Bayesian Analysis

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Spring 2021

Contents

1 Introduction

Ex. 1.1. *Assume that θ is the defective rate of a batch of products. The prior for θ is*

$$\pi(0.1) = 0.7, \quad \pi(0.2) = 0.3.$$

Suppose that 2 out of 8 randomly selected products from this batch are defective. Find the posterior distribution of θ.

Let the random variable X be the number of defective products, then $X \sim B(8, \theta)$. As a result,

$$\begin{aligned}\pi(\theta = 0.1|x) &= \frac{f(x|\theta = 0.1)\pi(\theta = 0.1)}{f(x|\theta = 0.1)\pi(\theta = 0.1) + f(x|\theta = 0.2)\pi(\theta = 0.2)} \\ &= \frac{C(8,2) \cdot 0.1^2 \cdot 0.9^6 \cdot 0.7}{C(8,2) \cdot 0.1^2 \cdot 0.9^6 \cdot 0.7 + C(8,2) \cdot 0.2^2 \cdot 0.8^6 \cdot 0.3} \\ &= 0.5418.\end{aligned} \tag{1}$$

Similarly,

$$\pi(\theta = 0.2|x) = 0.4582.$$

□

Ex. 1.2. *Assume that the number of defects on a tape follows a Poisson distribution $P(\lambda)$. The prior for λ is*

$$\pi(1.0) = 0.4, \quad \pi(1.5) = 0.6.$$

Suppose that 3 defects are found on a tape. Find the posterior distribution of λ.

Let the random variable X be the number of defects on a tape, then $X \sim P(\lambda)$. As a result,

$$\begin{aligned}\pi(\lambda = 1.0|x) &= \frac{f(x|\lambda = 1.0)\pi(\lambda = 0.1)}{f(x|\lambda = 1.0)\pi(\lambda = 1.0) + f(x|\lambda = 1.5)\pi(\lambda = 1.5)} \\ &= \frac{(e^{-1.0} \cdot 1.0^3)/(3!) \cdot 0.4}{(e^{-1.0} \cdot 1.0^3)/(3!) \cdot 0.4 + (e^{-1.5} \cdot 1.5^3)/(3!) \cdot 0.6} \\ &= 0.2457.\end{aligned} \tag{2}$$

Similarly,

$$\pi(\lambda = 1.5|x) = 0.7543.$$

□

Ex. 1.3. *Assume that θ is the defective rate of a batch of products. Suppose that 3 out of 8 randomly selected products from this batch are defective. The prior is:*

(1) $\theta \sim U(0, 1)$;

(2) $\theta \sim \pi(\theta) = \begin{cases} 2(1-\theta), & 0 < \theta < 1, \\ 0, & \text{otherwise.} \end{cases}$

Find the posterior distribution respectively.

Let the random variable X be the number of defective products, then $X \sim B(8, \theta)$. As a result,

(1)

$$\begin{aligned} \pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{C(8,3)\theta^3(1-\theta)^5 \cdot 1}{\int_0^1 C(8,3)\theta^3(1-\theta)^5 \cdot 1 d\theta} \\ &= \frac{\Gamma(10)}{\Gamma(4)\Gamma(6)}\theta^3(1-\theta)^5 \sim \text{Beta}(4,6). \end{aligned} \tag{3}$$

(2)

$$\begin{aligned} \pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{C(8,3)\theta^3(1-\theta)^5 \cdot 2(1-\theta)}{\int_0^1 C(8,3)\theta^3(1-\theta)^5 \cdot 2(1-\theta) d\theta} \\ &= \frac{\Gamma(11)}{\Gamma(4)\Gamma(7)}\theta^3(1-\theta)^6 \sim \text{Beta}(4,7). \end{aligned} \tag{4}$$

□

Ex. 1.4. *Suppose that $X_1, ..., X_n$ are samples from then density $p(x|\theta)$, and the prior is $\pi(\theta)$. Show that the posterior can be obtained as follows:*

(1) Find $\pi(\theta|x_1) \propto p(x_1|\theta)\pi(\theta)$ given $X_1 = x_1$;

(2) Regard $\pi(\theta|x_1)$ as the prior and find $\pi(\theta|x_1, x_2) \propto p(x_2|\theta)\pi(\theta|x_1)$ given $X_2 = x_2$;

(3) Repeat until we find $\pi(\theta|x_1, ..., x_{n-1}) \propto p(x_n|\theta)\pi(\theta|x_1, ..., x_{n-1})$ given $X_n = x_n$.

Let us go through this procedure:

$$\pi(\theta|x_1) = \frac{p(x_1|\theta)\pi(\theta)}{p(x_1)},$$

$$\pi(\theta|x_1, x_2) = \frac{p(x_2|\theta)\pi(\theta|x_1)}{p(x_2)} = \frac{p(x_2|\theta)p(x_1|\theta)\pi(\theta)}{p(x_1)p(x_2)},$$

$$\vdots$$

$$\pi(\theta|x_1, ..., x_{n-1}) = \frac{p(x_1|\theta)...p(x_n|\theta)\pi(\theta)}{p(x_1)...p(x_n)} = \frac{p(x_1, ..., x_n|\theta)\pi(\theta)}{p(x_1, ..., x_n)}.$$

□

Ex. 1.5. *Suppose that the time someone spends waiting for bus every morning follows a uniform distribution $U(0, \theta)$. The prior for θ is*

$$\pi(\theta) = \begin{cases} 192/\theta^4, & \theta \geq 4, \\ 0, & \theta < 4. \end{cases}$$

Say he waited 5, 5, and 8 minutes for the bus on three days. Find the posterior of θ.

Let the random variable X be the time spending to wait for bus. Then

$$\begin{aligned}\pi(\theta|x_1,x_2,x_3) &= \frac{f(x_1,x_2,x_3|\theta)\pi(\theta)}{\int_4^\infty f(x_1,x_2,x_3|\theta)\pi(\theta)d\theta} \\ &= \frac{1/\theta^3 \cdot 192/\theta^4 \cdot I(\theta \ge 8)}{\int_4^\infty 1/\theta^3 \cdot 192/\theta^4 \cdot I(\theta \ge 8)d\theta} \\ &= \begin{cases} \frac{1/\theta^3 \cdot 192/\theta^4}{\int_8^\infty 1/\theta^3 \cdot 192/\theta^4 d\theta}, & \theta \ge 8, \\ 0, & \theta < 8, \end{cases} \\ &= \begin{cases} 6 \cdot 8^6/\theta^7, & \theta \ge 8, \\ 0, & \theta < 8. \end{cases}\end{aligned} \tag{5}$$

□

Ex. 1.6. *Suppose that the random variable X follows a uniform distribution $U(\theta-1/2, \theta+1/2)$. The prior for θ is $U(10,20)$.*

(1) If an observation of X is 12, find the posterior;

(2) If we got 6 observations of X: 12.0, 11.5, 11.7, 11.1, 11.4, 11.9, find the posterior.

(1)

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{10}^{20} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{1/10 \cdot I(11.5 \le \theta \le 12.5)}{\int_{10}^{20} 1/10 \cdot I(11.5 \le \theta \le 12.5)d\theta} \\ &= \begin{cases} \frac{1/10}{\int_{11.5}^{12.5} 1/10 d\theta}, & 11.5 \le \theta \le 12.5, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} 1, & 11.5 \le \theta \le 12.5, \\ 0, & \text{otherwise}. \end{cases}\end{aligned} \tag{6}$$

(2)

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{10}^{20} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{1/10 \cdot I(11.5 \le \theta \le 11.6)}{\int_{10}^{20} 1/10 \cdot I(11.5 \le \theta \le 11.6)d\theta} \\ &= \begin{cases} \frac{1/10}{\int_{11.5}^{11.6} 1/10 d\theta}, & 11.5 \le \theta \le 11.6, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} 10, & 11.5 \le \theta \le 11.6, \\ 0, & \text{otherwise}. \end{cases}\end{aligned} \tag{7}$$

□

Ex. 1.7. *Suppose that the density of a random variable X is*

$$p(x|\theta) = \frac{2x}{\theta^2} \quad (0 < x < \theta < 1).$$

(1) If the prior for θ is $U(0,1)$, find the posterior;
(2) If the prior for θ is $\pi(\theta) = 3\theta^2$ $(0 < \theta < 1)$, find the posterior.

(1)

$$\begin{aligned}
\pi(\theta|x) &= \frac{p(x|\theta)\pi(\theta)}{\int_0^1 p(x|\theta)\pi(\theta)d\theta} \\
&= \frac{2x/\theta^2 \cdot 1 \cdot I(x<\theta)}{\int_0^1 2x/\theta^2 \cdot 1 \cdot I(x<\theta)d\theta} \\
&= \begin{cases} \frac{2x/\theta^2}{\int_x^1 2x/\theta^2 d\theta}, & x<\theta<1, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} x/[\theta^2(1-x)], & x<\theta<1, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned} \tag{8}$$

(2)

$$\begin{aligned}
\pi(\theta|x) &= \frac{p(x|\theta)\pi(\theta)}{\int_0^1 p(x|\theta)\pi(\theta)d\theta} \\
&= \frac{2x/\theta^2 \cdot 3\theta^2 \cdot I(x<\theta)}{\int_0^1 2x/\theta^2 \cdot 3\theta^2 \cdot I(x<\theta)d\theta} \\
&= \begin{cases} \frac{6x}{\int_x^1 6x d\theta}, & x<\theta<1, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1/(1-x), & x<\theta<1, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned} \tag{9}$$

□

Ex. 1.8. *Suppose that 3 out of 100 randomly selected products are found to be defective. The prior for the defective rate θ is $Be(2, 200)$. Find the posterior.*

Let the random variable X to be the number of defective products, then $X \sim B(100, \theta)$. As a result,

$$\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta} \\
&= \frac{C(100,3)\theta^3(1-\theta)^{97} \cdot \frac{\Gamma(202)}{\Gamma(1)\Gamma(199)}\theta(1-\theta)^{199}}{\int_0^1 C(100,3)\theta^3(1-\theta)^{97} \cdot \frac{\Gamma(202)}{\Gamma(1)\Gamma(199)}\theta(1-\theta)^{199}d\theta} \\
&= \frac{\Gamma(302)}{\Gamma(5)\Gamma(297)}\theta^4(1-\theta)^{296} \sim \text{Beta}(5, 297).
\end{aligned} \tag{10}$$

□

Ex. 1.9. *Transform Poisson distribution and Gamma distribution into the natural form.*

(1) The Poisson distribution is of the form:

$$p(x,\theta) = \frac{e^{-\lambda}\lambda^x}{x!} = \exp(-\lambda)\exp(x\log\lambda)(x!)^{-1}.$$

Let $\phi = \log\lambda$, then $\lambda = \exp(\phi)$, and as a result the natural form is

$$p(x,\phi) = \exp(-\exp(\phi))\exp(\phi x)(x!)^{-1}.$$

(2) The Gamma distribution is of the form:

$$f(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{x/\beta} = \frac{1}{\Gamma(\alpha)\beta^\alpha}\exp\left[\frac{x}{\beta} + (\alpha-1)\log x\right].$$

Let $\phi_1 = 1/\beta$ and $\phi_2 = \alpha - 1$, then $\beta = 1/\phi_1$ and $\alpha = \phi_2 + 1$. As a result the natural form is

$$f(x,\phi_1,\phi_2) = \frac{\phi_1^{\phi_2+1}}{\Gamma(\phi_2+1)}\exp\left[\phi_1 x + \phi_2\log x\right].$$

□

Ex. 1.10. *Denote the natural form of the exponential family by*

$$f(x,\theta) = C(\theta)\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x).$$

Show that

$$E_\theta(T_j(x)) = -\frac{\partial\log C(\theta)}{\partial\theta_j} = -\frac{1}{C(\theta)}\frac{\partial C(\theta)}{\partial\theta_j},$$
$$\operatorname{Cov}(T_j(x),T_s(x)) = -\frac{\partial^2\log C(\theta)}{\partial\theta_j\partial\theta_s}.$$

(1) Since

$$1 = \int_{\mathcal{X}} C(\theta)\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x)dx,$$

if we take the first derivative *w.r.t.* θ_j, then

$$0 = \int_{\mathcal{X}} \frac{\partial C(\theta)}{\partial\theta_j}\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x)dx + \int_{\mathcal{X}} T_j(x)C(\theta)\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x)dx.$$

The second term on the R.H.S is exactly $E_\theta(T_j(x))$. As a result, we have

$$E_\theta(T_j(x)) = -\frac{\partial C(\theta)}{\partial\theta_j}\frac{1}{C(\theta)}\int_{\mathcal{X}} C(\theta)\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x)dx = -\frac{\partial C(\theta)}{\partial\theta_j}\frac{1}{C(\theta)} = -\frac{\partial\log C(\theta)}{\partial\theta_j}.$$

(2) Now we know that

$$-\frac{\partial\log C(\theta)}{\partial\theta_j} = \int_{\mathcal{X}} T_j(x)C(\theta)\exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\}h(x)dx.$$

Take the first derivative $w.r.t.$ θ_s, we have

$$
\begin{aligned}
-\frac{\partial^2 \log C(\theta)}{\partial\theta_j\partial\theta_s} &= \int_{\mathcal{X}} T_j(x)\frac{\partial C(\theta)}{\partial\theta_s}\exp\left\{\sum_{j=1}^{k}\theta_j T_j(x)\right\}h(x)dx + \int_{\mathcal{X}} T_j(x)T_s(x)C(\theta)\exp\left\{\sum_{j=1}^{k}\theta_j T_j(x)\right\}h(x)dx \\
&= \frac{\partial C(\theta)}{\partial\theta_s}\frac{1}{C(\theta)}E_\theta(T_j(x)) + E_\theta(T_j(x)T_s(x)) \\
&= E_\theta(T_j(x)T_s(x)) - E_\theta(T_j(x))E_\theta(T_s(x)) \\
&= \mathrm{Cov}(T_j(x), T_s(x)).
\end{aligned}
\tag{11}
$$

□

Ex. 1.11. *Let $T = T(X)$ be a sufficient statistic, and $S(X) = G(T(X))$. Assume that $S = G(T)$ is a bijective mapping. Show that S is also a sufficient statistic.*

Since $T(X)$ is a sufficient statistic, by factorization theorem, we know

$$f(x,\theta) = g(T(x),\theta)\cdot h(x) = g(G^{-1}\circ S(x),\theta)\cdot h(x) = g^*(S(x),\theta)\cdot h(x).$$

Therefore, $S(X)$ is also a sufficient statistic. □

Ex. 1.12. *Suppose that $X = (X_1, ..., X_n) \sim^{\text{i.i.d.}} P(\lambda)$. Show that $T(X) = \sum_{i=1}^n X_i$ is a sufficient statistic by:*
(1) definition;
(2) factorization theorem.

(1) Note that $T(X) \sim P(n\lambda)$. The conditional distribution of X given $T(X)$ is

$$
\begin{aligned}
\Pr(x_1 = x_1, ..., X_n = x_n | T(X) = t; \lambda) &= \frac{\Pr(x_1 = x_1, ..., X_n = x_n, T(X) = t; \lambda)}{\Pr(T(X) = t; \lambda)} \\
&= \frac{\Pr(x_1 = x_1, ..., X_n = x_n; \lambda)}{\Pr(T(X) = t; \lambda)} \\
&= \left(\frac{e^{-n\lambda}\cdot\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}\right)\left(\frac{e^{-n\lambda}\cdot(n\lambda)^t}{t!}\right)^{-1} \\
&= \frac{(\sum_{i=1}^n x_i)!}{n^{\sum_{i=1}^n x_i}\prod_{i=1}^n x_i!}.
\end{aligned}
\tag{12}
$$

The last equation holds due to the fact $t = \sum_{i=1}^n x_i$. Since the conditional distribution is relative constant to λ, we can conclude that $T(X)$ is a sufficient statistic.

(2) The joint distribution of X is

$$p(x_1, ..., x_n; \lambda) = \prod_{i=1}^n \frac{e^{-\lambda}\cdot\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\cdot\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}.$$

Then $g(T(x),\lambda) = e^{-n\lambda}\cdot\lambda^{\sum_{i=1}^n x_i}$ and $h(x) = 1/(\prod_{i=1}^n x_i!)$, which implies that $T(X)$ is a sufficient statistic. □

Ex. 1.13. *Suppose that $X = (X_1, ..., X_n) \sim^{\text{i.i.d.}}$ Geometric(p). Show that $T(X) = \sum_{i=1}^n X_i$ is a sufficient statistic by:*
(1) definition;
(2) factorization theorem.

(1) Note that $T(X) \sim NB(n,p)$. The conditional distribution of X give $T(X)$ is

$$\begin{aligned}
\Pr(x_1 = x_1, ..., X_n = x_n | T(X) = t; p) &= \frac{\Pr(x_1 = x_1, ..., X_n = x_n, T(X) = t; p)}{\Pr(T(X) = t; p)} \\
&= \frac{\Pr(x_1 = x_1, ..., X_n = x_n; p)}{\Pr(T(X) = t; p)} \\
&= \left[p^n (1-p)^{\sum_{i=1}^n x_i - n}\right] \left[C(t-1, n-1) p^n (1-p)^{t-n}\right]^{-1} \\
&= \frac{1}{C(t-1, n-1)}.
\end{aligned} \tag{13}$$

The last equation holds due to the fact $t = \sum_{i=1}^n x_i$. Since the conditional distribution is relative constant to p, we can conclude that $T(X)$ is a sufficient statistic.

(2) The joint distribution of X is

$$p(x_1, ..., x_n; p) = \prod_{i=1}^n p(1-p)^{x_i - 1} = p^n (1-p)^{\sum_{i=1}^n x_i - n}.$$

Then $g(T(x), p) = p^n (1-p)^{\sum_{i=1}^n x_i - n}$ and $h(x) = 1$, which implies that $T(X)$ is a sufficient statistic. $\square$

Ex. 1.14. *Suppose that $X = (X_1, ..., X_n) \sim^{\text{i.i.d.}} U(\theta - 1/2, \theta + 1/2)$. Show that $(X_{(1)}, X_{(n)})$ is a sufficient statistic.*

The joint distribution of X is

$$f(x_1, ..., x_n; \theta) = 1 \cdot I\left(X_{(n)} - \frac{1}{2} \le \theta \le X_{(1)} + \frac{1}{2}\right).$$

Then $g(X_{(1)}, X_{(n)}, \theta)$ is the indicator function above, and $h(x) = 1$. Therefore, $(X_{(1)}, X_{(n)})$ is a sufficient statistic. $\square$

Ex. 1.15. *Suppose that $X_1, ..., X_m \sim^{\text{i.i.d.}} N(a, \sigma^2)$, $Y_1, ..., Y_n \sim^{\text{i.i.d.}} N(b, \sigma^2)$, and X_i's and Y_j's are independent as well. Let*

$$\bar{X} = \frac{1}{m}\sum_{i=1}^m X_i, \;\; \bar{Y} = \frac{1}{n}\sum_{j=1}^n Y_j, \;\; S^2 = \frac{1}{n+m-2}\left[\sum_{i=1}^m (X_i - \bar{X}) + \sum_{j=1}^n (Y_j - \bar{Y})^2\right].$$

Show that $(\bar{X}, \bar{Y}, S^2)$ is a complete sufficient statistic.

The joint distribution of X_i's and Y_j's is

$$\begin{aligned}
& f(x_1, ..., x_m, y_1, ..., y_n; a, b, \sigma) \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^m (X_i - a)^2 + \sum_{j=1}^n (Y_j - b)^2\right]\right\} \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^m X_i^2 - 2am\bar{X} + ma^2 + \sum_{j=1}^n Y_j^2 - 2bn\bar{Y} + nb^2\right]\right\} \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}} \exp\left\{-\frac{ma^2 + nb^2}{2\sigma^2}\right\} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2\right) + \frac{am}{\sigma^2}\bar{X} + \frac{bn}{\sigma^2}\bar{Y}\right\}.
\end{aligned} \tag{14}$$

As a result, the sampling distribution is a member of exponential family, and $(\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2, \bar{X}, \bar{Y})$ is a sufficient statistic. Now let $\phi_1 = -1/(2\sigma^2)$, $\phi_2 = am/\sigma^2$ and $\phi_3 = bn/\sigma^2$. The natural parameter space is

$$\Theta^* = \{(\phi_1, \phi_2, \phi_3) : -\infty < \phi_1 < 0, -\infty < \phi_2, \phi_3 < +\infty\},$$

so there exists interior points in it. We can conclude that $(\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2, \bar{X}, \bar{Y})$ is a complete sufficient statistic, and furthermore $(\bar{X}, \bar{Y}, S^2)$ is also a complete sufficient statistic. □

Ex. 1.16. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} f(x, \theta) = \frac{1}{2\theta} e^{-|x|/\theta} \quad (-\infty < x < +\infty,\ \theta > 0).$$

Show that $T = \sum_{i=1}^n |X_i|$ is a complete sufficient statistic of θ.

The joint distribution of $(X_1, ..., X_n)$ is

$$f(x_1, ..., x_n; \theta) = \frac{1}{(2\theta)^n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^n |X_i|\right\}.$$

As a result, the sampling distribution is a member of exponential family, and T is a sufficient statistic. Let $\phi = -1/\theta$, then then natural parameter space is

$$\Theta^* = \{\phi : -\infty < \theta < 0\},$$

so there exists interior points in it. We can conclude that $T = \sum_{i=1}^n |X_i|$ is a complete sufficient statistic of θ. □

Ex. 1.17. *Suppose that $X_1, ..., X_n \sim^{\text{i.i.d.}} B(1, p)$, where $p \in (0, 1)$ is unknown. Assume that $s \in (0, n)$ is an integer. Find*

(1) The UMVUE of p^s;

(2) The UMVUE of $p^s + (1-p)^{n-s}$.

(1) It is acknowledged that $T(X) = \sum_{i=1}^n X_i \sim B(n, p)$ is a complete sufficient statistic. Now we want an unbiased estimator $\delta(T)$ of p^s:

$$\sum_{t=0}^n C(n, t)\delta(t)p^t(1-p)^{n-t} = p^s.$$

Solve the equation and we have

$$\delta(T) = \frac{T(T-1)\cdots(T-s+1)}{n(n-1)\cdots(n-s+1)}.$$

Since the unbiased estimator $\delta(T)$ is a function of T, we can conclude that it is the UMVUE of p^s.

