Assinment 1

May 21, 2025

1 Requirements

- Read the material about how to find the roots of quadratic functions and analyze their monotonicity.
- Use this knowledge to solve the problems below.
- Write your answers in the space provided.

2 Introduction

A quadratic function is a second-degree polynomial function of the form:

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$, and $a, b, c \in \mathbb{R}$.

3 Finding Roots of Quadratic Functions

The roots of a quadratic equation $f(x) = 0$ are the solutions to:

$$ax^2 + bx + c = 0$$

3.1 Quadratic Formula

The roots can be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $\Delta = b^2 - 4ac$ is called the discriminant.

3.2 Discriminant Analysis

- $\Delta > 0$: Two distinct real roots
- $\Delta = 0$: One real root (double root)
- $\Delta < 0$: No real roots (complex conjugate roots)

3.3 Example

Find the roots of $2x^2 - 4x - 6 = 0$:

1. Identify coefficients: $a = 2$, $b = -4$, $c = -6$
2. Calculate discriminant:

$$\Delta = (-4)^2 - 4(2)(-6) = 16 + 48 = 64$$

3. Compute roots:

$$x = \frac{4 \pm \sqrt{64}}{4} = \frac{4 \pm 8}{4}$$
$$x_1 = 3, \quad x_2 = -1$$

4 Monotonicity Analysis

The monotonicity (increasing/decreasing behavior) of a quadratic function can be determined by its derivative.

4.1 Derivative of Quadratic Function

The derivative is:

$$f'(x) = 2ax + b$$

4.2 Critical Point

The vertex (critical point) occurs where $f'(x) = 0$:

$$2ax + b = 0 \Rightarrow x = -\frac{b}{2a}$$

4.3 Monotonicity Intervals

- For $a > 0$:
  - Decreasing on $\left(-\infty, -\frac{b}{2a}\right)$
  - Increasing on $\left(-\frac{b}{2a}, \infty\right)$
- For $a < 0$:
  - Increasing on $\left(-\infty, -\frac{b}{2a}\right)$
  - Decreasing on $\left(-\frac{b}{2a}, \infty\right)$

4.4 Monotonicity Around Zeros

For a quadratic function with two real roots $x_1 < x_2$:

- When $a > 0$:
  - Decreasing from $+\infty$ to x_1
  - Increasing from x_1 to x_2
  - Continuing to increase beyond x_2
- When $a < 0$:
  - Increasing from $-\infty$ to x_1
  - Decreasing from x_1 to x_2
  - Continuing to decrease beyond x_2

4.5 Example

Analyze $f(x) = x^2 - 4x + 3$:

1. Find derivative: $f'(x) = 2x - 4$
2. Critical point: $2x - 4 = 0 \Rightarrow x = 2$
3. Monotonicity:
   - Decreasing on $(-\infty, 2)$
   - Increasing on $(2, \infty)$
4. Roots: $x^2 - 4x + 3 = 0 \Rightarrow x_1 = 1$, $x_2 = 3$
5. Behavior around zeros:
   - Approaches $x_1 = 1$ while decreasing
   - Passes minimum at $x = 2$
   - Approaches $x_2 = 3$ while increasing

5 Conclusion

- Roots can be found using the quadratic formula and discriminant
- Monotonicity is determined by the derivative and the sign of a
- The vertex divides the function into increasing and decreasing regions
- Behavior around zeros depends on the function's concavity

Problem 1

Given the sets $A = \{x \mid 2 \le x < 6\}$, $B = \{x \mid 3 < x < 9\}$:

(1) Find $C_{\mathbb{R}}(A \cap B)$ and $(C_{\mathbb{R}}B) \cup B$;

(2) Given $C = \{x \mid a < x < a + 1\}$, if $C \subseteq B$, find the range of real number a.

Problem 2

Given the universal set $\mathbb{R}$, set $A = \{x \mid x^2 - 3x - 10 < 0\}$, and set $B = \{x \mid x^2 - (a+2)x + 2a > 0, a \in \mathbb{R}\}$.

(1) When $a = -1$, find $A \cap B$ and $A \cup B$;

(2) When $a < 2$, and $C_{\mathbb{R}}A \subseteq B$, find the range of real number a.

Problem 3

For a non-empty number set A, if its largest element is M and smallest element is m, then the spread of set A is defined as $T_A = M - m$. If set A has only one element, then $T_A = 0$.

(1) If $A = \{2, 3, 4, 5\}$, find T_A;

(2) If $A = \{1, 2, 3, \cdots, 9\}$, $A_i = \{a_i, b_i, c_i\} \subseteq A$,

$$A_i \cap A_j = \emptyset \quad (i, j = 1, 2, 3, i \neq j), \quad A_1 \cup A_2 \cup A_3 = A,$$

find the maximum value of $T_{A_1} + T_{A_2} + T_{A_3}$, and write one possible combination of A_1, A_2, A_3 that achieves this maximum value.