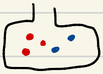
Lecture 07

- $Y_n \xrightarrow{P} Y$ (WLLN)
- $Y_n \Rightarrow Y$ (CLT)

Ex. 7 (Poly Urn)



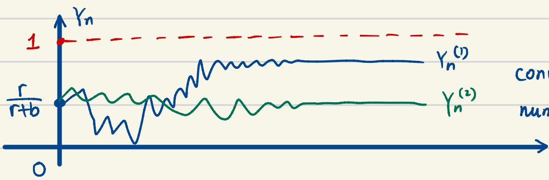
Repeat : ① Grab a ball at random
② Note its color and put it back
③ Add a new ball of the same color.

Draws are successively dependent.

$P(\text{R on 1st draw}) = \frac{r}{r+b}$

$P(\text{R on 100th draw}) = \frac{r}{r+b}$

Define Y_n = fraction of Red ball after n repeats



converge to different limits
numbers are distinct w/ prob. 1.

There exists a RV Y (on $(0,1)$) such that $Y_n \xrightarrow{P} Y$.

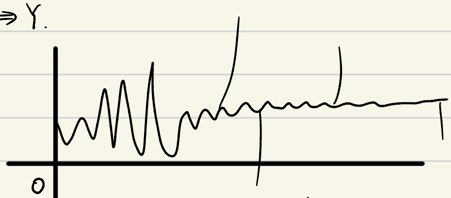
(which means $Y_n \Rightarrow Y$)

✱ $Y \sim Be(r, b)$ $P(Y_{1000} > \frac{1}{2}) \simeq 1 - F_{Be}(\frac{1}{2}, r, b)$

* Convergence almost surely $Y_n \rightarrow Y$ a.s. $P(\lim_{n\to\infty} Y_n = Y) = 1$

$\Downarrow$ $Y_n \xrightarrow{P} Y$

$\Downarrow$ $Y_n \Rightarrow Y$.



converge in probability.

Consider: $X_i \overset{iid}{\sim} Ber(1/2)$, $Y_n = \begin{cases} \bar{X}_n, & w/ \text{ prob. } 1-\frac{1}{n} \\ X_n, & w/ \text{ prob. } 1/n \end{cases}$

$P(|Y_n - \frac{1}{2}| > \varepsilon) \to 0$ Y_n converges in Probability, but not a.s.

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* Suppose $X_1, \dots, X_n, \dots \overset{iid}{\sim} f(x)$, $\mathbb{E}(X_i) = \mu$, $Var(X_i) = \sigma^2$

$\sqrt{n}(\bar{X}_n - \mu) \Rightarrow N(0, \sigma^2)$

Known:

$\underset{\smile}{Y_n \Rightarrow Y}$ iff $\varphi_{Y_n}(t) \longrightarrow \varphi_Y(t)$ pointwise $\forall t$ in an open interval around 0.

(assuming $\varphi_Y(t) < \infty$ in an open interval around 0).

$\Leftrightarrow$ $\mathbb{E}[g(Y_n)] \to \mathbb{E}[g(Y)]$

for every cont. func. $g(y)$

$\Leftrightarrow$ $\lim_{n\to\infty} P(Y_n \in A) \geq P(Y \in A)$

for all open sets $A \subset \mathbb{R}$.

$\varphi_Z(t) = \mathbb{E}[e^{tZ}]$, $\dot{\varphi}_Z(t) = \frac{d}{dt}\mathbb{E}[e^{tZ}]$

$\dot{\varphi}_Z(0) = \mathbb{E}(X)$, $\ddot{\varphi}_Z(0) = \mathbb{E}(Z^2)$, $\dddot{\varphi}_Z(0) = \mathbb{E}(Z^3)$, $\cdots$

Write using Taylor's : $\varphi_Z(t) = \varphi_Z(0) + t \cdot \dot{\varphi}_Z(0) + \frac{1}{2}t^2 \cdot \ddot{\varphi}_Z(0) + \frac{1}{3!}t^3 \cdot \dddot{\varphi}_Z(0) + \cdots$

Define $Z_i = X_i - \mu$, then $\mathbb{E}(Z_i) = 0$, $Var(Z_i) = \sigma^2$,

So $\varphi_{Z_i}(t) = 1 + 0 + \frac{1}{2}t^2\sigma^2 + \cdots$

$\varphi_{Z_1 + \cdots + Z_n}(t) = [\varphi_{Z_i}(t)]^n = (1 + \frac{1}{2}t^2\sigma^2 + \cdots)^n$