(2) Similarly, we want an unbiased estimator $\delta^*(T)$ of $p^s + (1-p)^{n-s}$. First we need to solve for the unbiased estimator $\delta^{'}(t)$ of $(1-p)^{n-s}$:

$$\sum_{t=0}^n C(n, t)\delta^{'}(t)p^t(1-p)^{n-t} = (1-p)^{n-s}.$$

The equation gives

$$\delta^{'}(t) = \frac{(n-T)(n-T-1)\cdots(s-T+1)}{n(n-1)\cdots(s+1)}.$$

As a result, the unbiased estimator $\delta^*(T) = \delta(T) + \delta^{'}(T)$ is a function of T. We conclude that it is the UMVUE of $p^s + (1-p)^{n-s}$. $\square$

Ex. 1.18. *Suppose that $X_1, ..., X_m \sim^{\text{i.i.d.}} N(a, \sigma^2)$, $Y_1, ..., Y_n \sim^{\text{i.i.d.}} N(a, 2\sigma^2)$, and X_i's and Y_j's are independent as well. Find the UMVUEs of a and σ^2.*

The joint distribution of X_i's and Y_j's is

$$\begin{aligned}
&f(x_1, ..., x_m, y_1, ..., y_n; a, \sigma) \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}(\sqrt{2})^n} \exp\left\{-\frac{1}{4\sigma^2}\left[\sum_{i=1}^m 2(X_i - a)^2 + \sum_{j=1}^n (Y_j - a)^2\right]\right\} \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}(\sqrt{2})^n} \exp\left\{-\frac{1}{4\sigma^2}\left[\sum_{i=1}^m 2X_i^2 - 4a\sum_{i=1}^m X_i + 2ma^2 + \sum_{j=1}^n Y_j^2 - 2a\sum_{j=1}^n Y_j + na^2\right]\right\} \\
&= \frac{1}{(\sqrt{2\pi}\sigma)^{m+n}(\sqrt{2})^n} \exp\left\{-\frac{(2m+n)a^2}{4\sigma^2}\right\} \exp\left\{-\frac{1}{4\sigma^2}\left(2\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2\right) + \frac{a}{2\sigma^2}\left(2\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j\right)\right\}.
\end{aligned} \tag{15}$$

As a result, the sufficient statistic is $(2\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2,\ 2\sum_{i=1}^m X_i + \sum_{j=1}^n)$. Now let $\phi_1 = -1/(4\sigma^2)$, and $\phi_2 = a/2\sigma^2$. The natural parameter space is

$$\Theta^* = \{(\phi_1, \phi_2) : -\infty < \phi_1 < 0,\ -\infty < \phi_2 < +\infty\},$$

so there exists interior points in it. We can conclude that $(2\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2,\ 2\sum_{i=1}^m X_i + \sum_{j=1}^n)$ is a complete sufficient statistic.

Now consider

$$\hat{a} = \frac{2\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j}{2m+n},$$

$$\hat{\sigma}^2 = \frac{1}{2m+2n-2}\left[2\sum_{i=1}^m X_i^2 + \sum_{j=1}^n Y_j^2 - \frac{(2\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j)^2}{2m+n}\right].$$

By simple algebra we know $E(\hat{a}) = a$ and $E(\hat{\sigma}^2) = \sigma^2$. Since the unbiased estimators are functions of complete sufficient statistic, we can conclude that $\hat{a}$ and $\hat{\sigma}^2$ the UMVUEs of their expectations respectively. $\square$

Ex. 1.19. *Suppose that $X_1, ..., X_n \sim^{\text{i.i.d.}} B(k, \theta)$. Find the UMVUE of $\theta(1-\theta)$ by Lehmann-Scheffe, find the variance of the UMVUE, and compare it with the Cramer-Rao bound.*

(1) It is acknowledged that $T(X) = \sum_{i=1}^n X_i \sim B(nk, \theta)$ is a complete sufficient statistic. Now we want an unbiased estimator $\delta(T)$ of $\theta(1-\theta)$:

$$\sum_{t=0}^{nk} C(nk, t)\delta(t)\theta^t(1-\theta)^{nk-t} = \theta(1-\theta).$$

Solve the equation and we have

$$\delta(T) = \frac{T(nk - T)}{nk(nk-1)}.$$

Since it is a function of T, we can conclude that it is the UMVUE of $\theta(1-\theta)$.

(2) Since $T \sim B(nk, \theta)$, we know:

- $E(T) = nk\theta$;
- $E(T^2) = \mathrm{Var}(T) + E(T)^2 = nk\theta(1-\theta) + (nk\theta)^2$;
- If we want $E(T^3)$, consider

  $$\sum_{t=0}^{nk} t^2 C(nk,t)\theta^t(1-\theta)^{nk-t} = nk\theta(1-\theta) + (nk\theta)^2.$$

  Take derivative w.r.t. θ, and by simple algebra, we have

  $$E(T^3) = \theta(1-\theta)(1-2\theta)(nk) + 3\theta^2(1-\theta)(nk)^2 + \theta^3(nk)^3.$$

- If we want $E(T^4)$, similarly, consider

  $$\sum_{t=0}^{nk} t^3 C(nk,t)\theta^t(1-\theta)^{nk-t} = \theta(1-\theta)(1-2\theta)(nk) + 3\theta^2(1-\theta)(nk)^2 + \theta^3(nk)^3.$$

  Take derivative w.r.t. θ, and by simple algebra, we have

  $$\begin{aligned} E(T^4) = {} & \theta(nk)E(T^3) \\ & + \left[\theta(1-\theta)^2(1-2\theta) - \theta^2(1-\theta)(1-2\theta) - 2\theta^2(1-\theta)^2\right](nk) \\ & + \left[6\theta^2(1-\theta)^2 - 3\theta^3(1-\theta)\right](nk)^2 \\ & + 3\theta^3(1-\theta)(nk)^3. \end{aligned} \tag{16}$$

As a result,

$$\begin{aligned} \mathrm{Var}[T(nk-T)] &= \mathrm{Var}[nkT - T^2] \\ &= (nk)^2\mathrm{Var}(T) + \mathrm{Var}(T^2) - 2(nk)\mathrm{Cov}(T, T^2) \\ &= (nk)^2\mathrm{Var}(T) + E(T^4) - E(T^2)^2 - 2(nk)(E(T^3) - E(T)E(T^2)). \end{aligned} \tag{17}$$

By some cumbersome calculations,

$$\begin{aligned} \mathrm{Var}[T(nk-T)] = {} & (-6\theta^4 + 12\theta^2 - 7\theta^2 + \theta)(nk) \\ & + (10\theta^4 - 2\theta^3 + 10\theta^2 - 2\theta)(nk)^2 \\ & + (-4\theta^4 + 10\theta^3 - 7\theta^2 + \theta)(nk)^3. \end{aligned} \tag{18}$$

Furthermore,

$$\mathrm{Var}[\delta(T)] = \frac{\mathrm{Var}[T(nk-T)]}{(nk)^2(nk-1)^2}.$$

(3) The density function of X_i is

$$p(x;\theta) = C(k,x)\theta^x(1-\theta)^{k-x}.$$

Therefore, the information number is

$$\begin{aligned} I(\theta) &= -E\left[\frac{\partial^2 \log p(X;\theta)}{\partial\theta^2}\right] \\ &= -E\left[\frac{\partial^2}{\partial\theta^2}\left(X\log\theta + (k-X)\log(1-\theta)\right)\right] \\ &= -E\left[\frac{\partial}{\partial\theta}\left(\frac{X}{\theta} - \frac{k-X}{1-\theta}\right)\right] \\ &= \frac{k(1+2\theta)}{\theta(\theta+1)}. \end{aligned} \tag{19}$$

The C-R bound is

$$\frac{\left[(\theta(1-\theta))''\right]^2}{nI(\theta)} = \frac{(1-2\theta)^2\theta(1+\theta)}{nk(1+2\theta)}.$$

Ex. 1.20. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} f(x,\theta) = \begin{cases} \theta^{-1}e^{-x/\theta}, & x>0, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that θ is an unknown parameter. Find the UMVUE by C-R inequality.

From the density function we know that $X_i \sim Exp(\theta)$, $E(X_i) = \theta$, and $\mathrm{Var}(X_i) = \theta^2$. As a result,

$$\hat{\theta} = \frac{\sum_{i=1}^n X_i}{n}$$

is an unbiased estimator of θ. The variance of it is θ^2/n.

On the other hand, the information number is

$$\begin{aligned} I(\theta) &= -E\left[\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}\right] \\ &= -E\left[\frac{\partial^2}{\partial\theta^2}\left(-\log\theta - \frac{X}{\theta}\right)\right] \\ &= -E\left[\frac{\partial}{\partial\theta}\left(-\frac{1}{\theta} + \frac{X}{\theta^2}\right)\right] \\ &= \frac{1}{\theta^2}. \end{aligned} \tag{20}$$

Thus the C-R bound is

$$\frac{1}{nI(\theta)} = \frac{\theta^2}{n}.$$

Since the lower bound is attained by $\hat{\theta}$, we can conclude that the estimator (sample mean) is the UMVUE.

Ex. 1.21. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} \Gamma(\alpha, \lambda),$$

where α is known, and $\lambda > 0$. Find the UMVUE of $g(\lambda) = 1/\lambda$ by C-R bound.

We can construct an estimator

$$\hat{g}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{\alpha}.$$

Then $E[\hat{g}(\lambda)] = E[X_i]/\alpha = 1/\lambda$, and $\text{Var}[\hat{g}(\lambda)] = \text{Var}[X_i]/(n\alpha^2) = 1/(n\alpha\lambda^2)$. The estimator is indeed an unbiased estimator.

On the other hand, since the density function of X_i is

$$f(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x},$$

the information number is

$$\begin{aligned} I(\lambda) &= -E\left[\frac{\partial^2 \log f(X; \lambda)}{\partial \lambda^2}\right] \\ &= -E\left[\frac{\partial^2}{\partial \lambda^2} \left(\alpha \log \lambda + (\alpha - 1) \log X - \lambda X\right)\right] \\ &= -E\left[\frac{\partial}{\partial \lambda} \left(\frac{\alpha}{\lambda} - X\right)\right] \\ &= \frac{\alpha}{\lambda^2}. \end{aligned} \tag{21}$$

As a result, the C-R bound is

$$\frac{1/\lambda^4}{nI(\lambda)} = \frac{1}{n\alpha\lambda^2}.$$

The lower bound is attained by $\hat{g}(\lambda)$, so we conclude that it is the UMVUE.

Ex. 1.22. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(0, \sigma^2).$$

Find the level α UMPT for

$$H_0 : \sigma^2 \le \sigma_0^2 \longleftrightarrow H_1 : \sigma^2 > \sigma_0^2.$$

The joint density is

$$f(x_1, ..., x_n; \sigma^2) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2\right\}.$$

Therefore,

$$Q(\sigma^2) = -\frac{1}{2\sigma^2}, \quad T(x) = \sum_{i=1}^{n} x_i^2.$$

Since $Q(\sigma^2)$ is a strictly increasing function, we can construct the test function

$$\phi(x) = \begin{cases} 1, & T(x) > c, \\ 0, & T(x) \le c. \end{cases}$$

Let

$$E[\phi(X)|\sigma = \sigma_0] = \Pr(T(X) > c|\sigma = \sigma_0) = \Pr\left(\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} > \frac{c}{\sigma_0^2}\right) = \alpha,$$

and as a result, we have $\chi^2_\alpha(n) = c/\sigma_0^2$, indicating that $c = \sigma_0^2\chi^2_\alpha(n)$.

In conclusion, the level α UMPT is

$$\phi(x) = \begin{cases} 1, & \sum_{i=1}^n x_i^2 > \sigma_0^2\chi^2_\alpha(n), \\ 0, & \sum_{i=1}^n x_i^2 \le \sigma_0^2\chi^2_\alpha(n). \end{cases}$$

Ex. 1.23. *Given an integer k. In order to test for the probability of an even ($p \le p_0$ or not), we independently repeat our experiment until the event occurs k times. Let the random variable X be the total number of experiments when we stop.*

(1) Show that the probability mass function of X is

$$\Pr(X = x) = C(x-1, k-1)p^k(1-p)^{x-k}$$

where $x = k,\ k+1,\ \ldots$

(2) Given p_0 and α, find the level α UMPT for

$$H_0 : p \le p_0 \ \longleftrightarrow\ H_1 : p > p_0.$$

(1) Suppose that our realization of X is x. Among the x experiments, k are considered as successes (if we call the occurrence of the event a success), and $x-k$ are considered as failures. The last experiment must by a success, so we need to choose another $k-1$ successes among the first $x-1$ experiments. Similar to the binomial experiment, we justify that the p.m.f. of X is

$$\Pr(X = x) = C(x-1, k-1)p^k(1-p)^{x-k},$$

and that the total number of experiments must be at least the number of successes. [In fact, the distribution is called the *negative binomial distribution*, and $X \sim NB(k, p)$]

(2) The p.m.f. of X is

$$\begin{aligned} p(x;p) &= C(x-1,k-1)\left(\frac{p}{1-p}\right)^k (1-p)^x \\ &= C(x-1,k-1)\left(\frac{p}{1-p}\right)^k \exp[\log(1-p)x]. \end{aligned} \tag{22}$$

Since $Q(p) = \log(1-p)$ is a strictly decreasing function, we can construct the test function

$$\phi(x) = \begin{cases} 1, & x < c, \\ r, & x = c, \\ 0, & x \ge c. \end{cases}$$

We can determine c by

$$\alpha_1 = \sum_{x=k}^{c-1} C(x-1,k-1)p_0^k(1-p_0)^{x-k} \le \alpha \le \sum_{x=k}^{c} C(x-1,k-1)p_0^k(1-p_0)^{x-k}.$$

Let

$$E[\phi(X)|p = p_0] = \Pr(X < c) + r \cdot \Pr(X = c) = \alpha,$$

then we can determine

$$r = \frac{\alpha - \alpha_1}{C(c-1, k-1)p_0^k(1-p_0)^{c-k}}.$$

Ex. 1.24. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} Exp(\lambda),$$

where $\lambda > 0$ is unknown. Given λ_0 and α, find the level α UMPT for

$$H_0 : \lambda \geq \lambda_0 \iff H_1 : \lambda < \lambda_0.$$

The joint density is

$$f(x_1, ..., x_n; \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\}.$$

Since $Q(\lambda) = -\lambda$ is a strictly decreasing function, and $T(x) = \sum_{i=1}^n x_i$, we can construct the test function

$$\phi(x) = \begin{cases} 1, & T(x) > c, \\ 0, & T(x) \leq c. \end{cases}$$

By $T(X) \sim \Gamma(n, \lambda)$ and

$$E[\phi(X)|\lambda = \lambda_0] = \Pr(T(X) > c|\lambda = \lambda_0) = \alpha,$$

we known that $c = \Gamma_\alpha(n, \lambda)$. Thus the level α UMPT is

$$\phi(x) = \begin{cases} 1, & \sum_{i=1}^n x_i > \Gamma_\alpha(n, \lambda), \\ 0, & \sum_{i=1}^n x_i \leq \Gamma_\alpha(n, \lambda). \end{cases}$$

2 The Prior Distributions

Ex. 2.1. *Not working on it.*

Ex. 2.2. *Not working on it.*

Ex. 2.3. *Suppose that the prior distribution for parameter θ is $Be(\alpha, \beta)$. The expectation and variance derived from the prior information are 1/3 and 1/45, respectively. Please find the prior distribution.*

If $X \sim Be(\alpha, \beta)$, then

$$E(X) = \frac{\alpha}{\alpha+\beta}, \ \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Let $E(X) = 1/3$ and $\mathrm{Var}(X) = 1/45$, we have $\alpha = 3$ and $\beta = 6$. Thus, the prior distribution is $Be(3, 6)$. □

Ex. 2.4. *Not working on it.*

Ex. 2.5. *Not working on it.*

Ex. 2.6. *The failure time of an electronic component follows the exponential distribution:*

$$f(x|\theta) = \theta^{-1}e^{-x/\theta} \quad (x > 0).$$

The prior distribution of the unknown parameter θ follows $\Gamma^{-1}(1, 100)$. Find the marginal probability of the component failing before time 200.

Consider:

$$\begin{aligned}
\Pr(X < 200) &= \int_0^{200} f(x)dx \\
&= \int_0^{200}\int_0^{+\infty} f(x|\theta)f(\theta)d\theta dx \\
&= \int_0^{200}\int_0^{+\infty} \theta^{-1}e^{-x/\theta}\frac{100}{\Gamma(1)}\theta^{-2}e^{-100/\theta}d\theta dx \\
&= 100\int_0^{200}\int_0^{+\infty} \theta^{-3}e^{-\frac{x+100}{\theta}}d\theta dx \\
&= 100\int_0^{200} \frac{\Gamma(2)}{(x+100)^2}dx \\
&= \frac{2}{3}.
\end{aligned} \tag{23}$$

□

Ex. 2.7. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} P(\theta_i) \quad i = 1, ..., n.$$

If $\theta_1, ..., \theta_n$ are samples from the Gamma distribution $\Gamma(r, \lambda)$, find the joint marginal distribution $m(\boldsymbol{x})$ for $\boldsymbol{X} = (X_1, ..., X_n)$.

Consider:

$$
\begin{aligned}
m(\mathbf{x}) &= \prod_{i=1}^{n} f(x_i) \\
&= \prod_{i=1}^{n} \int_0^{+\infty} f(x_i|\theta_i) f(\theta_i|r,\lambda) d\theta_i \\
&= \prod_{i=1}^{n} \int_0^{+\infty} \frac{e^{-\theta_i}\theta_i^{x_i}}{x_i!} \frac{\lambda^r}{\Gamma(r)} \theta_i^{r-1} e^{-\lambda\theta_i} d\theta_i \\
&= \prod_{i=1}^{n} \frac{\lambda^r}{x_i!\Gamma(r)} \int_0^{+\infty} \theta_i^{x_i+r-1} e^{-(\lambda+1)\theta_i} d\theta_i \\
&= \prod_{i=1}^{n} \frac{\lambda^r}{x_i!\Gamma(r)} \frac{\Gamma(x_i+r)}{(\lambda+1)^{x_i+r}}.
\end{aligned} \tag{24}
$$

□

Ex. 2.8. ***Cont'd Ex. 2.7****. Let $n = 3,\ x_1 = 3,\ x_2 = 0,\ x_3 = 5$. Find the ML-II prior.*

Let $\ell = \log m(\mathbf{x})$, then

$$
\begin{aligned}
\ell &= \log \prod_{i=1}^{n} \frac{\lambda^r}{x_i!\Gamma(r)} \frac{\Gamma(x_i+r)}{(\lambda+1)^{x_i+r}} \\
&= \sum_{i=1}^{n} \left[r\log\lambda - \log x_i! - \log\Gamma(r) + \log\Gamma(x_i+r) - (x_i+r)\log(\lambda+1)\right] \\
&= \sum_{i=1}^{n} \left[r\log\lambda - \log x_i! + \sum_{j=0}^{x_i-1}\log(r+j) - (x_i+r)\log(\lambda+1)\right].
\end{aligned} \tag{25}
$$

To solve for λ, let

$$
\frac{\partial\ell}{\partial\lambda} = \frac{nr}{\lambda} - \sum_{i=1}^{n} \frac{x_i+r}{\lambda+1} = 0,
$$

and we have $r = \lambda\bar{x}$. To solve for r, let

$$
\frac{\partial\ell}{\partial r} = n\log\lambda + \sum_{i=1}^{n}\sum_{j=0}^{x_i-1} \frac{1}{r+j} - n\log(\lambda+1) = 0,
$$

then we have

$$
n\log\frac{\lambda+1}{\lambda} = \sum_{i=1}^{n}\sum_{j=0}^{x_i-1} \frac{1}{\lambda\bar{x}+j},
$$

which is

$$
3\log\frac{\lambda+1}{\lambda} = \frac{2}{\lambda\bar{x}+0} + \frac{2}{\lambda\bar{x}+1} + \frac{2}{\lambda\bar{x}+2} + \frac{1}{\lambda\bar{x}+3} + \frac{1}{\lambda\bar{x}+4},
$$

where $\bar{x} = 8/3$. By numerical methods, we know $\lambda = 0.7018048$, and $r = 1.87147957$. Therefore, The ML-II prior is $\Gamma(1.87147957, 0.7018048)$. □

Ex. 2.9. ***Cont'd Ex. 2.7.*** *Show that*

$$\hat{r} = \frac{\bar{x}^2}{S^2 - \bar{x}}, \quad \hat{\lambda} = \frac{\bar{x}}{S^2 - \bar{x}}$$

using the moment method.

Since $\mu(\theta) = \theta$ and $\sigma^2(\theta) = \theta$, we know

$$E^{\theta|r,\lambda}[\mu(\theta)] = E^{\theta|r,\lambda}[\theta] = \frac{r}{\lambda},$$
$$E^{\theta|r,\lambda}[\sigma^2(\theta)] = E^{\theta|r,\lambda}[\theta] = \frac{r}{\lambda},$$
$$E^{\theta|r,\lambda}\left\{[\mu(\theta) - \mu_m(\lambda)]^2\right\} = E^{\theta|r,\lambda}\left\{\left[\theta - \frac{r}{\lambda}\right]^2\right\} = \frac{r}{\lambda^2}.$$

As a result,

$$\bar{x} = E^{\theta|r,\lambda}[\mu(\theta)] = \frac{r}{\lambda},$$
$$S^2 = E^{\theta|r,\lambda}[\sigma^2(\theta)] + E^{\theta|r,\lambda}\left\{[\mu(\theta) - \mu_m(\lambda)]^2\right\} = \frac{r}{\lambda} + \frac{r}{\lambda^2},$$

which implies that

$$\hat{r} = \frac{\bar{x}^2}{S^2 - \bar{x}}, \quad \hat{\lambda} = \frac{\bar{x}}{S^2 - \bar{x}}.$$

□

Ex. 2.10. *Suppose that X follows the exponential distribution $Exp(\theta)$, and the prior distribution for θ is $\Gamma(\alpha, \lambda)$. Let*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} m(x|\alpha, \lambda),$$

the sample mean be $\bar{X} = 2$, and the sample variance be $S^2 = 8$. Find the prior distribution using the moment method.

