Chap 4: Special Probability Distributions

Discrete Uniform Distribution

r.v. X assumes the values $x_1, x_2, \ldots, x_k$ $\bar{w}$ equal probability.

Probability Function

$$f(x) = \frac{1}{k}, \quad x = x_1, x_2, \ldots, x_k, \text{ and } 0 \text{ otherwise}$$

$$E(X) = \frac{1}{k}\sum_{i=1}^{k} x_i$$

$$V(X) = \frac{1}{k}\sum_{i=1}^{k}(x_i - \mu)^2 = \frac{1}{k}\left(\sum_{i=1}^{k} x_i^2\right) - \mu^2$$

Bernoulli Distribution (binomial distr when n = 1)

X has only 2 possible outcomes.

Collection of all probability distribution for diff values of the param is called a family of prob distr.

Known as a Parameter

Probability Function

$$f_X(x) = p^x(1-p)^{1-x}, \quad x = 0, 1, \quad 0 \le p \le 1$$
$$= p^x q^{1-x} \quad \text{and } 0 \text{ otherwise.}$$

$P(X=1) = p$

$P(X=0) = 1-p = q$

$E(X) = p$

$V(X) = p(1-p) = pq$

Negative Binomial Distribution $X \sim NB(k, p)$

X is the num of trials to produce k successes in a sequence of indep Bernoulli trials.

Probability Function

$$f_X(x) = \binom{x-1}{k-1} p^k q^{x-k} \quad \text{for } x = k, k+1, \ldots$$

$$E(X) = \frac{k}{p}$$

$$V(X) = \frac{(1-p)k}{p^2}$$

In an NBA championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

Let X be the number of games that Team A plays to win the series (i.e. Team A wins 4 games in the X games)

The probability of Team A winning a game is 0.55.

Therefore X follows a Negative Binomial Distribution with parameters $k = 4$ and $p = 0.55$.

That is $X \sim NB(4, 0.55)$

(a) $\Pr(X = 6) = \binom{5}{3} 0.55^4 (1 - 0.55)^{6-4} = 0.1853$

(b) $\Pr(X \ge 4) = \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$

$= \binom{3}{3} 0.55^4 (1-0.55)^{4-4} + \binom{4}{3} 0.55^4 (1-0.55)^{5-4} + \binom{5}{3} 0.55^4 (1-0.55)^{6-4} + \binom{6}{3} 0.55^4 (1-0.55)^{7-4}$

$= 0.6083.$ $P(B\text{ win}) = 1 - 0.6083$

Binomial Distribution $X \sim B(n, p)$

X is the num of successes that occurs in n indep Bernoulli trials.

Conditions for a Binomial Experiment:

1. It consists of n repeated Bernoulli trials.
2. Only 2 possible outcomes in each trial.
3. P(success) is constant in each trial.
4. Trials are independent.

Probability Function

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n; \ 0 \le p \le 1; \ n \in$$
$$= \binom{n}{x} p^x q^{n-x}$$

$E(X) = np$

$V(X) = np(1-p) = npq$

- An electronics manufacturer claims that at most 10% of its power supply units need services during the warranty period.
- To investigate this claim, technicians at a testing laboratory purchase 20 units and subject each one to accelerated testing to simulate use during the warranty period.
- Let p denote the probability that a power supply unit needs repair during the period (the proportion of all such units that need repair).
- The laboratory technicians must decide whether the data resulting from the experiment supports that claim the $p \le 0.10$.
- Let X denote the number of units among 20 sampled that need repair, so $X \sim B(20, p)$. (Why?)
- Consider the decision rule:
  - Reject the claim that $p \le 0.10$ in favour of the conclusion that $p > 0.10$ if $x \ge 5$, (where x is the observed value of X) and
  - consider the claim plausible if $x \le 4$. $\Rightarrow$ do not reject H_0

The probability that the claim is rejected when $p = 0.10$ (an incorrect conclusion) is

$$\Pr(X \ge 5 \text{ when } p = 0.10) = 0.0432$$

The probability that the claim is not rejected when $p = 0.20$ (a different type of incorrect conclusion) is

$$\Pr(X \le 4 \text{ when } p = 0.20) \quad p > 0.10$$
$$= 1 - \Pr(X \ge 5 \text{ when } p = 0.20) \quad \text{do not reject } p \le 0.10$$
$$= 1 - 0.3704 = 0.6296.$$

The first probability is rather small, but the second is intolerably large.

