Exercise 3.30

Xiaozhu Zhang

```r
library(nimble)
library(TeachingDemos)
library(ggplot2)
```

1. HPD for σ^2

The posterior for σ^2 is

$$\pi(\sigma^2|t) = \sigma^{-4}e^{-1/\sigma^2},$$

which is exactly the Inverse-Gamma distribution $\Gamma^{-1}(1,1)$. Therefore, the 95% HPD can be obtained by

```r
hpd(qinvgamma, shape = 1)
```

```
## [1]  0.09310459 19.50439050
```

2. HPD for σ

However, the posterior for σ is

$$\pi(\sigma|t) = 2\sigma^{-3}e^{-1/\sigma^2},$$

which is a rather uncommon distribution. In order to find the HPD, we first need to sample enough values from this distribution, using the accept-reject method.

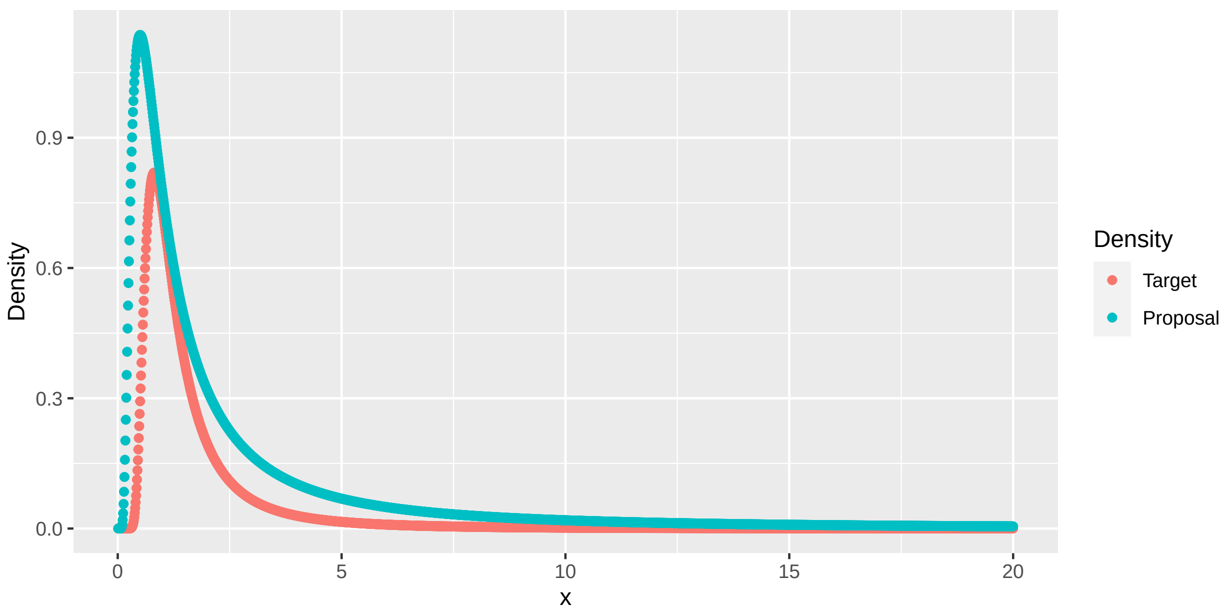
Step 1: the proposal distribution.

We can use the Inverse-Gamma distribution $\Gamma^{-1}(1,1)$ as the proposal distribution. Set $c = 2.1$. Such a choice is justified by the following test and graph.

```r
set.seed(54321)
# The target distribution
f <- function(x){
  2 * x^(-3) * exp(-1/x^2)
}

# Test the validity of proposal distribution
seq <- sapply(seq(0.01, 20, 0.01), f)
gam <- sapply(seq(0.01, 20, 0.01), function(x){
  dinvgamma(x, shape = 1)})
gam.n <- gam * 2.1

# Number of invalid points
sum(gam.n <= seq)
```

```
## [1] 0
```

```r
# Visualization
gp <- c(rep(0,length(seq)), rep(1, length(gam.n)))
df <- data.frame(x = c(seq(0.01, 20, 0.01), seq(0.01, 20, 0.01)),
                 y = c(seq, gam.n),
                 gp = as.factor(gp))

ggplot(data = df, aes(x = x, y = y, color = gp)) +
  geom_point() +
  scale_color_discrete(name = "Density", labels = c("Target", "Proposal")) +
  ylab("Density")
```



In a nutshell, there is no invalid points, and the graph above shows that the proposal distribution is completely above the target distribution.

Step 2: the accept-reject procedure.

```r
c = 2.1
n = 1e6
# Sample from the proposal distribution
x.star <- rinvgamma(n, shape = 1)

# Compute the acceot-reject threshold
ratio <- sapply(x.star, f) / (c * sapply(x.star, function(x){dinvgamma(x, shape = 1)}))

# Decide to accept or not
u <- runif(n)
ind <- u <= ratio

# Obtain the sample values
x.final <- x.star[ind]
length(x.final)
```

```
## [1] 476071
```

With the sampled values `x.final`, we can finaly find the 95% HPD of σ:

```r
emp.hpd(x.final)
```

```
## [1] 0.365027 4.444413
```

3. Conclusion

Since σ^2 and σ are from different distributions, we can presume that they have different HPD even without computing. Now that the HPD for σ^2 is about [0.093, 19.504], while the HPD for σ is about [0.365, 4.444], the presumption is indeed true. However, if we square the HPD for σ:

```r
emp.hpd(x.final)^2
```

```
## [1]  0.1332447 19.7528051
```

we can say that it is quite close to the interval for σ^2, though not exactly the same.