Here we assume that

$$f(X|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}.$$

Since $\mu(\theta) = \theta$ and $\sigma^2(\theta) = \theta^2$, we know

$$E^{\theta|\alpha,\lambda}[\mu(\theta)] = E^{\theta|\alpha,\lambda}[\theta] = \alpha\lambda,$$
$$E^{\theta|\alpha,\lambda}[\sigma^2(\theta)] = E^{\theta|\alpha,\lambda}[\theta^2] = \alpha\lambda^2 + \alpha^2\lambda^2,$$
$$E^{\theta|\alpha,\lambda}\left\{[\mu(\theta) - \mu_m(\lambda)]^2\right\} = E^{\theta|\alpha,\lambda}\left\{[\theta - \alpha\lambda]^2\right\} = \alpha\lambda^2.$$

As a result,

$$\bar{x} = E^{\theta|\alpha,\lambda}[\mu(\theta)] = \alpha\lambda = 2,$$
$$S^2 = E^{\theta|\alpha,\lambda}[\sigma^2(\theta)] + E^{\theta|\alpha,\lambda}\left\{[\mu(\theta) - \mu_m(\lambda)]^2\right\} = 2\alpha\lambda^2 + \alpha^2\lambda^2 = 8,$$

which implies that $\alpha = 2,\ \lambda = 1$. In other words,

$$f(\theta) = \theta e^{-\theta}.$$

□

Ex. 2.11. *Determine whether the following distribution families are location parameter families, scale parameter families or neither. Find a non-informative prior for each of them.*

(1) The uniform distribution $U(\theta-1,\theta+1)$*;*

(2) The Cauchy distribution $C(0,\beta)$*;*

(3) The T distribution $T(n,\mu,\sigma^2)$*, where* n *is fixed;*

(4) The Pareto distribution $Pa(x_0,\alpha)$ *where* α *is fixed.*

(1) If $X \sim U(\theta-1,\theta+1)$, then

$$f(x) = \frac{1}{2}\cdot I(\theta-1<x<\theta+1) = \frac{1}{2}\cdot I(-1<x-\theta<1) = g(x-\theta),$$

where $g(t) = 1/2 \cdot I(-1<t<1)$. So the uniform distribution family is a location parameter family, and the uninformative prior is $\pi(\theta) \equiv 1$.

(2) If $X \sim C(0,\beta)$, then

$$f(x) = \frac{1}{\pi}\cdot\frac{\beta}{\beta^2+x^2} = \frac{1}{\pi}\cdot\frac{1}{\beta\,(1+(x/\beta)^2)} = \beta^{-1}\phi(\frac{x}{\beta}),$$

where $\phi(t) = (1/\pi)\cdot(1+t^2)^{-1}$, and $\beta>0$. So the Cauchy distribution family is a scale parameter family, and the uninformative prior is $\pi(\beta) = 1/\beta$.

(3) If $X \sim T(n,\mu,\sigma^2)$, then

$$f(x) = \sigma^{-1} g(\frac{x-\mu}{\sigma}),$$

where

$$g(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}.$$

The T distribution family is a location-scale parameter distribution family.

If we assume that the non-informative priors of μ and of σ are independent, then the joint non-informative prior is $\pi(\mu,\sigma) = 1/\sigma$.

(4) If $X \sim Pa(x_0,\alpha)$, then

$$f(x) = \begin{cases} \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x \ge x_0, \\ 0, & x < x_0. \end{cases}$$

It is clear that the Pareto distribution family (w.r.t. x_0) is not a location parameter family, since it is not the form of $g(x-x_0)$. If we consider the Jeffreys prior, the problem arise since the probability density function of Pareto distribution does not satisfy the second regularity condition; more specifically, the support set depends on parameter x_0. A good paper Li, Sun and Peng (2019) gives the conclusion: $\pi(x_0) = 1/x_0$. □

Ex. 2.12. *Find the Jeffreys priors for the following distributions:*

(1) Poisson distribution $P(\lambda)$*;*

(2) Binomial distribution $B(n,\theta)$ *(*n *is known);*

(3) Negative binomial distribution $Nb(r,\theta)$ *(*r *is known);*

(4) Exponential distribution $Exp(1/\lambda)$*;*

(5) Gamma distribution $\Gamma(\alpha,\lambda)$ *(*α *is known);*

(6) Multinomial distribution $M(n,\boldsymbol{p})$*,* $\boldsymbol{p} = (p_1,...,p_k)$ *(*n *is known).*

(1) Poisson distribution $P(\lambda)$.
Step 1: the log-likelihood:

$$\ell = \sum_{i=1}^{n} \log f(x_i|\lambda) = \sum_{i=1}^{n} \log \frac{e^{\lambda}\lambda^{x_i}}{x_i!} = \sum_{i=1}^{n} \left[\lambda + x_i \log \lambda - \log x_i!\right].$$

Step 2: the Fisher information matrix:

$$I(\lambda) = E_{X|\lambda}\left[-\frac{\partial^2 \ell}{\partial \lambda^2}\right] = E_{X|\lambda}\left[\sum_{i=1}^{n} \frac{X_i}{\lambda^2}\right] = \frac{n}{\lambda}.$$

Step 3: the non-informative prior:

$$\pi(\lambda) = I(\lambda)^{1/2} \propto \frac{1}{\sqrt{\lambda}}.$$

(2) Binomial distribution $B(n, \theta)$ (n is known).
Step 1: the log-likelihood:

$$\begin{aligned}\ell &= \sum_{i=1}^{N} \log f(x_i|\theta) \\ &= \sum_{i=1}^{N} \log C(n, x_i)\theta^{x_i}(1-\theta)^{n-x_i} \\ &= \sum_{i=1}^{N} \left[\log C(n, x_i) + x_i \log \theta + (n - x_i)\log(1-\theta)\right].\end{aligned} \tag{26}$$

Step 2: the Fisher information matrix:

$$I(\theta) = E_{X|\theta}\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = E_{X|\theta}\sum_{i=1}^{N}\left[\frac{X_i}{\theta^2} + \frac{n - X_i}{(1-\theta)^2}\right] = \frac{nN}{\theta(1-\theta)}.$$

Step 3: the non-informative prior:

$$\pi(\theta) = I(\theta)^{1/2} \propto \frac{1}{\theta^{1/2}(1-\theta)^{1/2}}.$$

(3) Negative binomial distribution $Nb(r, \theta)$ (r is known).
Step 1: the log-likelihood:

$$\begin{aligned}\ell &= \sum_{i=1}^{n} \log f(x_i|\theta) \\ &= \sum_{i=1}^{n} \log C(x_i + r - 1, x_i)\theta^{r}(1-\theta)^{x_i} \\ &= \sum_{i=1}^{n} \left[\log C(x_i + r - 1, x_i) + r \log \theta + x_i \log(1-\theta)\right].\end{aligned} \tag{27}$$

Step 2: the Fisher information matrix:

$$I(\theta) = E_{X|\theta}\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = E_{X|\theta}\sum_{i=1}^{n}\left[\frac{r}{\theta^2} + \frac{X_i}{(1-\theta)^2}\right] = \frac{nr}{\theta^2(1-\theta)}.$$

Step 3: the non-informative prior:

$$\pi(\theta) = I(\theta)^{1/2} \propto \frac{1}{\theta(1-\theta)^{1/2}}.$$

(4) Exponential distribution $Exp(1/\lambda)$.
Step 1: the log-likelihood:

$$\ell = \sum_{i=1}^{n} \log f(x_i|\lambda) = \sum_{i=1}^{n} \log \frac{1}{\lambda} e^{-x_i/\lambda} = \sum_{i=1}^{n}\left[-\log\lambda - \frac{x_i}{\lambda}\right].$$

Step 2: the Fisher information matrix:

$$I(\lambda) = E_{X|\lambda}\left[-\frac{\partial^2 \ell}{\partial \lambda^2}\right] = E_{X|\lambda}\left[\sum_{i=1}^{n}\frac{X_i}{\lambda^2}\right] = \frac{n}{\lambda^2}.$$

Step 3: the non-informative prior:

$$\pi(\lambda) = I(\lambda)^{1/2} \propto \frac{1}{\lambda}.$$

(5) Gamma distribution $\Gamma(\alpha, \lambda)$ (α is known).
Step 1: the log-likelihood:

$$\begin{aligned}\ell &= \sum_{i=1}^{n} \log f(x_i|\lambda) \\ &= \sum_{i=1}^{n} \log \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \\ &= \sum_{i=1}^{n}\left[\alpha\log\lambda - \log\Gamma(\alpha) + (\alpha-1)\log x_i - \lambda x_i\right].\end{aligned} \tag{28}$$

Step 2: the Fisher information matrix:

$$I(\lambda) = E_{X|\lambda}\left[-\frac{\partial^2 \ell}{\partial \lambda^2}\right] = E_{X|\lambda}\left[\frac{n\alpha}{\lambda^2}\right] = \frac{n\alpha}{\lambda^2}.$$

Step 3: the non-informative prior:

$$\pi(\lambda) = I(\lambda)^{1/2} \propto \frac{1}{\lambda}.$$

(6) Multinomial distribution $M(n, \mathbf{p})$, $\mathbf{p} = (p_1, ..., p_k)$ (n is known).

Step 1: the log-likelihood:

$$\begin{aligned}
\ell &= \sum_{i=1}^{N} \log f(\mathbf{x}_i|\mathbf{p}) \\
&= \sum_{i=1}^{N} \log \left(\prod_{j=1}^{k} p_j^{x_{ij}}\right) \left(1 - \sum_{j=1}^{k} p_j\right)^{n-\sum_{j=1}^{k} x_{ij}} \\
&= \sum_{i=1}^{N} \left[\sum_{j=1}^{k} x_{ij} \log p_j + \left(n - \sum_{j=1}^{k} x_{ij}\right) \log \left(1 - \sum_{j=1}^{k} p_j\right)\right]
\end{aligned} \tag{29}$$

Step 2: the Fisher information matrix:

$$\begin{aligned}
I_{ab}(\mathbf{p}) &= E_{\mathbf{X}|\mathbf{p}} \left[-\frac{\partial^2 \ell}{\partial p_a \partial p_b}\right] \\
&= E_{\mathbf{X}|\mathbf{p}} \sum_{i=1}^{N} \frac{n - \sum_{j=1}^{k} x_{ij}}{(1 - \sum_{j=1}^{k} p_j)^2} \\
&= \frac{nN}{1 - \sum_{j=1}^{k} p_j};
\end{aligned} \tag{30}$$

$$\begin{aligned}
I_{aa}(\mathbf{p}) &= E_{\mathbf{X}|\mathbf{p}} \left[-\frac{\partial^2 \ell}{\partial p_a^2}\right] \\
&= E_{\mathbf{X}|\mathbf{p}} \sum_{i=1}^{N} \left[\frac{x_{ia}}{p_a^2} + \frac{n - \sum_{j=1}^{k} x_{ij}}{(1 - \sum_{j=1}^{k} p_j)^2}\right] \\
&= nN \left(\frac{1}{p_a} + \frac{1}{1 - \sum_{j=1}^{k} p_j}\right).
\end{aligned} \tag{31}$$

Step 3: the non-informative prior:

$$\begin{aligned}
\pi(\mathbf{p})^2 &= [\det \mathbf{I}(\mathbf{p})] \\
&= nN \begin{vmatrix}
p_1^{-1} + (1 - \sum_{j=1}^{k} p_j)^{-1} & (1 - \sum_{j=1}^{k} p_j)^{-1} & \cdots & (1 - \sum_{j=1}^{k} p_j)^{-1} \\
(1 - \sum_{j=1}^{k} p_j)^{-1} & p_2^{-1} + (1 - \sum_{j=1}^{k} p_j)^{-1} & \cdots & (1 - \sum_{j=1}^{k} p_j)^{-1} \\
\vdots & \vdots & & \vdots \\
(1 - \sum_{j=1}^{k} p_j)^{-1} & (1 - \sum_{j=1}^{k} p_j)^{-1} & \cdots & p_k^{-1} + (1 - \sum_{j=1}^{k} p_j)^{-1}
\end{vmatrix} \\
&\propto (p_1 p_2 \cdots p_k)^{-1}.
\end{aligned} \tag{32}$$

As a result,

$$\pi(\mathbf{p})^2 = (p_1 p_2 \cdots p_k)^{-1/2}.$$

□

Ex. 2.13. *Suppose that $X_j \sim f(x_j|\theta_j)$, and the Jeffreys priors for θ_j are $\pi_j(\theta_j)$ $(j = 1, ..., k)$. If the X_j's are independent, show that the Jeffreys prior for $\theta = (\theta_1, ..., \theta_k)$ is $\pi(\theta) = \prod_{i=j}^{k} \pi_j(\theta_j)$.*

The key to the problem is that for $\mathbf{X} = (X_1, ..., X_k)$:

$$f(\mathbf{x}|\theta) = \prod_{j=1}^{k} f(x_j|\theta_j).$$

First, we know that the Jeffrey prior for θ_j can be calculated as

$$\pi_j(\theta_j)^2 = \sum_{i=1}^{n} E_{X_j|\theta_j}\left[\frac{\partial^2}{\partial\theta_j^2}\log f(x_{ij}|\theta_j)\right].$$

Next, consider the log-likelihood of $\mathbf{X}$:

$$\begin{aligned}\ell &= \sum_{i=1}^{n}\log f(\mathbf{x}|\theta)\\ &= \sum_{i=1}^{n}\log\prod_{j=1}^{k} f(x_{ij}|\theta_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{k}\log f(x_{ij}|\theta_j).\end{aligned} \tag{33}$$

On the one hand, since $\partial^2\ell/(\partial\theta_a\partial\theta_b) = 0$, we know $I_{ab}(\theta) = 0$. On the other hand,

$$I_{aa}(\theta) = \sum_{i=1}^{n} E_{X_a|\theta_a}\left[\frac{\partial^2}{\partial\theta_a^2}\log f(x_{ia}|\theta_a)\right],$$

which is exactly $\pi_a(\theta_a)^2$. Since the Fisher information matrix is a diagonal matrix, we know

$$\det\mathbf{I}(\theta) = \prod_{a=1}^{k} I_{aa}(\theta) = \prod_{a=1}^{k}\pi_a(\theta_a)^2.$$

As a result, the non-informative prior for θ is

$$[\det\mathbf{I}(\theta)]^{1/2} = \sqrt{\prod_{j=1}^{k}\pi_j(\theta_j)^2} = \prod_{j=1}^{k}\pi_j(\theta_j).$$

$\square$

Ex. 2.14. *A location-scale density is of the form $\sigma^{-1}f((x-\theta)/\sigma)$. Show that $\pi(\theta,\sigma) = 1/\sigma^2$ is the non-informative prior for the location-scale parameter (θ,σ), using the invariance of the transformation group. The transformations are: $Y = cX + b$, $\eta = c\theta + b$ and $\xi = c\sigma$ $(b\in\mathbb{R},\ c>0)$.*

The transformations can be described as

$$\begin{gathered}X \to Y = cX + b,\\ (\theta,\sigma)\to(\eta,\xi) = (c\theta + b,\ c\sigma).\end{gathered}$$

Let π and π^* be the non-informative priors for (θ, σ) and (η, ξ), respectively. Due to the invariance of the translation-scale transformation group, we have

$$\pi(\tau) = \pi^*(\tau).$$

On the other hand, we know

$$\pi^*(\eta, \xi) = \pi(\theta, \sigma)|_{\theta=(\eta-b)/c,\ \sigma=\xi/c} \cdot |J| = \frac{1}{c^2}\pi(\frac{\eta - b}{c}, \frac{\xi}{c}),$$

where

$$J = \begin{bmatrix} \partial\theta/\partial\eta & \partial\theta/\partial\xi \\ \partial\sigma/\partial\eta & \partial\sigma/\partial\xi \end{bmatrix} = \begin{bmatrix} 1/c & 0 \\ 0 & 1/c \end{bmatrix}.$$

Put the results together, we have

$$\pi(\eta, \xi) = \pi^*(\eta, \xi) = \frac{1}{c^2}\pi(\frac{\eta - b}{c}, \frac{\xi}{c}).$$

without losing of generality, take $\eta = b$ and $\xi = c$,

$$\pi(b, c) = \frac{1}{c^2}\pi(0, 1).$$

Let $\pi(0, 1) = 1$, we conclude that

$$\pi(\theta, \sigma) = \frac{1}{\sigma^2} \quad (\sigma > 0).$$

□

Ex. 2.15. *Suppose that X follows the negative binomial distribution:*

$$f(x|p) = C(x - 1, k - 1)p^k(1 - p)^{x-k} \quad (x = k, k + 1, ...).$$

Show that the conjugate prior for p is of the Beta distribution.

Assume that

$$\pi(p|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1 - p)^{\beta-1}.$$

Then the posterior distribution of p is

$$\begin{aligned} \pi(p|x, \alpha, \beta) &= \frac{f(x|p)\pi(p|\alpha, \beta)}{\int_0^1 f(x|p)\pi(p|\alpha, \beta)dp} \\ &= \frac{p^k(1 - p)^{x-k}p^{\alpha-1}(1 - p)^{\beta-1}}{\int_0^1 p^k(1 - p)^{x-k}p^{\alpha-1}(1 - p)^{\beta-1}dp} \\ &= \frac{\Gamma(\alpha + x + \beta)}{\Gamma(k + \alpha)\Gamma(x - k + \beta)}p^{k+\alpha-1}(1 - p)^{x-k+\beta-1}, \end{aligned} \tag{34}$$

indicating that $p|x, \alpha, \beta \sim Beta(\alpha + k, \beta + x - k)$. Since both the prior and the posterior are of Beta distribution, we conclude that Beta distribution family is the conjugate prior for p. □

Ex. 2.16. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} Exp(\theta),$$

and the prior distribution of θ *is* $\Gamma(r, \lambda)$.

(1) If we known that the expectation of the prior is 0.0002, and the variance is 0.0001. Determine the values for hyper-parameters.

(2) Show that $\Gamma(r, \lambda)$ *is the conjugate prior distribution family for* θ.

(1) Consider

$$\pi(\theta|r, \lambda) = \frac{\lambda^r}{\Gamma(r)} \theta^{r-1} e^{-\lambda\theta}.$$

Since $E(\theta) = r/\lambda = 0.0002$, and $\text{Var}(\theta) = r/\lambda^2 = 0.0001$, we have

$$r = 0.0004, \quad \lambda = 2.$$

(2) The likelihood is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} = \theta^n e^{-n\theta\bar{x}}.$$

The posterior distribution of θ is

$$\begin{aligned}
\pi(\theta|\mathbf{x}, r, \lambda) &= \frac{f(\mathbf{x}|\theta)\pi(\theta|r, \lambda)}{\int_0^{+\infty} f(\mathbf{x}|\theta)\pi(\theta|r, \lambda) d\theta} \\
&= \frac{\theta^n e^{-n\theta\bar{x}} \theta^{r-1} e^{-\lambda\theta}}{\int_0^{+\infty} \theta^n e^{-n\theta\bar{x}} \theta^{r-1} e^{-\lambda\theta} d\theta} \\
&= \frac{(n\bar{x} + \lambda)^{n+r-1}}{\Gamma(n + r - 1)} \theta^{n+r-1} e^{-(n\bar{x}+\lambda)\theta},
\end{aligned} \tag{35}$$

indicating that $\theta|\mathbf{x}, r, \lambda \sim \Gamma(n+r-1, n\bar{x}+\lambda)$. Since both the prior and the posterior are of Gamma distribution, we conclude that Gamma distribution family is the conjugate prior for θ. □

Ex. 2.17. *Suppose that* X *follows a distribution from the exponential family, with the density*

$$f(x|\theta) = \exp\{a(\theta)b(x) + c(\theta) + d(x)\}.$$

Show that

$$h(\theta) = A \exp\{k_1 a(\theta) + k_2 c(\theta)\}$$

is the conjugate prior distribution of θ, *where* A *is a constant and* k_1, k_2 *are independent of* θ.

The posterior distribution is

$$\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)h(\theta)}{\int_\Theta f(x|\theta)h(\theta)d\theta} \\
&= \frac{\exp\{a(\theta)b(x) + c(\theta) + d(x)\}A \exp\{k_1 a(\theta) + k_2 c(\theta)\}}{\int_\Theta \exp\{a(\theta)b(x) + c(\theta) + d(x)\}A \exp\{k_1 a(\theta) + k_2 c(\theta)\}d\theta} \\
&= \frac{\exp\{(b(x) + k_1)a(\theta) + (1 + k_2)c(\theta)\}}{\int_\Theta \exp\{(b(x) + k_1)a(\theta) + (1 + k_2)c(\theta)\}d\theta}.
\end{aligned} \tag{36}$$

It we denote

$$A^* = \left[\int_\Theta \exp\{(b(x)+k_1)a(\theta)+(1+k_2)c(\theta)\}d\theta\right]^{-1},$$

then posterior is of the form

$$A^* \exp\{k_1^* a(\theta) + k_2^* c(\theta)\}$$

where $A^*,\ k_1^*,\ k_2^*$ are all independent of θ. □

Ex. 2.18. *Suppose that X follows the distribution*

$$f(x|\lambda) = \begin{cases} \lambda^{-1}e^{-x/\lambda}, & 0 < x < \infty, \\ 0, & t \le 0. \end{cases}$$

Show that the conjugate prior distribution for λ is the Inverse Gamma family.

