ENME665: Nonlinear Oscillations

- Material covered last class
  - ❖ Quantitative Analyses (continued)
    - ➢ Landau symbols and ordering
    - ➢ Straightforward expansions
    - ➢ Lindstedt-Poincaré technique
    - ➢ Method of multiple scales

  - ❖ Straightforward expansions (Section 2.3.1, Nayfeh and Mook, 1979) and limitations

    - ➢ Example i) Linear oscillator with weak damping
    - ➢ Example ii) Undamped, nonlinear oscillator with weak nonlinearity
    - ➢ Example iii) Nonlinear oscillator with weak damping and weak nonlinearity
    - ➢ Example iv) Forced nonlinear oscillator with damping
    - ➢ Example v) Coupled, nonlinear oscillators

ENME665: Nonlinear Oscillations

- Example i) Linear oscillator with weak damping (Strogatz, S. 1994, pp. 216—218).

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0;\ \ |\varepsilon| \ll 1;\ x(t=0) = 0 \text{ and } \dot{x}(t=0) = 1$$

$$\text{Exact Solution: } x(t) = (1-\varepsilon^2)^{-\frac{1}{2}} e^{-\varepsilon t} \sin[(1-\varepsilon^2)^{\frac{1}{2}} t]$$

$$\text{Seek a Straightforward Expansion: } x(t,\varepsilon) = x_o(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

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Straightforward Expansion: $x(t,\varepsilon) = x_o(t) + \varepsilon x_1(t) + O(\varepsilon^2)$

Substitute expansion into governing equation, collect terms at different orders of ε to obtain the following hierarchy of equations:

$O(\varepsilon^0): \ddot{x}_o + x_o = 0;\ x_o(t=0) = 0$ and $\dot{x}_o(t=0) = 1$

$O(\varepsilon): \ddot{x}_1 + x_1 = -2\dot{x}_o;\ x_1(t=0) = 0$ and $\dot{x}_1(t=0) = 0$

Solving the above equations, we obtain the approximation

$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$

Terms such as $t\sin t$ are called mixed secular terms, as they contain a combination of algebriac and trignometric terms. Secular terms grow unboundedly as $t \to \infty$. Note that $\varepsilon t \ll 1$, for approximation to be valid.

ENME665: Nonlinear Oscillations

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Exact Solution: $x(t) = (1-\varepsilon^2)^{-\frac{1}{2}} e^{-\varepsilon t} \sin[(1-\varepsilon^2)^{\frac{1}{2}} t]$

Straightforward Expansion: $x(t,\varepsilon) = x_o(t) + \varepsilon x_1(t) + O(\varepsilon^2)$

$$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$$

Comparing the straightforward expansion with the exact solution, we see that this perturbation expansion completely misrepresents the slow scale behavior. The sources of discrepancy come from the amplitude and frequency. First,

$$e^{-\varepsilon t} = 1 - \varepsilon t + O\left(\varepsilon^2 t^2\right)$$

Next,the frequency of oscillation is

$$\omega = (1-\varepsilon^2)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon^2 + \ldots$$

After long time $t \sim O\left(1/\varepsilon^2 \right)$, frequency error will have a significant cumulative effect.

ENME665: Nonlinear Oscillations

- Example ii) Undamped, nonlinear oscillator with weak nonlinearity

$\ddot{v} + v + \varepsilon \alpha v^3 = 0;\ |\varepsilon| \ll 1$ and $\alpha = O(1)$; Arbitrary initial conditions

Seek a straightforward Expansion: $v(t,\varepsilon) = v_o(t) + \varepsilon v_1(t) + O(\varepsilon^2)$

After substituting the expansion into the governing equation and solving the system at different orders of ε, we obtain

$$
\begin{aligned}
v(t,\varepsilon) &= a_o \cos(t+\beta_o) \\
&+ \varepsilon \left[a_1 \cos(t+\beta_1) - \frac{3}{8}\alpha a_o^3 t \sin(t+\beta_o) + \frac{1}{32}\alpha a_o^3 \cos(3t+3\beta_o) \right] \\
&+ \ldots.
\end{aligned}
$$

Here, the amplitudes a_i and phases β_i are determined by initial conditions. Again, the presence of a mixed secular term can be noted. The straightforward expansion is not valid for t $\geq O\left(1/\varepsilon\right)$. We need $\dfrac{v_1(t)}{v_o(t)}$ to be bounded as $t \to \infty$.

ENME665: Nonlinear Oscillations

- Example iii) Nonlinear oscillator with weak nonlinearity and weak damping

$\ddot{v} + v + 2\varepsilon\mu\dot{v} + \varepsilon\alpha v^3 = 0;\ \ |\varepsilon| \ll 1, \mu = O(1),$ and $\alpha = O(1)$; Arbitrary initial conditions

Seek a straightforward Expansion: $v(t,\varepsilon) = v_o(t) + \varepsilon v_1(t) + O(\varepsilon^2)$

After substituting the expansion into the governing equation andsolving the system at different orders of ε, we obtain

$$
\begin{aligned}
v(t,\varepsilon) &= a_o \cos\left(t+\beta_o\right) \\
&\quad + \varepsilon\left[a_1 \cos\left(t+\beta_1\right) + \mu a t \cos\left(t+\beta_o\right) - \frac{3}{8}\alpha a_o^3 t \sin\left(t+\beta_o\right) + \frac{1}{32}\alpha a_o^3 \cos\left(3t+3\beta_o\right) \right] \\
&\quad + \ldots.
\end{aligned}
$$

Here, the amplitudes a_i and phases β_i are determined by initial conditions. Again, the presence of mixed secular terms can be noted. The straightforward expansion is not valid for t $\geq O\left(1/\varepsilon\right)$. We need $\dfrac{v_1(t)}{v_o(t)}$ to be bounded as $t \to \infty$.

ENME665: Nonlinear Oscillations

•Material to be covered next class

- ❖ Quantitative Analyses (continued)
- ❖ Straightforward expansions and limitations (continued)

$$\ddot{v} + v + 2\varepsilon\mu\dot{v} + \varepsilon\alpha v^3 = F\cos(\Omega t);\ \left|\varepsilon\right| \ll 1,\ \mu = O(1),\ \alpha = O(1), \text{and } F = O(1)$$

Arbitrary initial conditions

Use straightforward expansions to pick up resonances $O(\varepsilon^0)$: Resonance at $\Omega = \omega_o$

$O(\varepsilon)$: Resonances at $\Omega = \dfrac{1}{3}\omega_o$ and $\Omega = 3\omega_o$

Resonances at $\Omega = \dfrac{1}{3}\omega_o$ and $\Omega = 3\omega_o$ particular to nonlinear system. For the present system, these are first-order resonances. At higher orders, additional resonances would be present.

Resonances cause straightforward expansions to breakdown.