W1

Linear System

§ A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

n variables $x_1, x_2, \dots, x_n$

$a_{11}, a_{12}, \dots, a_{mn}$ and $b_1, b_2, \dots, b_m$ are real constants

$$\begin{array}{l} x + y + z + w = 1 \\ 2x - y + 3z + 5w = 2 \\ x + 2y + 7z + 0w = 5 \\ 0x - 6y + 2z + 9w = 0 \\ 5x + 2y - 4z + 7w = 8 \end{array}$$

standard form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 5 \\ 1 & 2 & 7 & 0 \\ 0 & -6 & 2 & 9 \\ 5 & 2 & -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 8 \end{pmatrix} \quad\Longleftrightarrow\quad x\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 5 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 2 \\ -6 \\ 2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \\ 7 \\ 2 \\ -4 \end{pmatrix} + w\begin{pmatrix} 1 \\ 5 \\ 0 \\ 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 8 \end{pmatrix}$$

matrix equation form

vector equation form

coefficient matrix

variable matrix

constant matrix

Elementary Row Operations (EROs)

(1) Multiply a row by a non-zero constant — $cR_i,\ c \neq 0$

(2) Interchange two rows — $R_i \leftrightarrow R_j$

(3) Add a multiple of one row to another row — $R_i + cR_j,\ c \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right) \xrightarrow{\text{Multiply first row by 3}} \left(\begin{array}{ccc|c} 3 & 3 & 6 & 27 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add 2 times of first row to second row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 4 & 6 & 1 & 19 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

2 2 4 18
+2 4 -3 1

only second row is changed

Interchange second and third rows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \end{array}\right)$$

if order doesn't matter, can stack e.r.o.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[2R_2]{2R_1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Augmented Matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 5 & 2 \\ 1 & 2 & 7 & 0 & 5 \\ 0 & -6 & 2 & 9 & 0 \\ 5 & 2 & -4 & 7 & 8 \end{pmatrix}$$

e.g. $\left(\begin{array}{cc|c} a & b & b \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - aR_2} \left(\begin{array}{cc|c} 0 & b-a & b-a \\ 1 & 1 & 1 \end{array}\right)$ (can be 0)

$\to b = a \Rightarrow$ infinite

$\to b \neq a \Rightarrow$ unique

Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a 4×4 matrix and $\mathbf{b}$ is 4×1. Suppose that the reduced row echelon form of $(\mathbf{A} \mid \mathbf{b})$ is $(\mathbf{R} \mid \mathbf{d})$, where $\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

(a) (4 points) Assuming that $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions, write down all the possible forms of $\mathbf{R}$. You may use $*$ to denote an unknown number.

Solution: The last row of $\mathbf{R}$ must be zero and $\mathbf{R}$ must have exactly one non-pivot column. The possibilities are:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

When solving for RREF w k non-zero rows/non-pivot columns,
do not consider only the last k columns.
No. Of possibilities = n choose (n-k)

Linear Systems

xy-plane

$l_1: a_1x + b_1y = c_1$

$l_2: a_2x + b_2y = c_2$

no solution

→ inconsistent system

parallel lines

- REF has a row with non-zero last entry but zero elsewhere.
- the last column of REF is a pivot column.

 e.g. $\left(\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$

≥ 1 solution

→ consistent system

exactly 1 sol^n

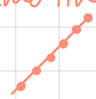
intersecting lines

- every column of REF is a pivot column, except the last column
- no. of variables in LS = no. of non-zero rows in REF

e.g. $\begin{array}{c} \\ \text{3 rows} \end{array} \overbrace{\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right)}^{\text{3 var}}$ $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

infinitely many sol^{n}s

same line



- there is a non-pivot column in the REF other than the last column.
- no. of variables in LS > no. of non-zero rows in REF

e.g. $\text{3 rows}\ \underbrace{\left(\begin{array}{cccc|c} 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)}_{\text{4 vars}}$ $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$

xyz-space

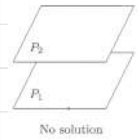
$p_1: a_1x + b_1y + c_1z = d_1$

$p_2: a_2x + b_2y + c_2z = d_2$

2 eqns, 3 vars

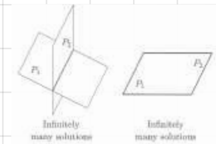
no sol^n

parallel planes



infinitely many sol^{n}s

- same plane
- intersect at a line



REF		Solutions	Geometrical interpretation
3 leading entries	$\left(\begin{array}{ccc|} \otimes & & \\ & \otimes & \\ & & \otimes \\ 0 & 0 & 0 \end{array}\right)$	0 parameter	Intersect at 1 point
2 leading entries	$\left(\begin{array}{ccc|} \otimes & & \\ & \otimes & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$	1 parameter	Intersect at a line
1 leading entry	$\left(\begin{array}{ccc|} \otimes & & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$	2 parameters	Intersect at a plane
0 leading entry	$\left(\begin{array}{ccc|} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$	3 parameters	NA

Row Equivalence

Def. 2 augmented matrices are row equivalent (to each other) if one can be obtained from the other by a series of EROs

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

^ Any 2 of the augmented matrices are row equivalent

no need be sq matrices

A & B are row-equivalent ⟺ $PA = B$, P inv

note pre-multiply

$AP = B$ means column-eqv.

Theorem 1.2.7.

If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.

Express each of the following by the set notation in both implicit and explicit form:

(a) the line in $\mathbb{R}^2$ passing through the points $(1,2)$ and $(2,-1)$.

(b) the plan in $\mathbb{R}^3$ containing the points $(0,1,-1)$, $(1,-1,0)$ and $(0,2,0)$.

(c) the line in $\mathbb{R}^3$ passing through the points $(0,1,-1)$ and $(1,-1,0)$.

(a) Substituting $(x,y) = (1,2)$ and $(2,-1)$ into the equation $ax + by = c$, we has a system of linear equations

$$\begin{cases} a + 2b - c = 0 \\ 2a - b - c = 0 \end{cases}$$

which implies $a = \frac{3}{5}c$ and $b = \frac{1}{5}c$. In set notation, the line is

$$\{(x,y) \mid 3x + y = 5\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{5-t}{3}, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

(b) Substituting $(x,y,z) = (0,1,-1)$, $(1,-1,0)$ and $(0,2,0)$ into the equation $ax + by + cz = d$, we has a system of linear equations

$$\begin{cases} b - c - d = 0 \\ a - b \quad - d = 0 \\ 2b \quad - d = 0 \end{cases}$$

which implies $a = \frac{3}{2}d$, $b = \frac{1}{2}d$ and $c = -\frac{1}{2}d$. In set notation, the plane is

$$\{(x,y,z) \mid 3x + y - z = 2\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{2-s+t}{3}, s, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

(c) In explicit form, the line is

$$\{(1,-1,0) + t(-1,2,-1) \mid t \in \mathbb{R}\} = \{(1-t, -1+2t, -t) \mid t \in \mathbb{R}\}.$$

To find the implicit form, we need to find two non-parallel planes containing the two points $(0,1,-1)$ and $(1,-1,0)$. The intersection of the two planes will give us the required line. Substituting $(0,1,-1)$ and $(1,-1,0)$ into $ax + by + cz = d$ we has a system of linear equations

$$\begin{cases} b - c - d = 0 \\ a - b \quad - d = 0 \end{cases}$$

We obtain $a = c + 2d$ and $b = c + d$. There are infinitely many such planes. For example, we can write the line implicitly as

→ sub $c=0, d=1$; $c=1, d=0$

$$\{(x,y,z) \mid x + y + z = 0 \text{ and } 2x + y = 1\}.$$

$$\begin{cases} 3x + 2y - z = 0 \\ x - 3y - 2z = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{11}(2 + 7t) \\ y = \frac{1}{11}(-3 - 5t) \quad \text{where } t \in \mathbb{R} \\ z = t \end{cases}$$

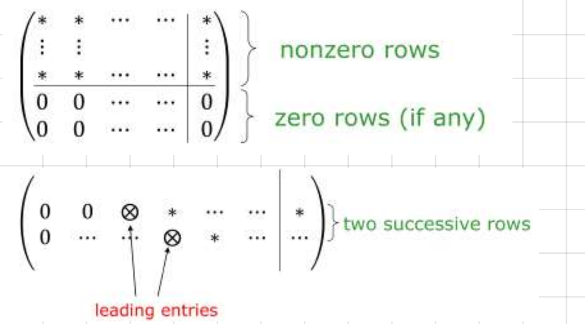
So $V \cap W = \{(\frac{2+7t}{11}, \frac{-3-5t}{11}, t) \mid t \in \mathbb{R}\}$.

Row-Echelon Form (REF) satisfies the following 2 properties:

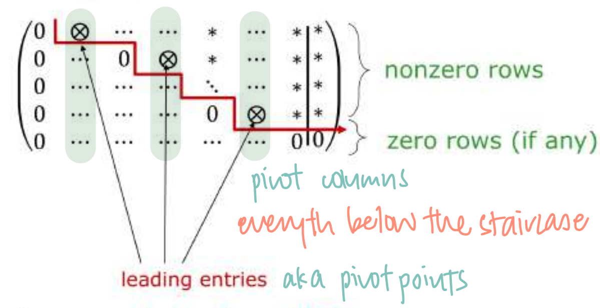
→ many variations

(1) If there are any rows that consist entirely of zeros, then they are group together at the bottom of the matrix.

(2) In any 2 successive non-zero rows, the 1st non-zero number in the lower row occurs farther away to the right than the 1st non-zero number in the higher row.



Combining properties 1 and 2:



This is a row-echelon form (REF)

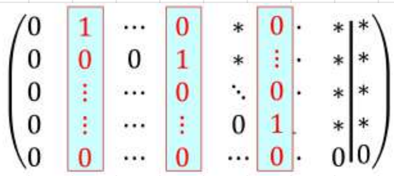
No. of pivot columns = no. of leading entries = no. of non-zero rows.

Reduced Row-Echelon Form (RREF) satisfies the following properties in addition to (1) & (2):

→ unique

(3) The leading entry of every non-zero row is 1.

(4) In each pivot column, except the pivot point, all other entries are zeros.



Theorem 2.2.11.1-3

Similar to ordinary numbers multiplication

1. $A(BC) = (AB)C$ — Associative Law
2. $A(B_1 + B_2) = AB_1 + AB_2$, $(C_1 + C_2)A = C_1A + C_2A$ — Distributive Law
3. $c(AB) = (cA)B = A(cB)$ — c is a scalar

Using REF/RREF to get solutions

§ More variables than equations -> infinitely many solutions.

§ Non-pivot columns -> free parameters s & t

e.g.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right) \end{array} \Rightarrow \begin{cases} 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ x_3 + x_4 + x_5 = 3 \\ 2x_5 = 4 \end{cases}$$

non-pivot columns

General solution:

$$\begin{cases} x_1 = s \\ x_2 = 2 + \frac{1}{2}t \\ x_3 = 1 - t \\ x_4 = t \\ x_5 = 2 \end{cases}$$

$\therefore$ infinitely many sol$^{\underline{n}}$

e.g. zero system

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

General solution:

$$\begin{cases} x_1 = r \\ x_2 = s \\ x_3 = t \end{cases}$$

$\therefore$ This system has infinitely many sol$^{\underline{n}}$s.