If the prior for λ is $IG(\alpha,\beta)$:

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{-\alpha-1}e^{-\beta/\lambda},$$

then the posterior is:

$$\begin{aligned} \pi(x|\lambda) &= \frac{f(x|\lambda)\pi(\lambda)}{\int_0^{+\infty} f(x|\lambda)\pi(\lambda)d\lambda} \\ &= \frac{\lambda^{-1}e^{-x/\lambda}\lambda^{-\alpha-1}e^{-\beta/\lambda}}{\int_0^{+\infty}\lambda^{-1}e^{-x/\lambda}\lambda^{-\alpha-1}e^{-\beta/\lambda}d\lambda} \\ &= \frac{(x+\beta)^{\alpha+1}}{\Gamma(\alpha+1)}\lambda^{-\alpha-2}e^{-(x+\beta)/\lambda}, \end{aligned} \tag{37}$$

which is also a member in the Inverse Gamma family. □

Ex. 2.19. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} U(0,\theta),$$

and the prior distribution of θ is the Pareto:

$$\pi(\theta) = \begin{cases} \alpha\theta_0^\alpha/\theta^{\alpha+1}, & \theta > \theta_0, \\ 0, & \theta \le \theta_0, \end{cases}$$

where $\theta_0 > 0,\ \alpha > 0$. Show that the Pareto distribution is the conjugate prior for θ.

The posterior is

$$\begin{aligned} \pi(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_\Theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{\prod_{i=1}^n \left[\frac{1}{\theta}I(0 \le x_i \le \theta)\right]\alpha\theta_0^\alpha/\theta^{\alpha+1}I(\theta>\theta_0)}{\int_{\Theta_0}\prod_{i=1}^n \left[\frac{1}{\theta}I(0 \le x_i \le \theta)\right]\alpha\theta_0^\alpha/\theta^{\alpha+1}I(\theta>\theta_0)d\theta} \\ &= \begin{cases} \frac{\alpha\theta_0^\alpha/\theta^{n+\alpha+1}}{\int_K^{+\infty}\alpha\theta_0^\alpha/\theta^{n+\alpha+1}d\theta}, & K \le \theta, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} (n+\alpha)\theta_0^{n+\alpha}/\theta^{n+\alpha+1}, & K \le \theta, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{38}$$

where $K = \max\{\theta_0, x_1, ..., x_n\}$. Thus, the posterior is also of the form of the Pareto distribution. □

Ex. 2.20. *Suppose that $X \sim N(\theta, 1)$, where $\theta > 0$ and the prior expectation of θ is μ. Show that the maximum entropy prior is $Exp(1/\mu)$.*

The constraint is

$$E^{\pi}(\theta) = \int_0^{+\infty} \theta\pi(\theta)d\theta = \mu.$$

According to the formula, the maximum entropy prior is

$$\tilde{\pi}(\theta) = \frac{\exp\{\lambda\theta\}}{\int_0^{+\infty}\exp\{\lambda\theta\}} = A^* \exp\{\lambda\theta\},$$

which is of the form of the exponential distribution $Exp(-\lambda)$. Therefore,

$$E^{\tilde{\pi}}(\theta) = -\frac{1}{\lambda}.$$

Let $\pi(\theta) = \tilde{\pi}(\theta)$, we have $\lambda = -1/\mu$, and the maximum entropy prior is $Exp(1/\mu)$. □

Ex. 2.21. *Suppose that θ is a scale parameter (and thus $\pi_0(\theta) = \theta^{-1}$). Assume that $\theta \in (a, b)$, and the median of the prior $\pi(\theta)$ is z:*

$$\int_a^z \pi(\theta)d\theta = \int_z^b \pi(\theta)d\theta = \frac{1}{2}.$$

Show that the maximum entropy prior is

$$\pi(\theta) = \begin{cases} \frac{1}{\theta}(2\log\frac{z}{a})^{-1}, & 0 < a < \theta < z, \\ \frac{1}{\theta}(2\log\frac{b}{z})^{-1}, & z < \theta < b. \end{cases}$$

The constraints are

$$\int_a^b I(\theta < z)\pi(\theta)d\theta = \frac{1}{2}, \quad \int_a^b I(\theta > z)\pi(\theta)d\theta = \frac{1}{2}.$$

According to the formula, the maximum entropy prior is

$$\begin{aligned} \tilde{\pi}(\theta) &= \frac{\frac{1}{\theta}\exp\{\lambda_1 I(\theta < z) + \lambda_2 I(\theta > z)\}}{\int_a^b \frac{1}{\theta}\exp\{\lambda_1 I(\theta < z) + \lambda_2 I(\theta > z)\}d\theta} \\ &= A^*\frac{1}{\theta}\exp\{\lambda_1 I(\theta < z) + \lambda_2 I(\theta > z)\} \\ &= \begin{cases} A^*/\theta \exp\{\lambda_1\}, & 0 < a < \theta < z, \\ A^*/\theta \exp\{\lambda_2\}, & z < \theta < b, \end{cases} \end{aligned} \tag{39}$$

where

$$A^* = \left[\int_a^b \frac{1}{\theta}\exp\{\lambda_1 I(\theta < z) + \lambda_2 I(\theta > z)\}d\theta\right]^{-1}.$$

Now let $\pi(\theta) = \tilde{\pi}(\theta)$:

$$\int_a^b I(\theta < z)\tilde{\pi}(\theta)d\theta = \int_a^z \frac{A^*}{\theta}\exp\{\lambda_1\} = A^*\exp\{\lambda_1\}\log\frac{z}{a} = \frac{1}{2},$$
$$\int_a^b I(\theta > z)\tilde{\pi}(\theta)d\theta = \int_z^b \frac{A^*}{\theta}\exp\{\lambda_2\} = A^*\exp\{\lambda_2\}\log\frac{b}{z} = \frac{1}{2},$$

then we can solve for the unknowns:

$$A^*\lambda_1 = \frac{1}{2}\log\frac{a}{z}, \quad A^*\lambda_2 = \frac{1}{2}\log\frac{z}{b}.$$

As a result, the maximum entropy prior is

$$\pi(\theta) = \begin{cases} \frac{1}{\theta}(2\log\frac{z}{a})^{-1}, & 0 < a < \theta < z, \\ \frac{1}{\theta}(2\log\frac{b}{z})^{-1}, & z < \theta < b. \end{cases}$$

□

Ex. 2.22. *Suppose that X_1 and X_2 are independent, and they follow the exponential distributions with expectations of μ_1 and μ_2, respectively. Assume that the interested parameter is $\phi_1 = \mu_2/\mu_1$, while $\phi_2 = \mu_2 \cdot \mu_1$ is the nuisance parameter. Show that the Reference prior for (ϕ_1, ϕ_2) is $\pi(\phi_1, \phi_2) = (\phi_1\phi_2)^{-1}$.*

Before we start to find the reference prior, let us find the Fisher information matrix first. Since

$$X_1 \sim \frac{1}{\mu_1}e^{-\frac{x_1}{\mu_1}}, \quad X_2 \sim \frac{1}{\mu_2}e^{-\frac{x_2}{\mu_2}},$$

the log-likelihood is

$$\begin{aligned} \ell &= \log f(x_1) + \log f(x_2) \\ &= -\log\mu_1 - \frac{x_1}{\mu_1} - \log\mu_2 - \frac{x_2}{\mu_2} \\ &= \log\phi_2 - x_1\sqrt{\frac{\phi_1}{\phi_2}} - x_2\sqrt{\frac{1}{\phi_1\phi_2}}. \end{aligned} \tag{40}$$

Consequently we can find the information matrix

$$\mathbf{I}(\phi_1, \phi_2) = \begin{bmatrix} 1/(2\phi_1^2) & 0 \\ 0 & 1/(2\phi_2^2). \end{bmatrix}$$

Step 1: The conditional Reference prior:

$$\pi(\phi_2|\phi_1) = \sqrt{\frac{1}{2\phi_2^2}} \propto \frac{1}{\phi_2}.$$

Step 2: Take the monotone-increasing subset $\Omega_i = L_i \times S_i$ from the parameter space $\Omega = \mathbb{R}^+ \times \mathbb{R}^+$, where $L_i = [l_{i1}, l_{i2}]$, $S_i = [s_{i1}, s_{i2}]$, such that $L_1 \subset L_2 \subset ...$, $S_1 \subset S_2 \subset ...$, $\bigcup_{i=1}^\infty L_i = \mathbb{R}^+$, and $\bigcup_{i=1}^\infty S_i = \mathbb{R}^+$. Let $\Omega_{i,\phi_1} = \{\phi_2 : (\phi_1, \phi_2) \in \Phi_i\} = S_i$, then

$$K_i(\phi_1) = \left[\int_{S_i} \pi(\phi_2|\phi_1)d\phi_2\right]^{-1} = \left[\int_{s_{i1}}^{s_{i2}} \frac{1}{\phi_2}d\phi_2\right]^{-1} = \frac{1}{\log s_{i2} - \log s_{i1}},$$

$$\pi_i(\phi_2|\phi_1) = K_i(\phi_1) \cdot \pi(\phi_2|\phi_1) \cdot I_{S_i}(\phi_2) = \frac{1}{(\log s_{i2} - \log s_{i1})\phi_2} \ (s_{i1} \le \phi_2 \le s_{i2}).$$

Step 3: The marginal Reference prior:

$$\begin{aligned}
\pi_i(\phi_1) &= \exp\left\{\frac{1}{2}\int_{S_i} \pi_i(\phi_2|\phi_1) \log \frac{|\mathbf{I}(\phi_1,\phi_2)|}{|I_{22}(\phi_1,\phi_2)|} d\phi_2\right\} \\
&= \exp\left\{\frac{1}{2}\int_{s_{i1}}^{s_{i2}} \frac{-\log 2\phi_1^2}{(\log s_{i2} - \log s_{i1})\phi_2} d\phi_2\right\} \\
&= \exp\left\{-\frac{1}{2}\log 2\phi_1^2\right\} \\
&= \frac{1}{\sqrt{2}\phi_1}.
\end{aligned} \tag{41}$$

Step 4: The limit:

$$\pi(\phi_1,\phi_2) = \lim_{i\to\infty} \frac{K_i(\phi_1)\pi_i(\phi_1)}{K_i(\phi_{10})\pi_i(\phi_{10})}\pi(\phi_2|\phi_1) \propto \pi_i(\phi_1)\pi(\phi_2|\phi_1) \propto (\phi_1\phi_2)^{-1}.$$

□

Ex. 2.23. *Suppose that*

$$X_i \sim^{\text{i.i.d.}} N(\theta_i, 900) \quad (i = 1, ..., p).$$

We believe that θ_i's are similar, and again assume that they are i.i.d samples form a distribution, and the mean of those θ_i's is about 100, while the standard deviation of the guess of the mean is about 20. The variance of θ_i's is an unknown constant, so it's set to be a non-informative constant prior. Find a reasonable multi-level prior model for the description above.

$$\begin{gathered}
X_i \sim^{\text{i.i.d.}} N(\theta_i, 900) \quad (i = 1, ..., p), \\
\theta_i \sim^{\text{i.i.d.}} N(k_i, C), \\
k_i \sim^{\text{i.i.d.}} N(100, 20^2).
\end{gathered}$$

□

3 Bayesian Statistical Inference

Ex. 3.1. *Suppose that* $X \sim B(n, \theta)$.

(1) If the prior is $\pi(\theta) = [\theta(1-\theta)]^{-1} I(0 < \theta < 1)$, *find the posterior of* θ *given* x, *when* $1 \le x \le n-1$.

(2) If $\pi(\theta) = I(0 < \theta < 1)$, *find the posterior of* θ *given* x.

(1) The posterior is

$$\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} \\
&= \frac{C(n,x)\theta^x(1-\theta)^{n-x}\theta^{-1}(1-\theta)^{-1}}{\int_0^1 C(n,x)\theta^x(1-\theta)^{n-x}\theta^{-1}(1-\theta)^{-1}d\theta} \\
&= \frac{\Gamma(n)}{\Gamma(x)\Gamma(n-x)}\theta^{x-1}(1-\theta)^{n-x-1}, \quad 1 \le x \le n-1.
\end{aligned} \tag{42}$$

(2) The posterior is

$$\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} \\
&= \frac{C(n,x)\theta^x(1-\theta)^{n-x}}{\int_0^1 C(n,x)\theta^x(1-\theta)^{n-x}d\theta} \\
&= \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^{x}(1-\theta)^{n-x}, \quad 0 < \theta < 1.
\end{aligned} \tag{43}$$

□

Ex. 3.2. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} P(\theta),$$

and the prior for θ *is* $\pi(\theta) = \theta^{-1} I(\theta > 0)$. *Find the posterior of* θ *given* $\boldsymbol{x}$, *when* $\boldsymbol{x} \neq (0, ..., 0)$.

The likelihood is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{n\bar{x}}}{\prod_{i=1}^n x_i!}.$$

Then the posterior is

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_0^{+\infty} f(\mathbf{x}|\theta)\pi(\theta)d\theta} \\
&= \frac{e^{-n\theta}\theta^{n\bar{x}}\theta^{-1}}{\int_0^{+\infty} e^{-n\theta}\theta^{n\bar{x}}\theta^{-1}d\theta} \\
&= \frac{n^{n\bar{x}}}{\Gamma(n\bar{x})}\theta^{n\bar{x}-1}e^{-n\theta}, \quad \theta > 0.
\end{aligned} \tag{44}$$

□

Ex. 3.3. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(\theta, \sigma^2)$$

where θ and σ^2 are unknown. Assume that the joint prior for (θ, σ^2) is

$$\pi(\theta, \sigma^2) = \sigma^{-2} I(\sigma^2 > 0).$$

Show that

(1) The posterior of (θ, σ^2) is

$$\pi(\theta, \sigma^2 | \boldsymbol{x}) = \pi_1(\theta | \sigma^2, \boldsymbol{x}) \pi_2(\sigma^2 | \boldsymbol{x}),$$

where $\pi_1(\theta|\sigma^2, \boldsymbol{x})$ is $N(\bar{x}, \sigma^2/n)$, and $\pi_2(\sigma|\boldsymbol{x})$ is of the Inverse Gamma distribution

$$\Gamma^{-1}\left(\frac{n-1}{2}, \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right).$$

(2) The marginal posterior of σ^2 is $\Gamma^{-1}\left(\frac{n-1}{2}, \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$ given $\boldsymbol{x}$.

(3) The marginal posterior of θ is

$$\mathcal{T}\left(n-1, \bar{x}, \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n(n-1)}\right),$$

given $\boldsymbol{x}$.

(1) Since $(\bar{X}, S^2)$ is the sufficient statistic of (θ, σ^2), where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\theta, \frac{\sigma^2}{n}), \quad S^2 = \frac{1}{\nu}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad \nu = n-1, \quad \frac{\nu S^2}{\sigma^2} \sim \chi^2_\nu,$$

the likelihood is

$$\begin{aligned} f(t|\theta, \sigma^2) &= f_1(\bar{x}|\theta, \sigma^2) f_2(s^2|\sigma^2) \\ &= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{-\frac{n}{2\sigma^2}(\bar{x} - \theta)^2\right\} \frac{\left(\frac{v}{2\sigma^2}\right)^{v/2}}{\Gamma(v/2)} (s^2)^{v/2-1} e^{-vs^2/2\sigma^2} \\ &\propto \sigma^{-\nu-1} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x} - \theta)^2\right]\right\}. \end{aligned} \tag{45}$$

Then the posterior is

$$\begin{aligned} \pi(\theta, \sigma^2 | t) &\propto f(t|\theta, \sigma^2) \pi(\theta, \sigma^2) \\ &\propto \sigma^{-\nu-3} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x} - \theta)^2\right]\right\}. \end{aligned} \tag{46}$$

In order to normalize the posterior, consider

$$\begin{aligned} K &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \sigma^{-\nu-3} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x} - \theta)^2\right]\right\} d\theta d\sigma^2 \\ &= \int_0^{+\infty} \sigma^{-\nu-3} \exp\left\{-\frac{\nu s^2}{2\sigma^2}\right\} \sqrt{\frac{2\pi\sigma^2}{n}} d\sigma^2 \\ &= \sqrt{\frac{2\pi}{n}} \frac{\Gamma(\nu/2)}{(vs^2/2)^{v/2}}, \end{aligned} \tag{47}$$

and therefore the posterior is

$$\pi(\theta, \sigma^2|t) = K^{-1}\sigma^{-\nu-3} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x}-\theta)^2\right]\right\}.$$

Of course $\pi(\theta, \sigma^2|\mathbf{x})$ can be written as $\pi_1(\theta|\sigma^2, \mathbf{x})\pi_2(\sigma^2|\mathbf{x})$. It is easy to find $\pi_1(\theta|\sigma^2, \mathbf{x})$ when the prior $\pi_1^*(\theta) \equiv 1$ and the σ^2 is known:

$$\begin{aligned}
\pi_1(\theta|\sigma^2, \bar{x}) &= \frac{f(\bar{x}|\theta, \sigma^2)\pi_1^*(\theta)}{\int_\Theta f(\bar{x}|\theta, \sigma^2)\pi_1^*(\theta)d\theta} \\
&= \frac{\sqrt{n/(2\pi\sigma^2)}\exp\left\{-n(\bar{x}-\theta)^2/(2\sigma^2)\right\}\cdot 1}{\int_{-\infty}^{+\infty}\sqrt{n/(2\pi\sigma^2)}\exp\left\{-n(\bar{x}-\theta)^2/(2\sigma^2)\right\}\cdot 1 d\theta} \\
&= \sqrt{\frac{n}{2\pi\sigma^2}}\exp\left\{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2\right\},
\end{aligned} \tag{48}$$

indicating that $\pi_1(\theta|\sigma^2, \mathbf{x})$ is $N(\bar{x}, \sigma^2/n)$. Consequently,

$$\begin{aligned}
\pi_2(\sigma^2|\mathbf{x}) &= \frac{\pi(\theta, \sigma^2|t)}{\pi_1(\theta|\sigma^2, \bar{x})} \\
&= \frac{(\nu s^2/2)^{\nu/2}}{\Gamma(\nu/2)}(\sigma^2)^{-(\nu/2+1)}\exp\left\{-\frac{v s^2}{2\sigma^2}\right\},
\end{aligned} \tag{49}$$

which is exactly $\Gamma^{-1}(\nu/2, vs^2/2)$. Plug the expressions of ν and s^2 in, we have

$$\Gamma^{-1}\left(\frac{n-1}{2}, \frac{1}{2}\sum_{i=1}^n (x_i-\bar{x})^2\right).$$

(2) The marginal posterior of σ^2 given $\mathbf{x}$ is

$$\begin{aligned}
\pi(\sigma^2|\mathbf{x}) &= \int_{-\infty}^{+\infty}\pi(\theta, \sigma^2|t)d\theta \\
&= \int_{-\infty}^{+\infty} K^{-1}\sigma^{-\nu-3}\exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x}-\theta)^2\right]\right\}d\theta \\
&= K^{-1}\sigma^{-\nu-3}\exp\left\{-\frac{\nu s^2}{2\sigma^2}\right\}\sqrt{\frac{2\pi\sigma^2}{n}} \\
&= \frac{(\nu s^2/2)^{\nu/2}}{\Gamma(\nu/2)}(\sigma^2)^{-(\nu/2+1)}\exp\left\{-\frac{v s^2}{2\sigma^2}\right\},
\end{aligned} \tag{50}$$

which is $\Gamma^{-1}\left(\frac{n-1}{2}, \frac{1}{2}\sum_{i=1}^n (x_i-\bar{x})^2\right)$ given $\mathbf{x}$.

(3) The marginal posterior of θ given $\mathbf{x}$ is

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \int_0^{+\infty}\pi(\theta, \sigma^2|t)d\sigma^2 \\
&= \int_0^{+\infty} K^{-1}\sigma^{-\nu-3}\exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x}-\theta)^2\right]\right\}d\sigma^2 \\
&= K^{-1}\Gamma(\frac{\nu+1}{2})\left\{\frac{\nu s^2 + n(\bar{x}-\theta)^2}{2}\right\}^{-\frac{\nu+1}{2}} \\
&= \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\frac{1}{\sqrt{\nu\pi}}\frac{\sqrt{n(n-1)}}{\sqrt{\nu s^2}}\left\{1 + \frac{1}{\nu}\frac{(\theta-\bar{x})^2 n(n-1)}{\nu s^2}\right\}^{-\frac{\nu+1}{2}},
\end{aligned} \tag{51}$$

which is exactly

$$\mathcal{T}\left(n-1, \bar{x}, \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n(n-1)}\right).$$

□

Ex. 3.4. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(\theta, 2^2),$$

and the prior for θ is of Gaussian.

(1) If $n=100$, show that the posterior standard deviation must be less than 1/5 regardless of the prior standard deviation.

(2) If the standard deviation of the prior for θ is 1, find the minimum sample size to guarantee that the posterior variance is less than or equal to 0.1.

It is acknowledged that the posterior is $N(u(\mathbf{x}), \eta^2)$, where

$$u(\mathbf{x}) = \frac{4/n}{4/n+\tau^2}\mu + \frac{\tau^2}{4/n+\tau^2}\bar{x}, \quad \eta^2 = \frac{4\tau^2}{4+n\tau^2}$$

if the prior is $N(\mu, \tau^2)$.

(1) If $n=100$, then

$$\eta^2 = \frac{4\tau^2}{4+n\tau^2} = \frac{4\tau^2}{4+100\tau^2} = \frac{1}{1/\tau^2+25} \leq \frac{1}{25},$$

so $\eta \leq 1/5$ for all τ.

(2) If $\tau = 1$, and let

$$\eta^2 = \frac{4\tau^2}{4+n\tau^2} = \frac{4}{4+n} \leq 0.1,$$

then $n \geq 36$. □

Ex. 3.5. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(\theta_1, \sigma^2).$$

Let $\theta_2 = 1/(2\sigma^2)$, and (θ_1, θ_2) enjoys the following assumptions:

(1) $\theta_1|\theta_2 \sim N(0, 1/(2\theta_2))$;

(2) $\theta_2 \sim \Gamma(\alpha, \lambda)$, where α and λ are known.

Find the joint posterior $\pi(\theta_1, \theta_2|\boldsymbol{x})$.

