Lecture 03

$$\begin{cases} X_1 \sim Ga(a_1, 1) \\ X_2 \sim Ga(a_2, 1) \end{cases} \quad \begin{cases} T = X_1 + X_2 \\ U = \frac{X_1}{X_1+X_2} \end{cases} \quad (T, U) \sim ?$$

* Change of variable

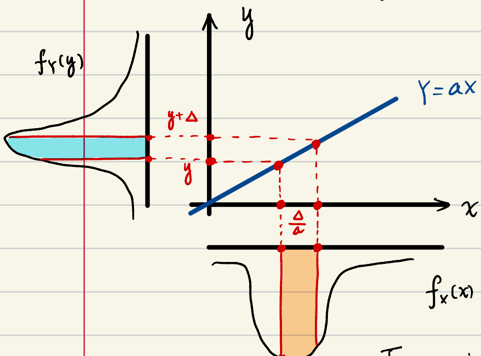
$x \sim f_X(x)$ ← pdf

$Y = aX \;, \quad X = \frac{Y}{a}$

$f_Y(y) = f_X(\frac{y}{a}) \cdot \frac{1}{a}$

Start with CDF: $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X \le \frac{y}{a}) = F_X(\frac{y}{a})$

So $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\frac{y}{a}) = f_X(\frac{y}{a}) \cdot \frac{1}{a}$



Area preservation.

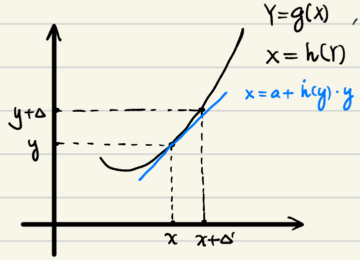
Two regions have the same area, since they represent the same event.

$$\Delta \cdot f_Y(y) = f_X(\frac{y}{a}) \cdot \frac{\Delta}{a} \Rightarrow f_Y(y) = f_X(\frac{y}{a}) \cdot \frac{1}{a}$$

More generally,

$Y = g(X) \;, \quad h = g^{-1}$

$X = h(Y)$



Therefore, $\Delta * f_Y(y) \doteq f_X(h(y)) \cdot \Delta'$

$\doteq f_X(h(y)) \cdot |h'(y)| * \Delta$

So $f_Y(y) \doteq f_X(h(y)) \cdot |h'(y)|$

$x + \Delta' = h(y+\Delta)$

$\approx h(y) + h'(y) \cdot \Delta$

$= x + h'(y) \cdot \Delta$

So $\Delta' = h'(y) * \Delta$

Ex. Suppose $X \sim U(0,1)$, $Y = X^k$, $k>0$

then $X = Y^{1/k}$, $h(y) = y^{1/k}$, $\dot{h}(y) = \frac{1}{k} y^{\frac{1}{k}-1}$

$$f_Y(y) = f_X(y^{1/k}) \cdot \frac{1}{k} y^{\frac{1}{k}-1}$$
$$= \frac{1}{k} \cdot y^{\frac{1}{k}-1}$$
$$= Be\left(\tfrac{1}{k}, 1\right), \quad 0<y<1$$

* Two dimensional case:

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $f_X(x_1, x_2)$ pdf on $\mathbb{R}^2$

Assume invertible

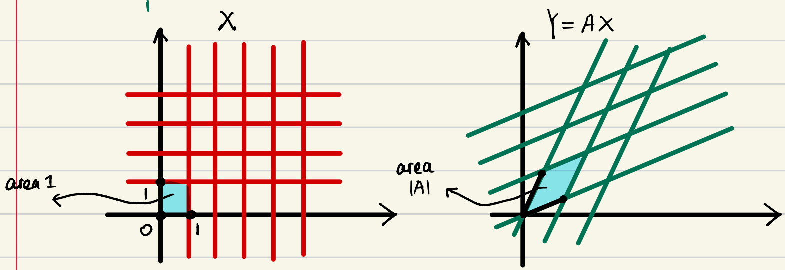
$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1,x_2) \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} h_1(Y_1, Y_2) \\ h_2(Y_1, Y_2) \end{pmatrix}$$

→ Linear:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad Y = AX, \quad X = A^{-1}Y$$

