Lecture 09

Cont' Lecture 08

$$g\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix} = \frac{e-ab}{\sqrt{c-a^2}\sqrt{d-b^2}}, \quad \mu = \begin{pmatrix}0\\0\\1\\1\\ \rho\end{pmatrix}, \quad g(\mu) = \rho.$$

$$\begin{pmatrix}\tilde{X}\\ \tilde{Y}\end{pmatrix} \Rightarrow \begin{pmatrix} X = \dfrac{\tilde{X}-m_X}{S_X} \\ Y = \dfrac{\tilde{Y}-m_Y}{S_Y}\end{pmatrix}$$

$$\dot{g} = \begin{pmatrix}\frac{\partial}{\partial a} g\\ \vdots \\ \frac{\partial}{\partial e} g\end{pmatrix} = \begin{bmatrix}0\\0\\-\rho/2\\-\rho/2\\1\end{bmatrix}:$$

$$\frac{\partial g}{\partial a} = -\frac{b}{\sqrt{c-a^2}\sqrt{d-b^2}} + \frac{e-ab}{\sqrt{d-b^2}} \times \left(-\frac{1}{2}\right)(c-a^2)^{-\frac{3}{2}}(-2a) = 0$$

$$\frac{\partial g}{\partial b} = -\frac{a}{\sqrt{c-a^2}\sqrt{d-b^2}} + \frac{e-ab}{\sqrt{c-a^2}} \times \left(-\frac{1}{2}\right)(d-b^2)^{-\frac{3}{2}}(-2b) = 0$$

$$\frac{\partial g}{\partial c} = \frac{e-ab}{\sqrt{d-b^2}} \times \left(-\frac{1}{2}\right)(c-a^2)^{-\frac{3}{2}} = -\frac{\rho}{2}$$

$$\frac{\partial g}{\partial d} = \frac{e-ab}{\sqrt{c-a^2}} \times \left(-\frac{1}{2}\right)(d-b^2)^{-\frac{3}{2}} = -\frac{\rho}{2}$$

$$\frac{\partial g}{\partial e} = \frac{1}{\sqrt{c-a^2}\sqrt{d-b^2}} = 1$$

Recall: $\Sigma = \begin{bmatrix} 1 & \rho & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 2 & 2\rho^2 & 2\rho^2 \\ & & & 2 & 2\rho \\ & & & & \rho^2+1 \end{bmatrix}$

For special case

$$(X, Y) \sim N_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$$

Then $$g(\mu)^T \Sigma g(\mu) = \begin{bmatrix}-\rho/2 & -\rho/2 & 1\end{bmatrix}\begin{bmatrix}2 & 2\rho^2 & 2\rho^2\\ 2\rho^2 & 2 & 2\rho\\ 2\rho & 2\rho & \rho^2+1\end{bmatrix}\begin{bmatrix}-\rho/2\\ -\rho/2\\ 1\end{bmatrix} = (1-\rho^2)^2$$

Conclusion:

$$\sqrt{n}(R_n - \rho) \Rightarrow N(0, (1-\rho^2)^2)$$

Additional problem 9.2

$$X = \begin{bmatrix}X_1\\ \vdots\\ X_d\end{bmatrix} \sim \text{Multi}(n, p), \quad p = (p_1, \dots, p_d)^T \in \mathring{\Delta}_d \quad \left(p_i > 0, \ \sum_{i=1}^{n} p_i = 1\right)$$

Consider $T = \sum_{i=1}^{d} \frac{(X_i - np_i)^2}{np_i}$

Bring in $z_j = \begin{bmatrix} z_{j1} \\ \vdots \\ z_{jd} \end{bmatrix} \overset{iid}{\sim} \text{Multi}(1, p), \quad j = 1, 2, \ldots$

Each z_j indicates where object j went,

Let $X \overset{d}{=} z_1 + \cdots + z_n$, $\quad \frac{1}{n} X = \bar{z}_n$

MV-CLT then $\sqrt{n}\left\{ \frac{1}{n} X - \mathbb{E}(z_1) \right\} \Rightarrow N(0, Var(z_1))$

where

$z_{1i} \in \{0, 1\}$

$$\mathbb{E}(z) = \begin{bmatrix} \mathbb{E}(z_{11}) \\ \vdots \\ \mathbb{E}(z_{1d}) \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_d \end{bmatrix}, \text{ since } p(z_{1i} = 1) = p(\text{obj } 1 \text{ in cat } i) = p_i$$

$$Var(z) = \begin{bmatrix} \cancel{Var(z_{11})} \; p_1(1-p_1) & & \\ & \ddots \quad Cov(z_{1i}, z_{1m}) & \\ & & \cancel{Var(z_{1d})} \; p_d(1-p_d) \end{bmatrix},$$

$$Cov(z_{1i}, z_{1m}) = \mathbb{E}(z_{1i} z_{1m}) - \mathbb{E}(z_{1i}) \cdot \mathbb{E}(z_{1m}) = 0 - p_i p_m$$

Consider

$$\mathbf{1}^T \left(\frac{1}{n} X\right) = 1 \quad \text{w/ prob } 1.$$

$$\mathbf{1}^T Var\left(\frac{1}{n} X\right) \mathbf{1} = 0 \quad \Rightarrow \quad \mathbf{1}^T \Sigma \mathbf{1} = 0.$$

Sum of all elements of Σ

$= \text{sum(diags)} + \text{sum(off diag)}$

$= 1 - \sum_{i=1}^{d} p_i^2 - 2 \sum_{i<m} p_i p_m = 0.$

Now: $\sqrt{n}\left(\frac{1}{n} X - p\right) = \frac{1}{\sqrt{n}}(X - np) = \frac{1}{\sqrt{n}} \begin{bmatrix} X_1 - np_1 \\ \cdots \\ X_d - np_d \end{bmatrix} = W_n$

Then $T = \frac{(X_1 - np_1)^2}{np_1} + \cdots + \frac{(X_d - np_d)^2}{np_d} = Y_1^2 + \cdots + Y_d^2$

where

$$Y_i = \frac{X_i - np_i}{\sqrt{np_i}}.$$

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_d \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{p_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{p_d}} \end{bmatrix} \cdot \left\{ \sqrt{n}\left(\frac{1}{n} X - p\right) \right\} = A W_n \Rightarrow N(0, A \Sigma A^T)$$

by CMT

$$T = Y_1^2 + \cdots + Y_d^2 \Rightarrow V_1^2 + \cdots + V_d^2, \text{ where } V = \begin{bmatrix} V_1 \\ \vdots \\ V_d \end{bmatrix} \sim N(0, A \Sigma A^T)$$

again by CMT.

and $A \Sigma A^T = I_d - q q^T$, where $q = (\sqrt{p_1}, \ldots, \sqrt{p_d})^T$.

$\downarrow$

This is idempotent, rank(H) = d-1

If $V \sim N_d(0, H)$, $V_1^2 + \cdots + V_d^2 \sim$?

idempotent : $H^2 = H$, $V_1^2 + \cdots + V_d^2 \sim \chi^2_{\text{rank}(H)}$