Since $(\bar{X}, S^2)$ is the sufficient statistic of (θ_1, σ^2), where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\theta_1, \frac{\sigma^2}{n}), \quad S^2 = \frac{1}{\nu}\sum_{i=1}^{n}(X_i-\bar{X})^2, \quad \nu = n-1, \quad \frac{\nu S^2}{\sigma^2} \sim \chi^2_\nu,$$

the likelihood is

$$\begin{aligned}
f(t|\theta_1, \sigma^2) &= f_1(\bar{x}|\theta_1, \sigma^2) f_2(s^2|\sigma^2) \\
&= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\theta_1)^2\right\} \frac{\left(\frac{v}{2\sigma^2}\right)^{v/2}}{\Gamma(v/2)} (s^2)^{v/2-1} e^{-vs^2/2\sigma^2} \\
&\propto \sigma^{-\nu-1} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x}-\theta_1)^2\right]\right\} \\
&\propto \theta_2^{(\nu+1)/2} \exp\left\{-\theta_2\left[\nu s^2 + n(\bar{x}-\theta_1)^2\right]\right\} \\
&\propto f(t|\theta_1, \theta_2).
\end{aligned} \tag{52}$$

Consequently, the posterior is

$$\begin{aligned}\pi(\theta_1,\theta_2|\mathbf{x}) &= \frac{f(t|\theta_1,\theta_2)\pi(\theta_1|\theta_2)\pi(\theta_2)}{\int_{\Theta_2}\int_{\Theta_1} f(t|\theta_1,\theta_2)\pi(\theta_1|\theta_2)\pi(\theta_2)d\theta_1 d\theta_2} \\ &= \frac{\theta_2^{(\nu+1)/2+1/2+\alpha-1}\exp\left\{-\theta_2\nu s^2-\lambda\theta_2\right\}\exp\left\{-\theta_2 n(\bar{x}-\theta_1)^2-\theta_2\theta_1^2\right\}}{\int_0^{+\infty}\int_{-\infty}^{+\infty}\theta_2^{(\nu+1)/2+1/2+\alpha-1}\exp\left\{-\theta_2\nu s^2-\lambda\theta_2\right\}\exp\left\{-\theta_2 n(\bar{x}-\theta_1)^2-\theta_2\theta_1^2\right\}d\theta_1 d\theta_2}.\end{aligned} \tag{53}$$

Since

$$\begin{aligned}&\int_{-\infty}^{+\infty}\exp\left\{-\theta_2 n(\bar{x}-\theta_1)^2-\theta_2\theta_1^2\right\}d\theta_1 \\ =&\int_{-\infty}^{+\infty}\exp\left\{-(n+1)\theta_2\left[\theta_1-\left(\frac{n\bar{x}}{n+1}\right)\right]^2\right\}\exp\left\{\left[\frac{(n\bar{x})^2}{n+1}-n\bar{x}^2\right]\theta_2\right\}d\theta_1 \\ =&\exp\left\{\left[\frac{(n\bar{x})^2}{n+1}-n\bar{x}^2\right]\theta_2\right\}\sqrt{\frac{\pi}{n+1}}\theta_2^{-1/2},\end{aligned} \tag{54}$$

the denominator of $\pi(\theta_1,\theta_2|\mathbf{x})$ is

$$\begin{aligned}K &= \sqrt{\frac{\pi}{n+1}}\int_0^{+\infty}\theta_2^{(\nu+1)/2+\alpha-1}\exp\left\{-\left[\nu s^2+\lambda+\frac{(n\bar{x})^2}{n+1}-n\bar{x}^2\right]\theta_2\right\}d\theta_2 \\ &= \sqrt{\frac{\pi}{n+1}}\frac{\left[\nu s^2+\lambda+\frac{(n\bar{x})^2}{n+1}-n\bar{x}^2\right]^{(\nu+1)/2+\alpha}}{\Gamma((\nu+1)/2+\alpha)}.\end{aligned} \tag{55}$$

Therefore, the joint posterior is

$$\pi(\theta_1,\theta_2|\mathbf{x}) = K^{-1}\theta_2^{(\nu+1)/2+1/2+\alpha-1}\exp\left\{-\left[\nu s^2+\lambda+n(\bar{x}-\theta_1)^2+\theta_1^2\right]\theta_2\right\}.$$

□

Ex. 3.6. *Suppose that*

$$\boldsymbol{X} = (X_1,...,X_n) \sim^{\text{i.i.d.}} Exp(1/\theta),$$

and the prior for θ is $\Gamma^{-1}(\alpha,\beta)$. Show that the posterior for θ is

$$\Gamma^{-1}\left(n+\alpha,\sum_{i=1}^n x_i+\beta\right),$$

given $\boldsymbol{x}$.

Since the likelihood is

$$f(x_1,...,x_n|\theta) = \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta} = \frac{1}{\theta^n}e^{-\sum_{i=1}^n x_i/\theta},$$

the posterior for θ is

$$\begin{aligned}\pi(\theta|\mathbf{x}) =& \frac{f(x_1,...,x_n|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x_1,...,x_n|\theta)\pi(\theta)d\theta} \\ =& \frac{\theta^{-n}\exp\left\{-\theta^{-1}\sum_{i=1}^n x_i\right\}\theta^{-(\alpha+1)}\exp\left\{\theta^{-1}\beta\right\}}{\int_{\mathbb{R}^+}\theta^{-n}\exp\left\{-\theta^{-1}\sum_{i=1}^n x_i\right\}\theta^{-(\alpha+1)}\exp\left\{\theta^{-1}\beta\right\}d\theta} \\ =& \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{\Gamma(\alpha+n)}\theta^{-(\alpha+n+1)}\exp\left\{\theta^{-1}\left(\sum_{i=1}^n x_i+\beta\right)\right\}.\end{aligned} \tag{56}$$

□

Ex. 3.7. *Suppose that*

$$\boldsymbol{X} = (X_1, ..., X_n) \sim^{\text{i.i.d.}} U(0, \theta),$$

and the prior for θ is $Pa(\theta_0, \alpha)$. Show that the posterior for θ is

$$Pa(\max\{\theta_0, x_1, ..., x_n\}, n + \alpha),$$

given $\boldsymbol{x}$.

The posterior is

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_\Theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\
&= \frac{\prod_{i=1}^n \left[\frac{1}{\theta} I(0 \le x_i \le \theta)\right] \alpha\theta_0^\alpha/\theta^{\alpha+1} I(\theta > \theta_0)}{\int_{\Theta_0} \prod_{i=1}^n \left[\frac{1}{\theta} I(0 \le x_i \le \theta)\right] \alpha\theta_0^\alpha/\theta^{\alpha+1} I(\theta > \theta_0)d\theta} \\
&= \begin{cases} \frac{\alpha\theta_0^\alpha/\theta^{n+\alpha+1}}{\int_K^{+\infty} \alpha\theta_0^\alpha/\theta^{n+\alpha+1}d\theta}, & K \le \theta, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} (n+\alpha)K^{n+\alpha}/\theta^{n+\alpha+1}, & K \le \theta, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{57}$$

where $K = \max\{\theta_0, x_1, ..., x_n\}$. Thus, the posterior is exactly $Pa(\max\{\theta_0, x_1, ..., x_n\}, n+\alpha)$. □

Ex. 3.8. *Suppose that $X \sim \Gamma(n/2, 1/(2\theta))$, and the prior for θ is $\Gamma^{-1}(\alpha, \beta/2)$. Show that the posterior for θ is $\Gamma^{-1}(n/2 + \alpha, x/2 + \beta/2)$ given $\boldsymbol{x}$.*

The posterior is

$$\begin{aligned}
\pi(\theta|x) =& \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x|\theta)\pi(\theta)d\theta} \\
=& \frac{(2\theta)^{-n/2}\exp\{-x/(2\theta)\}\theta^{-(\alpha+1)}\exp\{-\beta/(2\theta)\}}{\int_{\mathbb{R}^+}(2\theta)^{-n/2}\exp\{-x/(2\theta)\}\theta^{-(\alpha+1)}\exp\{-\beta/(2\theta)\}d\theta} \\
=& \frac{(x/2+\beta/2)^{n/2+\alpha}}{\Gamma(n/2+\alpha)}\theta^{-(n/2+\alpha+1)}\exp\left\{-\frac{x+\beta}{2\theta}\right\},
\end{aligned} \tag{58}$$

which is exactly $\Gamma^{-1}(n/2 + \alpha, x/2 + \beta/2)$. □

Ex. 3.9. *Suppose that*

$$\boldsymbol{X} = (X_1, ..., X_n) \sim^{\text{i.i.d.}} Nb(r, \theta),$$

and the prior for θ is $Be(\alpha, \beta)$. Show that the posterior for θ is $Be(\alpha+rn, \sum_{i=1}^n x_i - nr + \beta)$, given $\boldsymbol{x}$.

The likelihood is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n [C(x_i - 1, r - 1)\theta^r(1-\theta)^{x_i - r}] = \prod_{i=1}^n C(x_i - 1, r - 1)\theta^{rn}(1-\theta)^{\sum_{i=1}^n x_i - rn}. \tag{59}$$

Consequently, the posterior is

$$
\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_0^1 f(\mathbf{x}|\theta)\pi(\theta)d\theta} \\
&= \frac{\theta^{rn}(1-\theta)^{\sum_{i=1}^n x_i - rn}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 \theta^{rn}(1-\theta)^{\sum_{i=1}^n x_i - rn}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta} \\
&= \frac{\Gamma(\sum_{i=1}^n x_i + \alpha + \beta)}{\Gamma(\alpha+rn)\Gamma(\sum_{i=1}^n x_i - nr + \beta)}\theta^{\alpha+rn-1}(1-\theta)^{\sum_{i=1}^n x_i - nr + \beta - 1},
\end{aligned} \tag{60}
$$

which is exactly $Be(\alpha + rn, \sum_{i=1}^n x_i - nr + \beta)$. □

Ex. 3.10. *Suppose that $\boldsymbol{X} = (X_1, ..., X_p)$ follows a multivariate Gaussian $N_p(\theta, \Sigma)$, and the prior of θ is $N_P(\mu, A)$. Show that the posterior of θ given $\boldsymbol{x}$ is another p-Gaussian with the mean*

$$x - \Sigma(\Sigma + A)^{-1}(x - \mu)$$

and the variance

$$(\Sigma^{-1} + A^{-1})^{-1}.$$

The prior is

$$\pi(\theta) = \frac{1}{(2\pi)^{p/2}|A|^{1/2}} \exp\left\{-\frac{1}{2}(\theta-\mu)^T A^{-1}(\theta-\mu)\right\},$$

and the likelihood is

$$f(x|\theta) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\theta)^T \Sigma^{-1}(x-\theta)\right\}.$$

Therefore, the posterior is

$$
\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}} f(x|\theta)\pi(\theta)d\theta} \\
&\propto \exp\left\{-\frac{1}{2}(\theta-\mu)^T A^{-1}(\theta-\mu) - \frac{1}{2}(x-\theta)^T\Sigma^{-1}(x-\theta)\right\}.
\end{aligned} \tag{61}
$$

Let

$$
\begin{aligned}
N^* &= (\theta-\mu)^T A^{-1}(\theta-\mu) + (x-\theta)^T\Sigma^{-1}(x-\theta) \\
&= \theta^T(A^{-1} + \Sigma^{-1})\theta - 2(\mu^T A^{-1} + x^T\Sigma^{-1})\theta + Const_1 \\
&= \left[\theta^T - (\mu^T A^{-1} + x^T\Sigma^{-1})B^{-1}\right] B \left[\theta - B^{-1}(A^{-1}\mu + \Sigma^{-1}x)\right] + Const_2,
\end{aligned} \tag{62}
$$

where $B = A^{-1} + \Sigma^{-1}$, and $Const_1$, $Const_2$ are not functions of θ. Plug N^* in $\pi(\theta|x)$ and normalize it, we can conclude that the variance of the posterior is (by Sherman-Morrison-Woodbury)

$$B^{-1} = (A^{-1} + \Sigma^{-1})^{-1} = \Sigma - \Sigma(A+\Sigma)^{-1}\Sigma,$$

and the mean is

$$
\begin{aligned}
B^{-1}(A^{-1}\mu + \Sigma^{-1}x) &= (\Sigma - \Sigma(A+\Sigma)^{-1}\Sigma)(A^{-1}\mu + \Sigma^{-1}x) \\
&= \Sigma A^{-1}\mu + x - \Sigma(A+\Sigma)^{-1}\Sigma A^{-1}\mu - \Sigma(A+\Sigma)^{-1}x \\
&= \Sigma(\Sigma+A)^{-1}\mu + x - \Sigma(A+\Sigma)^{-1}x \\
&= x + \Sigma(\Sigma+A)^{-1}(\mu - x).
\end{aligned} \tag{63}
$$

The second to last equation holds because

$$A^{-1} - (A+\Sigma)^{-1}\Sigma A^{-1} = (\Sigma + A)^{-1}.$$

□

Ex. 3.11. *Suppose that*

$$\boldsymbol{X} = (X_1, ..., X_n) \sim^{\text{i.i.d.}} N(\theta, \sigma^2),$$

and the joint prior of (θ, σ^2) *is*

$$\pi(\theta, \sigma^2) = \pi_1(\theta|\sigma^2)\pi_2(\sigma^2),$$

where $\pi_1(\theta|\sigma^2) = N(\mu, \tau\sigma^2)$, *and* $\pi_2(\sigma^2) = \Gamma^{-1}(\alpha, \beta)$. *Show that*

(1) the joint posterior of (θ, σ^2) *given* $\boldsymbol{x}$ *is*

$$\pi(\theta, \sigma^2|\boldsymbol{x}) = \pi_1(\theta|\sigma^2, \boldsymbol{x})\pi_2(\sigma^2|\boldsymbol{x}),$$

where $\pi_1(\theta|\sigma^2, \boldsymbol{x}) = N(\mu(\boldsymbol{x}), \eta^2)$,

$$\mu(\boldsymbol{x}) = \frac{\mu + n\tau\bar{x}}{n\tau + 1}, \quad \eta^2 = \frac{\sigma^2}{n + \tau^{-1}}, \quad \bar{x} = \sum_{i=1}^n x_i,$$

and $\pi_2(\sigma^2|\boldsymbol{x})$ *is* $\Gamma^{-1}(\alpha + n/2, \tilde{\beta})$,

$$\tilde{\beta} = \left[\beta + \frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^2 + \frac{n(\bar{x} - \mu)^2}{2(1 + n\tau)}\right].$$

(2) The marginal posterior for σ^2 *given* $\boldsymbol{x}$ *is* $\Gamma^{-1}(\alpha + n/2, \tilde{\beta})$.

(3) The marginal posterior for θ *given* $\boldsymbol{x}$ *is*

$$\mathcal{T}\left(2\alpha + n, \mu(\boldsymbol{x}), \frac{\tilde{\beta}}{(\tau^{-1} + n)(\alpha + n/2)}\right).$$

(1) Since $(\bar{X}, S^2)$ is the sufficient statistic of (θ_1, σ^2), where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim N(\theta_1, \frac{\sigma^2}{n}), \quad S^2 = \frac{1}{\nu}\sum_{i=1}^n (X_i - \bar{X})^2, \quad \nu = n - 1, \quad \frac{\nu S^2}{\sigma^2} \sim \chi^2_\nu,$$

the likelihood is

$$\begin{aligned} f(t|\theta, \sigma^2) &= f_1(\bar{x}|\theta, \sigma^2) f_2(s^2|\sigma^2) \\ &= \sqrt{\frac{n}{2\pi\sigma^2}} \exp\left\{-\frac{n}{2\sigma^2}(\bar{x} - \theta)^2\right\} \frac{\left(\frac{v}{2\sigma^2}\right)^{v/2}}{\Gamma(v/2)} (s^2)^{v/2-1} e^{-vs^2/2\sigma^2} \\ &\propto \sigma^{-\nu-1} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu s^2 + n(\bar{x} - \theta)^2\right]\right\}. \end{aligned} \tag{64}$$

Therefore, the posterior is

$$\begin{aligned} \pi(\theta, \sigma^2|\mathbf{x}) &= \frac{f(t|\theta, \sigma^2)\pi_1(\theta|\sigma^2)\pi_2(\sigma^2)}{\int_{\mathbb{R}^+}\int_{\mathbb{R}} f(t|\theta, \sigma^2)\pi_1(\theta|\sigma^2)\pi_2(\sigma^2) d\theta d\sigma^2} \\ &\propto f(t|\theta, \sigma^2)\sigma^{-1} \exp\left\{-\frac{(\theta - \mu)^2}{2\tau\sigma^2}\right\} (\sigma^2)^{-\alpha-1} \exp\left\{-\frac{\beta}{\sigma^2}\right\} \\ &\propto (\sigma^2)^{-\nu/2-\alpha-2} \exp\left\{-\frac{1}{2\sigma^2}\left[n(\bar{x} - \theta)^2 + \frac{(\theta - \mu)^2}{\tau}\right]\right\} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{\nu s^2}{2} + \beta\right)\right\}. \end{aligned} \tag{65}$$

In order to normalize the posterior, consider

$$
\begin{aligned}
K &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} (\sigma^2)^{-\nu/2-\alpha-2} \exp\left\{-\frac{1}{2\sigma^2}\left[n(\bar{x}-\theta)^2 + \frac{(\theta-\mu)^2}{\tau}\right]\right\} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{\nu s^2}{2}+\beta\right)\right\} d\theta d\sigma^2 \\
&= \sqrt{\frac{2\pi}{n+\tau^{-1}}} \int_{\mathbb{R}^+} (\sigma^2)^{-\nu/2-\alpha-3/2} \exp\left\{-\frac{D^*}{\sigma^2}\right\} d\theta^2 \\
&= \sqrt{\frac{2\pi}{n+\tau^{-1}}} \frac{\Gamma(\nu/2+\alpha+1/2)}{D^{*(\nu/2+\alpha+1/2)}},
\end{aligned}
\tag{66}
$$

where

$$
D^* = \frac{\nu s^2}{2} + \beta + \frac{1}{2}\left[n\bar{x}^2 + \mu^2\tau^{-1} - \frac{(n\bar{x}+\mu\tau^{-1})^2}{n+\tau^{-1}}\right] = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n(\bar{x}-\mu)^2}{2(1+n\tau)}.
$$

So the posterior is

$$
\begin{aligned}
\pi(\theta,\sigma^2|\mathbf{x}) &= K^{-1}(\sigma^2)^{-\nu/2-\alpha-2} \exp\left\{-\frac{1}{2\sigma^2}\left[n(\bar{x}-\theta)^2 + \frac{(\theta-\mu)^2}{\tau}\right]\right\} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{\nu s^2}{2}+\beta\right)\right\} \\
&= K^{-1}(\sigma^2)^{-\nu/2-\alpha-2} \exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta - \frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\} \exp\left\{-\frac{D^*}{\sigma^2}\right\}.
\end{aligned}
\tag{67}
$$

Of course the joint posterior can be written as

$$
\pi(\theta,\sigma^2|\mathbf{x}) = \pi_1(\theta|\sigma^2,\mathbf{x})\pi_2(\sigma^2|\mathbf{x}).
$$

$\pi_1(\theta|\sigma^2,\mathbf{x})$ can be obtained by

$$
\begin{aligned}
\pi_1(\theta|\sigma^2,\mathbf{x}) &= \frac{f(\bar{x}|\theta,\sigma^2)\pi_1(\theta|\sigma^2)}{\int_\Theta f(\bar{x}|\theta,\sigma^2)\pi_1(\theta|\sigma^2)d\theta} \\
&\propto \exp\left\{-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right\} \exp\left\{-\frac{(\theta-\mu)^2}{2\tau\sigma^2}\right\} \\
&\propto \exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta - \frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\}.
\end{aligned}
\tag{68}
$$

As a result, $\pi_1(\theta|\sigma^2,\mathbf{x}) = N(\mu(\mathbf{x}),\eta^2)$,

$$
\mu(\mathbf{x}) = \frac{\mu+n\tau\bar{x}}{n\tau+1}, \quad \eta^2 = \frac{\sigma^2}{n+\tau^{-1}}, \quad \bar{x} = \sum_{i=1}^{n} x_i.
$$

$\pi_2(\sigma^2|\mathbf{x})$ can be obtained by

$$
\begin{aligned}
\pi_2(\sigma^2|\mathbf{x}) &= \frac{\pi(\theta,\sigma^2|\mathbf{x})}{\pi_1(\theta|\sigma^2,\mathbf{x})} \\
&= \frac{K^{-1}(\sigma^2)^{-\nu/2-\alpha-2} \exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta - \frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\} \exp\left\{-\frac{D^*}{\sigma^2}\right\}}{\sqrt{\frac{n+\tau^{-1}}{2\pi\sigma^2}} \exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta - \frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\}} \\
&= \frac{D^{*(n/2+\alpha)}}{\Gamma(n/2+\alpha)} (\sigma^2)^{-n/2-\alpha-1} \exp\left\{-\frac{D^*}{\sigma^2}\right\},
\end{aligned}
\tag{69}
$$

which is exactly $\pi_2(\sigma^2|\mathbf{x})$ is $\Gamma^{-1}(\alpha+n/2,\tilde{\beta})$,

$$\tilde{\beta} = \left[\beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n(\bar{x}-\mu)^2}{2(1+n\tau)}\right].$$

(2) The marginal posterior of σ^2 given $\mathbf{x}$ is

$$\begin{aligned}\pi(\sigma^2|\mathbf{x}) &= \int_{\mathbb{R}} \pi(\theta,\sigma^2|\mathbf{x})d\theta \\ &= K^{-1}(\sigma^2)^{-\nu/2-\alpha-2}\exp\left\{-\frac{D^*}{\sigma^2}\right\}\int_{\mathbb{R}}\exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta-\frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\}d\theta \\ &= \frac{D^{*(n/2+\alpha)}}{\Gamma(n/2+\alpha)}(\sigma^2)^{-n/2-\alpha-1}\exp\left\{-\frac{D^*}{\sigma^2}\right\}, \quad (\tilde{\beta}=D^*)\end{aligned} \tag{70}$$

which is exactly $\Gamma^{-1}(\alpha+n/2,\tilde{\beta})$.

