# How to derive (8.27)

Suppose that $z \sim N(H\beta, \sigma^2 I)$,
$\beta \sim N(0, \tau\Sigma)$.

Then the prior is:

$$\pi(\beta) = \frac{1}{(2\pi)^{p/2}|\tau\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\beta^T(\tau\Sigma)^{-1}\beta\right\},$$

and the log-likelihood is:

$$f(z|\beta) = \frac{1}{(2\pi)^{N/2}|\sigma^2 I|^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(z-H\beta)^T(z-H\beta)\right\}.$$

Therefore, the posterier is:

$$\pi(\beta|z) = \frac{f(z|\beta)\,\pi(\beta)}{\int_{-\infty}^{+\infty} f(z|\beta)\,\pi(\beta)\,d\beta},$$

$$= \frac{\text{Const} \cdot \exp\left\{-\frac{1}{2}\beta^T(\tau\Sigma)^{-1}\beta - \frac{1}{2\sigma^2}(z-H\beta)^T(z-H\beta)\right\}}{m(z)}.$$

Let

$$\begin{aligned} N^* &= \beta^T(\tau\Sigma)^{-1}\beta + \frac{1}{\sigma^2}(z-H\beta)^T(z-H\beta) \\ &= \beta^T(\tau\Sigma)^{-1}\beta + \frac{1}{\sigma^2}(\beta^T H^T H\beta - 2\beta^T H^T z + z^T z) \\ &= \beta^T\left[(\tau\Sigma)^{-1} + \frac{1}{\sigma^2}H^T H\right]\beta - \frac{2}{\sigma^2}\beta^T H^T z + \frac{z^T z}{\sigma^2}. \end{aligned}$$

Let $A = (\tau\Sigma)^{-1} + \frac{1}{\sigma^2}H^T H$, then

$$\begin{aligned} N^* &= \beta^T A\beta - \frac{2}{\sigma^2}\beta^T H^T z + \frac{z^T z}{\sigma^2} \\ &= \left(\beta^T - \frac{z^T H}{\sigma^2}A^{-1}\right)A\left(\beta - A^{-1}\frac{H^T z}{\sigma^2}\right) + \text{Const2} \end{aligned}$$

NOT a function of $\beta$

By normalizing $\exp\left\{-\frac{1}{2}N\right\}$, we have

$$\pi(\beta|z) \sim N\left(A^{-1}\frac{H^T z}{\sigma^2}, A^{-1}\right),$$

where $A^{-1}\frac{H^T z}{\sigma^2} = \left(H^T H + \frac{\sigma^2}{\tau}\Sigma\right)^{-1}H^T z$,

and $A^{-1} = \left(H^T H + \frac{\sigma^2}{\tau}\Sigma^{-1}\right)^{-1}\sigma^2$. $\square$