Modification in GE

Example → avoid rounding off errors

Standard

$$\begin{pmatrix} 4 & 3 & \cdots & \cdots \\ 1 & -2 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 3 & \cdots & \cdots \\ 0 & -11/4 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix}$$

Variation

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots \\ 4 & 3 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 11 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix}$$

Linear Systems with "unknown" terms

$$\left(\begin{array}{cc|c} 1 & a & b \\ 0 & (a+1)(b-2) & (a+2)(b-1) \end{array}\right)$$

- $\neq 0$ → Unique solution
- $= 0$ → No solution or infinite solutions
  - $(a + 2)(b - 1) \neq 0$ → No solution
  - $(a + 2)(b - 1) = 0$ → Infinite solutions

↳ ⊗ $\frac{1}{a} R_i$ ⇒ doesn't know if $a=0$

⊗ $a R_i$ ⇒

⊗ consider cases too early

<u>One solution</u>: $(a + 1)(b - 2) \neq 0$

$(a + 1) \neq 0$ AND $(b - 2) \neq 0$

$a \neq -1$ AND $b \neq 2$

<u>Infinite solutions</u>: $(a + 1)(b - 2) = 0$, $(a + 2)(b - 1) = 0$

$(a + 1) = 0$ OR $(b - 2) = 0$ AND $(a + 2) = 0$ OR $(b - 1) = 0$

$a = -1$ OR $b = 2$ AND $a = -2$ OR $b = 1$

Simplify as

$a = -1$ AND $b = 1$ OR $b = 2$ AND $a = -2$

<u>No solution</u>: $(a + 1)(b - 2) = 0$, $(a + 2)(b - 1) \neq 0$

$(a + 1) = 0$ OR $(b - 2) = 0$ AND $(a + 2) \neq 0$ AND $(b - 1) \neq 0$

$a = -1$ OR $b = 2$ AND $a \neq -2$ AND $b \neq 1$

Simplify as

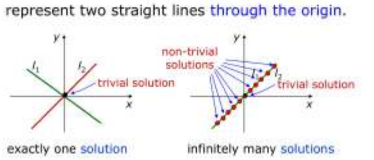
$a = -1$ AND $b \neq 1$ OR $b = 2$ AND $a \neq -2$

Linear Systems

- **Homogeneous System** (always consistent)
  - Trivial sol^n (zero sol^n)
    - always exists
  - Non-trivial sol^{n}s (w/ parameters)
    - may not exist
    - if exist → infinitely many sol^{n}s
- **Non-homogeneous System** (may not be consistent)

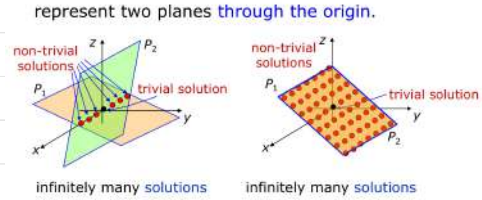
$$\begin{cases} a_1x + b_1y = 0 & (l_1) \\ a_2x + b_2y = 0 & (l_2) \end{cases}$$

represent two straight lines through the origin.



$$\begin{cases} a_1x + b_1y + c_1z = 0 & (P_1) \\ a_2x + b_2y + c_2z = 0 & (P_2) \end{cases}$$

represent two planes through the origin.



$a_1c_1 + a_2c_2 + \dots + a_nc_n = 0$

$a_1(tc_1) + a_2(tc_2) + \dots + a_n(tc_n)$

$= t(a_1c_1 + a_2c_2 + \dots + a_nc_n)$

$= 0$

$\therefore (tc_1, tc_2, \dots tc_n)$ is also a sol^n for $\forall t \in \mathbb{R}$

Summary

- a LS with the zero solution is called a homogeneous system.
- a homogeneous system is always consistent, as it always has the trivial solution.
- if a homogeneous system has a non-trivial solution, then it has infinitely many solution.
- a homogeneous system with <u>more variables than equations</u> has infinitely many solution.
- a homogeneous system with <u>more equations than variables</u> has one or many solution.

(Not in Lecture Slides)

For $Ax=b$, general solution of x:

$x = x_0 + x_p$

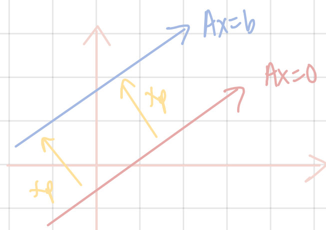
x_p: particular sol^n for $Ax=b$

x_0: general sol^n of $Ax=0$, then 1 or infinite sol^n

x has exactly one sol^n ⇒ x_0 is the trivial sol^n

⇒ $N(A) = \{0\}$ ⇒ A is invertible

⇒ $x = A^{-1}b$



Homogeneous System

- all the constant terms are zero.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

- only in homogeneous systems do we talk about trivial (the zero solution) and non-trivial solutions.

e.g. Given a quadratic surface w/ eq^{n} $ax^2+by^2+cz^2=d$ passes through $(1,1,-1)$, $(1,3,3)$ and $(-2,0,-2)$, find a formula for the quadratic surface. $(a,b,c,d \in \mathbb{R})$

$$\begin{cases} a + b + c = d \\ a + 9b + 9c = d \\ 4a + 4c = d \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 3 & 3 & -1 & 0 \\ -2 & 0 & 2 & -1 & 0 \end{array}\right)$$

$\Downarrow$

Homogeneous System

$$\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$$

General Sol^{n}:

$$\begin{cases} a = t \\ b = 3/4\, t \\ c = -3/4\, t \\ d = t \end{cases}$$

Particular Sol^{n}:

(Trivial) $t=0$: $a=b=c=d=0$

(Non-trivial) $t=4$: $a=4$, $b=3$, $c=-3$, $d=4$

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$A = (a_{ij})_{m \times n} = (a_{ij})_{i=1}^{m}\,_{j=1}^{n}$

Sub-matrix = Block matrix

- rows: m
- columns: n
- size: $m \times n$
- entries: a_{ij} or $a_{i,j}$

Square Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

↳ diagonal entries

- A is an $n \times n$ matrix
- $A = (a_{ij})$ is a square matrix of order n
- diagonal entries : $i=j$ ⇒ restricted to sq. matrices for MA1101R
- non-diagonal entries : $i \neq j$

Types of square matrices

Diagonal matrix $= \mathrm{diag}(d_1, d_2, \dots, d_n)$	all non-diagonal entries are zero	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$a_{ij} = 0$ whenever $i \neq j$
Scalar matrix $= \mathrm{diag}(c, c, \dots, c) = cI$	diagonal matrix with all diagonal entries the same	$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ c & \text{if } i = j \end{cases}$
Identity matrix $\boldsymbol{I}_n$	diagonal matrix with all diagonal entries equal 1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Zero matrix $\boldsymbol{0}_{m \times n}$	all entries equal to zero can be non-square	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$a_{ij} = 0$ for all i, j
Symmetric matrix	k^{th} row "equal" k^{th} column for all k	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}$	$a_{ij} = a_{ji}$ for all i, j
Upper triangular matrix	all entries below diagonals are zero	$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$	$a_{ij} = 0$ for all $i > j$
Lower triangular matrix	all entries above diagonals are zero	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 2 \end{pmatrix}$	$a_{ij} = 0$ for all $i < j$

strictly upper $a_{ij}=0 \ \forall i \geq j$ (i.e. diag also 0)
strictly lower $a_{ij}=0 \ \forall i \leq j$

Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ $\boldsymbol{B} = (b_{ij})_{m\times n}$ and c a real constant.

Matrix Equality	$\boldsymbol{A} = \boldsymbol{B}$	$\boldsymbol{A}$ and $\boldsymbol{B}$ have same size and same corresponding entries	$a_{ij} = b_{ij}$ for all i, j
Matrix Addition	$\boldsymbol{A} + \boldsymbol{B}$	addition of corresponding entries of $\boldsymbol{A}$ and $\boldsymbol{B}$	$(a_{ij} + b_{ij})_{m\times n}$
Matrix subtraction	$\boldsymbol{A} - \boldsymbol{B}$	subtraction of corresponding entries of $\boldsymbol{A}$ and $\boldsymbol{B}$	$(a_{ij} - b_{ij})_{m\times n}$
Scalar multiplication	$c\boldsymbol{A}$	multiply every entry of $\boldsymbol{A}$ by scalar c	$(ca_{ij})_{m\times n}$
Negative of matrix	$-\boldsymbol{A}$	attach negative sign to every entry of $\boldsymbol{A}$	$(-a_{ij})_{m\times n}$
Matrix Transpose	$\boldsymbol{A}^T$ (or $\boldsymbol{A}^t$)	interchanging the rows and columns of $\boldsymbol{A}$	$\boldsymbol{A}^T = (a_{ji})_{n\times m}$

A & B are of the same size

any size

$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ $\boldsymbol{A}^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

Properties

(i) commutative: $A+B = B+A$
(ii) Associative: $A+(B+C) = (A+B)+C = A+B+C$
(iii) Distributive: $a(A+B) = aA+aB$
(iv) Additive identity: $0_{m\times n} + A = A$
(v) Additive inverse: $A+(-A) = 0_{m\times n}$
(vi) Scalar Addition: $(a+b)A = aA+bA$
(vii) Scalar Multiplication: $(ab)A = a(bA) = b(aA)$
(viii) If $aA = 0_{m\times n}$, then $a=0$ or $A = 0_{m\times n}$

Matrix Multiplication Properties

$A: m\times n$

NOT commutative: $AB \neq BA$

(i) Associative: $(AB)C = A(BC) = ABC$
(ii) Distributive: $(A+B)C = AC+BC$
$(A+B)^2 = A^2+AB+BA+B^2$

(iii) Scalar multiplication is commutative: $c(AB) = cA(B) = A(cB)$
(iv) Multiplicative identity: $I_m A = A = AI_n$
(v) Zero Matrix: $A0_{n\times q} = 0_{m\times q}$
$0_{p\times m} A = 0_{p\times n}$
(vi) Zero divsor: $AB=0$ does NOT imply $A=0$ or $B=0$

- can only multiply two matrices A and B (AB) when no. of columns of A = no. of rows of B
(To prove, check size and corresponding entries)

Matrix Transpose

- A square matrix is symmetric iff $A = A^T$

Matrix Transpose Properties

(i) $(A^T)^T = A$
(ii) $(A+B)^T = A^T + B^T$
(iii) $(cA)^T = cA^T$
(iv) $(AB)^T = B^T A^T$

Exercise 2 Q24

T (a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two diagonal matrices of the same size, then $\boldsymbol{AB} = \boldsymbol{BA}$.
T (b) If $\boldsymbol{A}$ is a square matrix, then $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$ is symmetric.
F (c) For all matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$.
T (d) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{A} - \boldsymbol{B}$ is symmetric.
F (e) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{AB}$ is symmetric.
F (f) If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.
T (g) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA}^T = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

Suppose **A, B, C** are square matrices of the same size.