(3) The marginal posterior of θ given $\mathbf{x}$ is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &= \int_{\mathbb{R}+}\pi(\theta,\sigma^2|\mathbf{x}) \\ &= K^{-1}\int_{\mathbb{R}+}(\sigma^2)^{-\nu/2-\alpha-2}\exp\left\{-\frac{D^*}{\sigma^2}\right\}\exp\left\{-\frac{n+\tau^{-1}}{2\sigma^2}\left[\theta-\frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\}d\sigma^2 \\ &= \frac{\Gamma(\alpha+n/2+1/2)}{\Gamma(\alpha+n/2)\sqrt{(2\alpha+n)\pi}}\sqrt{\frac{(n+\tau^{-1})(\alpha+n/2)}{D^*}} \\ &\quad \left\{1+\frac{1}{2\alpha+n}\frac{(n+\tau^{-1})(\alpha+n/2)}{D^*}\left[\theta-\frac{n\bar{x}+\mu\tau^{-1}}{n+\tau^{-1}}\right]^2\right\}^{-(n/2+1/2+\alpha)},\end{aligned} \tag{71}$$

which is exactly

$$\mathcal{T}\left(2\alpha+n,\mu(\mathbf{x}),\frac{\tilde{\beta}}{(\tau^{-1}+n)(\alpha+n/2)}\right).$$

□

Ex. 3.12. *Cont'd Ex. 10 and Ex. 11(3). Find the posterior expectation estimator and the posterior mean square error.*

(1) **Ex. 10**

The posterior expectation estimator is

$$\hat{\theta}_E = x - \Sigma(\Sigma+A)^{-1}(x-\mu),$$

and the posterior MSE is

$$PMSE(\hat{\theta}_E) = V^{\pi}(x) = (\Sigma^{-1}+A^{-1})^{-1}.$$

(2) **Ex. 11(3)**

The posterior expectation estimator is

$$\hat{\theta}_E = \mu(\mathbf{x}) = \frac{\mu + n\tau\bar{x}}{n\tau + 1},$$

and the posterior MSE is

$$PMSE(\hat{\theta}_E) = V^{\pi}(\mathbf{x}) = \frac{2\alpha + n}{2\alpha + n - 2} \frac{\tilde{\beta}}{(\tau^{-1} + n)(\alpha + n/2)}.$$

□

Ex. 3.13. *Cont'd Ex. 2 and Ex. 6. Find the posterior mode estimator and the posterior mean square error.*

(1) **Ex. 2**
The posterior distribution is

$$\pi(\theta|\mathbf{x}) = \frac{n^{n\bar{x}}}{\Gamma(n\bar{x})}\theta^{n\bar{x}-1}e^{-n\theta}.$$

As a result, according to the formula, the mode estimator is

$$\hat{\theta}_{MD} = \bar{x} - \frac{1}{n},$$

and the posterior MSE is

$$\begin{aligned} PMSE(\hat{\theta}_{MD}) &= V^{\pi}(\mathbf{x}) + [\mu^{\pi}(\mathbf{x}) - \hat{\theta}_{MD}]^2 \\ &= \frac{\bar{x}}{n} + \frac{1}{n^2}. \end{aligned} \tag{72}$$

(2) **Ex. 6**
The posterior distribution is

$$\pi(\theta|\mathbf{x}) = \frac{(\sum_{i=1}^n x_i + \beta)^{\alpha+n}}{\Gamma(\alpha + n)}\theta^{-(\alpha+n+1)} \exp\left\{\theta^{-1}\left(\sum_{i=1}^n x_i + \beta\right)\right\}.$$

As a result, according to the formula, the mode estimator is

$$\hat{\theta}_{MD} = \frac{\sum_{i=1}^n x_i + \beta}{\alpha + n + 1},$$

and the posterior MSE is

$$\begin{aligned} PMSE(\hat{\theta}_{MD}) &= V^{\pi}(\mathbf{x}) + [\mu^{\pi}(\mathbf{x}) - \hat{\theta}_{MD}]^2 \\ &= \frac{(\sum_{i=1}^n x_i + \beta)^2}{(\alpha + n - 1)^2}\left[\frac{1}{\alpha + n - 2} - \frac{4}{(\alpha + n + 1)^2}\right]. \end{aligned} \tag{73}$$

□

Ex. 3.14. *Cont'd Ex. 1(2). Find the posterior median estimator and the posterior mean square error.*

Assume that $n = x = 1$. The posterior distribution is

$$\pi(\theta|x) = \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}.$$

As a result, we have to find the median of this distribution. Using numerical methods (the qbeta function in R) we know

$$\hat{\theta}_{ME} = 0.7071068.$$

According to the formula, the posterior MSE is

$$\begin{aligned} PMSE(\hat{\theta}_{ME}) &= V^\pi(\mathbf{x}) + [\mu^\pi(\mathbf{x}) - \hat{\theta}_{ME}]^2 \\ &= \frac{2}{3^2 \cdot 4} + [2/3 - 0.7071068]^2 \\ &= 0.05719. \end{aligned} \tag{74}$$

□

Ex. 3.15. *Suppose that X follows the distribution*

$$P(X = x) = \theta(1-\theta)^{x-1} \quad (x = 1, 2, ...),$$

and the prior for θ is $U(0,1)$. Show that

(1) If only one X (3) is observed, find the posterior expectation estimator of θ.

(2) If three X's (3, 2, 5) are observed, find the posterior expectation estimator of of θ.

The posterior is

$$\begin{aligned} \pi(\theta) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_0^1 \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^n(1-\theta)^{n\bar{x}-n}}{\int_0^1 \theta^n(1-\theta)^{n\bar{x}-n}d\theta} \\ &= \frac{\Gamma(n\bar{x}+2)}{\Gamma(n+1)\Gamma(n\bar{x}-n+1)}\theta^n(1-\theta)^{n\bar{x}-n}. \end{aligned} \tag{75}$$

As a result, the posterior expectation estimator is

$$\hat{\theta}_E = \frac{n+1}{n\bar{x}+2}.$$

(1) With $n = 1$, $\bar{x} = 3$, the posterior expectation estimator is 2/5.

(2) With $n = 3$, $\bar{x} = 10/3$, the posterior expectation estimator is 1/3. □

Ex. 3.16. *Suppose that X follows the distribution Exp(λ), where λ follows a Gamma distribution with the mean of 0.2 and the standard deviation of 1.0. For 20 observations the average $\bar{x} = 3.8$. Find the posterior expectation estimator for λ and λ^{-1}.*

(1) For the parameter λ, the prior is

$$\lambda \sim \Gamma(1/25, 1/5).$$

The posterior is

$$\pi(\lambda|\mathbf{x}) = \frac{\prod_{i=1}^n f(x_i|\lambda)\pi(\lambda)}{\int_{\mathbb{R}^+} \prod_{i=1}^n f(x_i|\lambda)\pi(\lambda)d\lambda} = \frac{\lambda^n e^{-\lambda n\bar{x}} \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_{\mathbb{R}^+} \lambda^n e^{-\lambda n\bar{x}} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda} = \frac{(n\bar{x}+\beta)^{\alpha+n}}{\Gamma(\alpha+n)} \lambda^{n+\alpha-1} e^{-(n\bar{x}+\beta)\lambda}. \tag{76}$$

As a result, the posterior expectation estimator is

$$\hat{\lambda}_E = \frac{n+\alpha}{n\bar{x}+\beta} = \frac{20+1/25}{20\cdot 3.8+1/5} = 0.263.$$

(2) For the parameter $t = \lambda^{-1}$, the prior is

$$t \sim \Gamma^{-1}(1/25, 1/5).$$

The posterior is

$$\pi(t|\mathbf{x}) = \frac{\prod_{i=1}^n f(x_i|t)\pi(t)}{\int_{\mathbb{R}^+} \prod_{i=1}^n f(x_i|t)\pi(t)dt} = \frac{t^{-n} e^{-n\bar{x}/t} t^{-\alpha-1} e^{-\beta/t}}{\int_{\mathbb{R}^+} t^{-n} e^{-n\bar{x}/t} t^{-\alpha-1} e^{-\beta/t} dt} = \frac{(n\bar{x}+\beta)^{n+\alpha}}{\Gamma(n+\alpha)} t^{-(n+\alpha+1)} e^{-(n\bar{x}+\beta)/t}. \tag{77}$$

As a result, the posterior expectation estimator is

$$\hat{t}_E = \frac{n\bar{x}+\beta}{n+\alpha-1} = \frac{20\cdot 3.8+1/5}{20+1/25-1} = 4.002.$$

□

Ex. 3.17. *Suppose that X follows the distribution $P(\lambda)$, where $\lambda \sim \Gamma(3,1)$. Three X's are observed: 2, 0, 6. Find the posterior expectation estimator and the posterior variance for λ.*

The posterior distribution is

$$\pi(\lambda|\mathbf{x}) = \frac{\prod_{i=1}^n f(x_i|\lambda)\pi(\lambda)}{\int_{\mathbb{R}^+} \prod_{i=1}^n f(x_i|\lambda)\pi(\lambda)d\lambda} = \frac{e^{-n\lambda} \lambda^{n\bar{x}} \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_{\mathbb{R}^+} e^{-n\lambda} \lambda^{n\bar{x}} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda} = \frac{(n+\beta)^{n\bar{x}+\alpha}}{\Gamma(n\bar{x}+\alpha)} \lambda^{n\bar{x}+\alpha-1} e^{-(n+\beta)\lambda}. \tag{78}$$

Therefore, the posterior expectation estimator is

$$\hat{\lambda}_E = \frac{n\bar{x}+\alpha}{n+\beta} = \frac{8+3}{3+1} = \frac{11}{4},$$

and the posterior variance is

$$V^{\pi}(\mathbf{x}) = \frac{n\bar{x}+\alpha}{(n+\beta)^2} = \frac{11}{16}.$$

□

Ex. 3.18. *Suppose that the prior for the defective rate θ is $Be(5,10)$. Find the posterior mode estimator and the posterior expectation estimator, given the following observations in order:*

(1) The first batch: 3 out of 20 products are defective.

(2) The second batch: 0 out of 20 products are defective.

(1) The posterior given the first batch is

$$\begin{aligned}
\pi_1(\theta|\mathbf{x}^{(1)}) &= \frac{\prod_{i=1}^{20} f(x_i|\theta)\pi_0(\theta)}{\int_0^1 \prod_{i=1}^{20} f(x_i|\theta)\pi_0(\theta)d\theta} \\
&= \frac{\theta^3(1-\theta)^{17}\theta^4(1-\theta)^9}{\int_0^1 \theta^3(1-\theta)^{17}\theta^4(1-\theta)^9 d\theta} \\
&= \frac{\Gamma(35)}{\Gamma(8)\Gamma(27)}\theta^7(1-\theta)^{26}.
\end{aligned} \tag{79}$$

As a result,

$$\begin{aligned}
\hat{\theta}_{MD} &= \frac{\alpha-1}{\alpha+\beta-2} = \frac{7}{33}, \\
\hat{\theta}_{E} &= \frac{\alpha}{\alpha+\beta} = \frac{8}{35}.
\end{aligned}$$

(2) The posterior given the second batch is

$$\begin{aligned}
\pi_2(\theta|\mathbf{x}^{(2)}) &= \frac{\prod_{i=1}^{20} f(x_i|\theta)\pi_1(\theta|\mathbf{x}^{(1)})}{\int_0^1 \prod_{i=1}^{20} f(x_i|\theta)\pi_1(\theta|\mathbf{x}^{(1)})d\theta} \\
&= \frac{\theta^0(1-\theta)^{20}\theta^7(1-\theta)^{26}}{\int_0^1 \theta^0(1-\theta)^{20}\theta^7(1-\theta)^{26} d\theta} \\
&= \frac{\Gamma(55)}{\Gamma(8)\Gamma(47)}\theta^7(1-\theta)^{46}.
\end{aligned} \tag{80}$$

As a result,

$$\begin{aligned}
\hat{\theta}_{MD} &= \frac{\alpha-1}{\alpha+\beta-2} = \frac{7}{53}, \\
\hat{\theta}_{E} &= \frac{\alpha}{\alpha+\beta} = \frac{8}{55}.
\end{aligned}$$

□

Ex. 3.19. *Suppose that θ is the rate of people that approve the policy A in a city. There are two candidates for the prior for θ:*

$$\begin{aligned}
A:&\ \ \pi_A(\theta) = 2\theta \quad (0<\theta<1); \\
B:&\ \ \pi_B(\theta) = 4\theta^3 \quad (0<\theta<1).
\end{aligned}$$

Assume that 1000 people are randomly sampled and 710 among them approve the policy A.

(1) Find the posterior for θ, with both priors respectively.

(2) Find the posterior expectation estimator for θ, with both priors respectively.

(3) Show that: with the sample size of 1000, the difference between the two posterior expectation estimators will be within 0.002, regardless of the number of people who approve the policy A.

(1) **For prior A:** the posterior is

$$\begin{aligned}\pi_A(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi_A(\theta)}{\int_0^1 \prod_{i=1}^n f(x_i|\theta)\pi_A(\theta)d\theta} \\ &= \frac{\theta^{710}(1-\theta)^{290}2\theta}{\int_0^1 \theta^{710}(1-\theta)^{290}2\theta d\theta} \\ &= \frac{\Gamma(1003)}{\Gamma(712)\Gamma(291)}\theta^{711}(1-\theta)^{290}.\end{aligned} \tag{81}$$

For prior B: the posterior is

$$\begin{aligned}\pi_B(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi_B(\theta)}{\int_0^1 \prod_{i=1}^n f(x_i|\theta)\pi_B(\theta)d\theta} \\ &= \frac{\theta^{710}(1-\theta)^{290}4\theta^3}{\int_0^1 \theta^{710}(1-\theta)^{290}4\theta^3 d\theta} \\ &= \frac{\Gamma(1005)}{\Gamma(714)\Gamma(291)}\theta^{713}(1-\theta)^{290}.\end{aligned} \tag{82}$$

(2) **For prior A:**

$$\hat{\theta}_E = \frac{\alpha}{\alpha+\beta} = \frac{712}{1003} = 0.70987.$$

For prior B:

$$\hat{\theta}_E = \frac{\alpha}{\alpha+\beta} = \frac{714}{1005} = 0.71045.$$

(3) With $n = 1000$, the posteriors are

$$\pi_A(\theta|\mathbf{x}) = \frac{\Gamma(1003)}{\Gamma(x+2)\Gamma(1001-x)}\theta^{x+1}(1-\theta)^{1000-x}$$

and

$$\pi_A(\theta|\mathbf{x}) = \frac{\Gamma(1005)}{\Gamma(x+4)\Gamma(1001-x)}\theta^{x+3}(1-\theta)^{1000-x}.$$

Therefore, the posterior expectation estimators are

$$\hat{\theta}_A = \frac{\alpha}{\alpha+\beta} = \frac{x+2}{1003},$$

and

$$\hat{\theta}_B = \frac{\alpha}{\alpha+\beta} = \frac{x+4}{1005}.$$

Since $x \in \{0, 1, ..., 1000\}$, the difference is

$$|\hat{\theta}_A - \hat{\theta}_B| = \frac{2002-2x}{1008015} \leq \frac{2002}{1008015} < 0.002.$$

$\square$

Ex. 3.20. *An insurance company wants to set up a new program for car accident. Suppose that the number of car accidents per thousand people per year, X, follows a distribution $P(\lambda)$, where the prior for λ is $\Gamma(35, 1)$. According to a survey, 85 car accidents occurred among 2000 people last year. If the insurance company pays \$1000 for one accident on average, the advertisement cost is \$500,000 a year, the selling price is \$50 per person, and 100,000 people a year buy the insurance. Find the profit the company can make in a year.*

Since $x = 42.5$, the posterior distribution is

$$\begin{aligned}\pi(\lambda|x) &= \frac{f(x|\lambda)\pi(\lambda)}{\int_{\mathbb{R}^+} f(x|\lambda)\pi(\lambda)d\lambda} \\ &= \frac{e^{-\lambda}\lambda^{42.5}\lambda^{34}e^{-\lambda}}{\int_{\mathbb{R}^+} e^{-\lambda}\lambda^{42.5}\lambda^{34}e^{-\lambda}d\lambda} \\ &= \frac{2^{77.5}}{\Gamma(77.5)}\lambda^{76.5}e^{-2\lambda}.\end{aligned} \tag{83}$$

Then the estimated number of car accidents is

$$\hat{\lambda}_E = \frac{\alpha}{\beta} = 38.75.$$

As a result, the profit is

$$100,000 \cdot \$50 - \$500,000 - 38.75 \cdot \$1000 = \$4,461,250.$$

$\square$

Ex. 3.21. *Suppose that*

$$(X_1, ..., X_5) \sim^{\text{i.i.d.}} Exp(1/\theta),$$

and $\theta \sim \Gamma^{-1}(10, 100)$. If the 5 observations are 5, 12, 14, 10, 12, find the posterior mode estimator and the posterior expectation estimator.

The posterior is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_{\mathbb{R}^+}\prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^{-5}e^{-53/\theta}\theta^{-11}e^{-100/\theta}}{\int_{\mathbb{R}^+}\theta^{-5}e^{-53/\theta}\theta^{-11}e^{-100/\theta}d\theta} \\ &= \frac{153^{15}}{\Gamma(15)}\theta^{-16}e^{-153/\theta}.\end{aligned} \tag{84}$$

As a result,

$$\hat{\theta}_{MD} = \frac{\beta}{\alpha+1} = \frac{153}{16} = 9.5625,$$

and

$$\hat{\theta}_E = \frac{\beta}{\alpha-1} = \frac{153}{14} = 10.9286.$$

$\square$

Ex. 3.22. *Suppose that*

$$X \sim \Gamma(\frac{n}{2}, \frac{1}{2\theta}),$$

and the prior for θ is $\Gamma^{-1}(\alpha, \beta)$.

(1) Find the posterior expectation estimator and the posterior variance.

(2) If $\boldsymbol{X} = (X_1, ..., X_n)$ are randomly sampled, find the posterior mode estimator and the posterior expectation estimator.

(1) The posterior is

$$\begin{aligned} \pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}+} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^{-n/2}e^{-x/(2\theta)}\theta^{-\alpha-1}e^{-\beta/\theta}}{\int_{\mathbb{R}+} \theta^{-n/2}e^{-x/(2\theta)}\theta^{-\alpha-1}e^{-\beta/\theta}d\theta} \\ &= \frac{(x/2+\beta)^{n/2+\alpha}}{\Gamma(n/2+\alpha)}\theta^{-n/2-\alpha-1}e^{-(x/2+\beta)/\theta}. \end{aligned} \tag{85}$$

Therefore,

$$\hat{\theta}_E = \frac{x/2+\beta}{n/2+\alpha-1},$$

and

$$V^\pi(\theta) = \frac{(x/2+\beta)^2}{(n/2+\alpha-1)^2(n/2+\alpha-2)}.$$

(2) The posterior is

$$\begin{aligned} \pi(\theta|\mathbf{x}) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}+} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^{-n^2/2}e^{-n\bar{x}/(2\theta)}\theta^{-\alpha-1}e^{-\beta/\theta}}{\int_{\mathbb{R}+} \theta^{-n^2/2}e^{-n\bar{x}/(2\theta)}\theta^{-\alpha-1}e^{-\beta/\theta}d\theta} \\ &= \frac{(n\bar{x}/2+\beta)^{n^2/2+\alpha}}{\Gamma(n^2/2+\alpha)}\theta^{-n^2/2-\alpha-1}e^{-(n\bar{x}/2+\beta)/\theta}. \end{aligned} \tag{86}$$

Therefore,

$$\hat{\theta}_{MD} = \frac{n\bar{x}/2+\beta}{n^2/2+\alpha+1},$$

and

$$\hat{\theta}_E = \frac{n\bar{x}/2+\beta}{n^2/2+\alpha-1}.$$

□

Ex. 3.23. *Suppose that*

$$\boldsymbol{X} = (X_1, ..., X_r) \sim^{\text{i.i.d.}} M(r, \theta).$$

The prior for $\theta = (\theta_1, ..., \theta_r)$ is $D(\alpha_1, ..., \alpha_r)$. Find the posterior mode estimator and the posterior expectation estimator.

