# ENME665: Nonlinear Oscillations

- Material covered in previous week and this week
  - ❖ Chapter 1, Nayfeh and Balachandran (1995, 2006)
    - ➢ Dissipation, Attraction, Attracting Sets
    - ➢ Lyapunov function based stability analyses

  - ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006)
    - ➢ Local stability analyses- Linearization
    - ➢ Fixed points of maps and differential equation systems; Classification
    - ➢ Why nonlinear analyses?

- Material to be covered today and next week
  - ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006)
    - ❖ Eigenspaces and manifolds
    - ❖ Fixed points of maps and stability
    - ❖ Local bifurcations of fixed points

❖ Hyperbolic Fixed Points (Chapter 2, N&B)

$$\dot{x}_1 = x_2 \tag{1.4.1}$$
$$\dot{x}_2 = -\,x_1 + x_1^3 - 2\mu x_2 \tag{1.4.2}$$

**Example 2.1.** For illustration, we consider the classification of the fixed points $(0,0)$, $(-1,0)$, and $(1,0)$ of (1.4.1) and (1.4.2). In the vicinity of a fixed point, we obtain the following system after linearization:

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1+3x_1^2 & -2\mu \end{bmatrix} \mathbf{y} \tag{2.1.8}$$

Hence, the eigenvalues of the Jacobian matrix are

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 1 + 3x_1^2} \text{ and } \lambda_2 = -\mu + \sqrt{\mu^2 - 1 + 3x_1^2} \tag{2.1.9}$$

For all three fixed points, both of the eigenvalues have nonzero real parts when $\mu \neq 0$. Hence, all three fixed points are hyperbolic fixed points.

In the vicinity of the fixed point $(0,0)$, (2.1.8) and (2.1.9) become

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1 & -2\mu \end{bmatrix} \mathbf{y} \tag{2.1.10}$$

and

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 1} \text{ and } \lambda_2 = -\mu + \sqrt{\mu^2 - 1} \tag{2.1.11}$$

❖ Hyperbolic Fixed Points (Chapter 2, N&B)

$$\dot{x}_1 = x_2 \tag{1.4.1}$$
$$\dot{x}_2 = -\,x_1 + x_1^3 - 2\mu x_2 \tag{1.4.2}$$

In the vicinity of either the fixed point $(-1, 0)$ or the fixed point $(1, 0)$, (2.1.8) and (2.1.9) become

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 2 & -2\mu \end{bmatrix} \mathbf{y} \tag{2.1.12}$$

and

$$\lambda_1 = -\mu - \sqrt{\mu^2 + 2} \text{ and } \lambda_2 = -\mu + \sqrt{\mu^2 + 2} \tag{2.1.13}$$

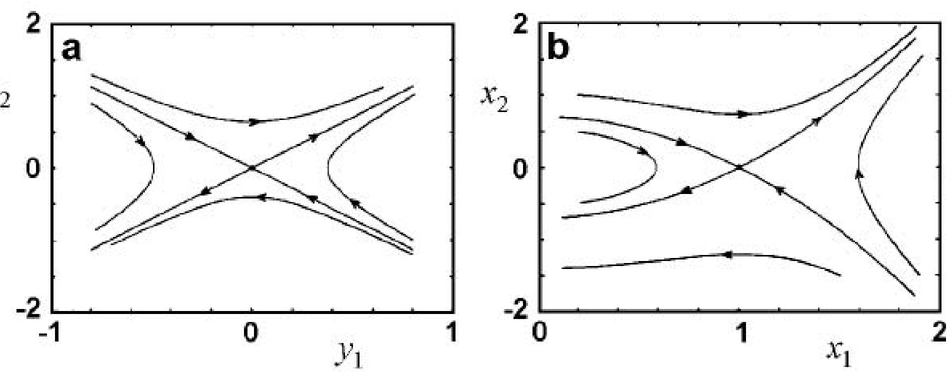


***Fig. 2.1.2*** (a) Flow in the vicinity of the saddle point $(0, 0)$ of the linear system (2.1.12) and (b) flow in the vicinity of the saddle point $(1, 0)$ of the nonlinear system (1.4.1) and (1.4.2). Both the flows are qualitatively similar.