$Y_n = \sqrt{n}(\bar{X}_n - \mu) = \sqrt{n}\,\bar{Z}_n = \frac{1}{\sqrt{n}}(Z_1 + \cdots + Z_n)$,

so $\varphi_{Y_n}(t) = \varphi_{Z_1 + \cdots + Z_n}(\frac{t}{\sqrt{n}})$

$= (1 + \frac{1}{2}\frac{t^2}{n}\sigma^2 + \frac{r(t)}{n^{3/2}})^n$

$\xrightarrow{n\to\infty} e^{\frac{1}{2}t^2\sigma^2}$ mGF of $N(0, \sigma^2)$

$\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a$

$\lim_{n\to\infty} (1 + \frac{a}{n} + \varepsilon_n)^n = e^a$ when $n\varepsilon_n \to 0$.

CLT requires the existence of σ^2 ($<\infty$).

* $S_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ (Assume $\mathbb{E}(X_i^4) < \infty$)

$= \overline{X_n^2} - \bar{X}_n^2 \xrightarrow{P} g(\mu, \sigma^2) - \mu^2$

WLLN $\left[\begin{array}{l} \overline{X_n^2} \xrightarrow{P} \mathbb{E}(X_i^2) = g(\mu, \sigma^2), \\ \bar{X}_n \xrightarrow{P} \mu. \end{array} \right.$

Continuous mapping theorem :

- If $Y_n \xrightarrow{P} Y$, then $g(Y_n) \xrightarrow{P} g(Y)$ for any cont. function $g(y)$.
- If $Y_n \Rightarrow Y$, then $g(Y_n) \Rightarrow g(Y)$, for any cont. function $g(y)$.

$\bar{X}_n \xrightarrow{P} \mu \overset{CMT}{\Longrightarrow} \bar{X}_n^2 \to \mu^2$

Suppose $Y_n = \begin{pmatrix} Y_{1n} \\ \vdots \\ Y_{dn} \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_d \end{pmatrix}$

* $Y_n \xrightarrow{P} Y$ if $\lim_{n\to\infty} P(\|Y_n - Y\| > \varepsilon) = 0$, $\forall \varepsilon > 0$

iff each $Y_{jn} \xrightarrow{P} Y_j$, $1 \le j \le d$.

* Similarly, $Y_n \Rightarrow Y$ if $P(Y_n \in A) \to P(Y \in A)$ $\forall A$ with $P(Y = \partial A) = 0$.

$\underset{\not\Leftarrow}{\Rightarrow}$ each $Y_{jn} \Rightarrow Y_j$, $1 \le j \le d$.

iff $t^T Y_n \Rightarrow t^T Y$, $\forall t \in \mathbb{R}^d$.

[WLLT, CLT, CMT hold in vector space].

Work through Ex. 7

7. (Pólya's Urn) From an urn consisting of only red and black balls, a ball is drawn at random and its color is noted. The ball drawn is then returned to the urn along with another ball of the same color. This process is then repeated *ad infinitum*. Assume the initial composition was b black and r red balls.

(a) For $n \in \mathbb{N}$, let p_n denote the probability of drawing a black ball on the n-th draw. Clearly, $p_1 = \frac{b}{b+r}$. An application of the law of total probability gives

$$p_2 = \frac{b}{b+r} \times \frac{b+1}{b+r+1} + \frac{r}{b+r} \times \frac{b}{b+r+1} = \frac{b}{b+r}.$$

Indeed, $p_n = \frac{b}{b+r}$ for every n. Prove this result with the help of the following. Define,

$$X_n = \#\text{black balls in urn prior to } n\text{-th draw}$$
$$Y_n = I(\text{black ball drawn on } n\text{-th draw}),$$

and argue that

$$p_{n+1} = \mathbb{E}(Y_{n+1}) = \frac{\mathbb{E}(X_{n+1})}{b+r+n+1} = \frac{\mathbb{E}(X_n)}{b+r+n} = p_n.$$

