Lecture 1/2

* Axioms

① $\mathbb{P}(A) \geq 0$

② $\mathbb{P}(\Omega) = 1$

③ $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if $A \cap B = \phi$.

Triple: $(\Omega, \mathcal{E}, \mathbb{P})$ $\mathbb{P}: \mathcal{E} \longrightarrow \mathbb{R}$ (probability map) assigns a numeric score to each event in $\mathcal{E}$

collects all outcomes = sample space (Ω); event space ($\mathcal{E}$) = a collection of subsets of Ω

Ex: Roll a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathbb{P}(\text{roll } 2) = \mathbb{P}(\{2\})$

$\mathbb{P}(\text{roll an even number}) = \mathbb{P}(\{2, 4, 6\})$

$\mathbb{P}(\text{roll a prime number}) = \mathbb{P}(\{2, 3, 5\})$

We can let $\mathcal{E} = \{\text{all subsets of } \Omega\} = 2^{\Omega}$

Event space $\mathcal{E}$ needs to be a σ-algebra

Axioms (again)

(1) $\mathbb{P}(A) \geq 0$

(2) $\mathbb{P}(\Omega) = 1$

(3) $\mathbb{P}(\underbrace{A_1 \cup A_2 \cup \cdots}) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$

disjoint, countable: either finite, or countably infinite

"coherent": closed under

(a) complementations

(b) countable unions

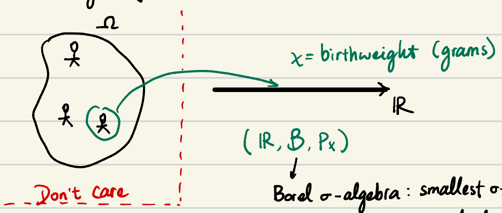
(c) countable intersections

$\Omega \in \mathcal{E}$, $\phi \in \mathcal{E}$

* Random Variables

$X: \Omega \rightarrow \mathbb{R}$

Birthweight of human babies



$(\mathbb{R}, \mathcal{B}, P_X)$

$\mathcal{B}$: Borel σ-algebra: smallest σ-algebra which includes all intervals of $\mathbb{R}$

$Y : \Omega \to \mathbb{R}$

Birth height

$(\mathbb{R}, \mathcal{B}, P_Y) \longrightarrow \mathbb{R}$

Then for $(X, Y) \in \mathbb{R}^2$,

the probability space is $(\mathbb{R}^2, \mathcal{B}_2, P)$

↓ Borel σ-algebra

* application : P_1 & P_2 on $(\mathbb{R}, \mathcal{B})$

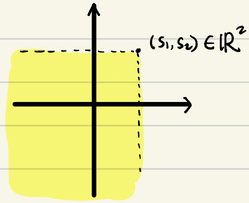
$P_1 = P_2 \iff P_1((a,b)) = P_2((a,b)) \quad \forall -\infty < a < b < \infty.$

$\iff P_1((-\infty, x]) = P_2((-\infty, x]), \quad \forall -\infty < x < +\infty.$

↘ CDF ↙

* CDF of $(\mathbb{R}^2, \mathcal{B}_2, P)$

$F(s_1, s_2) = P(\,\cdot\,)$



* High dimensional pmf : multinomial

For $(X_1, \ldots, X_d)$, $\quad f(x_1, \ldots, x_d) = \dfrac{n!}{x_1! \cdots x_d!} p_1^{x_1} p_2^{x_2} \cdots p_d^{x_d}$

where $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$, $p_i \geq 0$, $\sum_{i=1}^{d} p_i = 1$.

$x = (x_1, \ldots, x_d) \in \mathbb{N}_0^d$, $\quad \sum_{i=1}^{d} x_i = n$

* High dimensional pdf :

(1) MV Normal distribution

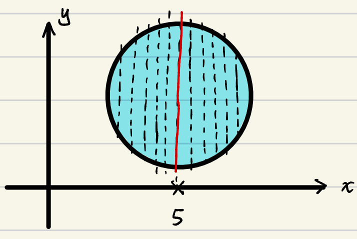
(2) Dirichlet distribution $\leftrightarrow$ Beta

$$f(x_1, \ldots, x_{d-1}) = x_1^{a_1 - 1} \times \cdots \cdot x_{d-1}^{a_{d-1}-1} \times (1 - x_1 - \cdots - x_{d-1})^{a_d - 1} \cdot \frac{1}{D(a_1, \ldots, a_d)}$$

where $x_i \geq 0$, $x_1 + \cdots + x_{d-1} \leq 1$,

$$D(a_1, \ldots, a_d) = \frac{\Gamma(a_1) \times \cdots \times \Gamma(a_d)}{\Gamma(a_1 + \cdots + a_d)}$$

*



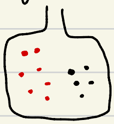
$Y \mid (X=5)$ is a part of partition

↓ a member of the slicing collection

The conditional distribution can be different by how you see the red segment as a part of partition.

* Illustration of $(\Omega, \mathcal{E}, \mathbb{P})$ v.s. $(\mathbb{R}, \mathcal{B}, P_X)$

Polya urn



(1) Grab a ball

(2) Note the color and put it back

(3) Add one more ball of the same color.

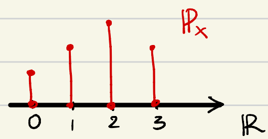
3 repeats

$\mathbb{P} : \mathcal{E} \to \mathbb{R}$ $\quad (\Omega, \mathcal{E}, \mathbb{P})$

$X : \Omega \to \mathbb{R}$ $\quad (\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$

Ω	$\mathbb{P}$	X = # reds
RRR	$\frac{7}{12} \times \frac{8}{13} \times \frac{9}{14}$	3
RRB	$\frac{7}{12} \times \frac{8}{13} \times \frac{5}{14}$	2
RBR	$\vdots$	2
BRR		2
RBB	$\frac{7}{12} \times \frac{5}{13} \times \frac{6}{14}$	1
BRB	$\vdots$	1
BBR		1
BBB	$\frac{5}{12} \times \frac{6}{13} \times \frac{7}{14}$	0

Don't need anymore



$\mathbb{P}_X(A) = \mathbb{P}(\{X \in A\})$, $\{X \in A\} \in \mathcal{E}$

* Dirichlet $\mathrm{Dir}(a_1, a_2, \dots, a_d)$

$\downarrow$ prob distn on

$$\Delta^d = \{ \underbrace{(x_1, \dots, x_d)}_{\text{prob. vector}} \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^{d} x_i = 1 \}$$