1. $\boldsymbol{AB} = \boldsymbol{BA}$
2. If $\boldsymbol{AB} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$
3. If $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$
4. If $\boldsymbol{A} = \boldsymbol{B}$, then $\boldsymbol{CA} = \boldsymbol{BC}$
5. If $\boldsymbol{AC} = \boldsymbol{BC}$, then $\boldsymbol{A} = \boldsymbol{B}$
6. $(\boldsymbol{AB})^n = \boldsymbol{A}^n\boldsymbol{B}^n$

Powers of a Matrix

A: square matrix
n: non-negative integer

- $A^n = \underbrace{AA\ldots A}_{n \text{ times}}$ $n \geq 1$
- $A^0 = I$
- $A^r A^s = A^{r+s}$
- $(AB)^n \neq A^n B^n$: $(AB)(AB)\ldots(AB) \neq (A\ldots A)(BB\ldots B)$

If A is invertible,
$A^{-n} = (A^{-1})^n = (A^n)^{-1}$
$= \underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{n \text{ times}}$

Matrix Multiplication

Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$ be two matrices.

The product $\boldsymbol{AB}$ is an $m \times n$ matrix

its (i, j)-entry is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$ summation notation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

$$\underset{A}{\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} \underset{b_1\ b_2\ b_3}{\overset{B}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}} = \overset{AB}{\begin{pmatrix} \underset{a_1b_1}{5} & \underset{a_1b_2}{7} & \underset{a_1b_3}{9} \\ \underset{a_2b_1}{14} & \underset{a_2b_2}{19} & \underset{a_2b_3}{24} \\ \underset{a_3b_1}{-9} & \underset{a_3b_2}{-12} & \underset{a_3b_3}{-15} \end{pmatrix}}$$

$$\boxed{\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} \quad \mathbf{B} = (\mathbf{b}_1 \ \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n) \Rightarrow \mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_n \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_n \end{pmatrix}} = \boxed{\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1\mathbf{B} \\ \mathbf{a}_2\mathbf{B} \\ \vdots \\ \mathbf{a}_m\mathbf{B} \end{pmatrix}} = \boxed{\mathbf{AB} = (\mathbf{Ab}_1 \ \ \mathbf{Ab}_2 \ \cdots \ \mathbf{Ab}_n)}$$

Expressing LS in matrix equation form

- A solution of the LS is represented by an (n*1) column matrix.
- u is a solution of Ax = b iff Au = b

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

$$\underset{\boldsymbol{A}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \underset{\boldsymbol{x}}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = \underset{\boldsymbol{b}}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$\underset{\mathbf{Ab}_1}{\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 8 \end{pmatrix}} \qquad \underset{\mathbf{Ab}_2}{\mathbf{A}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}} \qquad \underset{\mathbf{Ab}_3}{\mathbf{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}}$$

stacking

$$\mathbf{AB} = (\mathbf{Ab}_1 \ \ \mathbf{Ab}_2 \ \cdots \ \mathbf{Ab}_n)$$

splitting

$$\mathbf{A}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 0 \\ 3 & 7 & 2 \\ 8 & 2 & 4 \end{pmatrix}$$

Zipping of Matrices

zipped along the rows

$$A = (a_{ij})_{m\times p} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{array}{l} \text{1st row of } A \\ \text{2nd row of } A \\ \\ m^{th} \text{ row of } A \end{array}$$

a_i is a $(1\times p)$ row matrix

$$a_1 = (a_{11} \ \ a_{12} \ \ldots \ a_{1p})$$
$$a_2 = (a_{21} \ \ a_{22} \ \ldots \ a_{2p})$$
$$\vdots$$
$$a_m = (a_{m1} \ \ a_{m2} \ \ldots \ a_{mp})$$

zipped along the columns

$$B = (b_{ij})_{p\times n} = (\underset{\text{1st col}}{b_1} \ \ \underset{\text{2nd col}}{b_2} \ \ldots \ \underset{\text{nth col}}{b_n})$$

b_i is a $(p\times 1)$ column matrix

$$b_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix} \quad b_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{p2} \end{pmatrix} \quad \ldots \quad b_n = \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{pn} \end{pmatrix}$$

Invertible Matrix

(denoted as A^{-1})

A: square matrix of order n.

A is invertible if there exists a square matrix B of order n such that AB = I and BA = I.

B is called an inverse of A.

A square matrix is singular (non-invertible) if it has no inverse.

To show that A is invertible and B is the inverse of A, by Definition 2.3.2, we need to check AB = I and BA = I.
(By Theorem 2.4.12 in the next section, we shall see that we only need to check one of the two conditions: AB = I or BA = I.)

Only square matrices can be invertible
=> rref = I
=> unique solution
=> no. of eqn = no. of variables
=> no. of rows = no. of cols
=> square matrix

Theorem 2.3.5. (uniqueness of inverses):

If B and C are inverses of a square matrix A, then B=C.

i.e. every invertible matrix has exactly one inverse.

Cancellation Law for Matrices (invertible)

Let A be an invertible matrix.

$AB_1 = AB_2 \rightarrow B_1 = B_2$ $\qquad C_1A = C_2A \rightarrow C_1 = C_2$

If A is not invertible, then the Cancellation Law may not hold.

Theorem 2.4.12

Theorem 2.4.12 Let $\boldsymbol{A}$, $\boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{A}$, $\boldsymbol{B}$ are both invertible,

$$\boldsymbol{A}^{-1} = \boldsymbol{B}, \quad \boldsymbol{B}^{-1} = \boldsymbol{A} \quad \text{and} \quad \boldsymbol{BA} = \boldsymbol{I}.$$

Properties of Inverses

Theorem 2.3.9.

A, B: invertible matrices of the same size

c: non-zero scalar

1. cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.
2. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. A inv $\Leftrightarrow$ A^T inv
3. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
4. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. $\rightarrow$ product of inv matrices are inv.

(Remark 2.3.10)

By Part 4, if $A_1, A_2, \ldots, A_k$ are invertible matrices, then $A_1A_2\cdots A_k$ is invertible and
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$

$$A^{-n} = (A^{-1})^n = (A^n)^{-1} = \overbrace{A^{-1}A^{-1}\ldots A^{-1}}^{n \text{ times}}$$

Checking Invertibility of 2x2 Matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- A is invertible iff $ad - cd \neq 0$

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Elementary Matrices

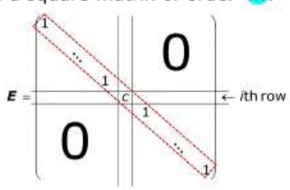
Def. A square matrix that can be obtained from an identity matrix by performing a single e.r.o. $I_n \xrightarrow{\text{e.r.o.}} E$

Every elementary matrix is of one of the 3 types below (invertible and their inverses are elementary matrices)

(1) Multiply a row by a constant

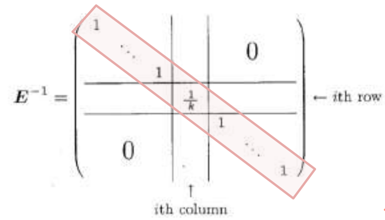
Let A be an $m \times n$ matrix.

Let E be a square matrix of order m:



EA : multiplying the ith row of A by c. cR_i

If $k \neq 0$, the matrix E is invertible and



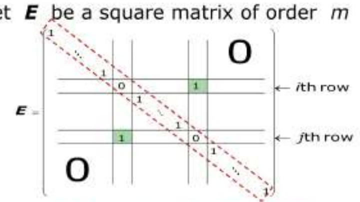
$\frac{1}{k}R_i$

corresponds to the row operation of multiplying the ith row of a matrix by $\frac{1}{k}$.

(2) Interchanging 2 rows

Let A be an $m \times n$ matrix.

Let E be a square matrix of order m :



EA : interchanging the ith and jth rows of A. $R_i \leftrightarrow R_j$

$E^{-1} = E$

$(i,i) \rightarrow (j,i)$

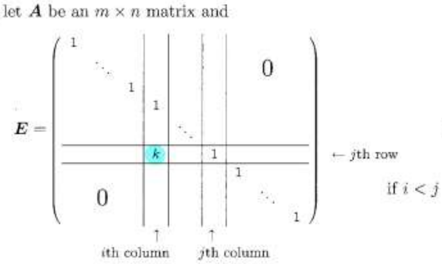
$(j,j) \rightarrow (i,j)$

post-multiply elementary matrix = elementary column operation

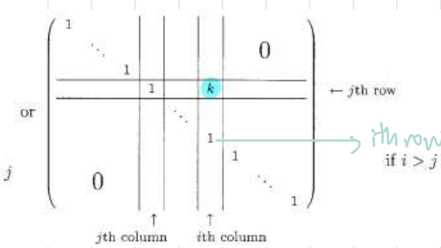
Pre-multiply EA		Post-multiply
$E : R_i + aR_j$	$\longleftrightarrow$	$E : C_j + aC_i$
$E : cR_i$	$\longleftrightarrow$	$E : cC_i$
$E : R_i \leftrightarrow R_j$	$\longleftrightarrow$	$E : C_i \leftrightarrow C_j$

(3) Adding a multiple of one row to another

In general, let A be an $m \times n$ matrix and



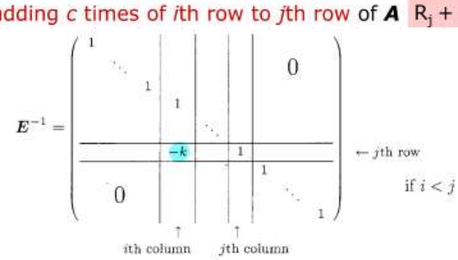
or



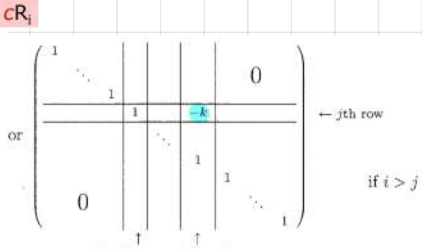
Below diagonal if i < j

Above diagonal if i > j

EA : adding c times of ith row to jth row of A $R_j + cR_i$



or



(j,i)-entry $= k$ (or c)

$E^{-1}A$: adding $-c$ times of ith row to jth row of A. $R_j - cR_i$

$\Rightarrow$ In E : jth row $- c \cdot$ ith row $\Rightarrow$ get I_n $\Rightarrow$ prove inverse post-multipling an elementary matrix to a matrix A is equivalent to do an elementary column operation on A.

Discussion 2.4.15 Elementary Column Operations (e.c.o.) (used less frequently)

Perform e.c.o. C to a matrix A is the same as post-multiply a certain square matrix E to A

(1) Multiply a column by a constant — resulting matrix = AE where E is the first elem. matrix ↑

(2) Interchange two columns — resulting matrix = AE " 2nd "

(3) Add a multiple of a column to another column — " = AE " 3rd "

$C_i + kC_j$

Row Equivalence and Elementary Matrices

If A & B are row equivalent (B can be obtained from A by performing a series of e.r.o. and vice versa), B can be obtained from A by pre-multiplying A with a series of elementary matrices and vice versa.