When $p = 0.20$, the manufacturer has grossly understated in percentage of units that need service, and the stated decision rule is used, 63% of all samples will result in the manufacturer's claim being judged plausible!

Geometric Distribution $X \sim Geom(p)$

- X is the num of trials to produce the 1st success in a sequence of indep Bernoulli trials.
- Special case of NB distr

- At a "busy time", a telephone exchange is very near capacity, so callers have difficulty placing their calls.
- It may be of interest to know the number of attempts necessary in order to make a connection.
- Suppose that we let $p = 0.05$ be the probability of connection during a busy time.
- We are interested in knowing the probability that 5 attempts are necessary for a successful call.

- Let X be the number of attempts are necessary for the **first** successful call.
- The probability of connecting is 0.05.
- Therefore X follows a Negative Binomial Distribution with parameters $k = 1$ and $p = 0.05$.
- (or X follows a Geometric Distribution with $p = 0.05$)
- That is $X \sim NB(1, 0.05)$ (or $X \sim Geom(0.05)$)

$$\Pr(X = 5) = \binom{5-1}{1-1} 0.05^1 (1 - 0.05)^{5-1}$$
$$= 0.05(0.95)^4 = 0.0407$$

Poisson Distribution

X is the num of successes occurring during a given time interval or in a specified region.

Properties:

1. The num of successes occurring in one time interval are indep of those occurring in any other disjoint time interval.
2. Probability of a single success occurring during a very short time interval is proportional to the length of the time interval.
3. Probability of >1 success occurring in such a short time interval is negligible.

Probability Function

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$ for $x \in \mathbb{Z}^+$ and λ = avg # sucesses in the given time interval

$E(X) = \lambda$

$V(X) = \lambda$

- As a check on the quality of the wooden doors produced by a company, its owner requested that each door undergoes inspection for defects before leaving the plant.
- The plant's quality control inspector found that on the average 1 square foot of door surface contains 0.5 minor flaw. $\lambda = 0.5$
- Subsequently, 1 square foot of each door's surface was examined for flaws. The owner decided to have all doors reworked that were found to have two or more minor flaws in the square foot of surface that was inspected.

(a) What is the probability that a door will fail inspection and be sent back for reworking?

(b) What is the probability that a door will pass inspection?

Let X be the number of flaws found in 1 square foot of door surface.

Then $X \sim P(\lambda)$, where $\lambda = E(X) = 0.5$.

(a) $\Pr(X \geq 2) = 1 - \Pr(X < 2)$
$$= 1 - \Pr(X = 0) - \Pr(X = 1)$$
$$= 1 - e^{-0.5} - \frac{(0.5)e^{-0.5}}{1!} = 0.0902.$$

(b) $\Pr(X < 2) = 1 - 0.0902 = 0.9098.$

Poisson Approximation to Binomial Distribution

$X \sim B(n, p)$

$$\lim_{\substack{p \to 0 \\ n \to \infty}} \Pr(X = x) = \frac{e^{-np}(np)^x}{x!}$$

If p -> 1, swap the def for success and failure to use approximation

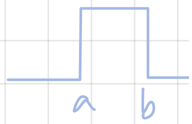
- In a manufacturing process in which glass items are being produced, defects or bubbles occur, occasionally rendering the piece undesirable for marketing.
- It is known that on the average 1 in every 1000 of these items produced has one or more bubbles.
- What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?
- Let X be the number of items processing bubbles.
- Then $X \sim B(8000, 0.001)$.
- Use the Poisson approximation to the binomial. Put $\lambda = np = 8000 \times 0.001 = 8$, and hence X approximately $\sim P(\lambda)$.
- The (approximate) probability is given by $\Pr(X < 7) = 1 - \Pr(X \geq 7) \approx 1 - 0.6866 = 0.3134$.

continuous r.v. from this point onwards

Continuous Uniform Distribution $X \sim U(a,b)$

Uniform distribution over the interval [a, b].

Aka rectangular distribution.