# ENME665: Nonlinear Oscillations

❖ Hyperbolic Fixed Points (Chapter 2, N&B)

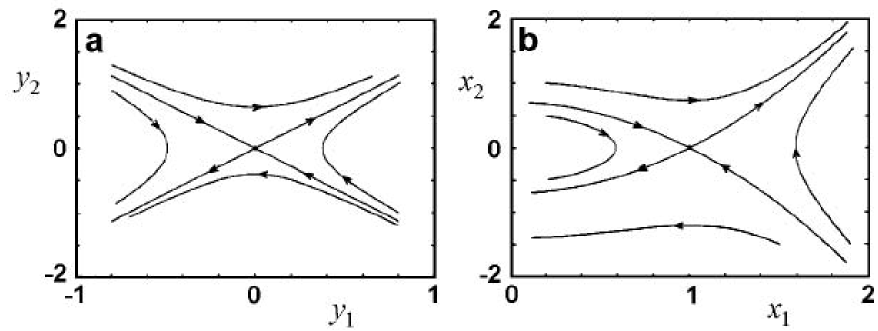


***Fig. 2.1.2*** (a) Flow in the vicinity of the saddle point $(0, 0)$ of the linear system (2.1.12) and (b) flow in the vicinity of the saddle point $(1, 0)$ of the nonlinear system (1.4.1) and (1.4.2). Both the flows are qualitatively similar.

Many theorems provide precise statements on what the stability of fixed–point solutions of the linearized system (2.1.5) imply for the stability of fixed–point solutions of the full nonlinear system (2.1.1). The **Hartman–Grobman theorem** (e.g., Arnold, 1988, Chapter 3; Wiggins, 1990, Chapter 2) is applicable to hyperbolic fixed points, whereas the **Shoshitaishvili theorem** (e.g., Arnold, 1988, Chapter 6) is applicable to nonhyperbolic fixed points. From these theorems, it follows that (a) the fixed point $\mathbf{x} = \mathbf{x}_0$ of the nonlinear system (2.1.1) is stable when the fixed point $\mathbf{y} = 0$ of the linear system (2.1.5) is asymptotically stable; (b) the fixed point $\mathbf{x} = \mathbf{x}_0$ of the nonlinear system (2.1.1) is unstable when the fixed point $\mathbf{y} = 0$ of the linear system (2.1.5) is unstable; and (c) linearization cannot determine the stability of neutrally stable fixed points (including centers) of (2.1.1). In the case of neutrally stable fixed points, a nonlinear analysis is necessary to determine the stability of $\mathbf{x}_0$. It will be necessary to retain quadratic and, sometimes, higher–order terms in the disturbance $\mathbf{y}$ in the Taylor–series expansion of (2.1.4).

# ENME665: Nonlinear Oscillations

- Eigenspaces and Invariant Manifolds (Chapter 2, N&B)

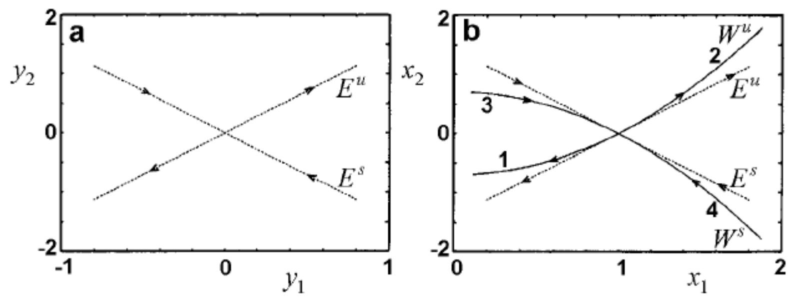


*Fig. 2.1.5* (a) Stable and unstable eigenspaces of the fixed point $(0,0)$ of (2.1.12) and (b) stable and unstable manifolds of the fixed point $(1,0)$ of (1.4.1) and (1.4.2).