Example (Take note of the order of the elementary inverses added)

$$\underset{A}{\begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}} \xrightarrow[E_1]{R_1 \leftrightarrow R_3} \underset{E_1A}{\begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix}} \xrightarrow[E_2]{R_2 + 2R_1} \underset{E_2E_1A}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix}}$$

$$\xrightarrow[E_3]{R_3 - 4R_2} \underset{E_3E_2E_1A}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}} \xrightarrow[E_4]{-\frac{1}{2}R_3} \underset{E_4E_3E_2E_1A}{\overset{B}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}}$$

if P is invertible, $P = E_n \dots E_1$ where E_i are elementary matrices

$A \overset{r}{\rightarrow} B \Leftrightarrow B = EA$

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$E_4E_3E_2E_1A = B$$

$$E_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}B = (E_1E_2E_3E_4)^{-1}B$$

to be continued

Map of LA — **A** is an n×n matrix

A is invertible	chapter 2	A is not invertible
det A ≠ 0	chapter 2	det A = 0
rref of A is identity matrix	chapter 1	rref of A has a zero row
AX= 0 has only the trivial solution	chapter 1	AX= 0 has non-trivial solutions
AX= B has a unique solution	chapter 1	AX= B has no solution or infinitely many solutions
rows (columns) of A are linearly independent	chapter 3	rows (columns) of A are linearly dependent
row (column) space of A = R^n	chapter 4	row (column) space of A ≠ R^n
rank(A) = n, nullity(A) = 0	chapter 4	rank(A) < n, nullity(A) > 0
0 is not an eigenvalue of A	chapter 6	0 is an eigenvalue of A

W3

Finding Inverse Matrix

For larger size

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{G.J.E.}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\boxed{(\boldsymbol{A} \mid \boldsymbol{I}) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\boldsymbol{I} \mid \boldsymbol{A}^{-1})}$$

$$\boldsymbol{A} \xrightarrow{\text{GJE}} \boldsymbol{I} \qquad \boldsymbol{I} \xrightarrow{\text{GJE}} \boldsymbol{A}^{-1}$$

$$E_k \cdots E_2E_1\boldsymbol{A} = \boldsymbol{I} \Rightarrow E_k \cdots E_2E_1\boldsymbol{I} = \boldsymbol{A}^{-1}$$

For smaller size

Theorem 2.5.25 Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}$ is invertible, then

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A}).$$

Checking Invertibility

By RREF:

- RREF = I -> invertible
- RREF ≠ I -> not invertible

By REF (usual method)

- has no zero row/Col -> invertible
- has zero rows/Col -> not invertible

By determinant (when det is easy to get)

- det ≠ 0 -> invertible
- det = 0 -> not invertible

A invertible ⇒ RREF **B** = **I** ⇒ det(**B**) = 1 ⇒ det(**A**) ≠ 0

A not invertible ⇒ RREF **B** has zero row ⇒ det(**B**) = 0 ⇒ det(**A**) = 0

$R = E_k \dots E_1 A$

$A_{n\times n} \rightarrow R$ (ref)

at least one zero row $R = \begin{pmatrix} Q \\ 0 \dots 0 \end{pmatrix}$

$\forall B,\ RB = \begin{pmatrix} QB \\ 0 \dots 0 \end{pmatrix} \neq I \Rightarrow R$ not inv

$\Rightarrow R = (\underbrace{E_k \dots E_1}_{\text{inv}})A$ not inv. $\Rightarrow A$ not inv

no zero row $\Rightarrow$ all pivot col $\Rightarrow R = I$

$\Leftrightarrow (E_k \dots E_1)A = I \Leftrightarrow A = (E_k \dots E_1)^{-1} = E_1^{-1} \dots E_k^{-1}$

$\Leftrightarrow A^{-1} = E_k \dots E_1$

Theorem 2.4.7 ***

Let A be a square matrix, the following statements are equivalent:

(1) A is invertible. ⇒ A has a right & left inv

(2) The linear system $Ax = 0$ has only the trivial solution. => $Ax = b$ has only one solution

(3) The RREF of A is an identity matrix (I).

(4) A can be expressed as a product of elementary matrices.

(Product of invertible matrices is invertible by Thm 2.3.9.)

(Product of a singular and an invertible matrices is singular)

$$\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}.$$

The inverse of the data matrix is $\begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}$ and hence

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 20 \end{pmatrix}.$$

Theorem 2.4.12

Theorem 2.4.12 Let $\boldsymbol{A}$, $\boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{A}$, $\boldsymbol{B}$ are both invertible,

$$\boldsymbol{A}^{-1} = \boldsymbol{B}, \quad \boldsymbol{B}^{-1} = \boldsymbol{A} \quad \text{and} \quad \boldsymbol{BA} = \boldsymbol{I}.$$

Theorem 2.4.14

A is singular ->

both AB and BA are singular

At least one zero-row

W4

Determinant (Square matrix)

Definition 2.5.2 Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix. Let $\boldsymbol{M}_{ij}$ be an $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column. Then the *determinant* of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij}).$$

The number A_{ij} is called the (i, j)-*cofactor* of $\boldsymbol{A}$.

The way we defined "determinant" above is known as the *cofactor expansion* (see also Theorem 2.5.6).

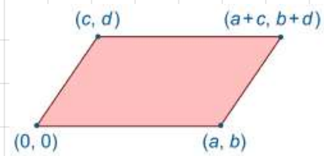
Notation 2.5.3 For an $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$, $\det(\boldsymbol{A})$ is usually written as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

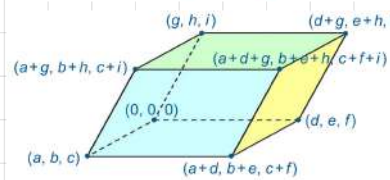
$A_{11} = \det(\boldsymbol{M_{11}})$ $A_{13} = \det(\boldsymbol{M_{13}})$ etc... cofactors of $\boldsymbol{A}$

$A_{12} = -\det(\boldsymbol{M_{12}})$ $A_{14} = -\det(\boldsymbol{M_{14}})$

Determinant alone cannot prove inconsistency of a system $Ax = b$



The area of the parallelogram is $ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.



The volume of the parallelepiped is

$$aei + bfg + cdh - ceg - afh - bdi = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Theorem 2.5.6 (Cofactors Expansions)

det(A) can be expressed as a cofactor expansion using any row or column of A.

$$\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

cofactor expansion along row 1

$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$ (i, j)-cofactor of $\boldsymbol{A}$

$$\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{k=1}^{n} a_{ik}A_{ik}, \ \forall i \in [1, n]$$

cofactor expansion along row i

for any $j = 1, 2, \ldots, n$

$$\det(\boldsymbol{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{k=1}^{n} a_{kj}A_{kj}, \ \forall j \in [i, n]$$

cofactor expansion along column j

alt. signs

(i, j)-cofactor: $A_{ij} = (-1)^{i+j}\det(M_{ij})$

$$\begin{pmatrix} \det(M_{11}) & -\det(M_{12}) & \det(M_{13}) \\ -\det(M_{21}) & \det(M_{22}) & -\det(M_{23}) \\ \det(M_{31}) & -\det(M_{23}) & \det(M_{33}) \end{pmatrix}$$

(i, j)-matrix minor: M_{ij} obtained from A by deleting the i^{th} row & j^{th} column

Example 2.5.7 We use Example 2.5.4.2 to illustrate Theorem 2.5.6. If we expand $\boldsymbol{B}$ along the second row,

Let $\boldsymbol{B} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$ $\det(\boldsymbol{B}) = -4\begin{vmatrix} -2 & 4 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -3 & 4 \\ 0 & 4 \end{vmatrix} - \begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix} = 34.$

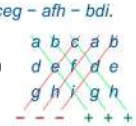
If we expand B along the third column,

$$\det(\boldsymbol{B}) = 4\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -2 \\ 4 & 3 \end{vmatrix} = 34.$$

Let $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Then $\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$.

The formula in can be easily remembered by using diagram on the right:



Warning: The method shown here cannot be generalized to higher order.

Theorem 2.5.8 (Triangular Matrix)

If A is a triangular matrix (diagonal matrix), then the determinant of A is equal to the product of the diagonal entries of A.

$$\begin{vmatrix} -1 & 3.5 & 14 \\ 0 & 5 & -26 \\ 0 & 0 & 2 \end{vmatrix} = (-1)\times 5\times 2 = -10$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 99 & 0 & 0 \\ 10 & -4.5 & 10 \end{vmatrix} = (-2)\times 0\times 10 = 0$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

Only B & C have the same determinant.

Theorem 2.5.10 (Transpose)

If $\boldsymbol{A}$ is a square matrix, then $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$.

Theorem 2.5.12 (identical rows/cols)

(1) The determinant of a square matrix with 2 identical rows is zero.

(2) The determinant of a square matrix with 2 identical columns is zero.

Theorem 2.5.15 (determinant and e.r.o.)

E.R.O	Determinant
$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}$	$\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{R_i + kR_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = \det(\boldsymbol{A})$

E : nxn elementary matrix

EA = B

⇒ det(EA) = det(E) det(A) = det(B)

$\det(\boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1) = \det(\boldsymbol{E}_k)\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)$

$\boldsymbol{A}$: nxn square matrix $\quad$ $\boldsymbol{E}$: nxn elementary matrix

Elementary column operation	Determinant
$\boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{B}$	$\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = \det(\boldsymbol{A})$

$\det(\boldsymbol{AE}) = \det(\boldsymbol{A})\det(\boldsymbol{E})$

$$\boldsymbol{A} \xrightarrow{R_3 + \frac{2}{9}R_1} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{4R_2} \boldsymbol{B} = \begin{bmatrix} 5 & 0 & 8 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Find $\det(\boldsymbol{A})$.

$$5\cdot(-2)\cdot 1\cdot\frac{1}{3} = \det(\boldsymbol{B}) = 1\cdot(-1)\cdot 4\det(\boldsymbol{A})$$

Thus $-\frac{10}{3} = -4\det(\boldsymbol{A})$ and hence $\det(\boldsymbol{A}) = \frac{5}{6}$.

Theorem 2.5.19

A square matrix A is invertible iff det(A) =/= 0.

Theorem 2.5.22

A, B: square matrices of order n

c: a scalar

1. $\det(c\boldsymbol{A}) = c^n\det(\boldsymbol{A})$ $\quad$ $\det(c\boldsymbol{A}) \neq c\det(\boldsymbol{A})$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ $\quad$ Multiplicative property
3. $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$
4. $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$ if $\boldsymbol{A}$ is invertible
5. $\det(\boldsymbol{A} + \boldsymbol{B}) \neq \det(\boldsymbol{A}) + \det(\boldsymbol{B})$

Adjoint

Let A be a square matrix of order n.

The adjoint of A is the n x n matrix.

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}$.

$(-1)^{i+j}\det(\boldsymbol{M}_{ij})$

Example

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\operatorname{adj}(\boldsymbol{B}) = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} -3 & 0 & 1 \\ 3 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}^T = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

Theorem 2.5.25

A: square matrix

If A is invertible, then $\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})$ $\Leftarrow$ For any square matrix: $A(\operatorname{adj}(A)) = \det(A)\, I$

Ex 2.60

$$\operatorname{adj}(\boldsymbol{A})^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{A}$$

Theorem 2.5.27 (Cramer's Rule)

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an $n \times n$ invertible matrix.

Let $\boldsymbol{A}_i$ be the matrix obtained from $\boldsymbol{A}$ by replacing the i^{th} column of $\boldsymbol{A}$ by $\boldsymbol{b}$.

