Lecture 14

Weekly food expenditure

$n = 22, \quad \bar{X} = 143.64, \quad S = 41.44$

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2), \quad \mu \in \mathbb{R}, \ \sigma > 0$

$H_0: \mu \le 140$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

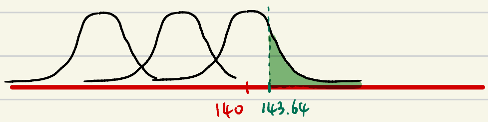
$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$

if $W \sim N(0,1)$, $V \sim \chi^2_k$, $W \perp\!\!\!\perp V$ then $\frac{W}{\sqrt{V/k}} \sim t(k)$

Options: (Fisher) $T_1 = \bar{X}, \quad T_2 = \bar{X} - 140, \quad T_3 = ?$

obs: $t_1 = 143.64, \quad t_2 = 3.64$

$$p_1 = \max_{\substack{\mu \le 140 \\ \sigma > 0}} \mathbb{P}(\bar{X} \ge 143.64 \mid \mu, \sigma^2) = \max_{\substack{\mu \le 140 \\ \sigma > 0}} \left\{ 1 - \Phi\left(\frac{143.64 - \mu}{\sigma/\sqrt{n}} \right) \right\}$$



$$p_1 = \max_{\sigma > 0} \mathbb{P}(T_1 \ge 143.64 \mid \mu = 140, \sigma)$$

$$= \max_{\sigma > 0} \left\{ 1 - \Phi\left(\frac{143.64 - 140}{\sigma/\sqrt{n}} \right) \right\}$$

$= \frac{1}{2}$ (attained in the limit as $\sigma \to +\infty$)

So

$$p_1 = p_1(t_1) = \begin{cases} 1/2, & t_1 \ge 140 \quad (\text{attained } \sigma \to \infty) \\ 1, & t_1 < 140 \quad (\text{attained } \sigma \to 0) \end{cases}$$

It doesn't quantify the evidence in the data in favor or against H_0.

(N-P) $T_1 = \bar{X}$

$H_0: \mu \le 140$

$H_1: \mu > 140$

$$\delta(X) = \begin{cases} \text{accept } H_1 \text{ if } \bar{X} \ge c \\ \text{accept } H_0 \text{ if } \bar{X} < c \end{cases}$$

$$\alpha = \max_{\substack{\mu \le 140 \\ \sigma > 0}} \mathbb{P}(\bar{X} \ge c \mid \mu, \sigma^2)$$

$$= \max_{\sigma > 0} \mathbb{P}(\bar{X} \ge c \mid \mu = 140, \sigma^2)$$

$$= \max_{\sigma > 0} \left\{ 1 - \Phi\left(\frac{c - 140}{\sigma/\sqrt{n}} \right) \right\} = \frac{1}{2} \quad (c \ge 140)$$

You cannot control α, so $T_1 = \bar{X}$ is not valid.

Similarly, $T_2 = \bar{X} - 140$ is also not valid. $(p_2 = p_1)$

(Fisher) Here comes out the $T_3 = \dfrac{\bar{X}-140}{S/\sqrt{n}}$, $t_3 = 0.41$

$$p_3 = \max_{\substack{\mu \le 140 \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-140}{S/\sqrt{n}} \geq t \,\middle|\, \mu, \sigma \right)$$

$$= \max_{\substack{\mu \le 140 \\ \sigma > 0}} \underbrace{\mathbb{P}\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} \geq t + \frac{140-\mu}{S/\sqrt{n}} \,\middle|\, \mu, \sigma \right)}_{\le \mathbb{P}\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} \geq t \,\middle|\, \mu,\sigma \right),\ \forall\, \mu \le 140}$$

$$= \max_{\sigma > 0} \mathbb{P}\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} \geq t \,\middle|\, \mu, \sigma \right)$$

$$= \max_{\sigma > 0} \left\{ 1 - \Phi_{n-1}(t) \right\} = 1 - \Phi_{n-1}(t)$$

Since $t_3 = 0.41$, $p_3 = 0.34$

(N-P) $H_0: \mu \le 140$, vs $H_1: \mu > 140$

$$\delta = \begin{cases} \text{acc } H_1, & \dfrac{\bar{X}-140}{S/\sqrt{n}} \geq c \\ \text{acc } H_0, & \dfrac{\bar{X}-140}{S/\sqrt{n}} < c \end{cases}$$

$$\alpha = \max_{\substack{\mu \le 140 \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-140}{S/\sqrt{n}} \geq c \,\middle|\, \mu, \sigma \right) = 1 - \Phi_{n-1}(c)$$

$\alpha = 1\%$, $\quad 0.01 = 1 - \Phi_{n-1}(c) \Rightarrow c = \Phi_{n-1}^{-1}(0.99) = t_{n-1,\,0.99}$

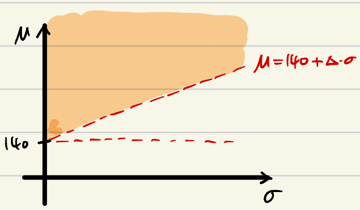
$$\beta = \max_{\substack{\mu > 140 \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-140}{S/\sqrt{n}} < c \,\middle|\, \mu, \sigma \right)$$

$$= \max_{\substack{\mu > 140 \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-140}{S/\sqrt{n}} < t_{n-1,\,0.99} \,\middle|\, \mu, \sigma \right) = 1 - \alpha = 0.99$$

So we have to consider a gap:

$$\beta = \max_{\substack{\mu > 140 + \Delta\cdot\sigma \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-140}{S/\sqrt{n}} < t_{n-1,\,0.99} \,\middle|\, \mu, \sigma \right)$$

[some fixed effect size $\Delta > 0$]



$$\beta = \max_{\substack{\mu > 140 + \Delta\cdot\sigma \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{n-1,\,0.99} + \frac{140-\mu}{S/\sqrt{n}} \,\middle|\, \mu, \sigma \right)$$

$P(A) \leq \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \leq \mathbb{P}(B) + \mathbb{P}(A \cap B^c)$

Now consider $T \sim t(n-1)$

$$\mathbb{P}\left(T < t - \frac{c}{s/\sqrt{n}}\right) \quad \text{where } c = \mu - 140 > \Delta \cdot \sigma$$

$$\leq \mathbb{P}(S > 2\sigma) + \mathbb{P}\left(T < t - \frac{c \cdot \sqrt{n}}{2\sigma}\right)$$

$$\leq \mathbb{P}(S > 2\sigma) + \mathbb{P}\left(T < t - \frac{\Delta\sqrt{n}}{2}\right)$$

So

$$\beta = \max_{\substack{\mu > 140 + \Delta \cdot \sigma \\ \sigma > 0}} \mathbb{P}\left(\frac{\bar{X} - \mu}{s/\sqrt{n}} < t_{n-1,\, 0.99} + \frac{140 - \mu}{s/\sqrt{n}} \,\middle|\, \mu, \sigma\right)$$

$$\leq \max_{\substack{\mu > 140 + \Delta \cdot \sigma \\ \sigma > 0}} \left\{ \underbrace{\mathbb{P}_{\mu,\sigma}(S > 2\sigma)}_{\text{free of } \mu,\sigma} + \underbrace{\mathbb{P}_{\mu,\sigma}\left(\frac{\bar{X} - \mu}{S/\sqrt{n}} < t - \frac{\Delta \cdot \sqrt{n}}{2}\right)}_{\text{free of } \mu,\sigma.} \right\}$$

$$\mathbb{P}_{\mu,\sigma}\left(\frac{(n-1)S^2}{\sigma^2} > 4(n-1)\right)$$