$$\begin{aligned} E^s &= \text{span } \{\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_s\} \\ E^u &= \text{span } \{\mathbf{p}_{s+1}, \mathbf{p}_{s+2}, \cdots, \mathbf{p}_{s+u}\} \\ E^c &= \text{span } \{\mathbf{p}_{s+u+1}, \mathbf{p}_{s+u+2}, \cdots, \mathbf{p}_{s+u+c}\} \end{aligned}$$

The spaces $E^s$, $E^u$, and $E^c$ are **invariant subspaces** of the corresponding linear system. A solution of the linear system initiated in an invariant subspace remains in this subspace for all times. Thus, solutions initiated in $E^s$ approach the fixed point as $t \to \infty$, solutions initiated in $E^u$ approach the fixed point as $t \to -\infty$, and solutions initiated in $E^c$ neither grow nor decay in time. The subspaces $E^s$, $E^u$, and $E^c$ are called **stable**, **unstable**, and **center subspaces** or **manifolds**, respectively, of the considered fixed point of the linear system.

The **stable manifold** of a fixed point of (2.1.1) is the set of all initial conditions such that the flow initiated at these points asymptotically approaches the fixed point as $t \to \infty$, whereas the **unstable manifold** of a fixed point of (2.1.1) is the set of all initial conditions such that the flow initiated at these points asymptotically approaches the fixed point as $t \to -\infty$. In a nonlinear system, a stable manifold is denoted by $W^s$, an unstable manifold is denoted by $W^u$, and a center manifold is denoted by $W^c$.

## ❖ Fixed Points of Maps and Stability (Chapter 2, N&B)

Here, we consider fixed points of the map

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \mathbf{M}) \tag{2.2.1}$$

A fixed point $\mathbf{x}_0$ of this map satisfies the condition

$$\mathbf{x}_0 = \mathbf{F}^m(\mathbf{x}_0; \mathbf{M}_0) \quad \text{for all} \quad m \in \mathcal{Z} \tag{2.2.2}$$

where $\mathbf{M} = \mathbf{M}_0$ is the value of the vector of control parameters. We note that an orbit of a map initiated at a fixed point of the map is the fixed point itself. Moreover, the fixed points of a map are examples of invariant sets.

To determine the stability of the fixed point $\mathbf{x}_0$, we superimpose on it a disturbance $\mathbf{y}$ and find from (2.2.1) that

$$\mathbf{x}_0 + \mathbf{y}_{k+1} = \mathbf{F}(\mathbf{x}_0 + \mathbf{y}_k; \mathbf{M}_0) \tag{2.2.3}$$

where $k \in \mathcal{Z}$. Expanding $\mathbf{F}$ in a Taylor series around $\mathbf{x}_0$, using (2.2.2), and linearizing in $\mathbf{y}_k$, we obtain

$$\mathbf{y}_{k+1} = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y}_k = A\mathbf{y}_k \tag{2.2.4}$$

where $D_{\mathbf{x}}\mathbf{F}$ is the matrix of the first partial derivatives of $\mathbf{F}$ evaluated at $(\mathbf{x}_0; \mathbf{M}_0)$.

# ENME665: Nonlinear Oscillations

❖ Fixed Points of Maps and Stability (Chapter 2, N&B)

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$$\mathbf{y}_{k+1} = D_\mathbf{x}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y}_k = A\mathbf{y}_k \tag{2.2.4}$$

where $D_\mathbf{x}\mathbf{F}$ is the matrix of the first partial derivatives of $\mathbf{F}$ evaluated at $(\mathbf{x}_0; \mathbf{M}_0)$. Next, we introduce the linear transformation

$$\mathbf{y} = P\mathbf{z} \tag{2.2.5}$$

into (2.2.4) and obtain

$$P\mathbf{z}_{k+1} = AP\mathbf{z}_k \tag{2.2.6}$$

Assuming that $P$ is nonsingular, we multiply (2.2.6) from the left by $P^{-1}$ and arrive at

$$\mathbf{z}_{k+1} = J\mathbf{z}_k, \quad J = P^{-1}AP \tag{2.2.7}$$

We choose $P$ as in the preceding section so that $J$ has a Jordan canonical form. If the eigenvalues $\rho_i$ of $A$ are distinct, $J$ is a diagonal matrix with entries $\rho_1, \ \rho_2, \ \cdots, \ \rho_n$.

