Department of Computer Science

Interval and polyhedral analysis

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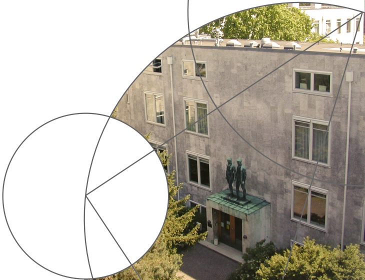
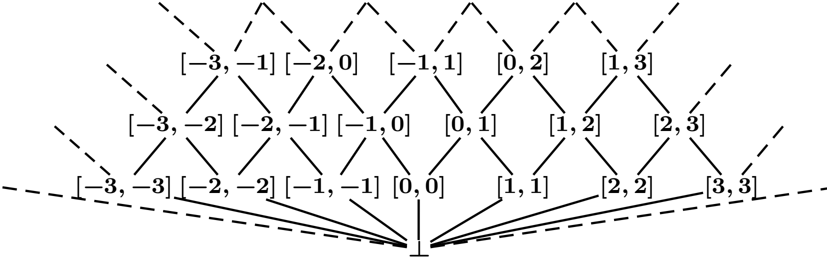


Plan

1. Interval analysis
2. Widening and narrowing
3. Polyhedral abstract interpretation

The lattice of intervals



The lattice of intervals

Elements :

$$\mathrm{Itv} \stackrel{\mathrm{def}}{=} \{\, [a,b] \mid a,b \in \overline{\mathbb{Z}},\ a \leqslant b \,\} \cup \{\bot\} \quad \text{with } \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$$

Order :

$$\frac{I \in \mathrm{Itv}}{\bot \sqsubseteq_{\mathrm{Itv}} I} \qquad \frac{c \leqslant a \qquad b \leqslant d \qquad a,b,c,d \in \overline{\mathbb{Z}}}{[a,b] \sqsubseteq_{\mathrm{Itv}} [c,d]}$$

Lattice operations :

$$\begin{aligned}
I \sqcup_{\mathrm{Itv}} \bot &\stackrel{\mathrm{def}}{=} I,\ \forall I \in \mathrm{Itv} \\
\bot \sqcup_{\mathrm{Itv}} I &\stackrel{\mathrm{def}}{=} I,\ \forall I \in \mathrm{Itv} \\
[a,b] \sqcup_{\mathrm{Itv}} [c,d] &\stackrel{\mathrm{def}}{=} [\min(a,c), \max(b,d)]
\end{aligned}$$

$$\begin{aligned}
I \sqcap_{\mathrm{Itv}} \bot &\stackrel{\mathrm{def}}{=} \bot,\ \forall I \in \mathrm{Itv} \\
\bot \sqcap_{\mathrm{Itv}} I &\stackrel{\mathrm{def}}{=} \bot,\ \forall I \in \mathrm{Itv} \\
[a,b] \sqcap_{\mathrm{Itv}} [c,d] &\stackrel{\mathrm{def}}{=} \rho_{\mathrm{Itv}}([\max(a,c), \min(b,d)])
\end{aligned}$$

Normalizer : $\rho_{\mathrm{Itv}} \in (\overline{\mathbb{Z}} \times \overline{\mathbb{Z}}) \to \mathrm{Itv}$ defined by

$$\rho_{\mathrm{Itv}}(a,b) = \begin{cases} [a,b] & \text{if } a \leqslant b, \\ \perp & \text{otherwise} \end{cases}$$

Least and greatest element :

$$\begin{aligned} \perp_{\mathrm{Itv}} &\stackrel{\text{def}}{=} \perp \\ \top_{\mathrm{Itv}} &\stackrel{\text{def}}{=} [-\infty, +\infty] \end{aligned}$$

Abstraction and concretisation :

$$\begin{array}{lcll} \alpha_{\mathrm{Itv}}(S) & \stackrel{\text{def}}{=} & \perp & \text{if } S = \emptyset \\ \alpha_{\mathrm{Itv}}(S) & \stackrel{\text{def}}{=} & [\min(S), \max(S)] & \text{otherwise} \end{array}$$

$$\begin{aligned} \gamma_{\mathrm{Itv}}(\perp) &\stackrel{\text{def}}{=} \emptyset \\ \gamma_{\mathrm{Itv}}([a,b]) &\stackrel{\text{def}}{=} \{\, z \in \mathbb{Z} \mid a \leqslant z \text{ and } z \leqslant b \,\} \end{aligned}$$

Abstraction of basic functions

All operators are *strict* : they return $\perp$ if one of their arguments is $\perp$.

$$
\begin{array}{rcl}
+^\sharp([a,b],[c,d]) & = & [a+c, b+d] \\
-^\sharp([a,b],[c,d]) & = & [a-d, b-c] \\
\times^\sharp([a,b],[c,d]) & = & [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]
\end{array}
$$

$$
\mathrm{const}(n)^\sharp = [n,n]
$$

Example :

$$
+^\sharp([2,\infty],[3,4]) = [5,\infty]
$$

Abstract comparison operators

Update knowledge about the value of variables.