Probability Function

$f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$, and 0 otherwise

$E(X) = \frac{a+b}{2}$

$V(X) = \frac{(b-a)^2}{12}$

$$\text{c.d.f.} = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & , x < a \\ \int_a^x \frac{1}{b-a}\,dt, & a \leq x \leq \\ 1 & , b < x \end{cases}$$

$$= \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ 1 & , b < x \end{cases}$$

Exponential Distribution $X \sim Exp(\alpha)$

Non-negative x

Probability Function

$f_X(x) = \alpha e^{-\alpha x}$, for $x > 0$ and 0 otherwise

$= \frac{1}{\mu} e^{-x/\mu}$

Note : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$E(X) = \frac{1}{\alpha} = \mu$

$V(X) = \frac{1}{\alpha^2} = \mu^2$

c.d.f. $= F_X(x) = \int_0^x \alpha e^{-\alpha t}\,dt = 1 - e^{-\alpha x} = P(X \leq x)$

$P(X > x) = e^{-\alpha x}$, $x > 0$

No Memory Property

$P(X > s+t \mid X > s) = P(X > t)$

- Suppose that the failure time, T, of a system is exponentially distributed, with a mean of 5 years.
- What is the probability that at least two out of five of these systems are still functioning at the end of 8 years?

<u>Solution</u> $P(T<8)$ = fails before 8 years Y = no. of successes out of 5 trials

$Y \sim B(5, p)$

- Since $E(T) = 5$, therefore $\alpha = 1/5$.
- Hence $T \sim Exp(1/5)$.

$$\Pr(T > 8) = \int_8^{\infty} \frac{1}{5} e^{-8/5}\,dx = e^{-8/5} \approx 0.2.$$

= probability of success

- Let X represent the number of systems out of the five systems still functioning after 8 years.
- Then, $X \sim B(5, 0.2)$.
- Hence, we obtain from the statistical table

$$\Pr(X \geq 2) = 0.2627.$$

Continuity Correction

Used when approximating Binomial dist with normal dist.

(a) $\Pr(X = k) \approx \Pr(k - ½ < X < k + ½)$.

(b) $\Pr(a \leq X \leq b) \approx \Pr(a - ½ < X < b + ½)$.

$\Pr(a < X \leq b) \approx \Pr(a + ½ < X < b + ½)$.

$\Pr(a \leq X < b) \approx \Pr(a - ½ < X < b - ½)$.

$\Pr(a < X < b) \approx \Pr(a + ½ < X < b - ½)$.

(c) $\Pr(X \leq c) = \Pr(0 \leq X \leq c) \approx \Pr(-½ < X < c + ½)$.

(d) $\Pr(X > c) = \Pr(c < X \leq n) \approx \Pr(c + ½ < X < n + ½)$

- A system is made up of 100 components, and each of which has a reliability equal to 0.90.
- These components function independently of one another, and the entire system functions only when at least 80 components function.
- What is the probability that the system functioning?

- Let X be the number of components functioning.
- Then $X \sim B(100, 0.9)$.
- Thus $E(X) = (100)(0.9) = 90$ and
  $V(X) = 100(0.9)(0.1) = 9$. $X \sim N(90, 9)$
- The system is functioning if $80 \leq X \leq 100$.

$\Pr(79.5 < X < 100.5)$

$$\Pr(80 \leq X \leq 100) = \Pr\left(\frac{79.5 - 90}{3} < \frac{X - 90}{3} < \frac{100.5 - 90}{3}\right)$$

$$\approx \Pr(-3.5 < Z < 3.5) \approx \Pr(Z < 3.5) - \Pr(Z < -3.5)$$

$$= 0.9995.$$

Normal Distribution $X \sim N(\mu, \sigma^2)$

Aka Gaussian distr

Properties: $P(X \geq x_\alpha) = P(X \leq -x_\alpha) = \alpha$

1. Symmetrical about the vertical line x = mean
2. Max point occurs at x = mean = $\frac{1}{\sqrt{2\pi}\,\sigma}$
3. x-axis is the asymptote
4. Area under the curve = 1 $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
5. Same variance, same shape
6. V(X) increases, curve flattens $P(X=x)=0$; $P(X \leq x) = P(X <$

Probability Function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty$$

$-\infty < \mu < \infty$

$\sigma > 0$

$E(X) = \mu$

$V(X) = \sigma^2$

Standardised Normal Distribution

$Z = \frac{(X-\mu)}{\sigma} \Rightarrow Z \sim N(0,1)$

$E(Z) = 0$

$V(Z) = 1$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

- Given $X \sim N(50, 10^2)$, find $\Pr(45 < X < 62)$.