Then $f_Y(y) = f_X(A^{-1}y) \cdot |A^{-1}| = f_X(A^{-1}y) \cdot |A|^{-1}$

Volume preservation



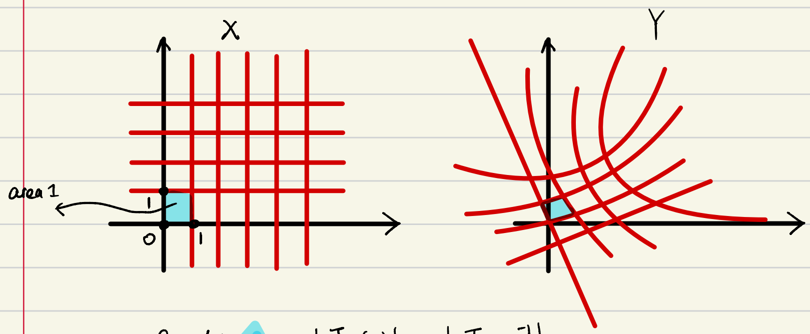
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

area of right ◇ $= |a_{11}a_{22} - a_{21}a_{12}|$

So $\Delta \cdot f_Y(y) \doteq \frac{\Delta}{|A|} f_X(A^{-1}y) \Rightarrow f_Y(y) = |A^{-1}| \cdot f_X(A^{-1}y)$

1-1 correspondence (invertible)

- Nonlinear $\quad Y = g(X) \quad X = h(Y) \quad \Rightarrow \begin{cases} Y_1 = g_1(X_1, X_2) \\ Y_2 = g_2(X_1, X_2) \end{cases}$ and $\begin{cases} X_1 = h_1(Y_1, Y_2) \\ X_2 = h_2(Y_1, Y_2) \end{cases}$



area of right ◇ $= |J_g(x)| = |J_h(y)^{-1}|$

Jacobian matrix

$$J_g(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1, x_2) & \frac{\partial g_2}{\partial x_1}(x_1, x_2) \\ \frac{\partial g_1}{\partial x_2}(x_1, x_2) & \frac{\partial g_2}{\partial x_2}(x_1, x_2) \end{pmatrix} \qquad J_h(y) = \begin{pmatrix} \frac{\partial h_1}{\partial y_1}(y_1, y_2) & \frac{\partial h_2}{\partial y_1}(y_1, y_2) \\ \frac{\partial h_1}{\partial y_2}(y_1, y_2) & \frac{\partial h_2}{\partial y_2}(y_1, y_2) \end{pmatrix}$$

So $\quad \Delta \cdot f_Y(y) \doteq \Delta \cdot |J_g(x)|^{-1} \cdot f_X(h(y)) \Rightarrow f_Y(y) = |J_h(y)| \cdot f_X(h(y))$

Ex. $\left.\begin{array}{l} X_1 \sim Ga(a_1, 1) \\ X_2 \sim Ga(a_2, 1) \end{array}\right]$ ind,

then

$$f_X(x_1, x_2) = \begin{cases} \frac{x_1^{a_1-1} e^{-x_1}}{\Gamma(a_1)} * \frac{x_2^{a_2-1} e^{-x_2}}{\Gamma(a_2)}, & \text{if } x_1, x_2 > 0. \\ 0, & \text{otherwise} \end{cases}$$

If we want the distn of $T = X_1 + X_2$ (non-invertible)

Consider:

$$(X_1, X_2) \to (T, U) \text{ by } \begin{cases} T = X_1 + X_2 \\ U = \frac{X_1}{X_1 + X_2} \end{cases} \text{ or } \begin{cases} X_1 = TU \\ X_2 = T(1-U) \end{cases}$$

↑ inverse function h

where $T > 0$, $0 < U < 1$.

Then

$$J_h = \begin{bmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} \\ \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \end{bmatrix} = \begin{bmatrix} u & 1-u \\ t & -t \end{bmatrix} = -ut - t(1-u) = -t$$

So

$$\begin{aligned} f_{u,T}(u,t) &= f_{X_1,X_2}(tu,\, t(1-u)) \cdot |-t| \\ &= \frac{1}{\Gamma(a_1)} \cdot (ut)^{a_1-1} e^{-ut} \cdot \frac{1}{\Gamma(a_2)} \cdot [t(1-u)]^{a_2-1} e^{-t(1-u)} \cdot t \\ &= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} u^{a_1-1}(1-u)^{a_2-1} \cdot \frac{1}{\Gamma(a_1+a_2)} t^{a_1+a_2-1} \cdot e^{-t} \end{aligned}$$

where $t>0$, $0<u<1$.

Conclusion:

$T \perp\!\!\!\perp U$, $T \sim \text{Gamma}(a_1+a_2, 1)$, $U \sim Be(a_1, a_2)$