The posterior is

$$
\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_{\theta_1}\cdots\int_{\theta_r} f(\mathbf{x}|\theta)\pi(\theta)d\theta_1...d\theta_r} \\
&= \frac{\theta_1^{x_1}...\theta_r^{x_r}\theta_1^{\alpha_1-1}...\theta_r^{\alpha_r-1}}{\int_{\theta_1}\cdots\int_{\theta_r} \theta_1^{x_1}...\theta_r^{x_r}\theta_1^{\alpha_1-1}...\theta_r^{\alpha_r-1}d\theta_1...d\theta_r} \\
&= \frac{\Gamma(\sum_{i=1}^r(x_i+\alpha_i))}{\prod_{i=1}^r\Gamma(x_i+\alpha_i)}\theta_1^{x_1+\alpha_1-1}...\theta_r^{x_r+\alpha_r-1}.
\end{aligned} \tag{87}
$$

Therefore,

$$
\hat{\theta}_{iMD} = \frac{x_i+\alpha_i-1}{\sum_{j=1}^r(x_j+\alpha_j)-r}
$$

and

$$
\hat{\theta}_{iE} = \frac{x_i+\alpha_i}{\sum_{j=1}^r(x_j+\alpha_j)}.
$$

□

Ex. 3.24. *Three(3) observations (2, 4, 3) are obtained from the distribution $N(\theta,1)$. The prior for θ is $N(3,1)$. Find the 0.95 Bayes credible interval.*

It is known that the posterior is

$$
\pi(\theta|\mathbf{x}) \sim N(\mu_n, \eta_n^2)
$$

where

$$
\mu_n = \frac{\sigma^2/n}{\sigma^2/n+\tau^2}\mu + \frac{\tau^2}{\sigma^2/n+\tau^2}\bar{x} = 3, \quad \eta_n^2 = \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2} = 0.25.
$$

Therefore, the 0.95 Bayes credible interval is

$$
[3-1.96\cdot 0.5,\ 3+1.96\cdot 0.5] = [2.02,\ 3.98].
$$

□

Ex. 3.25. *Five(5) products are sampled from a batch. Suppose that the number of defective products, X, follows $B(5,\theta)$. The prior for θ is $Beta(1,9)$. If $x=0$, find the 95% HPD credible interval.*

The posterior is

$$
\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} \\
&= \frac{\theta^0(1-\theta)^5\theta^0(1-\theta)^8}{\int_0^1 \theta^0(1-\theta)^5\theta^0(1-\theta)^8 d\theta} \\
&= \frac{\Gamma(15)}{\Gamma(1)\Gamma(14)}(1-\theta)^{13}.
\end{aligned} \tag{88}
$$

Since the posterior density is not symmetric, we can use the hpd function in the TeachingDemos R package to find the HPD interval:

$$
[0.0453,\ 0.1926].
$$

□

Ex. 3.26. *Suppose the random variable X follows $P(\theta)$, and the prior for θ is the non-informative $\pi(\theta) = \theta^{-1}I(\theta > 0)$. Find the 90% HPD credible interval using normal approximation.*

The posterior is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_\Theta f(\mathbf{x}|\theta)\pi(\theta)d\theta} \\ &= \frac{e^{-n\theta}\theta^{n\bar{x}}\theta^{-1}}{\int_{\mathbb{R}^+} e^{-n\theta}\theta^{n\bar{x}}\theta^{-1}d\theta} \\ &= \frac{(n\theta)^{n\bar{x}}}{\Gamma(n\bar{x})}\theta^{n\bar{x}-1}e^{-n\theta}.\end{aligned} \tag{89}$$

Therefore

$$\mu^\pi(\mathbf{x}) = \frac{\bar{x}}{\theta}, \quad V^\pi(\mathbf{x}) = \frac{\bar{x}}{n\theta^2}.$$

The normal approximation of the 90% HPD credible interval is

$$\left[\frac{\bar{x}}{\theta} - 1.64 \cdot \sqrt{\frac{\bar{x}}{n\theta^2}},\ \frac{\bar{x}}{\theta} + 1.64 \cdot \sqrt{\frac{\bar{x}}{n\theta^2}}\right].$$

□

Ex. 3.27. *Five(5) observations (1.2, 1.6, 1.3, 1.4, 1.4) are obtained from the distribution $N(\theta, \sigma^2)$. The prior for (θ, σ^2) is the non-informative prior $\pi(\theta, \sigma^2) = \sigma^{-1}I(\sigma^2 > 0)$. Find the 90% HPD credible interval for θ.*

It is known that the posterior for θ is

$$\mathcal{T}\left(n-1,\ \bar{x},\ \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n(n-1)}\right),$$

which is

$$\mathcal{T}\left(4,\ 1.38,\ 0.0044\right).$$

Therefore,

$$\mu^\pi(\mathbf{x}) = 1.38, \quad V^\pi(\mathbf{x}) = \frac{\nu}{\nu-2}\tau^2 = 0.0088.$$

The 90% HPD credible interval is

$$\left[\mu^\pi(\mathbf{x}) - 2.13 \cdot \sqrt{V^\pi(\mathbf{x})},\ \mu^\pi(\mathbf{x}) + 2.13 \cdot \sqrt{V^\pi(\mathbf{x})}\right],$$

which is

$$[1.1802,\ 1.5798].$$

□

Ex. 3.28. *Suppose that $X \sim N(\theta, 1)$. The observation is $x = 6$.*

(1) If the prior is $N(0, 2.19)$, find the 90% HPD credible interval for θ.

(2) If the prior is $C(0, 1)$, find the 95% HPD credible interval for θ.

(1) It is known that the posterior is

$$\pi(\theta|\mathbf{x}) \sim N(\mu(x), \eta^2)$$

where

$$\mu(x) = \frac{\sigma^2}{\sigma^2+\tau^2}\mu + \frac{\tau^2}{\sigma^2+\tau^2}x = 4.119, \quad \eta^2 = \frac{\sigma^2\tau^2}{\tau^2+\sigma^2} = 0.68652.$$

Therefore, the 90% HPD credible interval is

$$[4.119 - 1.64 \cdot 0.829,\ 4.119 + 1.64 \cdot 0.829] = [2.759,\ 5.479].$$

(2) The posterior distribution is

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}} f(x|\theta)\pi(\theta)d\theta} \\ &\propto \exp\left\{-\frac{1}{2}(x-\theta)^2\right\}\frac{1}{1+\theta^2}.\end{aligned} \tag{90}$$

Since this is not easy to estimate analytically, we should resort to computational packages and Monte-Carlo methods. Details can be found here with reproducible codes.

The 95% HPD credible interval is [3.63, 7.68]. □

Ex. 3.29. *Ex. 21 Cont'd. Find the 95% HPD credible set using normal approximation.*

Since the posterior is

$$\pi(\theta|\mathbf{x}) = \frac{153^{15}}{\Gamma(15)}\theta^{-16}e^{-153/\theta},$$

we known

$$\mu^\pi(\mathbf{x}) = \frac{153}{15-1} = 10.929, \quad V^\pi(\mathbf{x}) = \frac{153^2}{(15-1)^2(15-2)} = 9.1872.$$

The normal approximation of the 95% HPD credible interval is

$$[10.929 - 1.96 \cdot 3.031,\ 10.929 + 1.96 \cdot 3.031] = [4.988,\ 16.870].$$

□

Ex. 3.30. *Suppose that $Q^2/\sigma^2 \sim \chi^2(n)$, and the prior for σ is $\pi(\sigma) = \sigma^{-1}$. Show that:*
(1) The non-informative prior for σ^2 is $\pi^*(\sigma^2) = \sigma^{-2}$;
(2) If $n = 2,\ Q^2 = 2$, the 95% HPD credible intervals for σ and σ^2 are different.

(1) Since

$$\frac{Q^2}{\sigma^2} \sim \chi^2(n) = \Gamma(\frac{n}{2}, \frac{1}{2}),$$

we know

$$Q^2 \sim \Gamma(\frac{n}{2}, \frac{1}{2\sigma^2}).$$

Therefore, the distribution of Q^2 can be written as $(2\sigma^2)^{-1}\phi(\frac{t}{2\sigma^2})$, where

$$\phi(t) = \frac{1}{\Gamma(n/2)}t^{n/2-1}e^{-t},$$

indicating that the non-informative prior for σ^2 is $\pi^*(\sigma^2) = \sigma^{-2}$.

(2) The posterior for σ is

$$\begin{aligned}
\pi(\sigma|t) &= \frac{f(t|\sigma)\pi(\sigma)}{\int_{\mathbb{R}^+} f(t|\sigma)\pi(\sigma)d\sigma} \\
&= \frac{\sigma^{-n}e^{-t/(2\sigma^2)}\sigma^{-1}}{\int_{\mathbb{R}^+} \sigma^{-n}e^{-t/(2\sigma^2)}\sigma^{-1}d\sigma} \\
&= \frac{t^{n/2}}{2^{n/2-1}\Gamma(n/2)}\sigma^{-n-1}e^{-\frac{t}{2\sigma^2}} \\
&= 2\sigma^{-3}e^{-\frac{1}{\sigma^2}}.
\end{aligned} \tag{91}$$

While the posterior for σ^2 is

$$\begin{aligned}
\pi(\sigma^2|t) &= \frac{f(t|\sigma^2)\pi(\sigma^2)}{\int_{\mathbb{R}^+} f(t|\sigma^2)\pi(\sigma^2)d\sigma^2} \\
&= \frac{(\sigma^2)^{-n/2}e^{-t/(2\sigma^2)}(\sigma^2)^{-1}}{\int_{\mathbb{R}^+} (\sigma^2)^{-n/2}e^{-t/(2\sigma^2)}(\sigma^2)^{-1}d\sigma^2} \\
&= \frac{(t/2)^{n/2}}{\Gamma(n/2)}(\sigma^2)^{-n/2-1}e^{-\frac{t}{2\sigma^2}} \\
&= \sigma^{-4}e^{-1/\sigma^2}.
\end{aligned} \tag{92}$$

Since the HPD intervals are not easy to find analytically, we have to resort to statistical software packages and Monte-Carlo methods. Details can be found here with reproducible codes.

Ex. 3.31. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(0, \sigma^2),$$

and the prior for σ^2 is $\Gamma^{-1}(\alpha, \lambda)$. Find the 0.90 Bayes upper credible limit for σ^2.

It is known that the posterior is

$$\pi(\sigma^2|t) = \frac{(t/2+\lambda)^{n/2+\alpha}}{\Gamma(n/2+\alpha)}(\sigma^2)^{-n/2-\alpha-1}\exp\left\{-\frac{t/2+\lambda}{\sigma^2}\right\},$$

where

$$t = \sum_{i=1}^{n} x_i^2.$$

Therefore, the 0.90 Bayes upper credible limit, $\hat{\sigma}_U^2(t)$, is the 90% quantile of the distribution $\Gamma^{-1}(n/2+\alpha, t/2+\lambda)$. □

Ex. 3.32. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} U(0, \theta),$$

and the prior for θ is $Pa(\theta_0, \alpha)$. Find the $1-\alpha$ Bayes upper credible limit for θ.

It is known that the posterior is

$$\pi(\theta|\mathbf{x}) = \begin{cases} (n+\alpha)K^{n+\alpha}/\theta^{n+\alpha+1}, & \theta \geq K, \\ 0, & \text{otherwise}, \end{cases}$$

where $K = \max\{\theta_0, x_1, ..., x_n\}$. Therefore, the $1-\alpha$ credible limit for θ is the $1-\alpha$ quantile of the distribution $Pa(\max\{\theta_0, x_1, ..., x_n\}, n+\alpha)$. □

Ex. 3.33. *Cont'd Ex. 25. Consider the hypothesis test*

$$H_0 : \theta \leq 0.1 \leftrightarrow H_1 : \theta > 0.1.$$

Find the posterior probabilities of the hypotheses, the posterior odds ratio, and the Bayes factor.

The prior is

$$\pi(\theta) = \frac{\Gamma(10)}{\Gamma(1)\Gamma(9)}(1-\theta)^8$$

and the posterior is

$$\pi(\theta|x) = \frac{\Gamma(15)}{\Gamma(1)\Gamma(14)}(1-\theta)^{13}.$$

Therefore,

$$\begin{aligned}
\alpha_0 = \Pr(\theta \leq 0.1|x) &= \int_0^{0.1} \pi(\theta|x)d\theta = 0.7712321, \\
\alpha_1 = \Pr(\theta > 0.1|x) &= \int_{0.1}^{1} \pi(\theta|x)d\theta = 0.2287679, \\
\frac{\alpha_0}{\alpha_1} &= 3.371243, \\
\frac{\pi_0}{\pi_1} = \frac{\Pr(\theta \leq 0.1)}{\Pr(\theta > 0.1)} &= \frac{0.6125795}{0.3874205} = 1.581175, \\
B^\pi(x) &= \frac{\alpha_0/\alpha_1}{\pi_0/\pi_1} = 2.132113.
\end{aligned}$$

□

Ex. 3.34. *Cont'd Ex. 27. Consider the hypothesis test*

$$H_0 : \theta \leq 0.1 \leftrightarrow H_1 : \theta > 0.1.$$

Find the posterior probabilities of the hypotheses, and the posterior odds ratio.

The posterior is

$$\mathcal{T}\left(n-1,\ \bar{x},\ \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n(n-1)}\right) = \mathcal{T}\,(4,\ 1.38,\ 0.0044)$$

Therefore,

$$\begin{aligned}
\alpha_0 = \Pr(\theta \leq 0.1|\mathbf{x}) &= \int_{-\infty}^{0.1} \pi(\theta|\mathbf{x})d\theta \approx 0, \\
\alpha_1 = \Pr(\theta > 0.1|\mathbf{x}) &= \int_{0.1}^{+\infty} \pi(\theta|\mathbf{x})d\theta \approx 1, \\
\frac{\alpha_0}{\alpha_1} &= 0.
\end{aligned}$$

□

Ex. 3.35. *Suppose that*

$$f(x,\theta)=\begin{cases} e^{-(x-\theta)}, & x>\theta,\\ 0, & x\le\theta,\end{cases}$$

and

$$\pi(\theta)=\begin{cases} e^{-\theta}, & \theta>0,\\ 0, & \theta\le 0.\end{cases}$$

Do the hypothesis test:

$$H_0:\theta\le 1\leftrightarrow H_1:\theta>1.$$

The posterior is

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x|\theta)\pi(\theta)d\theta}\\ &= \frac{e^{-(x-\theta)}e^{-\theta}I(0<\theta<x)}{\int_0^x e^{-(x-\theta)}e^{-\theta}d\theta}\\ &= \frac{1}{x}I(0<\theta<x).\end{aligned} \tag{93}$$

Therefore, if $\Pr(\theta\le 1|x)\le\Pr(\theta>1|x)$, then accept the null hypothesis; otherwise reject it. □

Ex. 3.36. *Suppose that*

$$X_1,...,X_m\sim^{\text{i.i.d.}} N(a,1),\quad Y_1,...,Y_n\sim^{\text{i.i.d.}} N(b,1),$$

and X's and Y's independent. Assume that a and b are independent, and that $a\sim N(\mu_1,\tau_1^2)$, $b\sim N(\mu_2,\tau_2^2)$. Do the hypothesis test:

$$H_0:a-b\le 0\leftrightarrow H_1:a-b>0.$$

It is known that the posterior for a and b are

$$a|X_1,...,X_m\sim N(\mu_a,\eta_a^2),\quad \mu_a=\frac{1/m}{1/m+\tau_1^2}\mu_1+\frac{\tau_1^2}{1/m+\tau_1^2}\bar{x}_m,\quad \eta_a^2=\frac{\tau_1^2}{m\tau_1^2+1}$$

and

$$b|Y_1,...,Y_n\sim N(\mu_b,\eta_b^2),\quad \mu_b=\frac{1/n}{1/n+\tau_2^2}\mu_2+\frac{\tau_2^2}{1/n+\tau_2^2}\bar{x}_n,\quad \eta_b^2=\frac{\tau_2^2}{n\tau_2^2+1}.$$

Therefore, let $s=a-b$, then the difference between a and b has the distribution

$$\pi(s|\mathbf{x})=N(\mu_a-\mu_b,\eta_a^2+\eta_b^2).$$

Consider

$$\Pr(H_0|\mathbf{x})=\Pr(s\le 0|\mathbf{x})=\Pr\left(\frac{s-(\mu_a-\mu_b)}{\sqrt{\eta_a^2+\eta_b^2}}\le\frac{\mu_b-\mu_a}{\sqrt{\eta_a^2+\eta_b^2}}\right)=\Phi\left(\frac{\mu_b-\mu_a}{\sqrt{\eta_a^2+\eta_b^2}}\right).$$

As a result, if $\mu_b-\mu_a>0$, accept the null hypothesis; otherwise, reject it. □

Ex. 3.37. *Suppose that $X \sim N(\theta, 1)$, and five(5) observations are 4.9, 5.6, 5.1, 4.6, 3.6. The prior is*

$$\pi(\theta) = \begin{cases} 0.5, & \theta = 4.01, \\ 0.5N(4.01, 1), & \text{otherwise.} \end{cases}$$

Do the hypothesis test:

$$H_0 : \theta = 4.01 \leftrightarrow H_1 : \theta \neq 4.01.$$

The likelihood is

$$f(\mathbf{x}|\theta) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right\} = \frac{1}{(2\pi)^{5/2}} \exp\left\{-\frac{1}{2}(5\theta^2 - 47.6\theta + 115.5)\right\},$$

and

$$\begin{aligned} m_1(\mathbf{x}) &= \int_{\theta\neq\theta_0} f(\mathbf{x}|\theta)g_1(\theta)d\theta \\ &= \int_{\theta\neq 4.01} \frac{1}{(2\pi)^{5/2}} \exp\left\{-\frac{1}{2}(5\theta^2 - 47.6\theta + 115.5)\right\} \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(\theta - 4.01)^2\right\} d\theta \\ &= \frac{1}{(2\pi)^3} \exp\{-1.340375\} \int_{\theta\neq 4.01} \exp\left\{-\frac{6}{2}(\theta - 4.635)^2\right\} \\ &= \frac{1}{(2\pi)^3} \exp\{-1.340375\} \sqrt{\frac{3}{\pi}} \\ &= 0.001031166. \end{aligned} \tag{94}$$

Therefore,

$$B^{\pi}(\mathbf{x}) = \frac{f(\mathbf{x}|\theta_0)}{m_1(\mathbf{x})} = \frac{0.0008193963}{0.001031166} = 0.7946308 < 1.$$

H_0 should be rejected. □

Ex. 3.38. *Suppose that $X \sim U(0, \theta)$, and the prior for θ is $Pa(5, 3)$. The five(5) observations are 10, 3, 2, 5, 14. Do the hypothesis test*

$$H_0 : 0 \leq \theta \leq 15 \leftrightarrow H_1 : \theta > 15.$$

Find the posterior probabilities, the posterior odds ratio, and the Bayes factor.

It is known that the posterior is

$$\begin{aligned} \pi(\theta|\mathbf{x}) &= \begin{cases} (n+\alpha)K^{n+\alpha}/\theta^{n+\alpha+1}, & \theta \geq \max\{\theta_0, x_1, ..., x_n\}, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 8 \cdot 14^8/\theta^9, & \theta \geq 14, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{95}$$

Therefore,

$$
\begin{gathered}
\alpha_0 = \Pr(\Theta_0|\mathbf{x}) = \Pr(0 \le \theta \le 15) = 8 \cdot 14^8 \int_{14}^{15} \frac{1}{\theta^9} d\theta = 0.4241701, \\
\alpha_1 = \Pr(\Theta_1|\mathbf{x}) = \Pr(\theta > 15) = 8 \cdot 14^8 \int_{15}^{+\infty} \frac{1}{\theta^9} d\theta = 0.5758299, \\
\frac{\alpha_0}{\alpha_1} = 0.736624, \\
\pi_0 = 3 \cdot 5^3 \int_{5}^{15} \frac{1}{\theta^4} d\theta = 0.962963, \\
\pi_1 = 3 \cdot 5^3 \int_{15}^{+\infty} \frac{1}{\theta^4} d\theta = 0.037037, \\
B^\pi(\mathbf{x}) = \frac{\alpha_0/\alpha_1}{\pi_0/\pi_1} = 0.02833166.
\end{gathered}
$$

The Bayes factor strongly supports the alternative hypothesis, so we should accept H_1. $\square$

4 Bayesian Decision Theory

Ex. 4.1. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(\theta, 1)$$

where the prior for θ is $N(0,1)$. Find the Bayes estimator under the square error loss.

The posterior $\pi(\theta|\mathbf{x})$ is $N(\mu(\mathbf{x}), \eta_n^2)$ where

$$\mu(\mathbf{x}) = \frac{1}{1/n+1}\bar{x} = \frac{n}{1+n}\bar{x}, \quad \eta_n^2 = \frac{1/n}{1/n+1} = \frac{1}{1+n}.$$

Therefore, the Bayes estimator is

$$\hat{\theta}_E = \frac{n}{1+n}\bar{x}.$$

□

Ex. 4.2. *Suppose that X follows the distribution*

$$P(X = x) = \theta(1-\theta)^{x-1} \quad (x = 1, 2, ...),$$

and the prior for θ is $Be(\alpha, \beta)$. Find the Bayes estimator under the square error loss.

The posterior is

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta(1-\theta)^{x-1}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 \theta(1-\theta)^{x-1}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta} \\ &= \frac{\Gamma(\alpha+\beta+x)}{\Gamma(\alpha+1)\Gamma(x+\beta-1)}\theta^{\alpha}(1-\theta)^{x+\beta-2}.\end{aligned} \tag{96}$$

Therefore, the Bayes estimator is

$$\hat{\theta}_E = \frac{\alpha+1}{\alpha+\beta+x}.$$

□

Ex. 4.3. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} P(\theta)$$

where the prior for θ is $Exp(\lambda)$. Find the Bayes estimator under the square error loss.

The posterior is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_{\mathbb{R}^+}\prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{e^{-n\theta}\theta^{n\bar{x}}\lambda e^{-\lambda\theta}}{\int_{\mathbb{R}^+} e^{-n\theta}\theta^{n\bar{x}}\lambda e^{-\lambda\theta}d\theta} \\ &= \frac{(n+\lambda)^{n\bar{x}+1}}{\Gamma(n\bar{x}+1)}\theta^{n\bar{x}}e^{-(n+\lambda)\theta}.\end{aligned} \tag{97}$$

Therefore, the Bayes estimator is

$$\hat{\theta}_E = \frac{b\bar{x}+1}{n+\lambda}.$$

□

Ex. 4.4. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} Exp(\theta),$$

where the prior for θ *is* $\Gamma(\alpha, \beta)$*. Find the Bayes estimator for* θ *and* $1/\theta$ *under the square error loss.*

The posterior is

$$\begin{aligned}\pi_\theta(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_{\mathbb{R}^+}\prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^n e^{-n\theta\bar{x}}\theta^{\alpha-1}e^{-\beta\theta}}{\int_{\mathbb{R}^+}\theta^n e^{-n\theta\bar{x}}\theta^{\alpha-1}e^{-\beta\theta}d\theta} \\ &= \frac{(n\bar{x}+\beta)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}e^{-(n\bar{x}+\beta)\theta}.\end{aligned} \tag{98}$$

Therefore, the Bayes estimator for θ is

$$\hat{\theta}_E = \frac{n+\alpha}{n\bar{x}+\beta}.$$

As for the Bayes estimator for $1/\theta$, we have to find the posterior for $t = 1/\theta$:

$$\pi_t(t|\mathbf{x}) = \pi_\theta(1/t|\mathbf{x})/t^2 = \frac{(n\bar{x}+\beta)^{n+\alpha}}{\Gamma(n+\alpha)}t^{-(n+\alpha+1)}e^{-\frac{(n\bar{x}+\beta)}{t}}.$$

Therefore, the Bayes estimator is

$$\hat{\left(\frac{1}{\theta}\right)}_E = \frac{n\bar{x}+\beta}{n+\alpha-1}.$$

□

Ex. 4.5. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} U(0, \theta),$$

where the prior for θ *is* $U(0, a)$*,* $a > 0$*. Find the Bayes estimator for* θ *under the square error loss.*

The posterior is

$$\begin{aligned}\pi_\theta(\theta|\mathbf{x}) &= \frac{\prod_{i=1}^n f(x_i|\theta)\pi(\theta)}{\int_{\mathbb{R}^+}\prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta} \\ &= \frac{1/\theta^n I(0 < x_{\max} < \theta)1/aI(0 < \theta < a)}{\int_\Theta 1/\theta^n I(0 < x_{\max} < \theta)1/aI(0 < \theta < a)d\theta}.\end{aligned} \tag{99}$$

When $n = 1$,

$$\pi_\theta(\theta|\mathbf{x}) = \begin{cases}\log\frac{x_{\max}}{a}\cdot\frac{1}{\theta}, & x_{\max} < \theta < a, \\ 0, & \text{otherwise.}\end{cases} \tag{100}$$

When $n > 1$

$$\pi_\theta(\theta|\mathbf{x}) = \begin{cases}\frac{n-1}{x_{\max}^{1-n}-a^{1-n}}\cdot\frac{1}{\theta^n}, & x_{\max} < \theta < a, \\ 0, & \text{otherwise.}\end{cases} \tag{101}$$

Therefore, the Bayes estimator is

$$\hat{\theta}_E = \begin{cases} (a - x_{\max}) \log \frac{x_{\max}}{a}, & n = 1, \\ \frac{n-1}{n-2} \cdot \frac{x_{\max}^{2-n} - a^{2-n}}{x_{\max}^{1-n} - a^{1-n}}, & n > 1. \end{cases}$$

□

Ex. 4.6. *Suppose that*

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(0, \tau),$$

where the prior for τ is $\Gamma^{-1}(\alpha, \beta)$. Find the Bayes estimator for τ under the weighted square error loss $L(\tau, \hat{\tau}) = (\tau - \hat{\tau})/\tau^2$.

