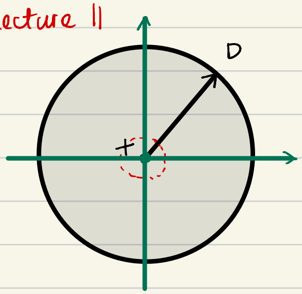
Lecture 11



[One line of argument]

Whether should I believe in this theory?

H: Uniform over the board

$P(\text{land within } 1'' \text{ of the center}) = \frac{1}{144} < 1\%$

surprising

[Another line of argument]

Which theory is more plausible?

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if } (x,y) \in D \\ 0, & \text{o.w.} \end{cases}, \xrightarrow[\text{variable}]{\text{change of}} f_{R\Theta}(r,\theta) = \frac{r}{\pi}, \quad \begin{matrix} 0<r<1 \\ 0<\theta<2\pi \end{matrix}$$

$$f_R(r) = \frac{r}{\pi}\int_0^{2\pi} d\theta = 2r \equiv Be(2,1)$$

$\mathcal{U} \sim$ unif insider unit $\underbrace{\text{hyper-sphere}}_{\text{center at } 0}$ in $\mathbb{R}^p$,

$\Rightarrow$

$$\|\mathcal{U}\| \sim Be(p,1) \longrightarrow \mathbb{E}\|\mathcal{U}\| = \frac{p}{p+1}$$

$$\mathrm{Var}\|\mathcal{U}\| = \frac{p}{(p+1)^2(p+2)} = \frac{1}{p^2}$$

$\Rightarrow$ curse of dimensionality

Uniformly distributed points in high-dimensional sphere are all close to the shell.

$$H^*: f^*_{R\Theta}(r,\theta) = \overbrace{\frac{1-r}{\pi}}^{\text{Beta}(1,2)}, \quad \begin{matrix} 0<r<1 \\ 0<\theta<2\pi \end{matrix}$$

$$f^*_{XY}(x,y) = \frac{1-\sqrt{x^2+y^2}}{\pi\sqrt{x^2+y^2}}, \quad (x,y) \in D.$$

For H, $f_{R\Theta}(\frac{1}{12},\theta) = \frac{1}{12\pi} \approx$ prob

For H^*, $f^*_{R\Theta}(\frac{1}{12},\theta) = \frac{11}{12\pi} \approx$ prob

$\rightarrow$ Score: likelihood $X \sim P$, $P \in \mathcal{P}$

assume: either all P are discrete
OR all P are cont.
$P \leftrightarrow p(x)$ pdf/pmf

$X = x_{obs}$, $\quad L(P) = \text{const} \times p(x_{obs})$

$$\frac{L(P_1)}{L(P_2)} = \frac{p_1(x_{obs})}{p_2(x_{obs})}$$

$\bar{x} = 143$
$s^2 = 40$ $\quad (x_1,\dots,x_n) \mid (\bar{x}=143,\ s^2=40) \leftarrow N(\mu,\sigma^2)$

→ Food expenditure

$X:\ x_1,\dots,x_n \overset{iid}{\sim} N(\mu,\sigma^2),\quad \mu\in\mathbb{R},\quad \sigma^2>0$

joint pdf :

$$p(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\},\quad (x_1,\dots,x_n)^T\in\mathbb{R}^n$$
$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$
$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left[(n-1)s^2+n(\bar{x}-\mu)^2\right]\right\}$$

Any summary of data = statistic

→ Sufficient statistic : $T=T(X)$ is suff. if

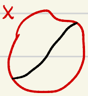
the condition distn of X given T is the same under every $P\in\mathcal{P}$.

$$Y=g(X),\quad f_{Y|X}(y|x) = \begin{cases} 1, & \text{if } y=g(x) \\ 0, & \text{ow} \end{cases}$$

Now consider $T=T(X)$.

$$f_X(x|T=t) = \frac{f_T(t|X=x)\cdot f_X(x)}{f_T(t)} = \begin{cases} \dfrac{f_X(x)}{\underbrace{\cancel{f_T(t)}}_{\text{const}}}, & \text{if } T(x)=t \\ 0, & \text{o.w.} \end{cases}$$

$f_T(t)$ is still a function of θ.

X  $T(X)=t$

Work on Normal

$$f_X(x|\bar{x}=a,(n-1)s^2=b^2) = \text{const}\cdot f_X(x) \quad \text{on } \bar{x}=a,\ \sum_{i=1}^{n}(x_i-\bar{x})^2=b$$
$$= \text{const}\cdot(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}(b^2+n(a-\mu)^2)\right\}$$

Denote $S_{a,b} = \{x=(x_1,\dots,x_n)^T\in\mathbb{R}^n:\ \bar{x}=a,\ \sum_{i=1}^{n}(x_i-\bar{x})^2=b\}$

Then

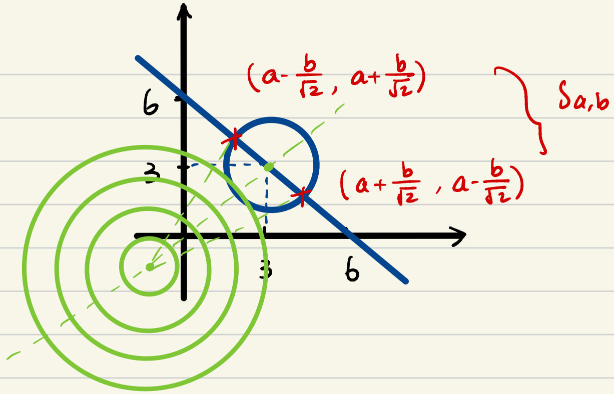
$$X\mid(\bar{x}=a,(n-1)s^2=b^2) \sim \text{Unif}(S_{a,b}) \quad \text{For every } (\mu,\sigma^2)$$

$$f_X(x|\bar{x}=a,(n-1)s^2=b^2,\mu,\sigma^2) = \begin{cases} \dfrac{1}{\text{Vol}(S_{a,b})}, & x\in S_{a,b} \\ 0, & \text{ow.} \end{cases}$$

$n=2,\quad X=(x_1,x_2),\quad S_{a,b} = \{(x_1,x_2):\ x_1+x_2=2a,\ (x_1-a)^2+(x_2-a)^2=b^2\}$

$$\begin{cases} a=3 \\ b=2 \end{cases}$$

$\mu = -1$
$\sigma = 1$



$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$$