<u>Solution</u>

$\Pr(45 < X < 62)$

$= \Pr[(45 - 50)/10 < Z < (62 - 50/10)]$ if 1.045 → take avg of 1.04 & 1.05

also means 0.5 σ below μ $= \Pr(-0.5 < Z < 1.2)$ 1.2 σ above mean

$\Pr(Z < 1.2) - \Pr(Z < -0.5)$ symmetrical

$= 1 - \Pr(Z \geq 1.2) - \Pr(Z > 0.5)$

$= 1 - 0.1151 - 0.3085 = 0.5764.$

Microsoft Excel Worksheet

$P(X$ w/i $1\sigma) = 0.6826$

$P(X$ w/i $2\sigma) = 0.9544$

$P(X$ w/i $3\sigma) = 0.9974$

Normal Approximation to Binomial Distribution

$n \to \infty$ and $p \to ½$

Rule of thumb :

$np > 5$ and $n(1-p) > 5$

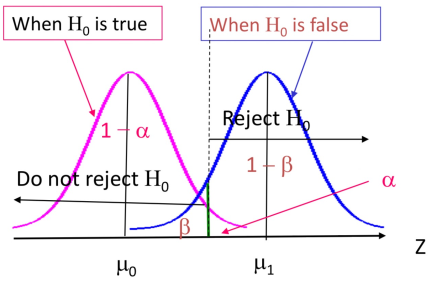
$X \sim B(n,p) \Rightarrow \mu = np$ and $\sigma^2 = np(1-p)$

$Z = \frac{X - np}{\sqrt{npq}}$ is approximately $\sim N(0,1)$

$Y \sim N(np, npq)$

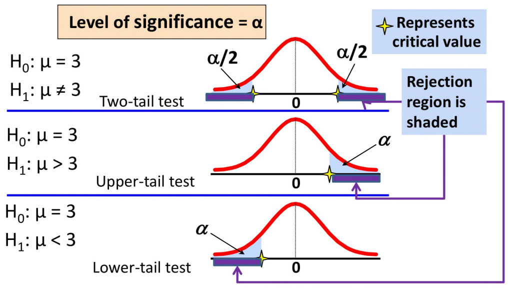
Decision	State of Nature: H_0 is true	State of Nature: H_0 is false
Reject H_0	**Type I error** Pr(Reject H_0 given that H_0 is true) = α	Correct decision Pr(Reject H_0 given that H_0 is false) = $1 - \beta$
Do not reject H_0	Correct decision Pr(Do not reject H_0 given that H_0 is true) = $1 - \alpha$	**Type II error** Pr(Do not reject H_0 given that H_0 is false) = β

Test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$



α = level of significance
= Pr(type I error)
= Pr(reject H_0 when H_0 is true)
= Pr(reject H_0 | H_0).
α is set by the researcher in advance
α is usually set at 5% or 1%

$1-\alpha$: confidence interval



A null hypothesis concerning a population parameter will always be stated to specify an exact value of the parameter.

H_0 is accepted if the C.I. covers μ_0

$$\Pr\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$\Leftrightarrow \quad \Pr\left(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

p-value = $2 \times \min\{P(Z < z_{-\alpha/2}), P(Z > z_{\alpha/2})\}$
- if p-value < α, reject H_0
- if p-value $\geq \alpha$, do not reject H_0

β = Pr(type II error)
= Pr(do not reject H_0 when H_0 is false)
= Pr(do not reject H_0 | H_1).

$1 - \beta$ = Power of a test = Pr(reject H_0 | H_1)

Known Variance & Normal Distribution (or n > 30)

2-sided Z - Test

- The director of a factory wants to determine if a new machine A is producing cloths with a breaking strength of 35 kg with a standard deviation of 1.5 kg.
- A random sample of 49 pieces of cloths is tested and found to have a mean breaking strength of 34.5 kg.
- Is there evidence that the machine is not meeting the specifications for mean breaking strength?
- Use $\alpha = 0.05$

Step 1
- Let μ be the mean breaking strength of cloths manufactured by the new machine.
- Test H_0: $\mu = 35$ kg vs H_1: $\mu \neq 35$ kg. (Why?)

