Lecture 08

$$\begin{bmatrix} X_i \\ Y_i \end{bmatrix} \overset{iid}{\sim} N_2\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$$

$1 \le i \le n$

$$R_n = \frac{S_{n,xy}}{S_{n,x} \cdot S_{n,y}},$$

$$S_{n,xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

$$S_{n,x} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{x})^2$$

$$S_{n,y} = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

① $R_n \xrightarrow{P} \rho$

② $\sqrt{n}(R_n-\rho) \Rightarrow N(0,\tau^2)$

③ τ^2 ?

Know: $S_{n,x}^2 = \overline{X_n^2} - (\overline{X_n})^2$

$S_{n,Y}^2 = \overline{Y_n^2} - (\overline{Y_n})^2$

$S_{n,xy} = \overline{(XY)}_n - (\overline{X_n})(\overline{Y_n})$

$$\sqrt{n}\left\{ \begin{bmatrix} \overline{X_n} \\ \overline{Y_n} \\ \overline{X_n^2} \\ \overline{Y_n^2} \\ \overline{(XY)}_n \end{bmatrix} - \begin{bmatrix} \mathbb{E}(X_1) \\ \mathbb{E}(Y_1) \\ \mathbb{E}(X_1^2) \\ \mathbb{E}(Y_1^2) \\ \mathbb{E}(X_1Y_1) \end{bmatrix} \right\} \Rightarrow N_5(0, \underset{5\times5}{\Sigma})$$

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1,Y_1) & Cov(X_1,X_1^2) & Cov(X_1,Y_1^2) & Cov(X_1,X_1Y_1) \\ & Var(Y_1) & Cov(Y_1,X_1^2) & Cov(Y_1,Y_1^2) & Cov(Y_1,X_1Y_1) \\ & & Var(X_1^2) & Cov(X_1^2,Y_1^2) & Cov(X_1^2,X_1Y_1) \\ & & & Var(Y_1^2) & Cov(Y_1^2,X_1Y_1) \\ & & & & Var(X_1Y_1) \end{bmatrix}$$

Work on the special case $\mu_1=\mu_2=0$, $\sigma_1^2=\sigma_2^2=1$.

$X_i \to \frac{X_i-\mu_1}{\sigma_1}$

$Y_i \to \frac{Y_i-\mu_2}{\sigma_2}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

* First consider the mean vector:

$$\begin{bmatrix} \mathbb{E}(X_1) \\ \mathbb{E}(Y_1) \\ \mathbb{E}(X_1^2) \\ \mathbb{E}(Y_1^2) \\ \mathbb{E}(X_1Y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \rho \end{bmatrix}$$

* Then consider the covariance matrix

$$\mathrm{Var}(X^2) = \mathbb{E}(X^4) - \mathbb{E}^2(X^2) = 3 - 1 = 2 = \mathrm{Var}(Y^2)$$

$$\mathrm{Cov}(X, X^2) = \mathbb{E}(X^3) - \mathbb{E}(X)\,\mathbb{E}(X^2) = 0$$

$\to$ Let $U = Y - \rho X \sim N(0, 1-\rho^2)$

$$\mathbb{E}(U) = \mathbb{E}(Y - \rho X) = 0,$$

$$\mathrm{Var}(U) = \mathrm{Var}(Y - \rho X) = \mathrm{Var}(Y) + \rho^2 \mathrm{Var}(X) - 2\rho\,\mathrm{Cov}(X,Y) = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2.$$

$\to$ $\mathrm{Cov}(X, U) = \mathrm{Cov}(X, Y - \rho X) = \mathrm{Cov}(X,Y) - \rho\,\mathrm{Cov}(X,X) = \rho - \rho\cdot 1 = 0.$

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} X \\ Y - \rho X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1-\rho^2 \end{bmatrix} \right) \qquad X \perp\!\!\!\perp U$$

$\to$ $\mathrm{Cov}(X, Y^2) = \mathbb{E}(XY^2) - \mathbb{E}(X)\cdot\mathbb{E}(Y^2) = \mathbb{E}(XY^2)$

$$\mathrm{Cov}(Y, X^2) = \mathbb{E}(X^2Y) - \mathbb{E}(X^2)\cdot\mathbb{E}(Y) = \mathbb{E}(X^2Y) = \mathbb{E}(X^2(\rho X + U)) = \rho\,\cancel{\mathbb{E}(X^3)}^{\,0} + \mathbb{E}(X^2)\cdot\cancel{\mathbb{E}(U)}^{\,0} = 0.$$

$$\mathrm{Cov}(X, XY) = \mathbb{E}(X^2Y) - \mathbb{E}(X)\cdot\mathbb{E}(XY) = 0$$

$$\mathrm{Cov}(X^2, XY) = \mathbb{E}(X^3Y) - \cancel{\mathbb{E}(X^2)}^{\,1}\cdot\cancel{\mathbb{E}(XY)}^{\,\rho} = \mathbb{E}(X^3(\rho X + U)) - \rho = \rho\,\mathbb{E}(X^4) + \mathbb{E}(X^3)\,\cancel{\mathbb{E}(U)}^{\,0} - \rho = 2\rho$$

$$\mathrm{Cov}(X^2, Y^2) = \mathbb{E}(X^2Y^2) - \mathbb{E}(X^2)\cdot\mathbb{E}(Y^2) = \mathbb{E}(X^2(\rho X + U)^2) - 1$$

$$= \mathbb{E}(X^2(\rho^2X^2 + 2\rho UX + U^2)) - 1 = \rho^2\,\mathbb{E}(X^4) + 2\rho\,\mathbb{E}(U)\,\mathbb{E}(X^3) + \mathbb{E}(U^2)\,\mathbb{E}(X^2) - 1$$

$$= 3\rho^2 + (1-\rho^2) - 1 = 2\rho^2$$

$$\mathrm{Var}(XY) = \mathbb{E}(X^2Y^2) - \mathbb{E}^2(XY) = 2\rho^2 + 1 - \rho^2 = \rho^2 + 1$$

$$\Sigma = \begin{bmatrix} \cancel{\mathrm{Var}(X_1)}^{\,1} & \cancel{\mathrm{Cov}(X_1,Y_1)}^{\,\rho} & \cancel{\mathrm{Cov}(X_1,X_1^2)}^{\,0} & \cancel{\mathrm{Cov}(X_1,Y_1^2)}^{\,0} & \cancel{\mathrm{Cov}(X_1,X_1Y_1)}^{\,0} \\ & \cancel{\mathrm{Var}(Y_1)}^{\,1} & \cancel{\mathrm{Cov}(Y_1,X_1^2)}^{\,0} & \cancel{\mathrm{Cov}(Y_1,Y_1^2)}^{\,0} & \cancel{\mathrm{Cov}(Y_1,X_1Y_1)}^{\,0} \\ & & \cancel{\mathrm{Var}(X_1^2)}^{\,2} & \cancel{\mathrm{Cov}(X_1^2,Y_1^2)}^{\,2\rho^2} & \cancel{\mathrm{Cov}(X_1^2,X_1Y_1)}^{\,2\rho} \\ & & & \cancel{\mathrm{Var}(Y_1^2)}^{\,2} & \cancel{\mathrm{Cov}(Y_1^2,X_1Y_1)}^{\,2\rho} \\ & & & & \cancel{\mathrm{Var}(X_1Y_1)}^{\,\rho^2+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \rho & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 2 & 2\rho^2 & 2\rho^2 \\ & & & 2 & 2\rho \\ & & & & \rho^2+1 \end{bmatrix}$$

As a result,

$$\sqrt{n}\left\{\begin{bmatrix}\bar{X}\\ \bar{Y}\\ \overline{X^2}\\ \overline{Y^2}\\ \overline{XY}\end{bmatrix}-\mu\right\} \Rightarrow N_5(0,\Sigma)$$

$$\underbrace{\begin{bmatrix}\bar{X}\\ \bar{Y}\\ \overline{X^2}\\ \overline{Y^2}\\ \overline{XY}\end{bmatrix}}_{= T_n}$$

$$R_n = \frac{\overline{XY}-\bar{X}\cdot\bar{Y}}{\sqrt{(\overline{X^2}-\bar{X}^2)(\overline{Y^2}-\bar{Y}^2)}}$$

$$= g(\bar{X},\bar{Y},\overline{X^2},\overline{Y^2},\overline{XY})$$

where $g(a,b,c,d,e) = \dfrac{e-ab}{\sqrt{(c-a^2)\cdot(d-b^2)}}$

$T_n \xrightarrow{P} \mu$.

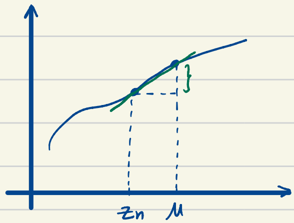
By CMT, we have $g(T_n) \xrightarrow{P} g(\mu)$

* The Delta theorem:

Assume $\sqrt{n}(z_n-\mu) \Rightarrow N(0,\Sigma)$

function $g(z)$ is smooth, then

$$\sqrt{n}(g(z_n)-g(\mu)) \Rightarrow N(0, \dot{g}(\mu)^T\Sigma\,\dot{g}(\mu))$$



$$g(z_n)-g(\mu) = \dot{g}(\tilde{z}_n)(z_n-\mu)$$

$$\tilde{z}_n \in (z_n,\mu)$$

$$\sqrt{n}\{g(z_n)-g(\mu)\} = \dot{g}(\tilde{z}_n)\times\{\sqrt{n}(z_n-\mu)\} \Rightarrow N(0,\sigma^2\dot{g}(\mu)^2)$$

$z_n \xrightarrow{P} \mu$, $\Rightarrow \tilde{z}_n \xrightarrow{P} \mu$, $\Rightarrow \dot{g}(\tilde{z}_n) \xrightarrow{P} \dot{g}(\mu)$

* Slutsky's theorem:

If $Y_n \Rightarrow Y$, $X_n \xrightarrow{P} a$, $V_n \xrightarrow{P} b$

then $X_n + V_nY_n \Rightarrow a+bY$

Since $\sqrt{n}(T_n-\mu) \Rightarrow N_5(0,\Sigma)$

then $\sqrt{n}(R_n-\rho) = \sqrt{n}(g(T_n)-g(\mu)) \Rightarrow N_5(0, \underbrace{\dot{g}(\mu)^T\Sigma\,\dot{g}(\mu)}_{\tau^2})$