Then the system has a unique solution

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \det(\boldsymbol{A}_2) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$$

$x_1 = \frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})}$

$x_2 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})}$

$x_n = \frac{\det(\boldsymbol{A}_n)}{\det(\boldsymbol{A})}$

Example

What is Cramer's rule?

Example 2.5.28

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y = 3 \end{cases}$$

Cramer's rule says:

$$x = \frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})} = \frac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{132}{60} = 2.2$$

$$y = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-24}{60} = -0.4$$

$$z = \frac{\det(\boldsymbol{A}_3)}{\det(\boldsymbol{A})} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-36}{60} = -0.6$$

this gives the unique solution of the system

n-vector (n-tuples)

An n-vector or ordered n-tuple of real numbers has the form

$$(u_1, u_2, \ldots, u_i, \ldots, u_n)$$

where $u_1, u_2, \ldots, u_n$ are real numbers.

The number u_i in the i^{th} position of an n-vector is called the i^{th} component or the i^{th} coordinate of the n-vector.

Theorem 3.1.6

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be n-vectors and c, d real numbers.

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$.
3. $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u} = \boldsymbol{0} + \boldsymbol{u}$.
4. $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$.
5. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.
6. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
7. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$.
8. $1\boldsymbol{u} = \boldsymbol{u}$.

Euclidean n-space

Def. the set of all n-vectors of real numbers

Denoted by $\mathbb{R}^n$

$\boldsymbol{u} \in \mathbf{R}^n \longleftrightarrow \boldsymbol{u}$ is an n-vector $\longleftrightarrow \boldsymbol{u} = (u_1, u_2, \ldots, u_n)$

Euclidean 2-space $\mathbf{R}^2$

all the 2-vectors (as points) in xy-plane

Euclidean 3-space $\mathbf{R}^3$

all the 3-vectors (as points) in xyz-space

For a finite set S, we denote the number of elements of S by |S|

$$\mathbb{R}^n = \{(u_1, u_2, \ldots, u_n) \mid u_1, u_2, \ldots, u_n \in \mathbb{R}\}$$

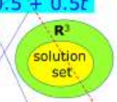
Example

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases}$$

x = 2, y = -1, z = -1 $\Updownarrow$ (2, -1, -1) a 3-vector

general solution: $\begin{cases} x = 0.5 - 1.5t \\ y = -0.5 + 0.5t \\ z = t \end{cases}$

solution set subset of $\mathbf{R}^3$



Explicit form

$\{ (0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbf{R} \}$

Implicit form

$\{ (x, y, z) \mid x + y + z = 0 \text{ and } x - y + 2z = 1 \}$

Set notation for subsets of $\mathbf{R}^n$

Implicit form

{ general n-tuple $(u_1, u_2, \dots, u_n)$ | conditions satisfied by $u_1, u_2, \dots, u_n$ }

e.g. $S = \{ (u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4 \}$

Explicit form — Not always possible to express in explicit form

{ n-tuples with explicit form | range of parameters appearing on the left }

e.g. $S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$

Don't write $\{a, b \in \mathbf{R} \mid (0, a, b, a)\}$

defined by
(i) a point (a, b, c) on the line, and
(ii) an arrow (u, v, w) parallel to the line

Explicit form:
(a, b, c) + t(u, v, w)

Set notations for lines and planes

Lines in xy-plane

Implicit form: $\{ (x, y) \mid ax + by = c \}$

Explicit form: $\left\{ \left(\frac{c - bt}{a}, t \right) \mid t \in \mathbf{R} \right\}$

Planes in xyz-space

Implicit form: $\{ (x, y, z) \mid ax + by + cz = d \}$

Explicit form: $\left\{ \left(\frac{d - bs - ct}{a}, s, t \right) \mid s, t \in \mathbf{R} \right\}$

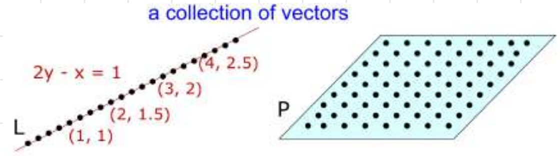
Lines in xyz-space

Implicit form: { (x, y, z) | eqn of two planes } Does not exist

Explicit form: { (general solution) | 1 parameter }

A line in R3 cannot be represented by a single line equation.
- need 2 eqn: intersection of 2 planes

A line(plane) in the xy-plane (xyz-space) can be regarded as a collection of points.



L is a subset of $\mathbf{R}^2$ P is a subset of $\mathbf{R}^3$

W5

Linear Combination

Definition 3.2.1 Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$. For any real numbers $c_1, c_2, \dots, c_k$, the vector

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$$

is called a *linear combination* of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$.

Example $\boldsymbol{u}_1 = (2, 1, 0)$ $\boldsymbol{u}_2 = (-3, 0, 1)$

$c_1 = 1,\ c_2 = 1$

$1(2, 1, 0) + 1(-3, 0, 1) = (-1, 1, 1)$ a specific linear combination

$c_1 = s,\ c_2 = t$

$s(2, 1, 0) + t(-3, 0, 1)$ general linear combination with parameters s and t

Standard Basis Vectors

Every vector in $\mathbf{R}^3$ is a linear combination of the following vectors — Directional vectors of the x-axis, y-axis, z-axis

$\boxed{\mathbf{e}_1 = (1, 0, 0),\ \mathbf{e}_2 = (0, 1, 0),\ \mathbf{e}_3 = (0, 0, 1)}$

Take a general 3-vector (x, y, z)

$$\begin{aligned}(x, y, z) &= (x, 0, 0) + (0, y, 0) + (0, 0, z)\\ &= x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)\\ &= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3\end{aligned}$$

e.g. $(1, 2, 5) = 1\mathbf{e}_1 + 2\mathbf{e}_2 + 5\mathbf{e}_3$

Linear Span

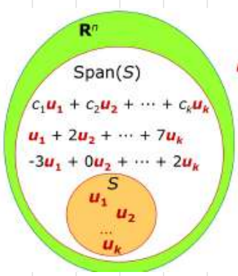
Definition 3.2.3 Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Then the set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$,

$$\{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\},$$

is called the *linear span* of S (or the *linear span* of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$) and is denoted by $\mathrm{span}(S)$ (or $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$).

"Linear span" is always used w.r.t. a set of vectors

denoted by $\mathrm{span}\{u_1, u_2, \dots, u_k\}$



$\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k \in \mathbf{R}^n$

$S \subseteq \mathbf{R}^n$

$\mathrm{span}(S) \subseteq \mathbf{R}^n$

$S \subseteq \mathrm{span}(S)$

span(S) can be equal to $\mathbf{R}^n$ but not always.

Checking for Linear Combinations

LS has solution -> v is a linear combi of other vectors

$\boldsymbol{v} = (3, 3, 4, 0)$

$(3, 3, 4, 0) = a(2, 1, 3, 1) + b(1, -1, 2, 2) + c(3, 0, 5, 1)$

$$\begin{cases} 2a + b + 3c = 3\\ a - b = 3\\ 3a + 2b + 5c = 4\\ a + 2b + c = 0\end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3\\ 1 & -1 & 0 & 3\\ 3 & 2 & 5 & 4\\ 1 & 2 & 1 & 0\end{array}\right) \xrightarrow{\text{G.E}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3\\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

LS has no solution -> not a linear combi

$\boldsymbol{v} = (3, 3, 4, 1)$

$(3, 3, 4, 1) = a(2, 1, 3, 1) + b(1, -1, 2, 2) + c(3, 0, 5, 1)$

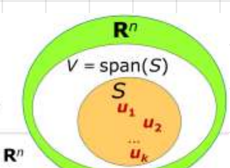
$$\begin{cases} 2a + b + 3c = 3\\ a - b = 3\\ 3a + 2b + 5c = 4\\ a + 2b + c = 1\end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3\\ 1 & -1 & 0 & 3\\ 3 & 2 & 5 & 4\\ 1 & 2 & 1 & 1\end{array}\right) \xrightarrow{\text{G.E}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3\\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{array}\right)$$

$\mathrm{span}(S) = \mathrm{span}\{(1, 0, 0, -1), (0, 1, 1, 0)\}$ linear span form

$= \{ a(1, 0, 0, -1) + b(0, 1, 1, 0) \mid a, b \in \mathbf{R} \}$

$= \{ (a, b, b, -a) \mid a, b \in \mathbf{R} \}$ explicit form

The word "Span"



$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} \subseteq \mathbf{R}^n$

Let $V = \mathrm{span}(S) = \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$

- V is a linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$
- V is a linear span of S
- V is spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$
- V is spanned by S
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ spans V
- S spans V

If v is a linear combination of $u_1, u_2, \dots, u_k$, $v \in \mathrm{span}(S)$.

If w is not a linear combination of $u_1, u_2, \dots, u_k$, $w \notin \mathrm{span}(S)$.

span{} = {0} = span{0}

Checking if a linear span = $\mathbf{R}^n$

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ column form

Check $\text{span}(S) \subseteq \mathbf{R}^n$ AND $\mathbf{R}^n \subseteq \text{span}(S)$

- $\text{span}(S) \subseteq \mathbf{R}^n$ is automatic (nothing to check)
- To check $\mathbf{R}^n \subseteq \text{span}(S)$
  - Take a general vector $\mathbf{x} \in \mathbf{R}^n$ $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{pmatrix}$
  - Check $(\ \mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_k \mid \mathbf{x})$ is consistent

It is enough to check:
REF of $(\ \mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_k)$ has no zero row

Consider the linear system — general vector $\mathbf{x} \in \mathbf{R}^n$

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{k1} & x \\ a_{12} & a_{22} & \cdots & a_{k2} & y \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} & z \end{array}\right)}_{\mathbf{A}} \xrightarrow{\text{G.E.}} \underbrace{\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \ddots & * & \vdots & \vdots \\ 0 & \cdots & 0 & * & * \end{array}\right)}_{\mathbf{R}} \text{ REF}$$

$s_1 \quad s_2 \quad s_k$

R has no zero row
⇒ system is always consistent
⇒ $\text{span}\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} = \mathbf{R}^n$

R has a zero row
⇒ system may be inconsistent
⇒ $\text{span}\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} \neq \mathbf{R}^n$

If the zero row = non-zero constant
Case 1: infinitely many solutions
Case 2: no solution

If $k < n$, then $\text{span}\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\} \neq \mathbf{R}^n$

If $k \geq n$, $\text{span}\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ may or may not be equal to $\mathbf{R}^n$

Theorem 3.2.7

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.
If $k < n$, then S cannot span $\mathbf{R}^n$. $\text{span}(S) \neq \mathbf{R}^n$

More rows than columns

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{k1} & x \\ a_{12} & a_{22} & \cdots & a_{k2} & y \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} & z \end{array}\right)}_{\mathbf{A}} \xrightarrow{\text{.E.}} \underbrace{\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \ddots & * & \vdots & \vdots \\ 0 & \ldots & & 0 & * \end{array}\right)}_{\mathbf{R}} \text{ REF}$$

The REF **R** of **A** must have a zero row,
so the system may be inconsistent,
and $\text{span}(\ S\ ) \neq \mathbf{R}^n$.