❖ Fixed Points of Maps and Stability (Chapter 2, N&B)

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We choose $P$ as in the preceding section so that $J$ has a Jordan canonical form. If the eigenvalues $\rho_i$ of $A$ are distinct, $J$ is a diagonal matrix with entries $\rho_1, \ \rho_2, \ \cdots, \ \rho_n$. Then, (2.2.7) can be rewritten as

$$z_{k+1}^{(m)} = \rho_m z_k^{(m)}, \quad m = 1, 2, \cdots, n \tag{2.2.8}$$

where $z^{(m)}$ is the $m$th component of $\mathbf{z}$. It follows from (2.2.8) that as $k \to \infty$,

$$\begin{aligned}
z_k^{(m)} &\to 0 && \text{if } |\rho_m| < 1 \\
z_k^{(m)} &\to \infty && \text{if } |\rho_m| > 1 \\
z_k^{(m)} &= z_0^{(m)} && \text{if } \rho_m = 1 \\
z_{2k+1}^{(m)} &= -z_0^{(m)} && \text{if } \rho_m = -1 \\
z_{2k}^{(m)} &= z_0^{(m)} && \text{if } \rho_m = -1
\end{aligned}$$

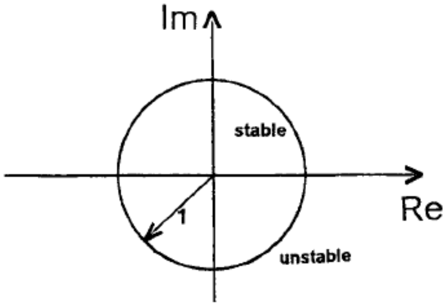


***Fig. 2.2.1*** Unit circle in the complex plane.

# ENME665: Nonlinear Oscillations

❖ Fixed Points of Maps and Stability (Chapter 2, N&B)

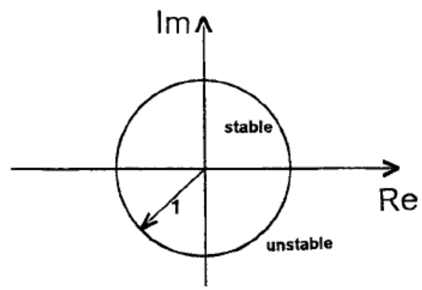


*Fig. 2.2.1* Unit circle in the complex plane.

Therefore, to ascertain the stability of the fixed point $x_0$, we examine the location of the eigenvalues of $A$ in the complex plane with respect to the unit circle shown in Figure 2.2.1. If all of the eigenvalues of $A$ are such that they are either inside the unit circle or outside the unit circle, the corresponding fixed point is called a **hyperbolic fixed point**. A hyperbolic fixed point is called a **saddle point** if some eigenvalues are within the unit circle and the rest of them are outside the unit circle. A hyperbolic fixed point is called a **sink** if all of the eigenvalues are within the unit circle. Similarly, a **source** corresponds to the case where all of the eigenvalues are outside the unit circle. If one or more eigenvalues of $A$ lie on the unit circle, the corresponding fixed point is called a **nonhyperbolic fixed point**. The **Hartman–Grobman theorem** is also applicable to fixed points of maps. From this theorem, it follows that linearization of a map is sufficient to determine the stability of a hyperbolic fixed point. If all of the eigenvalues of $A$ lie within the unit circle, the fixed point $x_0$ is said to be asymptotically stable. If at least one eigenvalue of $A$ lies outside the unit circle, the fixed point $x_0$ is unstable, as depicted in Figure 2.2.1. If none of the eigenvalues of $A$ lie outside the unit circle, a linear analysis is not sufficient to determine the stability of a nonhyperbolic fixed point and nonlinear terms have to be included on the right–hand side of (2.2.4).

# ENME665: Nonlinear Oscillations

❖ Fixed Points of Maps and Stability (Chapter 2, N&B)

A solution $\mathbf{x}_0$ that satisfies the condition

$$\mathbf{x}_0 = \mathbf{F}^k(\mathbf{x}_0; \mathbf{M}_0) \tag{2.2.9}$$

where $k \geq 1$ is called a **period–$k$ point** or **periodic point of order** $k$ of the map $\mathbf{F}$. This point is a fixed point of the map $\mathbf{G}$, which is formed by $k$ successive iterations of $\mathbf{F}$; that is,

$$\mathbf{G}(\mathbf{x}; \mathbf{M}) = \mathbf{F}^k(\mathbf{x}; \mathbf{M})$$

Thus, the stability of period-$k$ points of $\mathbf{F}$ can be studied by investigating the stability of the fixed points of $\mathbf{F}^k$.

# ENME665: Nonlinear Oscillations

❖ Local bifurcations of fixed points of continuous systems (Chapter 2, N&B)

$$\dot{x} = F(x;\mu) = \mu - x^2 \tag{2.3.1}$$

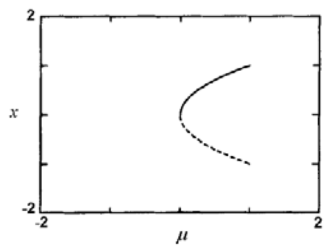


*Fig. 2.3.1* Scenario in the vicinity of a saddle-node bifurcation.

$$\dot{x} = \mu x - x^2 \tag{2.3.3}$$

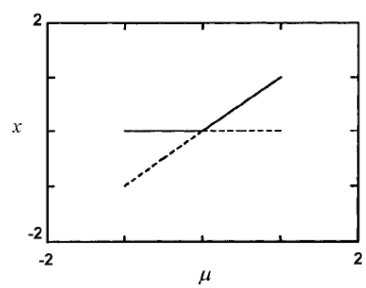


*Fig. 2.3.3* Scenario in the vicinity of a transcritical bifurcation.

$$\dot{x} = F(x;\mu) = \mu x + \alpha x^3 \tag{2.3.2}$$

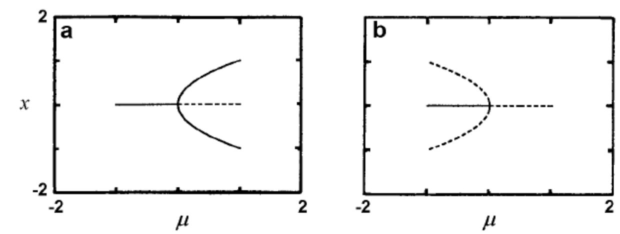


*Fig. 2.3.2* Local scenarios: (a) supercritical pitchfork bifurcation and (b) subcritical pitchfork bifurcation.

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2) \tag{2.3.5}$$
$$\dot{y} = \omega x + \mu y + (\beta x + \alpha y)(x^2 + y^2) \tag{2.3.6}$$

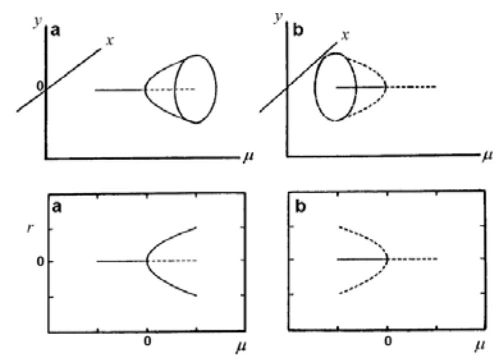


*Fig. 2.3.5* Local scenarios: (a) supercritical Hopf bifurcation and (b) subcritical Hopf bifurcation.

# ENME665: Nonlinear Oscillations

•Material to be covered next class

- ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006) – Local bifurcations to be continued