Step 2
- Set $\alpha = 0.05$.

Step 3
- Since σ is known, the test statistic

$$Z = \frac{(\bar{X} - \mu_0)}{\sigma/\sqrt{n}}$$

is used.
- $z_{\alpha/2} = z_{0.025} = 1.96$.
- Critical region $z < -1.96$ or $z > 1.96$, where

$$z = \frac{(\bar{x} - \mu_0)}{\sigma/\sqrt{n}}$$

Step 4
- Computations: $\bar{x} = 34.5$ kg, $n = 49$, and hence

$$z = \frac{34.5 - 35}{1.5/\sqrt{49}} = -2.3333$$

Step 5
- Conclusion: Since the observed z value = –2.3333 falls inside the critical region (i.e. $z < z_{0.025} = -1.96$), hence H_0: $\mu = 35$ kg is rejected at the 5% level of significance.

Pr(Z < -2.33) = 0.0099
Pr(Z > -2.33) = 0.9901

p-value
= 2 min{Pr(Z < -2.33), Pr(Z >-2.33) }
= 2(0.0099) = .0198

Here: *p*-value = .0198
α = .05
Since .0198 < .05, we reject the null hypothesis

1-sided Z - Test

- A manufacturer of sports equipment has developed a new synthetic fishing line that he claims has a mean breaking strength better than the market average strength of 8 kilograms.
- Suppose the breaking strength of this type of fishing lines has a standard deviation of 0.5 kg.
- A random sample of 50 lines is tested and found to have a mean breaking strength of 8.2 kg.
- Test the manufacturer's claim.
- Use a 0.01 level of significance.

Step 1
- Let μ be the mean breaking strength of the new type of fishing lines.
- Test H_0: $\mu = 8$ against H_1: $\mu > 8$. (Why?)

Step 2
- Set $\alpha = 0.01$.

Step 3
- Since σ is known, the test statistic

$$Z = \frac{(\bar{X} - 8)}{0.5/\sqrt{50}}$$

is used.
- $z_\alpha = z_{0.01} = 2.326$.
- Critical region $z > 2.326$, where

$$z = \frac{(\bar{x} - 8)}{0.5/\sqrt{50}}$$

Step 4
- Computations: $\bar{x} = 8.2$, hence

$$z = \frac{8.2 - 8}{0.5/\sqrt{50}} = 2.828.$$

- *p*-value = Pr(Z > 2.828) ≈ 0.00233.

Step 5
- Conclusion: Since the observed z value = 2.828 falls inside the critical region (i.e. $z > z_{0.01} = 2.326$), hence H_0: $\mu =$ 8 kg is rejected at the 1% level of significance.
- Conclusion based on *p*-value: Since *p*-value ≈ 0.00233 is less than 0.01, hence H_0 is rejected at the 1% level of significance.

Unknown Variance

(1) Two sided test

- Test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$.
- Let

$$T = \frac{(\bar{X} - \mu_0)}{S/\sqrt{n}}$$

where S^2 is the sample variance.

- Then H_0 is rejected if the observed value of T, say t, $> t_{n-1;\alpha/2}$ or $< -t_{n-1;\alpha/2}$.

2-sided T - Test

(2) One sided test

- Test $H_0: \mu = \mu_0$ against $H_1: \mu > \mu_0$.
- Then H_0 is rejected if $t > t_{n-1;\alpha}$.
- Test $H_0: \mu = \mu_0$ against $H_1: \mu < \mu_0$.
- Then H_0 is rejected if $t < -t_{n-1;\alpha}$.

1-sided T - Test

- The average length of time for students to register for summer classes at a certain college has been 50 minutes.
- A new registration procedure is being tried.
- A random sample of 12 students had an average registration time of 42 minutes with a standard deviation of 11.9 minutes under the new system.
- Test the hypothesis that the population mean is now less than 50, using a level of significance of 0.05.
- Assume the population of times to be normal.

Step 1

- Let μ be the mean registration time.
- Test $H_0: \mu = 50$ against $H_1: \mu < 50$. (Why?)

Step 2

- Set $\alpha = 0.05$.

Step 3

- Since σ is unknown, the test statistic

$$T = \frac{(\bar{X} - \mu_0)}{S/\sqrt{n}}$$

is used.

- $n = 12$ implies that $t_{11;0.05} = 1.796$
- **Critical region** $t < -1.796$, where

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}}$$

Step 4

- **Computations:** $\bar{x} = 42$, $s = 11.9$, $n = 12$, and hence

$$t = \frac{42 - 50}{11.9/\sqrt{12}} = -2.329$$

- p-value $= \Pr(T < -2.329) = 0.0199$.

[or p-value is between 0.025 and 0.01 since 2.329 is between $t_{11;0.025} = 2.201$ and $t_{11;0.01} = 2.718$ if statistical table is used.]

Step 5

- **Conclusion:** Since the observed $t = -2.329$ falls inside the critical region (i.e. $t < t_{0.05} = -1.796$), hence $H_0: \mu = 50$ minutes is rejected at the 5% level of significance and we conclude that the true mean is likely to be less than 50 minutes.
- **Conclusion based on *p*-value:** Since p-value = 0.0199 is less than 0.05, hence H_0 is rejected at the 5% level of significance and we conclude that the true mean is likely to be less than 50 minutes.

Hypotheses Testing between 2 means

Known Variance & Normal Distribution (or n > 30)

2-Sample Z - Test

- Analysis of a random sample consisting of $n_1 = 20$ specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of $\bar{x}_1 = 29.8$ ksi.
- A second random sample of $n_2 = 25$ two-side galvanized steel specimens gave a sample average strength of $\bar{x}_2 = 34.7$ ksi.
- Assuming that the two yield strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$,
- does the data indicate that the corresponding true average yield strengths μ_1 and μ_2 are different?
- Use $\alpha = 0.01$.

Step 1

- Let μ_1 and μ_2 be the mean strength of cold-rolled steel and two-side galvanized steel respectively.
- Test $H_0: \mu_1 - \mu_2 = 0$ against $H_1: \mu_1 - \mu_2 \neq 0$.

Step 2

- Set $\alpha = 0.01$.

Step 3

- Since σ_1^2 and σ_2^2 are known, therefore the test statistic
$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$
is used.
- $\alpha = 0.01$ implies $z_{\alpha/2} = z_{0.005} = 2.5728$.
- Critical region: $z < -2.5758$ or $z > 2.5758$, where
$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

Step 4

- **Computations:** $\bar{x}_1 = 29.8, \bar{x}_2 = 34.7, \sigma_1^2 = 16, \sigma_2^2 = 25$, $n_1 = 20$ and $n_2 = 25$, so
$$z = \frac{[(29.8 - 34.7) - 0]}{\sqrt{16/20 + 25/25}} = -3.652.$$
- ***p*-value** $= 2 \times \min\{\Pr(Z > -3.652), \Pr(Z < -3.652)\} = 2(0.00013) = 0.00026$.

Step 5

- Conclusion: Since $z = -3.652$ falls inside the critical region, hence $H_0: \mu_1 = \mu_2$ is rejected at the 1% level of significance and conclude that the sample data strongly suggest that the true average yield strength for cold-rolled steel differs from that for galvanized steel.
- Conclusion based on p-value: Since p-value = 0.00026 is less than the level of significance 0.01, hence H_0 is rejected at the 1% level of significance.

Unknown Variance & Unequal Variance

2-Sample Z - Test

- In selecting a sulfur concrete for roadway construction in regions that experience heavy frost,
- it is important that the chosen concrete have a low value of thermal conductivity in order to minimize subsequent damage due to changing temperatures.
- Suppose two types of concrete, a graded aggregate and a no-fines aggregate, are being considered for a certain road.
- The following table summarizes data from an experiment carried out to compare the two types of concrete.

Type	Sample size	Sample average conductivity	Sample s.d.
Graded	35	0.497	0.187
No-fines	35	0.359	0.158

- Does this information suggest that the true conductivity for the graded concrete exceeds that for the no-fines concrete?
- Use $\alpha = 0.01$.

Step 1

- Let μ_1 and μ_2 be the mean conductivity of graded and no-fines concretes respectively.
- Test $H_0: \mu_1 - \mu_2 = 0$ against $H_1: \mu_1 - \mu_2 > 0$.

Step 2

- Set $\alpha = 0.01$.

Step 3

- Since σ_1^2 and σ_2^2 are unknown and the sample sizes are large, therefore the test statistic
$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}$$
is used.
- $\alpha = 0.01$ implies $z_\alpha = z_{0.01} = 2.3263$.
- **Critical region**: $z > 2.3263$, where
$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

Step 4

- **Computations**: $\bar{x}_1 = 0.497, \bar{x}_2 = 0.359, s_1^2 = 0.187^2, s_2^2 = 0.158^2, n_1 = n_2 = 35$, so
$$z = \frac{[(0.497 - 0.359) - 0]}{\sqrt{0.187^2/35 + 0.158^2/35}} = 3.335.$$
- p-value $= \Pr(Z > 3.335) = 0.00043$.

Step 5

- **Conclusion:** Since $z = 3.335$ falls inside the critical region, hence $H_0: \mu_1 = \mu_2$ is rejected at the 1% level of significance and conclude that the sample data argue strongly that the true average thermal conductivity for the graded concrete does exceed that for the no-fines concrete.
- **Conclusion based on *p*-value**: Since p-value = 0.00043 is less than the level of significance 0.01, hence H_0 is rejected at the 1% level of significance.

Unknown Variance & Equal Variance
Small Sample, Normal Dist

2-Sample T - Test (Pooled)

- A course in mathematics is taught to 12 students by the conventional classroom procedure.
- A second group of 10 students was given the same course by means of programmed materials.
- At the end of the semester the same examination was given to each group.
- The 12 students meeting in the classroom made an average grade of 85 with a standard deviation of 4,
- while the 10 students using programmed materials made an average of 81 with a standard deviation of 5.
- Test the hypothesis that the two methods of learning are equal using a 0.10 level of significance.
- Assume the populations to be approximately normal with equal variances

Step 1

- Let μ_1 and μ_2 be the average grades students taking this course by the classroom and programmed presentations, respectively.
- Test $H_0: \mu_1 - \mu_2 = 0$ against $H_1: \mu_1 - \mu_2 \neq 0$.

Step 2

- Set $\alpha = 0.1$.

Step 3

- $n_1 = 12$ and $n_2 = 10$ implies $t_{n_1+n_2-2;\alpha} = t_{20;0.05} = 1.725$.
- **Critical region :** $t < -1.725$ or $t > 1.725$, where

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

with

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2].$$

Step 4

- **Computations:**

$\bar{x}_1 = 85$ and $\bar{x}_2 = 81$,

$s_1^2 = 16, s_2^2 = 25$,

$n_1 = 12$ and $n_2 = 10$, so

$$s_p^2 = \frac{[11(16) + 9(25)]}{(12 + 10 - 2)} = 20.05$$

and $s_p = 4.478$.

- Hence

$$t = \frac{[(85 - 81) - 0]}{\sqrt{20.05(1/12 + 1/10)}} = 2.086$$

- p-value $= 2 \times \min\{\Pr(T_{20} > 2.086), \Pr(T_{20} < 2.806)\}$
  $= 2(0.025) = 0.05$.

Step 5

- **Conclusion**: Since the observed t-value = 2.086 which falls inside the critical region, hence $H_0: \mu_1 = \mu_2$ is rejected at the 10% level of significance

  and conclude that the two methods of learning are not equal.
- Since p-value = 0.05 is less than 0.10, therefore we reject H_0 at the 10% level of significance and conclude that the two methods of learning are not equal.

Hypotheses Testing on Variance

One Variance Case - Norm Dist

INVCHI

- A manufacturer of car batteries claims that the life of his batteries is approximately normally distributed with a standard deviation equal to 0.9 year.
- If a random sample of 10 of these batteries has a standard deviation of 1.2 years,
- do you think that σ > 0.9 year
- Use a 0.05 level of significance.

Step 1

- Let σ^2 be the variance of the battery life.
- Test $H_0: \sigma^2 = 0.81$. $H_1: \sigma^2 > 0.81$.

Step 2

- Set $\alpha = 0.05$.