$\subseteq$

Example

$\text{span}\{\ \mathbf{u}\ \} \neq \mathbf{R}^2$ since $k = 1 < n = 2$

$\text{span}\{\ \mathbf{u}\ \} \neq \mathbf{R}^3$ since $k = 1 < n = 3$

$\text{span}\{\mathbf{u}_1, \mathbf{u}_2\ \} \neq \mathbf{R}^3$ since $k = 2 < n = 3$

Suppose $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbf{R}^3$.

Determine which sets below span $\mathbf{R}^3$ as well.

$S_1 = \{\mathbf{u}, \mathbf{v}\}$ $\quad$ $S_2 = \{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$

$S_4 = \{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$

$S_4 = \{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ show $S_4 \subseteq \text{span}\{u, v, w\}$ & $S_4 \supseteq \text{span}\{u, v, w\}$

alr is — S_4 can be expressed as LC of u, v, w.

show u, v, w can be expressed as LC of vectors in S_4

i.e. $u, v, w \in \text{span}(S_4)$

$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u}$

$\mathbf{w} = (\mathbf{u} + \mathbf{v} + \mathbf{w}) - (\mathbf{u} + \mathbf{v})$

$\Rightarrow \mathbf{u}, \mathbf{v}, \mathbf{w} \in \text{span}(S_4)$

$\Rightarrow \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq \text{span}(S_4)$ $\quad$ So $\text{span}(S_4) = \mathbf{R}^3$.

Theorem 3.2.9

Theorem 3.2.9 Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \in \mathbb{R}^n$.

1. $\mathbf{0} \in \text{span}(S)$.
2. For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in \text{span}(S).$$

Consequent of theorem

- Closure Property under Vector Addition:

if u and $v \in \text{span}(S)$, then $(u + v) \in \text{span}(S)$. Closure property under vector addition

- Closure Property under Scalar Multiplication:

if $u \in \text{span}(S)$ and $c \in \mathbb{R}$, then $cu \in \text{span}(S)$.

Theorem 3.2.10

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be subsets of $\mathbf{R}^n$.

Every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ belongs to $\text{span}(S_2)$

Then

$\text{span}(S_1) \subseteq \text{span}(S_2)$

if and only if

each $\mathbf{u}_i$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

Every $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ belongs to $\text{span}(S_2)$

Checking span(S) ⊆ span(T)

$S = \{s_1, s_2, \ldots, s_n\}$

$T = \{t_1, t_2, \ldots, t_m\}$

$(\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m\ |\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n)$

Every vector of S is a linear combination of vectors in T.

Check that this multiple-augmented matrix is consistent:

- REF has no pivot columns among the augmented columns

Example 3.2.11.1

$\boldsymbol{u}_1 = (1, 0, 1)$ $\boldsymbol{v}_1 = (1, 2, 3)$
$\boldsymbol{u}_2 = (1, 1, 2)$ $\boldsymbol{v}_2 = (2, -1, 1)$
$\boldsymbol{u}_3 = (-1, 2, 1)$

We can solve the three systems simultaneously:

$$\begin{array}{c} \\ \\ \end{array}\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c|c} 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array}$$

All the three systems are consistent.

This shows each $\boldsymbol{u}_i$ can be written as $a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ for some real number a and b,

So span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Theorem 3.2.9.2

By solve the three systems, we get:

$\boldsymbol{u}_1 = \frac{1}{5}\boldsymbol{v}_1 + \frac{2}{5}\boldsymbol{v}_2 \quad \boldsymbol{u}_2 = \frac{3}{5}\boldsymbol{v}_1 + \frac{1}{5}\boldsymbol{v}_2 \quad \boldsymbol{u}_3 = \frac{3}{5}\boldsymbol{v}_1 - \frac{4}{5}\boldsymbol{v}_2$

Checking span(S) = span(T)

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ $\quad T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ column form

Check span(S) ⊆ span(T) AND span(T) ⊆ span(S)

Every vector of S is a linear combination of vectors in T

$(\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m\ |\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n)$

And every vector of T is a linear combination of vectors in S

$(\mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_n\ |\mathbf{t}_1|\mathbf{t}_2|\ldots|\mathbf{t}_m)$

Check that both multiple-augmented matrices are consistent

Show that Span(S) = W, a subspace of $\mathbf{R}^n$

If dim(W) = m and S has m vectors,

- Check S is linearly independent
- Check each $\mathbf{s}_i \in W$
  - This will imply S is a basis for W
  - This will imply span(S) = W

Checking span(S) ≠ span(T)

$S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ $\quad T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ column form

Check span(S) ⊄ span(T) OR span(T) ⊄ span(S)

Some vector of S is not a linear combination of vectors in T

$(\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_m\ |\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n)$

Or some vector of T is not a linear combination of vectors in S

$(\mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_n\ |\mathbf{t}_1|\mathbf{t}_2|\ldots|\mathbf{t}_m)$

Check that at least one of the multiple-augmented matrices is inconsistent

2. Let $\boldsymbol{u}_1 = (1,0,0,1)$, $\boldsymbol{u}_2 = (0,1,-1,2)$, $\boldsymbol{u}_3 = (2,1,-1,4)$ and $\boldsymbol{v}_1 = (1,1,1,1)$, $\boldsymbol{v}_2 = (-1,1,-1,1)$, $\boldsymbol{v}_3 = (-1,1,1,-1)$. Show that

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$$

but

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

Solution To show that each $\boldsymbol{u}_i$, $i = 1, 2, 3$, is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, we follow the same procedure as the previous example and solve the three linear systems together (what are the three systems?):

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 2 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since all three systems are consistent, all $\boldsymbol{u}_i$ are linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. So

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

On the other hand,

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 1 & -1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since not all three systems are consistent, some $\boldsymbol{v}_i$'s are not linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$. So span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \not\subseteq$ span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and hence

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

Theorem 3.2.12

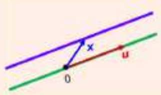
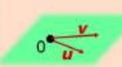
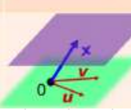
Theorem 3.2.12 Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$. If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}.$$

u_k is the "redundant" vector in span$\{u_1, u_2, \ldots, u_{k-1}, u_k\}$

# of arbitray params = # of redundant vectors

Geometrical Meaning of a Linear Span

Objects	Geometrical	Span	Set notation
Line through origin		span$\{\boldsymbol{u}\}$	$\{t\boldsymbol{u} \mid t \in \mathbf{R}\}$ $\{(cu_1, cu_2) \mid c \in \mathbb{R}\} = \{(x,y) \mid u_2x - u_1y = 0\}$
Line not through origin		$\boldsymbol{x}$ + span$\{\boldsymbol{u}\}$	$\{\boldsymbol{x} + t\boldsymbol{u} \mid t \in \mathbf{R}\}$ $\{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}\}$
Plane through origin		span$\{\boldsymbol{u}, \boldsymbol{v}\}$	$\{t\boldsymbol{u} + s\boldsymbol{v} \mid t, s \in \mathbf{R}\}$ $\{(x,y,z) \mid ax+by+cz=0\}$
Plane not through origin		$\boldsymbol{x}$ + span$\{\boldsymbol{u}, \boldsymbol{v}\}$	$\{\boldsymbol{x} + t\boldsymbol{u} + s\boldsymbol{v} \mid t, s \in \mathbf{R}\}$ $\{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u},\boldsymbol{v}\}\}$

A linear span of 3 non-coplanar vectors in $\mathbb{R}^3$ is the entire $\mathbb{R}^3$ space.

Take $\boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r \in \mathbb{R}^n$.

The set $Q = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}\}$ is called a k-plane in $\mathbb{R}^n$

where k is the "dimension" of span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ which will be studied in Section 3.6.

Subspaces

Definition 3.3.2

no condition

Let V be a subset of $\mathbf{R}^n$

condition applies

V is called a subspace of $\mathbf{R}^n$ provided ...

there is a set $S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \}$ of $\mathbf{R}^n$ such that V = span(S)

condition of subspace

i.e. V can be expressed in linear span form.

Every subspace of $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$.

Not every subset of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

Any subset that is not a subspace is non- subspace.

1. $\{\mathbf{0}\}$ is a subspace of $\mathbf{R}^n$. zero space

   Take S = $\{\mathbf{0}\}$

   $\{\mathbf{0}\}$ = span$\{\mathbf{0}\}$ Trivial subspace

2. $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$.

   Take S to be standard basis vectors for $\mathbf{R}^3$

   $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$

   $\mathbf{R}^3$ = span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ Refer to Example 3.2.2

   $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

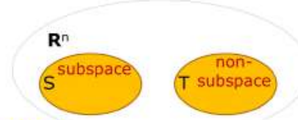
   Take S to be standard basis vectors for $\mathbf{R}^n$

   $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, ..., $\mathbf{e}_n = (0, \ldots, 0, 1)$

   $\mathbf{R}^n$ = span$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

S is a subset of $\mathbf{R}^n$

T is a subset of $\mathbf{R}^n$



Two types of subsets of $\mathbf{R}^n$

Subspaces
- Can be written as linear span
  S = span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$
- Satisfy closure properties

Non-subspaces
- Cannot be written as linear span
  T ≠ span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$
- Violate closure properties

Subspaces of $\mathbb{R}^2$:

① $\{0\}$ (span$\{0\}$)

② any lines passing through the origin (span$\{u\}$ for non-zero $u \in \mathbb{R}^2$)

③ $\mathbb{R}^2$ (spanned by 2 non-parallel vectors u & v e.g. span$\{(0,1),(1,0)\}$)

$\mathbf{R}^3$

Subspaces
- span$\{\mathbf{0}\}$
  just one point (origin)
- span$\{\mathbf{v}_1\}$
  a line that passes through origin
- span$\{\mathbf{v}_1, \mathbf{v}_2\}$
  a plane that contains the origin
- span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
  the entire 3D space

Non-subspaces
- A point that is not the origin
- Line or plane that does not contain the origin
- A (space) curve or a bended surface (e.g. paraboloid)
- A rectangular block
- etc

Checking Subspaces

S can be expressed as a linear span

-> S is a subspace

To show that V is closed under linear combinations

☆ S is not a subspace if one of the conditions below occurs:

point of reference

- the zero vector is not in S
- $v \in S$ and a scalar c such that $cv \notin S$ (give examples of c & v)
- $u, v \in S$ but $(u + v) \notin S$ (give examples of u & v)

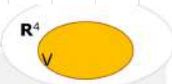
closed under linear combinations

ease of identificat^n

To show a subset S of $\mathbf{R}^n$ is a subspace:
- Express S as a linear span
- Show that S is the solution set of a homogeneous system
- (For $\mathbf{R}^2$ and $\mathbf{R}^3$) show that S represents a line or plane through origin.

To show a subset S of $\mathbf{R}^n$ is not a subspace:
- Show that the zero vector is not in S
- Find $\mathbf{u}$, $\mathbf{v} \in$ S such that $\mathbf{u} + \mathbf{v} \notin$ S
- Find $\mathbf{v} \in$ S and a scalar c such that $c\mathbf{v} \notin$ S
- (For $\mathbf{R}^2$ and $\mathbf{R}^3$) show that S is not a line or plane through origin.

