: Let $c_1, c_2, n_0 \in \mathbb{R}^+$, Let $n = max(\frac{\ln \frac{1}{c_1}}{\ln 4} + 1, n_0)$, we want to prove that $n \geq n_0$ and $c_1 * 4^{n^2+n} > 4^{n^2}$, or $n \geq n_0$ and $c_2 * 4^{n^2+n} < 4^{n^2}$. We will prove that $n \geq n_0$ and $c_1 * 4^{n^2+n} > 4^{n^2}$. Let us divide the proof in two part.
Part 1: We want to prove that $n \geq n_0$.
Since $n = max(\frac{\ln \frac{1}{c_1}}{\ln 4} + 1, n_0)$, then $n \geq n_0$.
Part 2: We want to prove that $c_1 * 4^{n^2+n} > 4^{n^2}$. since $n = max(\frac{\ln \frac{1}{c_1}}{\ln 4} + 1, n_0)$, then $n \geq \frac{\ln \frac{1}{c_1}}{\ln 4} + 1$

$$n \geq \frac{\ln \frac{1}{c_1}}{\ln 4} + 1$$

$$n > \frac{\ln \frac{1}{c_1}}{\ln 4}$$

$$4^n > 4^{\frac{\ln \frac{1}{c_1}}{\ln 4}} \text{ (since } 4^n \text{ is non-decreasing and use Definition 1)}$$

$$\ln 4^n > \ln 4^{\frac{\ln \frac{1}{c_1}}{\ln 4}} \text{ (since } \ln x \text{ is non-decreasing and use Definition 1)}$$

$$\ln 4^n > \frac{\ln \frac{1}{c_1}}{\ln 4} * \ln 4$$

$$\ln 4^n > \ln \frac{1}{c_1}$$

$$4^n > \frac{1}{c_1} \text{ (since } \ln x \text{ is non-decreasing and use Definition 1)}$$

$$c_1 * 4^n > 1 \text{ (since } c_1 \in \mathbb{R}^+)$$

$$c_1 * 4^{n^2} * 4^n > 4^{n^2} \text{ (since } 4^{n^2} > 0$$

$$c_1 * 4^{n^2+n} > 4^{n^2}$$

so the statement is false.

(d) *Proof.* Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Let $n \in \mathbb{N}$. Assume that f is non-decreasing and $f(n) = n^2$ for every $n \in \mathbb{N}$.
If we want to prove that $f \in \Theta(n^2)$, we can prove that $f \in O(n^2)$ and $f \in \Omega(n^2)$.
We will divide the proof in two parts.
Part 1: We want to prove that $f \in O(n^2)$.
Let c = 4, $n_0 = 0.1$ and $g(n) = 4n^2$. We want to prove that when $n \geq n_0, f(n) \leq 4n^2$, which is $f(n) \leq g(n)$, so that f $\in O(n^2)$.
Since n is natural number, we always have that $n \geq 1 > 0.1 = n_0$.
Let us divide the proof in two cases.
Case 1: n is a power of two, which means that n = 2^k, where $k \in \mathbb{N}$.
Because n is a power of two, from the question, we can know that $f(n) = n^2 \leq 4n^2$
Thus, f $\in O(n^2)$.
Case 2: n is not a power of two.
Since n is not the power of two and $n \in \mathbb{N}$, $n > 2$. Thus there must be a $k \in \mathbb{N}$ so that $2^k <$ n $< 2^{k+1}$.
Since f is non-decreasing, $f(2^k) < f(n) < f(2^{k+1})$ (from Definition 1).
From the question, we can know that:
$f(2^k) = (2^k)^2 = 2^{2k}$
$f(2^{k+1}) = (2^{k+1})^2 = 2^{2k+1}$ (since $f(n) = n^2$ for every $n \in \mathbb{N}$ that is a power of two)
So $2^{2k} \leq f(n) \leq 2^{2k+1}$.
Also, we can easily know that $g(n) = 4n^2$ is a non-decreasing function.
Thus, $g(2^k) \leq g(n)$ (since $2^k \leq n$).

$g(2^k) = 4 * (2^k)^2 = 4 * 2^{2k} = 2^2 * 2^{2k} = 2^{2k+2}$
Thus, $2^{2k+2} \leq g(n)$
Thus, $2^{2k} \leq f(n) \leq 2^{2k+1} \leq g(n)$.
So, $f(n) \leq g(n) = 4n^2$
So $f \in O(n^2)$

Part 2: We want to prove that $f \in \Omega(n^2)$.
Let c $= \frac{1}{4}$, $n_0 = 0.1$ and $g(n) = \frac{1}{4}n^2$. We want to prove that when $n \geq n_0$, $f(n) \geq \frac{1}{4}n^2$, which is $f(n) \geq g(n)$, so that f $\in \Omega(n^2)$.
Since n is natural number, we always have that $n \geq 1 > 0.1 = n_0$.
Let us divide the proof in two cases.
Case 1: n is a power of two, which means that there is a $k \in \mathbb{N}$ so that n $= 2^k$.
Because n is a power of two, from the question, we can know that $f(n) = n^2 \geq \frac{1}{4}n^2$
Thus, $f \in \Omega(n^2)$.
Case 2: n is not a power of two.
Since n is not the power of two and $n \in \mathbb{N}$, $n > 2$. Thus there must be a $k \in \mathbb{N}$ so that $2^k <$ n $< 2^{k+1}$.
Since f is non-decreasing, $f(2^k) < f(n) < f(2^{k+1})$ (from Definition 1).
From the question, we can know that:
$f(2^k) = (2^k)^2 = 2^{2k}$
$f(2^{k+1}) = (2^{k+1})^2 = 2^{2k+1}$ (since $f(n) = n^2$ for every $n \in \mathbb{N}$ that is a power of two)
So $2^{2k} \leq f(n) \leq 2^{2k+1}$.
We can easily know that $g(n) = \frac{1}{4}n^2$ is a non-decreasing function.
Thus, $g(n) \leq g(2^{k+1})$ (since $n \leq 2^{k+1}$).
$g(2^{k+1}) = \frac{1}{4} * (2^{k+1})^2 = \frac{1}{4} * 2^{2k+2} = 2^{-2} * 2^{2k+2} = 2^{2k}$
Thus, $g(n) \leq 2^{2k}$
Thus, $g(n) \leq 2^{2k} \leq f(n) \leq 2^{2k+1}$.
So, $f(n) \geq g(n) = \frac{1}{4}n^2$
So $f \in \Omega(n^2)$