(a) $p_{n+1} = P(\mathbb{1}\{\text{black ball drawn on } (n+1)\text{-th draw}\})$

$$= \mathbb{E}(Y_{n+1})$$
$$= \mathbb{E}(\mathbb{E}(Y_{n+1} \mid X_{n+1}))$$
$$= \mathbb{E}(P(\{\text{draw back ball}\} \mid \{\text{have } X_{n+1} \text{ black balls}\}))$$
$$= \mathbb{E}\left(\frac{X_{n+1}}{b+r+n+1}\right)$$
$$= \frac{\mathbb{E}(\mathbb{E}(X_{n+1} \mid X_n))}{b+r+n+1}$$
$$= \frac{1}{b+r+n+1} \mathbb{E}\left[(X_n+1) \cdot \frac{X_n}{b+r+n} + X_n \cdot \frac{b+r+n-X_n}{b+r+n}\right]$$
$$= \frac{1}{b+r+n+1} \mathbb{E}\left[\frac{(b+r+n+1) X_n}{b+r+n}\right]$$
$$= \frac{\mathbb{E}(X_n)}{b+r+n} = p_n.$$

(b) Now consider the first n ball draws. For a $k \in \{0, 1, \ldots, n\}$, let $p_{n,k}$ denote the probability that the first k balls drawn were all black and the next $n-k$ were all red. It is straightforward to see that, with $s = b+r$ denoting the initial ball count in the urn,

$$p_{n,k} = \frac{b}{s} \times \cdots \times \frac{b+k-1}{s+k-1} \times \frac{r}{s+k} \times \cdots \times \frac{r+n-k-1}{s+n-1}$$
$$= \frac{\frac{\Gamma(b+k)}{\Gamma(b)} \times \frac{\Gamma(r+n-k)}{\Gamma(r)}}{\frac{\Gamma(s+n)}{\Gamma(s)}}$$

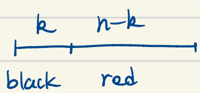
where the second equality follows by repeatedly apply the gamma function identity $\Gamma(a+1) = a\Gamma(a)$. Argue that we can express

$$p_{n,k} = \int_0^1 v^k (1-v)^{n-k} f_{b,r}(v) dv$$

where $f_{b,r}(v)$ is the pdf of the $\mathsf{Beta}(b, r)$ distribution.

(b)

$$\int_0^1 v^k (1-v)^{n-k} f_{b,r}(v)\, dv$$
$$= \int_0^1 v^k (1-v)^{n-k} \frac{\Gamma(b+r)}{\Gamma(b) \cdot \Gamma(r)} v^{b-1} (1-v)^{r-1} dv$$
$$= \frac{\Gamma(b+r)}{\Gamma(b) \cdot \Gamma(r)} \int_0^1 v^{k+b-1} (1-v)^{n-k+r-1} dv$$
$$= \frac{\Gamma(b+r)}{\Gamma(b)\, \Gamma(r)} \cdot \frac{\Gamma(k+b) \cdot \Gamma(n-k+r)}{\Gamma(n+r+b)}$$
$$= p_{n,k}.$$



(c) Let T_n denote the number of black balls drawn in the first n draws. Argue that

$$P(T_n = k) = \binom{n}{k} p_{n,k}.$$

Also argue that the probability distribution of T_n is the same as the marginal distribution of Z in the pair (V, Z) described as : $V \sim \mathsf{Beta}(b, r)$, $Z|(V = v) \sim \mathsf{Binomial}(n, v)$.

(d) The fraction of black balls drawn, namely $V_n = \frac{T_n}{n}$, converges in distribution. Can you speculate on what the limit distribution should be? You don't have to give a formal argument, but the main ideas should be clearly sketched out.

(c) If $T_n = k$, we need to pick k slots out of n to denote the positions of black balls.
There are $\binom{n}{k}$ of such choices.
The prob. of each choice is the same: $p_{n,k}$.
So $P(T_n = k) = \binom{n}{k} p_{n,k}$.

$$\begin{aligned} P(T_n = k) &= \binom{n}{k} \int_0^1 v^k (1-v)^{n-k} f_{b,r}(v)\, dv \\ &= \int_0^1 P_Z(Z=k \mid V=v) \cdot f_{b,r}(v)\, dv \\ &= \int_0^1 P_{Z,V}(Z=k, v)\, dv \\ &= P_Z(Z=k) \rightsquigarrow \text{marginal distn.} \end{aligned}$$

(d) Conjecture: $V_n \Rightarrow \mathrm{Beta}(b, r)$

$V_n \overset{d}{=} \frac{T_n}{n} \overset{d}{=} \frac{Z}{n}$ is an estimate of v.
Since $V \sim \mathrm{Beta}(b,r)$, we reason that $V_n \Rightarrow V$.