Subspaces (Example 1)



V = { (a, b, a+b, 0) | a, b ∈ **R** }

Is V a subspace of $\mathbf{R}^4$? Can we write V = span{ $\mathbf{u}_1$, $\mathbf{u}_2$, ...} ?

general vector: (a, b, a+b, 0) = a(1, 0, 1, 0) + b(0, 1, 1, 0)

So V = span{ (1, 0, 1, 0), (0, 1, 1, 0) } ⇒ V is a subspace

W = { (a, b, a+b, 1) | a, b ∈ **R** }

Is W a subspace of $\mathbf{R}^4$? ⇒ W is not a subspace

(1,1,2,1) ∈ W, (1,0,1,1) ∈ W

but (1,1,2,1) + (1,0,1,1) = (2,1,3,2) ∉ W

closure property under addition not satisfied

$V_3 = \{ (1, a) \mid a \text{ in } \mathbf{R} \}$ subset of $\mathbf{R}^2$

(1, a) = (1, 0) + (0, a) = (1, 0) + a(0, 1)

not a general linear combination

V_3 is not a linear span of "any" set of vectors

"So" V_3 is not a subspace of $\mathbf{R}^2$

There is an easier way: Use theorem 3.2.9.1

$(0, 0) \notin V_3 = \{ (1, a) \mid a \text{ in } \mathbf{R} \}$

⇒ not a subspace of $\mathbf{R}^2$

If a subset of $\mathbf{R}^n$ does not contain the zero vector $\mathbf{0}$, then it is not a linear span.

Theorem 3.3.6

$\boldsymbol{Ax} = \mathbf{0}$

The solution set of a homogeneous linear system in n variables is a subspace of $\mathbf{R}^n$.

The solution set of every homogeneous LS can be written as a linear span

We call it the solution space of the system.

$Ax = b$ where $b \neq 0$

The solution set of non-homogeneous LS is not a subspace of $\mathbf{R}^n$.

Ax = b linear system with n variables

system is homogeneous **Ax = 0**	system is non-homogeneous **Ax = b**
Solution space — solution set is a subspace of $\mathbf{R}^n$	solution set is not a subspace of $\mathbf{R}^n$
If **u**, **v** are solutions of **Ax = 0**,	If **u**, **v** are solutions of **Ax = b**,
• **u** + **v** also a solution	• **u** + **v** not a solution
• c**u** also a solution	• c**u** not a solution

Example

Homogeneous system → general solution

$$\begin{cases} x - 2y + 3z = 0 \\ 2x - 4y + 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases} \longrightarrow \begin{cases} x = 2s - 3t \\ y = s \\ z = t \end{cases}$$

subspace of $\mathbf{R}^3$

linear span form = solution set

span{(2, 1, 0), (-3, 0, 1)} = {(2s − 3t, s, t) | s, t in **R**}

s(2, 1, 0) + t(-3, 0, 1)

general linear combination

Remark 3.3.8 Let V be a non-empty subset of $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if

$$\text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \text{ and } c, d \in \mathbb{R},\ c\boldsymbol{u} + d\boldsymbol{v} \in V.$$

W6

Linear Independence

Definition 3.4.2 Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Consider the equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}, \tag{3.3}$$

where $c_1, c_2, \ldots, c_k$ are variables. Note that $c_1 = 0, c_2 = 0, \ldots, c_k = 0$ satisfies Equation (3.3) and hence is a solution to Equation (3.3). This solution is called the *trivial solution*. (See also Definition 1.5.1.)

1. S is called a *linearly independent set* and $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are said to be *linearly independent* if Equation (3.3) has only the trivial solution.
2. S is called a *linearly dependent set* and $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are said to be *linearly dependent* if Equation (3.3) has non-trivial solutions, i.e. there exist real numbers $a_1, a_2, \ldots, a_k$, not all of them are zero, such that $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \cdots + a_k\boldsymbol{u}_k = \mathbf{0}$.

Theorem 3.4.4

Theorem 3.4.4 Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$ where $k \geq 2$. Then

1. S is linearly dependent if and only if at least one vector $\boldsymbol{u}_i$ in S can be written as a linear combination of other vectors in S, i.e.

(Redundant vector) $$\boldsymbol{u}_i = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \cdots + a_{i-1}\boldsymbol{u}_{i-1} + a_{i+1}\boldsymbol{u}_{i+1} + \cdots + a_k\boldsymbol{u}_k$$

   for some real numbers $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k$; and

2. S is linearly independent if and only if no vector in S can be written as a linear combination of other vectors in S. (No redundant vector)

Linearly independent set:
§ no redundant vectors in the set

Linearly dependent set:
§ redundant vectors in the set

§ If u = 0, then c can be non-zero. So S = {u} is linearly dependent.
§ If u ≠ 0, then c must be zero. So S = {u} is linearly independent.

Let S = {u, v} be a set with two vectors.
§ If u and v are scalar multiples of each other, S is linearly dependent.
§ If u and v are not scalar multiples of each other, S is linearly independent.

Let S be a finite subset of Rn.
§ If 0 ∈ S , then S is linearly dependent.

Empty set is linearly independent.

Checking Linear Independence

- Standard method

Form the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots c_n\mathbf{v}_n = \mathbf{0}$ Homogeneous system

> If $c_1 = 0$, $c_2 = 0$, ... $c_n = 0$ is the unique solution, then they are linearly independent

> If there are non-trivial solutions, then they are linearly dependent.

- Redundancy (Not always easy to find the redundant vector)

> If some $\mathbf{v}_i$ is a linear combination of the others, then they are linearly dependent

> If every $\mathbf{v}_i$ is not a linear combination of the others, then they are linearly independent.

Determine whether the vectors
(1, −2, 3), (5, 6, −1), (3, 2, 1)
are linearly independent.

Set up the vector equation:

$$c_1\begin{pmatrix}1\\-2\\3\end{pmatrix}+c_2\begin{pmatrix}5\\6\\-1\end{pmatrix}+c_3\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{ccc|c}1&5&3&0\\-2&6&2&0\\3&-1&1&0\end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c}1&5&3&0\\0&16&8&0\\0&0&0&0\end{array}\right)$$

There are infinitely many solutions for c_1, c_2, c_3.
i.e. There exist non-trivial solutions.
So (1, −2, 3), (5, 6, −1), (3, 2, 1) are linearly dependent.

- Use the column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to form matrix $\mathbf{A}$

Check whether every column in the r.e.f. of $\mathbf{A}$ is a pivot column.

- Use the row vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to form matrix $\mathbf{A}$

Check whether every row in the r.e.f. of $\mathbf{A}$ is non-zero.

Special Methods: (only work under certain circumstances)

- The column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbf{R}^n$ form a square matrix $\mathbf{A}$.

> If $\det(\mathbf{A}) = 0$, then they are linearly dependent.

> If $\det(\mathbf{A}) \neq 0$, then they are linearly independent.

- There are only two vectors $\mathbf{v}_1, \mathbf{v}_2$ in the set.

> If $\mathbf{v}_1, \mathbf{v}_2$ are scalar multiple of each other, then they are linearly dependent.

> If $\mathbf{v}_1, \mathbf{v}_2$ are not scalar multiple of each other, then they are linearly independent.

- Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbf{R}^m$.

> If $n > m$, then they are linearly dependent.

> If $n \leq m$, it can be linearly independent or dependent.

The more vectors you have, the more likely for them to be linearly dependent

- If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is orthogonal and non-zero, then the set is linearly independent.

Determine whether the vectors
(1, 0, 0, 1), (0, 2, 1, 0), (1, −1, 1, 1)
are linearly independent.

Set up the vector equation:

$$c_1\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+c_2\begin{pmatrix}0\\2\\1\\0\end{pmatrix}+c_3\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c}1&0&1&0\\0&2&-1&0\\0&0&1&0\\0&0&0&0\end{array}\right)$$

Convert into augmented matrix

There is only one solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So the vectors are linearly independent.

Theorem 34.7

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.

If $k > n$, then S is linearly dependent.

If S ⊆ $\mathbf{R}^n$ and S has more than n elements, then S is linearly dependent.

- more variables than equations
(>= 1 non-pivot column -> free param)
- the system has non-trivial solutions
- S is linearly dependent.

$a(\boldsymbol{u}-\boldsymbol{v})+b(\boldsymbol{v}-\boldsymbol{w})+c(\boldsymbol{w}+\boldsymbol{u})=\mathbf{0} \Leftrightarrow (a+c)\boldsymbol{u}+(-a+b)\boldsymbol{v}+(-b+c)\boldsymbol{w}=\mathbf{0}$.
Since $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, we have

$$\begin{cases} a \quad + c = 0 \\ -a + b \quad = 0 \\ \quad -b + c = 0. \end{cases}$$

The system has only the trivial solution $a = 0$, $b = 0$, $c = 0$. Thus S_3 is linearly independent.

Theorem 34.10 (Extending a linearly indep set)

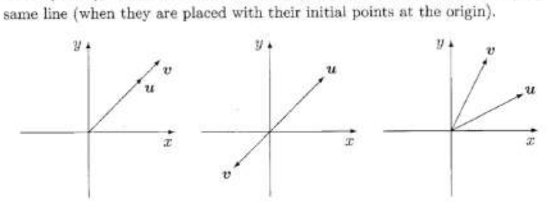
$\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ are linearly independent

$\boldsymbol{u}_{k+1}$ is not redundant

If $\boldsymbol{u}_{k+1}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$
then $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly independent.

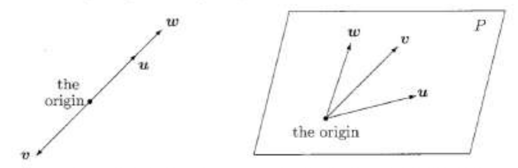
Geometrical Representation

1. In $\mathbb{R}^2$ (or $\mathbb{R}^3$), two vectors $\boldsymbol{u}, \boldsymbol{v}$ are linearly dependent if and only if they lie on the same line (when they are placed with their initial points at the origin).



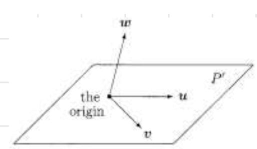
$\boldsymbol{u}, \boldsymbol{v}$ are linearly dependent | $\boldsymbol{u}, \boldsymbol{v}$ are linearly dependent | $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent

2. In $\mathbb{R}^3$, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent if and only if they lie on the same line or the same plane (when they are placed with their initial points at the origin).



$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent ($P = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$)



$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent ($P' = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$)

Linear Span v.s. Linear Independence

If S is linearly independent, then
- u ∉ span(S) ⇔ S ∪ {u} is linearly independent.
- u ∈ span(S) ⇔ S ∪ {u} is linearly dependent

Let {u, v} ∈ R2.
{u, v} is linearly independent ⇔ span{u, v} = R2

Let {u, v, w} ∈ R3.
{u, v, w} is linearly independent ⇔ span{u, v, w} = R3

Linear Span v.s. Linear Independence

$S = \{u_1, u_2, \ldots, u_k\}$ is a basis for $\mathbf{R}^n$

Linear independence VS Span

Given that: $S = \{u_1, u_2, \ldots, u_k\}$ is a subset of $\mathbf{R}^n$

To Show: $S = \{u_1, u_2, \ldots, u_k\}$ spans $\mathbf{R}^n$	To Show: $S = \{u_1, u_2, \ldots, u_k\}$ is lin. indep.
same as: span$(S) = \mathbf{R}^n$	
$c_1u_1 + c_2u_2 + \cdots + c_ku_k = v$ $\begin{pmatrix} x \\ y \\ \vdots \\ z \end{pmatrix}$	$c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$ $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
v is any general vector in $\mathbf{R}^n$	0 is the zero vector in $\mathbf{R}^n$
check whether the system is always consistent	check whether the system has non-trivial solution
yes → spans; no → does not span	yes → lin.dep; no → lin.indep

Vector Space

A set V is called a vector space if:

- either $V = \mathbf{R}^n$
- or V is a subspace of $\mathbf{R}^n$.