Example : if $x \mapsto [2,5]$ and $y \mapsto [3,7]$ and we know that the test $x = y$ succeeds then we can refine our description of x and y :

$$[\![=]\!]\downarrow^{\sharp}_{\text{comp}}\ ([2,5],[3,7]) = ([3,5],[3,5])$$

In general :

$$\begin{array}{lcl}
[\![=]\!]\downarrow^{\sharp}_{\text{comp}}\ ([a,b],[c,d]) & = & ([a,b] \sqcap_{\text{Itv}} [c,d], [a,b] \sqcap_{\text{Itv}} [c,d]) \\
[\![<]\!]\downarrow^{\sharp}_{\text{comp}}\ ([a,b],[c,d]) & = & ([a,b] \sqcap_{\text{Itv}} [-\infty, d-1], [a+1,+\infty] \sqcap_{\text{Itv}} [c,d]) \\
[\![\leqslant]\!]\downarrow^{\sharp}_{\text{comp}}\ ([a,b],[c,d]) & = & ([a,b] \sqcap_{\text{Itv}} [-\infty, d], [a,+\infty] \sqcap_{\text{Itv}} [c,d]) \\
[\![\neq]\!]\downarrow^{\sharp}_{\text{comp}}\ ([a,b],[c,d]) & = & ?\ \textit{exercise...}
\end{array}$$

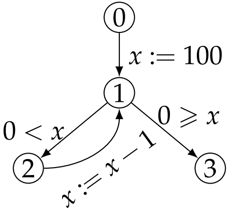
Abstract interpretation with intervals

```
x := 100;

while 0 < x {

  x := x − 1;
}
```

$$X_1 = [100, 100] \sqcup_{\mathrm{Itv}} \left(X_2 -^{\sharp} [1,1]\right)$$

$$X_2 = [1, +\infty] \sqcap_{\mathrm{Itv}} X_1$$

$$X_3 = [-\infty, 0] \sqcap_{\mathrm{Itv}} X_1$$



Example : fixpoint iteration

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\mathrm{Itv}} \left(X_2 -^\sharp [1,1]\right)\\
X_2 &= [1,+\infty] \sqcap_{\mathrm{Itv}} X_1\\
X_3 &= [-\infty,0] \sqcap_{\mathrm{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{aligned}
X_1^0 &= \bot \quad & X_1^{n+1} &= [100,100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^\sharp [1,1]\right)\\
X_2^0 &= \bot \quad & X_2^{n+1} &= [1,+\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1}\\
X_3^0 &= \bot \quad & X_3^{n+1} &= [-\infty,0] \sqcap_{\mathrm{Itv}} X_1^{n+1}
\end{aligned}
$$

X_1	$\bot$	$\cdots$
X_2	$\bot$	$\cdots$
X_3	$\bot$	$\cdots$

Example : fixpoint iteration

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\mathrm{Itv}} \left(X_2 -^\sharp [1,1]\right)\\
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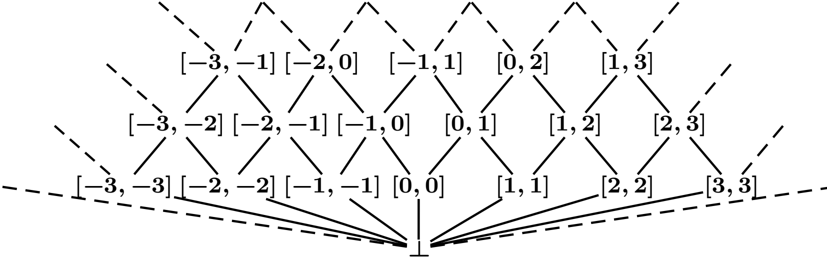
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = [100,100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^\sharp [1,1]\right)\\
X_2^0 = \bot & X_2^{n+1} = [1,+\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1}\\
X_3^0 = \bot & X_3^{n+1} = [-\infty,0] \sqcap_{\mathrm{Itv}} X_1^{n+1}
\end{array}
$$

X_1	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[0,100]$
X_2	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[1,100]$
X_3	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\bot$	$[0,0]$

Convergence problem



The lattice of intervals has infinite ascending chains.

$$\perp \sqsubset [0,0] \sqsubset [0,1] \sqsubset \cdots \sqsubset [0,n] \sqsubset \cdots .$$

.

Solution : dynamic approximation

- we extrapolate the limit thanks to a widening operator ∇

$$\perp \sqsubset [0,0] \sqsubset [0,1] \sqsubset [0,2] \sqsubset [0,+\infty] = [0,2]\nabla[0,3]$$

Plan

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2. Widening and narrowing
3. Polyhedral abstract interpretation

Fixpoint approximation

Lemma

Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice and f a monotone operator on A. If a is a post-fixpoint of f (i.e. $f(a) \sqsubseteq a$), then

$$\mathrm{lfp}(f) \sqsubseteq a$$

.

We may want to over-approximate $\mathrm{lfp}(f)$ when :

- The lattice does not satisfies the ascending chain condition, the iteration $\bot, f(\bot), \ldots, f^n(\bot), \ldots$ may never terminate.
- The ascending chain condition is satisfied but the iteration chain is too long to allow an efficient computation.
- The underlying lattice is not complete, so the limits of the ascending iterations do not necessarily belong to the abstraction domain.

Widening

Idea : the standard iteration is of the form

$$x^0 = \bot, \quad x^{n+1} = F(x^n) = x^n \sqcup F(x^n)$$

We will replace it by something of the form

$$y^0 = \bot, \quad y^{n+1} = y^n \nabla F(y^n)$$

such that

(i) (y^n) is increasing,

(ii) $x^n \sqsubseteq y^n$, for all n,

(iii) and (y^n) stabilizes after a finite number of steps.

Widening : definition

A widening is an operator $\nabla : L \times L \to L$ such that

- $\forall x, x' \in L, x \sqcup x' \sqsubseteq x \nabla x'$ (implies (i) & (ii))
- If $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$ is an increasing chain, then the increasing chain $y^0 = x^0, y^{n+1} = y^n \nabla x^{n+1}$ stabilizes after a finite number of steps (implies (iii)).

Usage : we replace $x^0 = \bot, x^{n+1} = F(x^n)$
by $y^0 = \bot, y^{n+1} = y^n \nabla F(y^n)$

Widening : theorem

Theorem

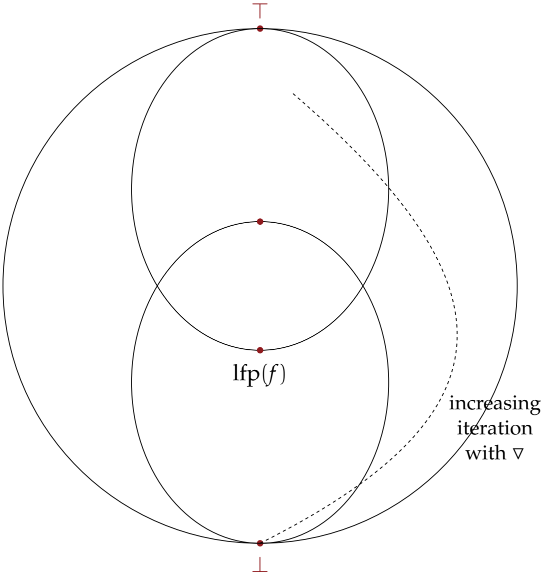
Let

- L *be a complete lattice,*
- $F : L \to L$ *be a monotone function and*
- $\nabla : L \times L \to L$ *a widening operator.*

Then, the chain $y^0 = \bot, y^{n+1} = y^n \nabla F(y^n)$ *stabilizes after a finite number of steps at a post-fixpoint* y *of* F*.*

Corollary : $\mathrm{lfp}(F) \sqsubseteq y$.

Scheme



Example : widening on intervals

Idea : as soon as a bound is not stable, we extrapolate it by $+\infty$ (or $-\infty$).
After such an extrapolation, the bound can't move any more.

Definition :

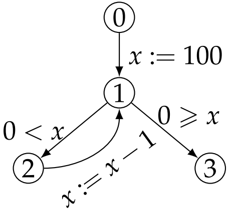
$$\begin{array}{rcl}
[a,b]\nabla_{\mathrm{Itv}}[a',b'] & = & [\quad \text{if } a' < a \text{ then } -\infty \text{ else } a, \\
 & & \quad\;\; \text{if } b' > b \text{ then } +\infty \text{ else } b\,] \\
\perp\nabla_{\mathrm{Itv}}[a',b'] & = & [a',b'] \\
I\ \nabla_{\mathrm{Itv}}\ \perp & = & I
\end{array}$$

Examples :

$[-3,4]\nabla_{\mathrm{Itv}}[-3,2] = [-3,4]$
$[-3,4]\nabla_{\mathrm{Itv}}[-3,5] = [-3,+\infty]$

Example

```
x := 100;

while 0 < x {

  x := x − 1;
}
```



$$X_1 = [100, 100] \sqcup_{\text{Itv}} \left(X_2 -^\sharp [1, 1]\right)$$

$$X_2 = [1, +\infty] \sqcap_{\text{Itv}} X_1$$

$$X_3 = [-\infty, 0] \sqcap_{\text{Itv}} X_1$$

Example : without widening

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\mathrm{Itv}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\mathrm{Itv}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\mathrm{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$

$$
\begin{array}{ll}
X_1^0 = \perp & X_1^{n+1} = [100, 100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^\sharp [1, 1]\right) \\
X_2^0 = \perp & X_2^{n+1} = [1, +\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1} \\
X_3^0 = \perp & X_3^{n+1} = [-\infty, 0] \sqcap_{\mathrm{Itv}} X_1^{n+1}
\end{array}
$$

X_1	$\perp$	$\cdots$
X_2	$\perp$	$\cdots$
X_3	$\perp$	$\cdots$

Example : without widening

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\mathrm{Itv}} \left(X_2 -^\sharp [1,1]\right) \\
X_2 &= [1,+\infty] \sqcap_{\mathrm{Itv}} X_1 \\
X_3 &= [-\infty,0] \sqcap_{\mathrm{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = [100,100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^\sharp [1,1]\right) \\
X_2^0 = \bot & X_2^{n+1} = [1,+\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1} \\
X_3^0 = \bot & X_3^{n+1} = [-\infty,0] \sqcap_{\mathrm{Itv}} X_1^{n+1}
\end{array}
$$

X_1	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[0,100]$
X_2	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[1,100]$
X_3	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\bot$	$[0,0]$

Example : with widening at each node of the cfg

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Itv}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Itv}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \perp & X_1^{n+1} = X_1^n \nabla_{\text{Itv}} \left([100, 100] \sqcup_{\text{Itv}} \left(X_2^n -^\sharp [1, 1]\right)\right) \\
X_2^0 = \perp & X_2^{n+1} = X_2^n \nabla_{\text{Itv}} \left([1, +\infty] \sqcap_{\text{Itv}} X_1^{n+1}\right) \\
X_3^0 = \perp & X_3^{n+1} = X_3^n \nabla_{\text{Itv}} \left([-\infty, 0] \sqcap_{\text{Itv}} X_1^{n+1}\right)
\end{array}
$$

X_1	$\perp$
X_2	$\perp$
X_3	$\perp$

Example : with widening at each node of the cfg

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\mathrm{Itv}} \left(X_2 -^{\sharp} [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\mathrm{Itv}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\mathrm{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \perp & X_1^{n+1} = X_1^n \nabla_{\mathrm{Itv}} \left([100, 100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^{\sharp} [1, 1]\right)\right) \\
X_2^0 = \perp & X_2^{n+1} = X_2^n \nabla_{\mathrm{Itv}} \left([1, +\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1}\right) \\
X_3^0 = \perp & X_3^{n+1} = X_3^n \nabla_{\mathrm{Itv}} \left([-\infty, 0] \sqcap_{\mathrm{Itv}} X_1^{n+1}\right)
\end{array}
$$

X_1	$\perp$	$[100, 100]$	$[-\infty, 100]$
X_2	$\perp$	$[100, 100]$	$[-\infty, 100]$
X_3	$\perp$	$\perp$	$[-\infty, 0]$

Improving fixpoint approximation

Idea : iterating a little more may help...

Theorem

Let $(A, \sqsubseteq, \sqcup, \sqcap)$ *be a complete lattice,* f *a monotone operator on* A *and* a *a post-fixpoint of* f.

The chain $(x_n)_n$ *defined by*

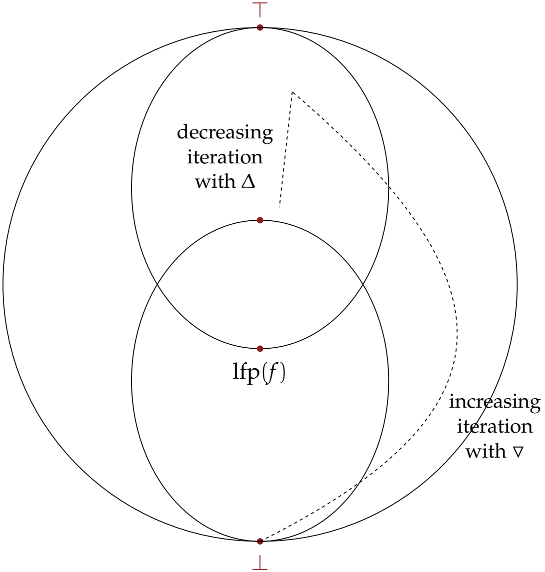
$$\begin{cases} x_0 & = & a \\ x_{k+1} & = & f(x_k) \end{cases}$$

admits for limit ($\bigsqcap\{x_n\}$) *the greatest fixpoint of* f *smaller than* a *(written* $\mathrm{gfp}_a(f)$*).*

- $\mathrm{lfp}(f) \sqsubseteq \bigsqcap\{x_n\}$.
- *Each intermediate step is a correct approximation :*

$$\forall k,\ \mathrm{lfp}(f) \sqsubseteq \mathrm{gfp}_a(f) \sqsubseteq x_k \sqsubseteq a$$

Scheme


$\top$
decreasing
iteration
with Δ
$\mathrm{lfp}(f)$
increasing
iteration
with ∇
$\bot$

Narrowing on intervals

Intuition : improve infinite bounds.

$$
\begin{array}{rcl}
[a,b]\,\Delta_{\mathrm{Itv}}\,[c,d] & = & [\text{if } a = -\infty \text{ then } c \text{ else } a \;\; ; \;\; \text{if } b = +\infty \text{ then } d \text{ else } b] \\
I \;\; \Delta_{\mathrm{Itv}} \;\; \bot & = & \bot \\
\bot \;\; \Delta_{\mathrm{Itv}} \;\; I & = & \bot
\end{array}
$$

In practice : a few standard iterations already improve a lot the result that has been obtained after widening.

- Assignments by constants and conditional guards make the decreasing iterations efficient : they *filter* the (too big) approximations computed by the widening

Example : with narrowing at each node of the cfg

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\mathrm{Itv}} \left(X_2 -^\sharp [1,1]\right)\\
X_2 &= [1,+\infty] \sqcap_{\mathrm{Itv}} X_1\\
X_3 &= [-\infty,0] \sqcap_{\mathrm{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = [-\infty,100] & X_1^{n+1} = X_1^n \Delta_{\mathrm{Itv}} \left([100,100] \sqcup_{\mathrm{Itv}} \left(X_2^n -^\sharp [1,1]\right)\right)\\
X_2^0 = [-\infty,100] & X_2^{n+1} = X_2^n \Delta_{\mathrm{Itv}} \left([1,+\infty] \sqcap_{\mathrm{Itv}} X_1^{n+1}\right)\\
X_3^0 = [-\infty,0] & X_3^{n+1} = X_3^n \Delta_{\mathrm{Itv}} \left([-\infty,0] \sqcap_{\mathrm{Itv}} X_1^{n+1}\right)
\end{array}
$$

X_1	$[-\infty,100]$
X_2	$[-\infty,100]$
X_3	$[-\infty,0]$

Example : with narrowing at each node of the cfg

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Itv}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Itv}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Itv}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = [-\infty, 100] & X_1^{n+1} = X_1^n \Delta_{\text{Itv}} \left([100, 100] \sqcup_{\text{Itv}} \left(X_2^n -^\sharp [1, 1]\right)\right) \\
X_2^0 = [-\infty, 100] & X_2^{n+1} = X_2^n \Delta_{\text{Itv}} \left([1, +\infty] \sqcap_{\text{Itv}} X_1^{n+1}\right) \\
X_3^0 = [-\infty, 0] & X_3^{n+1} = X_3^n \Delta_{\text{Itv}} \left([-\infty, 0] \sqcap_{\text{Itv}} X_1^{n+1}\right)
\end{array}
$$

X_1	$[-\infty, 100]$	$[-\infty, 100]$	$[0, 100]$
X_2	$[-\infty, 100]$	$[1, 100]$	$[1, 100]$
X_3	$[-\infty, 0]$	$[-\infty, 0]$	$[0, 0]$

The particular case of an equation system

Consider a system with $f_1, \ldots, f_n$ monotone.

$$\begin{cases} x_1 & = & f_1(x_1, \ldots, x_n) \\ & \vdots & \\ x_n & = & f_n(x_1, \ldots, x_n) \end{cases}$$

Standard iteration :

$$\begin{array}{rcl} x_1^{i+1} & = & f_1(x_1^i, \ldots, x_n^i) \\ x_2^{i+1} & = & f_2(x_1^i, \ldots, x_n^i) \\ & \vdots & \\ x_n^{i+1} & = & f_n(x_1^i, \ldots, x_n^i) \end{array}$$

Standard iteration with widening :

$$\begin{array}{rcl} x_1^{i+1} & = & x_1^i \nabla f_1(x_1^i, \ldots, x_n^i) \\ x_2^{i+1} & = & x_2^i \nabla f_2(x_1^i, \ldots, x_n^i) \\ & \vdots & \\ x_n^{i+1} & = & x_n^i \nabla f_n(x_1^i, \ldots, x_n^i) \end{array}$$

The particular case of an equation system

$$\begin{cases} x_1 & = & f_1(x_1, \ldots, x_n) \\ & \vdots & \\ x_n & = & f_n(x_1, \ldots, x_n) \end{cases}$$

It is sufficient (and generally more precise) to use ∇ for a selection of index W provided that each cycle in the system has at least one point in W.

$$\forall k = 1..n,\ x_k^{i+1} = \begin{array}{ll} x_k^i \nabla f_k(x_1^i, \ldots, x_n^i) & \text{if } k \in W \\ f_k(x_1^i, \ldots, x_n^i) & \text{otherwise} \end{array}$$

Chaotic iteration : at each step, we use only one equation, without forgetting one for ever.
Beware : this time the iteration strategy may affect the precision of the obtained post-fixpoint!
Delayed widening : It is generally better to wait a few standard iterations before launching the widenings.

Plan

1. Interval analysis
2. Widening and narrowing
3. Polyhedral abstract interpretation

The need for relational program analysis

Consider the program

```
i := 1;
t := 0;
while i < 100 {
  t := t + 1;
  i := i + 2;
}
```

where `t` is inserted to argue termination of the loop.

Interval analysis with widening and narrowing will

- find precise bounds for `i` ($[1, 100]$)
- but not for `t` (only finds $[0, \infty]$).

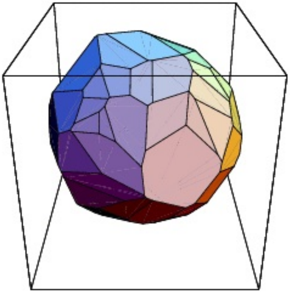
Need to know that the two variables are related : $\mathtt{i} = 2\mathtt{t} + 1$.

Polyhedral abstract interpretation

Polyhedral analysis seeks to discover invariants of **linear equality and inequality** relations (such as $x = y$ or $x \leqslant 2y + z$) among the variables of an imperative program.

A convex polyhedron can be defined

- algebraically as the set of solutions of a system of linear inequalities.
- geometrically, as a finite intersection of half-spaces.

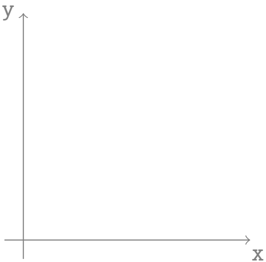


The classical reference :

Automatic discovery of linear restraints among variables of a program. P. Cousot and N. Halbwachs. POPL'78.

Polyhedral analysis

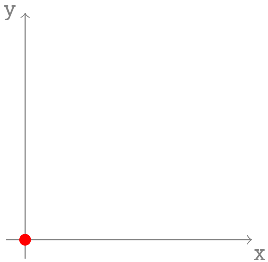
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;

while (x<6) {
  if (?) {
    y = y+2;
  };
  x = x+1;
}
```

Polyhedral analysis

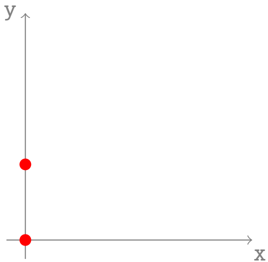
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
        {x = 0 ∧ y = 0}

while (x<6) {
  if (?) {
        {x = 0 ∧ y = 0}
    y = y+2;

  };



  x = x+1;

}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.



At junction points, we over-approximates union by a convex union.

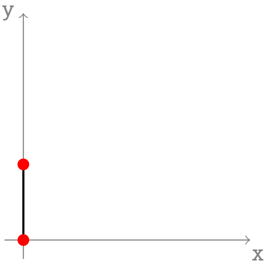
```
x = 0; y = 0;
```
$\{x = 0 \wedge y = 0\}$

```
while (x<6) {
  if (?) {
```
$\{x = 0 \wedge y = 0\}$
```
    y = y+2;
```
$\{x = 0 \wedge y = 2\}$
```
  };
```
$\{x = 0 \wedge y = 0\} \uplus \{x = 0 \wedge y = 2\}$

```
  x = x+1;

}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

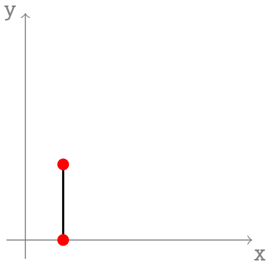


At junction points, we over-approximates union by a convex union.

```
x = 0; y = 0;
        {x = 0 ∧ y = 0}

while (x<6) {
  if (?) {
        {x = 0 ∧ y = 0}
    y = y+2;
        {x = 0 ∧ y = 2}
  };
        {x = 0 ∧ 0 ⩽ y ⩽ 2}

  x = x+1;

}
```

Polyhedral analysis

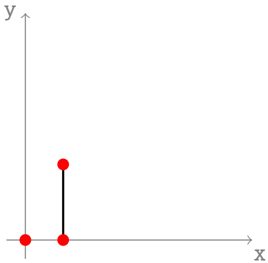
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
        {x = 0 ∧ y = 0}

while (x<6) {
  if (?) {
        {x = 0 ∧ y = 0}
    y = y+2;
        {x = 0 ∧ y = 2}
  };
        {x = 0 ∧ 0 ⩽ y ⩽ 2}

  x = x+1;
        {x = 1 ∧ 0 ⩽ y ⩽ 2}
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.



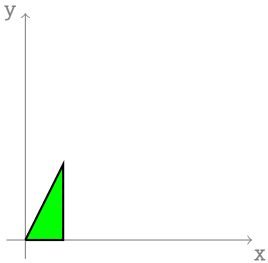

```
x = 0; y = 0;
```
$\{x = 0 \wedge y = 0\} \uplus \{x = 1 \wedge 0 \leqslant y \leqslant 2\}$

```
while (x<6) {
  if (?) {
```
$\{x = 0 \wedge y = 0\}$
```
    y = y+2;
```
$\{x = 0 \wedge y = 2\}$
```
  };
```
$\{x = 0 \wedge 0 \leqslant y \leqslant 2\}$

```
  x = x+1;
```
$\{x = 1 \wedge 0 \leqslant y \leqslant 2\}$
```
}
```

Polyhedral analysis

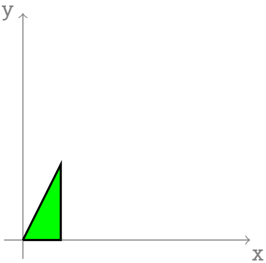
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
        {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}

while (x<6) {
  if (?) {
        {x = 0 ∧ y = 0}
    y = y+2;
        {x = 0 ∧ y = 2}
  };
        {x = 0 ∧ 0 ⩽ y ⩽ 2}

  x = x+1;
        {x = 1 ∧ 0 ⩽ y ⩽ 2}
}
```

Polyhedral analysis

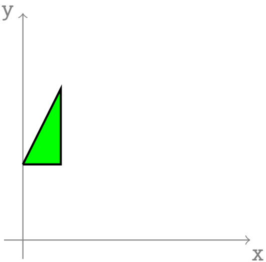
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
      {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}

while (x<6) {
  if (?) {
      {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}
    y = y+2;
      {x = 0 ∧ y = 2}
  };
      {x = 0 ∧ 0 ⩽ y ⩽ 2}

  x = x+1;
      {x = 1 ∧ 0 ⩽ y ⩽ 2}
}
```

Polyhedral analysis

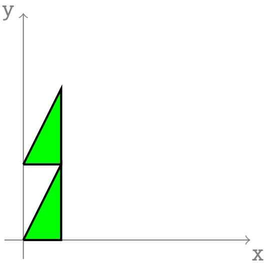
State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
        {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}

while (x<6) {
  if (?) {
        {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}
    y = y+2;
        {x ⩽ 1 ∧ 2 ⩽ y ⩽ 2x + 2}
  };
        {x = 0 ∧ 0 ⩽ y ⩽ 2}

  x = x+1;
        {x = 1 ∧ 0 ⩽ y ⩽ 2}
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
```

$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```

$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
    y = y+2;
```

$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x+2\}$

```
  };
```

$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$
$\uplus\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x+2\}$

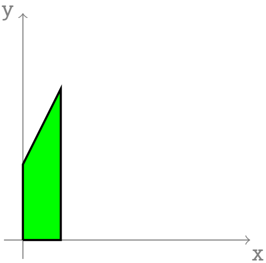
```
  x = x+1;
```

$\{x = 1 \wedge 0 \leqslant y \leqslant 2\}$

```
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.



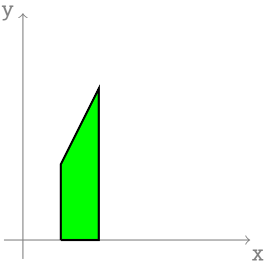

```
x = 0; y = 0;
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$
```
    y = y+2;
```
$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x + 2\}$
```
  };
```
$\{0 \leqslant x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x + 2\}$

```
  x = x+1;
```
$\{x = 1 \wedge 0 \leqslant y \leqslant 2\}$
```
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
    y = y+2;
```
$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x+2\}$

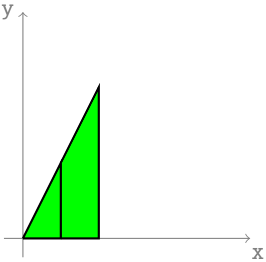
```
  };
```
$\{0 \leqslant x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x+2\}$

```
  x = x+1;
```
$\{1 \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$

```
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

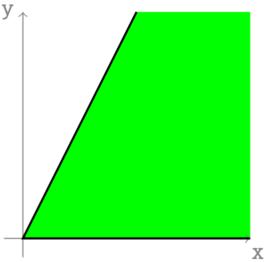
$\nabla \{x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$
```
    y = y+2;
```
$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x + 2\}$
```
  };
```
$\{0 \leqslant x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x + 2\}$
```
  x = x+1;
```
$\{1 \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$
```
}
```

At loop headers, we use heuristics (widening) to ensure finite convergence.

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;
      {0 ⩽ y ⩽ 2x}

while (x<6) {
  if (?) {
      {x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x}
    y = y+2;
      {x ⩽ 1 ∧ 2 ⩽ y ⩽ 2x + 2}
  };
      {0 ⩽ x ⩽ 1 ∧ 0 ⩽ y ⩽ 2x + 2}

  x = x+1;
      {1 ⩽ x ⩽ 2 ∧ 0 ⩽ y ⩽ 2x}
}
```

At loop headers, we use heuristics (widening) to ensure finite convergence.

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

By propagation we obtain a post-fixpoint

```
x = 0; y = 0;
        {0 ⩽ y ⩽ 2x}

while (x<6) {
  if (?) {
        {0 ⩽ y ⩽ 2x ∧ x ⩽ 5}
    y = y+2;
        {2 ⩽ y ⩽ 2x + 2 ∧ x ⩽ 5}
  };
        {0 ⩽ y ⩽ 2x + 2 ∧ 0 ⩽ x ⩽ 5}

  x = x+1;
        {0 ⩽ y ⩽ 2x ∧ 1 ⩽ x ⩽ 6}
}
        {0 ⩽ y ⩽ 2x ∧ 6 ⩽ x}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

By propagation we obtain a post-fixpoint which is enhanced by downward iteration.

```
x = 0; y = 0;
```

$\{0 \leqslant y \leqslant 2x \wedge x \leqslant 6\}$

```
while (x<6) {
  if (?) {
```

$\{0 \leqslant y \leqslant 2x \wedge x \leqslant 5\}$

```
    y = y+2;
```

$\{2 \leqslant y \leqslant 2x + 2 \wedge x \leqslant 5\}$

```
  };
```

$\{0 \leqslant y \leqslant 2x + 2 \wedge 0 \leqslant x \leqslant 5\}$

```
  x = x+1;
```

$\{0 \leqslant y \leqslant 2x \wedge 1 \leqslant x \leqslant 6\}$

```
}
```

$\{0 \leqslant y \leqslant 2x \wedge 6 = x\}$

Polyhedral analysis

A more complex example.

The analysis accepts to replace some constants by parameters.

```
x = 0; y = A;
```
$\{A \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A \wedge x \leqslant N\}$

```
while (x<N) {
  if (?) {
```
$\{A \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A \;\wedge\; x \leqslant N - 1\}$
```
    y = y+2;
```
$\{A + 2 \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A + 2 \;\wedge\; x \leqslant N - 1\}$
```
  };
```
$\{A \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A + 2 \;\wedge\; 0 \leqslant x \leqslant N - 1\}$

```
  x = x+1;
```
$\{A \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A \;\wedge\; 1 \leqslant x \leqslant N\}$
```
}
```
$\{A \leqslant \mathrm{y} \leqslant 2\mathrm{x} + A \;\wedge\; N = x\}$

The four polyhedra operations

- $\uplus \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: convex union
  - over-approximates the concrete union at junction points
- $\cap \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: intersection
  - over-approximates the concrete intersection after a conditional instruction
- $[\![\mathtt{x} := e]\!] \in \mathbb{P}_n \to \mathbb{P}_n$: affine transformation
  - over-approximates the assignment of a variable by a linear expression
- $\nabla \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: widening
  - ensures (and accelerates) convergence of (post-)fixpoint iteration
  - includes heuristics to infer loop invariants

```
x = 0; y = 0;
```

$$P_0 = [\![\mathtt{y} := \mathtt{0}]\!]\,[\![\mathtt{x} := \mathtt{0}]\!]\,(\mathbb{Q}^2)\ \nabla\ P_4$$

```
while (x<6) {
  if (?) {
```

$$P_1 = P_0 \cap \{x < 6\}$$

```
    y = y+2;
```

$$P_2 = [\![\mathtt{y} := \mathtt{y} + \mathtt{2}]\!]\,(P_1)$$

```
  };
```

$$P_3 = P_1 \uplus P_2$$

```
  x = x+1;
```

$$P_4 = [\![\mathtt{x} := \mathtt{x} + \mathtt{1}]\!]\,(P_3)$$

```
}
```

$$P_5 = P_0 \cap \{x \geqslant 6\}$$

More about abstraction of assignments

Distinguish between two types of assignments `x := exp`

- invertible assignments, where `x` appears in `exp`
  Example : `x = 2x + y`
- non-invertible assignments, when the new value of `x` does not depend on its old value.
  Example : `x = y + 42`

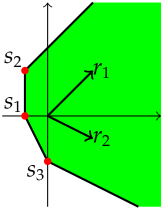
Invertible assignments can be abstracted precisely in two steps :

1. express the old value of `x` as an expression involving the new value (written `x'`)
   Example : for `x := x + 1`, we have `x'-1 = x`
2. in the constraints, replace all occurrences of `x` by the inverted expression for `x`.
   Example : $y \geqslant 2x$ becomes $y \geqslant 2(x-1)$.

For non-invertible assignments : all information about old `x` is lost, so we remove all constraints involving `x`, and add $x = exp$.

Library for manipulating polyhedra

- Parma Polyhedra Library [1] (PPL), NewPolka : complex C/C++ libraries
- They rely on the Double Description Method
  - polyhedra are managed using two representations in parallel



    - by set of inequalities

$$P = \left\{ (x,y) \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} x \geqslant -1 \\ x - y \geqslant -3 \\ 2x + y \geqslant -2 \\ x + 2y \geqslant -4 \end{array} \right\}$$

    - by set of generators

$$P = \left\{ \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \mu_1 r_1 + \mu_2 r_2 \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 \in \mathbb{R}^+ \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{array} \right\}$$

  - operations efficiency strongly depends on the chosen representations, so they keep both

1. Previous tutorial on polyhedra partially comes from http://www.cs.unipr.it/ppl/

Other relational domains

Polyhedral analysis uses all linear relations ($\bigwedge_j \sum_i a_{ij}x_i \geqslant c_j$).
It is precise but has exponential complexity.

Other examples of (weakly) relational analyses

- linear equalities ($x = \sum_i a_i x_i$)
- zones and octagons.

Octagons : restrict constraints to be of form $\pm X_1 \pm X_2 \leqslant C$.

- At most two variables are related in one constraint.
- Only allowed coefficients are -1,1.

Efficient polynomial-time operations and good precision.

Example (Miné p. 131) : least upper bound of two boxes with intervals and with octagons :