Step 3

INVCHI
AREA RIGHT: 0.05
DF: 9

- $n = 10$ implies $\chi^2_{n-1;\,\alpha} = \chi^2_{9;0.05} = 16.919$.
- Critical region $\chi^2 > 16.919$, where

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2},$$

with $n = 10$ and $\sigma_0^2 = 0.81$.

Step 4

- **Computations:**

$s^2 = 11.44$, and $n = 10$, so

$1.2^2 = 1.44$

$$\chi^2 = \frac{9(1.44)}{0.81} = 16.0.$$

- p-value $= \Pr(\chi_9^2 > 16) = 0.0669$. [or it is between 0.05 and 0.10 from the statistical table]

Step 5

- **Conclusion:** Since the observed χ^2-value = 16, which falls outside the critical region, hence $H_0: \sigma^2 = 0.81$ is not rejected at the 5% level of significance and conclude that there is no reason to doubt that the standard deviation is 0.9 year. Or
- Since p-value is greater than 0.05, we do not reject H_0.

H_0	Test Statistic
$\sigma^2 = \sigma_0^2$	$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$

$H_0: \sigma^2 = \sigma_0^2$ is rejected if the observed χ^2-value

H_1	Critical Region
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{n-1;\alpha}$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{n-1;1-\alpha}$
$\sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi^2_{n-1;1-\alpha/2}$ or $\chi^2 > \chi^2_{n-1;\alpha/2}$

where $\Pr\left(W > \chi^2_{n-1;\alpha}\right) = \alpha$ with $W \sim \chi^2(n-1)$

Hypotheses Testing on Ratio of Variances

Unknown Means & Norm Dist

INVF or 2-Sample F Test

• An experiment was performed to compare the abrasive wear of two different laminated materials.
• 11 pieces of Material 1 were tested, by exposing each piece to a machine measuring wear.
• 9 pieces of Material 2 were similarly tested.
• In each case, the depth of wear was observed.
• The samples of Material 1 gave an average (coded) wear of 85 units with a standard deviation of 4,
• while the samples of Material 2 gave an average of 81 and a standard deviation of 5.
• Assume that the two unknown populations to be approximately normal,
• test the two variances are equal.
• Use a 0.10 level of significance

Step 1

- Let σ_1^2 and σ_2^2 be the variances of the abrasive wear made from Materials 1 and 2 respectively.
- Test: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \neq \sigma_2^2$.

Step 2

- Set $\alpha = 0.1$.

INVF
AREA RIGHT: 0.05
D1: 10
D2: 8

INVF
AREA RIGHT: 0.95
D1: 10
D2: 8

Step 3

- $n_1 = 11, n_2 = 9$ implies $F_{n_1-1\,,n_2-1\,;\alpha/2} = F_{10,8;0.05} = 3.35$ and
- $F_{n_1-1,n_2-1;1-\alpha/2} = F_{10,8;0.95} = 1/F_{8,10;0.05} = 1/3.07 = 0.326$.
- Critical region: $F > 3.35$ or $F < 0.326$, where $F = s_1^2/S_2^2$

Step 4

- **Computations:**
  $s_1^2 = 16, s_2^2 = 25$, so $F = 16/25 = 0.64$.

Step 5

- **Conclusion:** Since the observed F-value = 0.64 which falls outside the critical region, hence H_0: $\sigma_1^2 = \sigma_2^2$ is not rejected at the 10% level of significance and we conclude that we were justified in assuming the unknown variances equal.

$\mathbf{H_0}$	Test Statistic
$\sigma_1^2 = \sigma_2^2$	$F = \dfrac{S_1^2}{S_2^2}$

H_0: $\sigma_1^2 = \sigma_2^2$ is rejected if the observed F-value falls in the critical region

$\mathbf{H_1}$	Critical Region
$\sigma_1^2 > \sigma_2^2$	$F > F_{(n_1-1,n_2-1\,;\alpha)}$
$\sigma_1^2 > \sigma_2^2$	$F < F_{(n_1-1,n_2-1\,;1-\alpha)}$
$\sigma_1^2 \neq \sigma_2^2$	$F < F_{(n_1-1,n_2-1\,;1-\alpha/2)}$ or $F > F_{(n_1-1,n_2-1\,;\alpha/2\,)}$

where $\Pr\left(W > F_{v_1,v_2;\alpha}\right) = \alpha$ with $W \sim F(v_1, v_2)$