It is known that the posterior, $\pi(\tau|\mathbf{x})$, is $\Gamma^{-1}(n/2 + \alpha, t + \beta)$ where

$$t = \frac{1}{2} \sum_{i=1}^{n} x_i^2.$$

Therefore, the Bayes estimator is

$$\hat{\tau}_B = \frac{E(\tau w(\tau)|\mathbf{x})}{E(w(\tau)|\mathbf{x})}, \tag{102}$$

where $w(\tau) = \tau^{-2}$, and

$$E(\tau w(\tau)|\mathbf{x}) = \int_{\mathbb{R}^+} \tau^{-1} \pi(\tau|\mathbf{x}) d\tau = \frac{n/2 + \alpha}{t + \beta},$$

$$E(w(\tau)|\mathbf{x}) = \int_{\mathbb{R}^+} \tau^{-2} \pi(\tau|\mathbf{x}) d\tau = \frac{(n/2 + \alpha + 1)(n/2 + \alpha)}{(t + \beta)^2}.$$

As a result, the Bayes estimator is

$$\hat{\tau}_B = \frac{t + \beta}{n/2 + \alpha + 1}.$$

□

Ex. 4.7. *Suppose that $X \sim B(n, \theta)$, and $\theta \sim Be(\alpha, \beta)$. Find the Bayes estimator for θ under the weighted square error loss $L(a, \theta) = (a - \theta)^2/[\theta(1 - \theta)]$.*

It is known that the posterior is $Be(\alpha + x, n - x + \beta)$. Therefore, the Bayes estimator is

$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)},$$

where $w(\theta) = 1/[\theta(1 - \theta)]$, and

$$E(\theta w(\theta)|x) = \int_0^1 (1 - \theta)^{-1} \pi(\theta|x) d\theta = \frac{n + \alpha + \beta - 1}{n - x + \beta - 1},$$

$$E(w(\theta)|x) = \int_0^1 \theta^{-1} (1 - \theta)^{-1} \pi(\theta|x) d\theta = \frac{(n + \alpha + \beta - 1)(n + \alpha + \beta - 2)}{(x + \alpha - 1)(n - x + \beta - 1)}.$$

As a result, the Bayes estimator is

$$\hat{\theta}_B = \frac{x + \alpha - 1}{n + \alpha + \beta - 2}.$$

□

Ex. 4.8. *Suppose that $X \sim \Gamma(n/2, (2\theta)^{-1})$, and $\theta \sim \Gamma^{-1}(\alpha, \beta/2)$. Find the Bayes estimator for θ under the weighted square error loss $L(a, \theta) = (a - \theta)^2/\theta^2$.*

It is known that the posterior, $\pi(\theta|x)$, is $\Gamma^{-1}(n/2+\alpha, (x+\beta)/2)$. Therefore, the Bayes estimator is

$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)},$$

where $w(\theta) = \theta^{-2}$, and

$$E(\theta w(\theta)|x) = \int_{\mathbb{R}^+} \theta^{-1}\pi(\theta|x)d\theta = \frac{n+2\alpha}{x+\beta},$$
$$E(w(\theta)|x) = \int_{\mathbb{R}^+} \theta^{-2}\pi(\theta|x)d\theta = \frac{(n+2\alpha+2)(n+2\alpha)}{(x+\beta)^2}.$$

As a result, the Bayes estimator is

$$\hat{\theta}_B = \frac{x+\beta}{n+2\alpha+2}.$$

□

Ex. 4.9. *Suppose that θ, X and a are all real numbers, and the posterior $\pi(\theta|x)$ is symmetric and uni-modal. The loss function $L(\theta - a)$ is the increasing function of $|\theta - a|$. Show that the Bayes estimator is the mode of $\pi(\theta|x)$.*

First, notice that for a symmetric and uni-modal distribution, the mode is exactly the median. Let the mode and median of $\pi(\theta|x)$ be M.

Next, let the loss function be

$$L(\theta - a) = g(|\theta - a|),$$

where $g(\cdot)$ is an differentiable increasing function.

As a result, the posterior risk can be expressed as

$$R(a|x) = \int_{\mathbb{R}^+} L(\theta, a)\pi(\theta|x)d\theta = \int_{-\infty}^{a} g(a-\theta)\pi(\theta|x)d\theta + \int_{a}^{+\infty} g(\theta - a)\pi(\theta|x)d\theta.$$

Take the first derivative of $R(a|x)$ w.r.t. a,

$$\frac{\partial R(a|x)}{\partial a} = \int_{-\infty}^{a} g'(a-\theta)\pi(\theta|x)d\theta + g(0)\pi(a|x) - \int_{-\infty}^{a} g'(a-\theta)\pi(\theta|x)d\theta - g(0)\pi(a|x) = 0,$$

we have

$$\int_{-\infty}^{a} g'(a-\theta)\pi(\theta|x)d\theta = \int_{a}^{+\infty} g'(\theta - a)\pi(\theta|x)d\theta.$$

Now we want to find the a that satisfies the equation above. Without losing of generality, let $a = M$. Since $\forall\theta \in (-\infty, M)$,

$$g'(M-\theta) = g'((2M-\theta) - M)$$

and

$$\pi(\theta|x) = \pi(2M - \theta|x)$$

where $2M-\theta \in (M,+\infty)$, and the mapping from $(-\infty, M)$ to $(M,+\infty)$

$$\theta \mapsto 2M-\theta$$

is bijective, we can conclude that

$$\int_{-\infty}^{M} g'(M-\theta)\pi(\theta|x)d\theta = \int_{M}^{+\infty} g'(\theta-M)\pi(\theta|x)d\theta.$$

Therefore, M is the Bayes estimator that minimizes the posterior risk. □

Ex. 4.10. *Suppose that the random variable X follows the distribution*

$$P(X=x|p) = C(x+r-1, r-1)p^x(1-p)^r, \quad x=0,1,2,...;\ 0<p<1.$$

Let $\theta = p/(1-p)$, and the loss function be $L(d,\theta) = (d-\theta)^2/[\theta(1+\theta)]$:

(1) If $\pi(\theta)\equiv 1$, find the Bayes estimator using the loss function above, and compare it with the MLE.

(2) If the prior for θ is

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1+\theta)^{-(\alpha+\beta)},$$

find the posterior for θ, and the Bayes estimator.

The likelihood is

$$f(x|\theta) = C(x+r-1, r-1)\theta^x(1+\theta)^{-(r+x)}.$$

(1) The posterior is

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^x(1+\theta)^{-(r+x)}}{\int_{\mathbb{R}^+}\theta^x(1+\theta)^{-(r+x)}d\theta} \\ &= \frac{\Gamma(r+x)}{\Gamma(x+1)\Gamma(r-1)}\theta^x(1+\theta)^{-(r+x)}.\end{aligned} \tag{103}$$

Since the loss function is a weighted square error loss, with $w(\theta) = [\theta(1+\theta)]^{-1}$, we have

$$\begin{aligned}E(\theta w(\theta)|x) &= \int_{\mathbb{R}^+}(1+\theta)^{-1}\pi(\theta|x)d\theta = \frac{r-1}{r+x}, \\ E(w(\theta)|x) &= \int_{\mathbb{R}^+}[\theta(1+\theta)]^{-1}\pi(\theta|x)d\theta = \frac{r(r-1)}{(r+x)x},\end{aligned}$$

and as a result,

$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)} = \frac{x}{r}.$$

On the other hand, consider the log-likelihood

$$\log f(x|\theta) = \log C(x+r-1, r-1) + x\log\theta - (r+x)\log(1+\theta).$$

Take the first derivative of the log-likelihood w.r.t. θ, and set it to 0:

$$\frac{x}{\theta} = \frac{r+x}{1+\theta},$$

we have

$$\hat{\theta}_{MLE} = \frac{x}{r},$$

which is the same as the Bayes estimator.

(2) The posterior is

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\theta^x(1+\theta)^{-(r+x)}\theta^{\alpha-1}(1+\theta)^{-(\alpha+\beta)}}{\int_{\mathbb{R}^+} \theta^x(1+\theta)^{-(r+x)}\theta^{\alpha-1}(1+\theta)^{-(\alpha+\beta)}d\theta} \\ &= \frac{\Gamma(r+x+\alpha+\beta)}{\Gamma(x+\alpha)\Gamma(r+\beta)}\theta^{x+\alpha-1}(1+\theta)^{-(r+x+\alpha+\beta)}.\end{aligned} \tag{104}$$

With the weighted square error loss and $w(\theta) = [\theta(1+\theta)]^{-1}$, we have

$$E(\theta w(\theta)|x) = \int_{\mathbb{R}^+} (1+\theta)^{-1}\pi(\theta|x)d\theta = \frac{r+\beta}{r+x+\alpha+\beta},$$

$$E(w(\theta)|x) = \int_{\mathbb{R}^+} [\theta(1+\theta)]^{-1}\pi(\theta|x)d\theta = \frac{(r+\beta+1)(r+\beta)}{(r+x+\alpha+\beta)(x+\alpha-1)},$$

and as a result,

$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)} = \frac{x+\alpha-1}{r+\beta+1}.$$

□

Ex. 4.11. *Suppose that $X \sim N(\theta, 100)$, and the prior for θ is $N(100, 225)$. Find the Bayes estimator using the linear error loss function*

$$L(d,\theta) = \begin{cases} 3(\theta-d), & d<\theta, \\ d-\theta, & d \geq \theta. \end{cases}$$

It is known that the posterior is

$$\pi(\theta|\mathbf{x}) \sim N(\mu(x), \eta^2)$$

where

$$\mu(x) = \frac{\sigma^2}{\sigma^2+\tau^2}\mu + \frac{\tau^2}{\sigma^2+\tau^2}x = \frac{9x+400}{13}, \quad \eta^2 = \frac{\sigma^2\tau^2}{\tau^2+\sigma^2} = 67.164.$$

Therefore, the Bayes estimator is 0.75 quantile of $N(\mu(x), \eta^2)$, which is

$$\mu(x) + Z_{0.75}\cdot\eta = 0.6923077x + 36.29692.$$

□

Ex. 4.12. *Suppose that*

$$\boldsymbol{X} = (X_1, ..., X_n) \sim (\theta, \Sigma),$$

and denote the estimator of $\theta = (\theta_1, ..., \theta_P)^T$ *by* $a = (a_1, ..., a_p)^T$. *If the prior of* θ *is* $\pi(\theta)$, *and the loss function is*

$$L(a, \theta) = (a - \theta)^T D(a - \theta),$$

where D *is a* $p \times p$ *positive definite matrix. Show that the Bayes estimator is*

$$\hat{\theta}_B = E(\theta|\boldsymbol{x}).$$

The posterior risk is

$$R(\theta|\mathbf{x}) = \int_{\Theta} (a - \theta)^T D(a - \theta)\pi(\theta|\mathbf{x})d\theta.$$

In order to find the a that minimizes $R(\theta|\mathbf{x})$, take the first derivative of $R(\theta|\mathbf{x})$ w.r.t. a, and set it as 0:

$$\nabla_a R(\theta|\mathbf{x}) = \int_{\Theta} (2Da - 2D\theta)\pi(\theta|x)d\theta = 0.$$

So we have

$$\hat{\theta}_B = \hat{a} = \int_{\Theta} \theta\pi(\theta|\mathbf{x})d\theta = E(\theta|x).$$

□

Ex. 4.13. *Suppose that there are* N *products in a batch, among which there are* M *(unknown) defects. If we randomly select* n *products from this batch, we will get* x *defects. Assume that the prior for* M *is the uniform distribution:*

$$P(M = k) = \frac{1}{N+1} \ (k = 0, 1, ..., N).$$

Find the Bayes estimator of the defective rate $p = M/N$ *using the square error loss.*

The likelihood follows a hyper-geometric distribution:

$$f(x|M) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}.$$

As a result, the posterior is

$$\begin{aligned}\pi(M|x) &= \frac{f(x|M)\pi(M)}{\sum_{M=0}^{N} f(x|M)\pi(M)} \\ &= \frac{\binom{M}{x}\binom{N-M}{n-x}}{\sum_{M=0}^{N}\binom{M}{x}\binom{N-M}{n-x}} \\ &= Z\binom{(M-x)+x}{x}\binom{(n-x)+(N-n)-(M-x)}{n-x},\end{aligned} \tag{105}$$

where Z is the normalization factor. Note that if let

$$y = M - x, \quad a = x, \quad b = n - x, \quad c = N - n,$$

Table 1: Parameter transformation

Original	ABC distribution	Beta-binomial
x	a	$\alpha - 1$
$n - x$	b	$\beta - 1$
$N - n$	c	n

then we have

$$\sum_{y=0}^{c} \binom{a+y}{a} \binom{b+c-y}{b} = \binom{a+b+c+1}{a+b+1},$$

indicating that $Z^{-1} = \binom{N+1}{n+1}$. Furthermore, such a distribution is also a beta-binomial distribution with the transformations shown in table 1.

Therefore,

$$E(y|x) = \frac{n\alpha}{\alpha+\beta} = \frac{(N-n)(x+1)}{n+2},$$

and

$$E(p|x) = E(M/N|x) = (E(y|x)+x)/N = \frac{Nx+N-n+2x}{(n+2)N},$$

which is exactly the Bayes estimator. $\square$

Ex. 4.14. *Suppose that $X \sim B(5,\theta)$, and the prior for θ is $Be(1,9)$. If $x = 1$, find the Bayes estimator using the following loss functions:*

(1) $L(a,\theta) = (\theta - a)^2$;

(2) $L(a,\theta) = |\theta - a|$;

(3) $L(a,\theta) = (\theta - a)^2/[\theta(1-\theta)]$;

(4) $L(a,\theta) = \begin{cases} \theta - a, & \theta > a, \\ 2(a-\theta), & \theta \le a. \end{cases}$

The posterior is

$$\begin{aligned} \pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} \\ &= \frac{\Gamma(15)}{\Gamma(x+1)\Gamma(14-x)}\theta^x(1-\theta)^{13-x} \\ &= \frac{\Gamma(15)}{\Gamma(2)\Gamma(13)}\theta(1-\theta)^{12}. \end{aligned} \tag{106}$$

(1) With the square error loss,

$$\hat{\theta}_B = E(\theta|x) = \frac{x+1}{15} = \frac{2}{15}.$$

(2) With the absolute error loss,

$$\hat{\theta}_B = \text{Median}(\theta|x) = 0.1170221.$$

(3) With the weighted square error loss, and the weight $w(\theta) = [\theta(1-\theta)]^{-1}$, we have

$$E(\theta w(\theta)|x) = \int_0^1 (1-\theta)^{-1}\pi(\theta|x)d\theta = \frac{14}{12},$$

$$E(w(\theta)|x) = \int_0^1 [\theta(1-\theta)]^{-1}\pi(\theta|x)d\theta = \frac{13\cdot 14}{12},$$

and as a result,

$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)} = \frac{1}{13}.$$

(4) With the linear error loss function,

$$\hat{\theta}_B = \frac{1}{3}\text{Quantile}(\theta|\text{x}) = 0.08437303.$$

Ex. 4.15. *Consider the Linex loss:*

$$L(d, \theta) = e^{c(\theta-d)} - c(\theta - d) - 1.$$

Show that:

(1) $L(d, \theta) > 0$;

(2) Draw the plot of Linex loss as function of $\theta - d$ when c = 0.1, 0.5, 1.2, respectively;

(3) Find the Bayes estimator based on the Linex loss;

(4) Find the Bayes estimator when

$$X_1, ..., X_n \sim^{\text{i.i.d.}} N(\theta, 1)$$

and $\pi(\theta) \equiv 1$.

(1) According to the Taylor expansion,

$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!} > 1 + x,$$

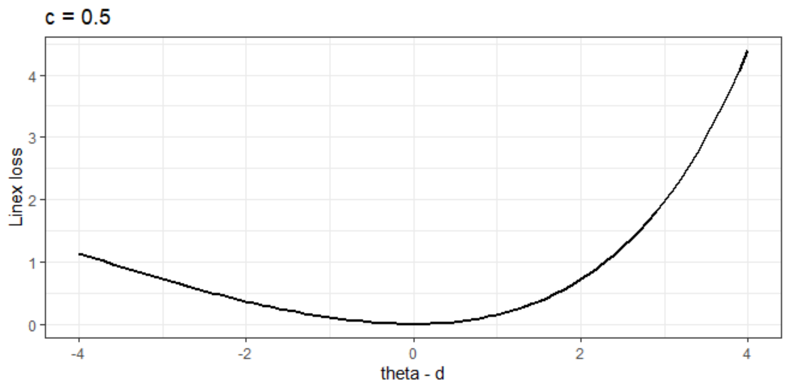
so

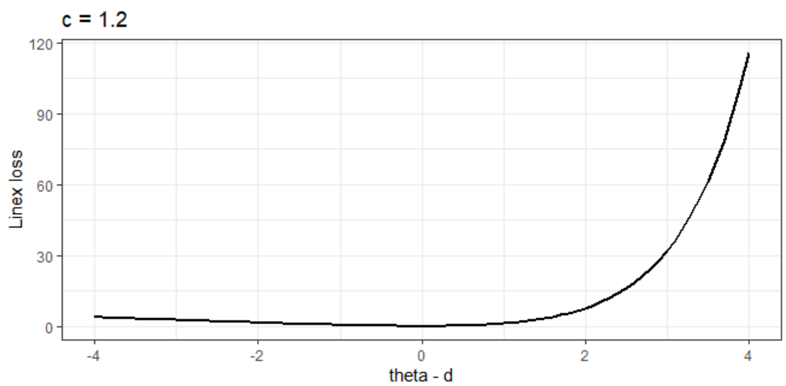
$$e^{c(\theta-d)} > 1 + c(\theta - d),$$

which is

$$L(d, \theta) = e^{c(\theta-d)} - c(\theta - d) - 1 > 0.$$

(2)





(3) The posterior risk can be expressed as

$$\begin{aligned} R(d|x) &= \int_{\mathbb{R}} L(d,\theta)\pi(\theta|x)d\theta \\ &= \int_{\mathbb{R}} \left[e^{c(\theta-d)} - c(\theta-d) - 1\right] \pi(\theta|x)d\theta. \end{aligned} \tag{107}$$

Take the first derivative of $R(d|x)$ w.r.t. d,

$$\begin{aligned} \frac{\partial R(d|x)}{\partial d} &= -c\int_{\mathbb{R}} e^{c\theta}e^{-cd}\pi(\theta|x)d\theta + c\int_{\mathbb{R}} \pi(\theta|x)d\theta \\ &= -ce^{-cd}E(e^{c\theta}|x) + c \\ &= 0, \end{aligned} \tag{108}$$

we have

$$\hat{\theta}_B = d^* = \frac{1}{c}\log E(e^{c\theta}|x).$$

(4) It is known that the posterior on θ is $N(\bar{x}, 1/n)$. As a result,

$$\begin{aligned} E(e^{c\theta}|x) &= \int_{\mathbb{R}} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}(\theta-\bar{x})^2\right\} \exp\{c\theta\}\, d\theta \\ &= \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} \exp\left\{-\frac{n}{2}(\theta-\bar{x}-\frac{c}{n})^2\right\} \exp\left\{c\bar{x}+\frac{c^2}{2n}\right\} d\theta \\ &= \exp\left\{c\bar{x}+\frac{c^2}{2n}\right\}. \end{aligned} \tag{109}$$

Thus,
$$\hat{\theta}_B = \frac{1}{c}\log E(e^{c\theta}|x) = \bar{x} + \frac{c}{2n}.$$

□

Ex. 4.16. *Suppose that $X \sim \Gamma(\alpha, 1/\theta)$. The prior on θ is $\pi(\theta) = 1/\theta \cdot I(0 < \theta < +\infty)$. Find the Bayes estimator for θ using the weighted square error loss $L(d,\theta) = (d-\theta)^2/\theta^2$.*

The posterior is
$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int_{\mathbb{R}^+} f(x|\theta)\pi(\theta)d\theta} = \frac{x^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-x/\theta}.$$

With the weight $w(\theta) = \theta^{-2}$, we have
$$E(\theta w(\theta)|x) = \int_{\mathbb{R}^+} \theta^{-1}\pi(\theta|x)d\theta = \frac{\alpha}{x},$$
$$E(w(\theta)|x) = \int_{\mathbb{R}^+} \theta^{-2}\pi(\theta|x)d\theta = \frac{(\alpha+1)\alpha}{x^2},$$

and as a result,
$$\hat{\theta}_B = \frac{E(\theta w(\theta)|x)}{E(w(\theta)|x)} = \frac{x}{\alpha+1}.$$

□