(any span is automatically a subspace)

e.g. The following subspaces of $\mathbb{R}^3$ are also the vector spaces

- $\mathbb{R}^3$
- $\{0\}$
- span$\{(1,2,3)\}$
- span$\{(1,2,3), (2,1,4)\}$
- sol^n space of a LS $\bar{w}$ 3 vars

Basis

Definition 3.5.4 Let $S = \{u_1, u_2, \ldots, u_k\}$ be a subset of a vector space V. Then S is called a *basis* (plural *bases*) for V if

1. S is linearly independent and no redundant vectors in S
2. S spans V. span$\{u_1, u_2, \ldots, u_k\} = V$

1. A basis for V gives a "coordinate system" for V.
2. A basis for "Rn" is not a basis for "subspace of Rn"

(1) A basis for a vector space V contains the smallest possible number of vectors that can span V
largest linearly indep set

(2) The empty set $\varnothing$ is the basis for the zero space, $\{0\}$ span$\{\varnothing\} = \{0\}$

(3) All vector space (except the zero space) has infinitely many different bases.

e.g. The following set are all bases of $\mathbb{R}^n$

- $\{(1,0,0), (0,1,0), (0,0,1)\}$
- $\{(2,0,0), (0,2,0), (0,0,2)\}$
- $\{(1,2,1), (2,9,0), (3,3,4)\}$

same dim

$\{0\}$ = span$\{0\}$ -> linearly dep

-> not basis for the zero space

Standard basis vectors for $\mathbf{R}^3$

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

- span$\{e_1, e_2, e_3\} = \mathbf{R}^3$ building block
- $\{e_1, e_2, e_3\}$ is linearly independent No redundant vectors

$S = \{e_1, e_2, e_3\}$ is called a basis for $\mathbf{R}^3$

S is a smallest possible subset of $\mathbf{R}^3$ so that every vector in $\mathbf{R}^3$ is a linear combination of the elements in S.

If S is a set of vectors that spans V but S is not a basis, every vector in V be represented in more than one way.

- redundant vectors present
- >1 way to express the vector (w or w/o the redundant vector)

e.g. plane $x = y$

$$\left.\begin{aligned} x &= s \\ y &= s \\ z &= t \end{aligned}\right\} s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

basis $= \{(1,1,0), (0,0,1)\}$

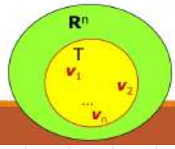
dim $= 2$

T is not a basis for V

Bases for $\mathbf{R}^n$ VS Bases for its subspaces

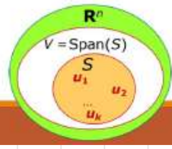
Bases for $\mathbf{R}^n$

- The basis T is the smallest set of "building blocks" for $\mathbf{R}^n$.
- T is linearly independent and span(T) = $\mathbf{R}^n$.
- Every vector in $\mathbf{R}^n$ can be expressed as a linear combination of T in a unique way.



Bases for subspace V

- The basis S is the smallest set of "building blocks" for V.
- S is linearly independent and span(S) = V.
- Every vector in V can be expressed as a linear combination of S in a unique way.



Checking basis

To show S is a basis for Rn

- Check S has n vectors (non-redundant)
- Check S is linearly independent

To show S is a basis for a subspace V of Rn

- Check S is linearly independent
- Check span(S) = V

Basis for Solution Space

homogeneous system

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0 \end{cases}$$

implicit form

Gaussian Elimination ↓

general solution

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix}$$

explicit form

separate parameters ↓

linear combination of linearly independent vectors

$$s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{u_1} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{u_2}$$

↓

basis for solution space $\{u_1, u_2\}$

linear span form

solution space = span$\{u_1, u_2\}$

Example

Show that $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ is a basis for $\mathbf{R}^3$.

(i) S is linearly independent:

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}2\\9\\0\end{pmatrix} + c_3\begin{pmatrix}3\\3\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Gaussian Elimination (details skipped) $\Rightarrow$ $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$

The system only has the trivial solution.
So S is linearly independent.

(ii) span(S) = $\mathbf{R}^3$:

Let (x, y, z) be any (general) vector in $\mathbf{R}^3$.

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}2\\9\\0\end{pmatrix} + c_3\begin{pmatrix}3\\3\\4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

Gaussian Elimination (details skipped) $\Rightarrow$ system is consistent for any values of x, y, z.

So span $(S) = \mathbf{R}^3$.

By (i) and (ii), we conclude S is a basis for $\mathbf{R}^3$.

Let $S = \{u_1, u_2, u_3\}$ be a basis for some vector space V.
Is T = $\{u_1, u_1 + u_2, u_1 + u_2 + u_3\}$ also a basis for V?

Every vector in T belongs to span(S). So span(T) ⊆ span(S).

$u_1 \in$ span(T)

$u_2 = (u_1 + u_2) - u_1 \in$ span(T)

$u_3 = (u_1 + u_2 + u_3) - (u_1 + u_2) \in$ span(T)

Every vector in S belongs to span(T). So span(S) ⊆ span(T).

So span(T) = span(S) = V

Consider $a\,u_1 + b(u_1 + u_2) + c(u_1 + u_2 + u_3) = \mathbf{0}$ (*)

Does (*) have non-trivial scalars for a, b, c ?

Rewrite (*): $(a+b+c)u_1 + (b+c)u_2 + c\,u_3 = \mathbf{0}$ (**)

(**) has only trivial scalars for a+b+c, b+c, c

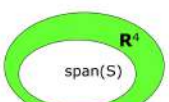
$$\left.\begin{aligned} a + b + c &= 0 \\ b + c &= 0 \\ c &= 0 \end{aligned}\right\} \text{Solve: } a = b = c = 0$$

So (*) has only trivial scalars for a, b, c
So T is linearly independent.

Is $S = \{(1, 1, 1, 1), (0, 0, 1, 2), (-1, 0, 0, 1)\}$ a basis for $\mathbf{R}^4$?

A basis for $\mathbf{R}^n$ must have n elements

- S is linearly independent
- span(S) $\neq \mathbf{R}^4$ $(|S| < 4)$
- So S is not a basis for $\mathbf{R}^4$.



span(S) is a subspace of $\mathbf{R}^4$

S is a basis for this subspace span(S)

$V = \{(a, a + b, b) \mid a, b \text{ in } \mathbf{R}\}$ is a subspace of $\mathbf{R}^3$

Show that $S = \{(1, 3, 2), (1, 2, 1)\}$ is a basis for V

Check S is linearly independent
(1, 3, 2) and (1, 2, 1) are not scalar multiples of each other

Check $\boxed{\text{span}(S) = V}$ $V = \text{span}\{(1, 1, 0), (0, 1, 1)\}$

span{(1,3,2), (1,2,1)} = span{(1,1,0), (0,1,1)} (*)

To show (*), refer: Example 3.2.11

$$\left(\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

CANNOT conclude that since span(S) = span(T) and the number of vectors in S and in T are equal, T is also a basis

- Both span(S) and span(T) contain infinitely many vectors.
- We cannot compare "infinity" like finite numbers.
- There is also no guarantee that T is linearly independent by having the same span for S and T.

Theorem 3.5.7 (unique expression in terms of basis)

Let $S = \{ u_1, u_2, \ldots, u_k \}$ be a basis for a vector space V.

Then every vector $v \in V$ can be expressed in the form

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

in exactly one way, where $c_1, c_2, \ldots, c_k$ are real numbers.

unique set of $c_1, c_2, \ldots, c_k$

Consequence of linear independence of S

Coordinate Vectors

Definition 3.5.8 Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ be a basis for a vector space V and $\boldsymbol{v}$ a vector in V. By Theorem 3.5.7, $\boldsymbol{v}$ is expressed uniquely as a linear combination

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}.$$

The coefficients $c_1, c_2, \ldots, c_k$ are called the *coordinates* of $\boldsymbol{v}$ relative to the basis S. The vector

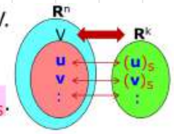
$$(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$$

ordered basis

is called the *coordinate vector* of $\boldsymbol{v}$ relative to the basis S. (Here we assume the vectors of S are in a fixed order so that $\boldsymbol{u_1}$ is the first vector, $\boldsymbol{u_2}$ is the second vector, etc.)

for consistency : $(\ )_S$: row vector ; $[\ ]_S$: column vector

Let S be a basis for a vector space V.



1. For any $\mathbf{u}, \mathbf{v} \in V$,
   $\mathbf{u} = \mathbf{v}$ if and only if $(\mathbf{u})_S = (\mathbf{v})_S$.
2. For any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbf{R}$,
   $(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r)_S$
   $= c_1(\mathbf{v}_1)_S + c_2(\mathbf{v}_2)_S + \ldots + c_r(\mathbf{v}_r)_S$.

coordinate vector of linear combination
= linear combination of coordinate vectors

Theorem 3.5.1 1

Theorem 3.5.11 Let S be a basis for a vector space V where $|S| = k$. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}$ be vectors in V. Then

1. $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}$ are linearly dependent (respectively, independent) vectors in V if and only if $(\boldsymbol{v_1})_S, (\boldsymbol{v_2})_S, \ldots, (\boldsymbol{v_r})_S$ are linearly dependent (respectively, independent) vectors in $\mathbb{R}^k$; and
2. $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\} = V$ if and only if $\text{span}\{(\boldsymbol{v_1})_S, (\boldsymbol{v_2})_S, \ldots, (\boldsymbol{v_r})_S\} = \mathbb{R}^k$.

If $S = \{u_1, \ldots, u_k\}$ is a basis for $\mathbb{R}^n$

$\Leftrightarrow$ $k = n$ and $A = (u_1 \ldots u_k)$ is invertible

S span $\mathbb{R}^n \Rightarrow k \geq n$ & REF has no zero row

$\Rightarrow$ sq matrix $\Rightarrow$ inv

S linearly indep $\Rightarrow k \leq n$ & REF has no non-pivot

$S = \{u_1, \ldots, u_n\}$ is linearly indep $\Leftrightarrow$ S spans $\mathbb{R}^n$

Theorem 3.6.1 (no. of vectors in subset v.s. basis)

Let V be a vector space which has a basis with k vectors. Then,

(1) any subset of V with more than k vectors is always linearly dependent.

(2) any subset of V with less than k vectors cannot span V.

(Relate to Thm 3.4.7 and Thm 3.2.7)

> k: too many vectors to be a basis

< k: too few vectors to be a basis

§ All bases for a vector space have the same no. of vectors.

30. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ a square matrix or order n.

    (a) Show that if $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

    (b) Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

    (i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent.