Studio #1: Review of Laplace transform, partial fraction expansion, and solution of differential equations using Laplace transform and Matlab

1. Laplace Transform Table

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

2. Laplace Transform Theorems

TABLE 2.2 Laplace transform theorems

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$		Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

3. Partial Fraction Expansion (PFE)

We are familiar with combining fractions over common denominators:

$$\frac{2}{x+1}+\frac{3}{x+2}=\frac{2x+4+3x+3}{(x+1)(x+2)}=\frac{5x+7}{x^2+3x+2}$$

The inverse of this procedure is PFE.

(1) Steps for PFE

1) Check that the degree of the numerator must be less than the degree of the denominator. This will be mostly the case in ENME462.
2) Factor out the denominator into 1^{st}-order and 2^{nd}-order rational factors.
3) Find numerator coefficients.

Example 1: Find the PFE of $\frac{5x+7}{x^2+3x+2}$.

Step 1: degree of the numerator < degree of denominator

Step 2: factor the denominator

$$x^2+3x+2=(x+1)(x+2)$$

Step 3:

$$\frac{5x+7}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}$$

Cross multiply the terms on the right side and equate the numerator of right and left:

$$A(x+2)+B(x+1)=5x+7$$

Note that this equality is valid for all values of x. Carefully choosing values for x yields the values for A and B as follows:

$$x=-1,\rightarrow\ A=2$$
$$x=-2,\rightarrow\ B=3$$

(2) Factors Anticipated

I. For Linear terms $ax+b$ we have:

$$\frac{A}{ax+b}$$

II. For quadratic terms (quadratics that don't have real roots)[1]:

$$\frac{Ax+B}{ax^2+bx+c}$$

III. For repeated roots $(ax+b)^3$:

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

[1] Note that it is possible to factor a quadratic polynomial with complex roots and use method I. since the two complex roots are distinct. Nonetheless, it is easier to keep the quadratic polynomial and complete the square then use the Laplace transform table to find the inverse directly.

Example 2: Repeated roots:

$$\frac{2x^2+1}{x^3-x^2-8x+12}=\frac{2x^2+1}{(x-2)^2(x+3)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}$$

Example 3: One simple root and complex roots

$$\frac{x^2+1}{(x^2+x+2)(x+7)}=\frac{Ax+b}{x^2+x+2}+\frac{C}{x+7}$$

Example 4: Repeated complex roots

$$\frac{1}{(x^2+2x+5)^2(x-1)(x+2)}=\frac{Ax+B}{x^2+2x+5}+\frac{Cx+D}{(x^2+2x+5)^2}+\frac{E}{x-1}+\frac{F}{x+2}$$

Example 5: Repeated roots

$$\frac{1}{(x^2-3)^2}=\frac{1}{\left(x-\sqrt{3}\right)^2\left(x+\sqrt{3}\right)^2}=\frac{A}{\left(x-\sqrt{3}\right)}+\frac{B}{\left(x-\sqrt{3}\right)^2}+\frac{C}{\left(x+\sqrt{3}\right)}+\frac{D}{\left(x+\sqrt{3}\right)^2}$$

<u>(3) Determination of Numerator Coefficients</u>

Method 1

Multiply both sides by the denominator and choose values for the variable to derive coefficients.

Example 6:

$$\frac{2x-1}{(x+2)^2(x-3)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-3}$$

$$2x-1=A(x+2)(x-3)+B(x-3)+C(x+2)^2$$

Now we choose values for the variable x:

$$x=3,\rightarrow C=\frac{1}{5}$$

$$x=-2,\rightarrow B=1$$

$$x=0,\rightarrow A=-\frac{1}{5}$$

Method 2

Multiply both sides by the denominator and place the coefficients of the variable equal on both sides of the equation:

Example 7:

$$2x-1=Ax^2-Ax-6A+Bx-3B+Cx^2+4Cx+4C$$

$$2x-1=(A+C)x^2+(-A+B+4C)x+(-6A-3B+4C)$$

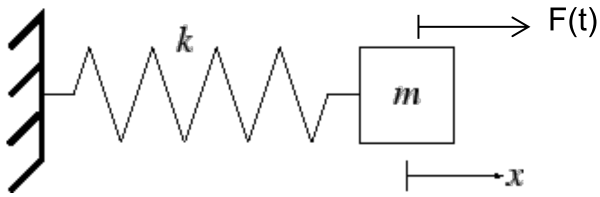
$$0=A+C$$

$$2=-A+B+4C$$

$$-1=-6A-3B+4C$$

4. Solution to an Ordinary Differential Equations (ODE): A Mass-Spring System Example: Writing ODE model and its Solution

The objective is to compare different ways to solve 1-DoF vibration problem:
(i) hand calculation,
(ii) use of ODE function in MATLAB, and
(iii) use of Laplace transform and SIMULINK



Spring-mass diagram.

(i) Hand calculation via convolution integral
The equation of motion is given by:

$$m\ddot{x}(t) + kx(t) = F(t) \tag{1}$$

The response of the system to an external force input can be obtained using the convolution integral:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \tag{2}$$

For the case where the force input F(t) is a step input, i.e. F(t)=u(t), equation (2) reduces to:

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau = \frac{1}{m\omega_n}\frac{1}{\omega_n}\cos\omega_n(t-\tau)\big|_0^t = \frac{1}{m\omega_n^2}(1-\cos\omega_n t) \\ &= \frac{1}{k}(1-\cos\omega_n t) \end{aligned} \tag{3}$$

The above response can be graphically represented using the following MATLAB commands:

```matlab
clear all;
clc

t         = 0:0.01:10; % Time in Seconds with increments of 0.01
k         = 2;         % Spring Stiffness (N/m)
m         = 1;         % mass [kg]
omega     = sqrt(k/m); % radians/sec

x = (1/k) * (1 - cos(omega*t));

plot(t,x)
axis([0,10,-0.01,1.2])
xlabel 'Time [s]'
ylabel 'Displacement [m]'
```

(ii) Use of MATLAB ODE function
To use the ODE solvers in MATLAB, the differential equation must be reformulated into a set of first-order differential equations, this is known as state-space representation. For this purpose, define the following variables:

$$x_1(t) = x(t) \qquad (4)$$
$$x_2(t) = \dot{x}(t)$$

Where $\dot{x}(t)$ is the first derivative of x(t).
Then equation (1) can be rewritten into as:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) + \dfrac{1}{m}F(t) \end{cases} \qquad (5)$$

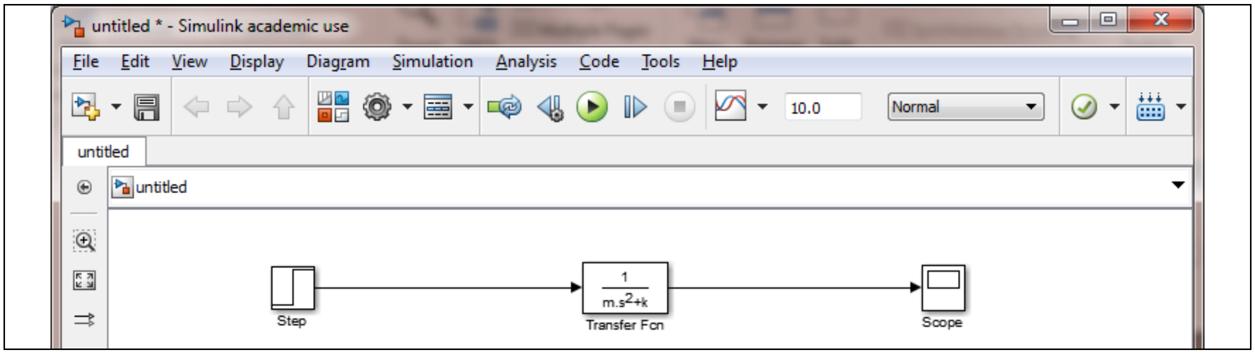
Using equation (5), the following MATLAB script can be used to solve equation (1) numerically:

```
clear all;
clc

x_o = [0,0];
t_o = 0;
t_f = 10;

[t,x] = ode45('S_M_ODE',[t_o,t_f],x_o);

plot(t,x(:,1))
axis([0,10,-0.01,1.2])
xlabel 'Time [s]'
ylabel 'Displacement [m]'

function [ xp ] = S_M_ODE( t , x )

xp=zeros(2,1);

k          = 2;            % Spring Stiffness [N/m]
m          = 1;            % Mass [kg]
F          = 1;

xp(1) = x(2);
xp(2) = -k/m*x(1)+1/m*F;

end
```

(iii) Use of Laplace transform in SIMULINK
Taking Laplace transform of both sides of equation (1) yields:

$$m\ddot{x} + kx = F(t) \rightarrow ms^2X(x) + kX(s) = F(s) \rightarrow X(s) = \frac{1}{ms^2 + k}F(s) \quad (3)$$

Using MATLAB's SIMULINK, the following block diagram can be constructed to numerically simulate the response of the 1-DoF system to step input.



Assignment

Implement the MATLAB/SIMULINK codes for the mass-spring system example discussed above and determine the displacement time response, i.e., x(t), using the three methods (i)-(iii) and compare the plots of x(t) obtained by each methods. Show that the responses derived from different approaches are identical.

Each student will be checked off during studio session. For remote students and students who do not finish in time to be checked off during the studio session itself, you must upload a PDF document to Canvas, use naming convention: LastName_FirstName_Studio01.pdf, within one week of the the studio session. Your submission shall include:

1. A screen shot of your Matlab/Simulink code for each of the methods.
2. A plot comparing the responses of the mass-spring system derived from the 3 approaches mentioned above.