ENME665: Nonlinear Oscillations

- Material covered last class
  - ❖ Quantitative Analyses (continued)
  - ❖ Straightforward expansions (Section 2.3.1, Nayfeh and Mook, 1979) and limitations

- Material to be covered today
  - ❖ Straightforward expansions and limitations (continued)
  - ❖ Introduction to Lindstedt-Poincaré method (Section 2.3.2, Nayfeh and Mook, 1979)

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➢ Example ii) Undamped, nonlinear oscillator with weak nonlinearity

$\ddot{v} + v + \varepsilon \alpha v^3 = 0;\ |\varepsilon| \ll 1$ and $\alpha = O(1)$; Arbitrary initial conditions

Seek a straightforward Expansion: $v(t, \varepsilon) = v_o(t) + \varepsilon v_1(t) + O(\varepsilon^2)$

After substituting the expansion into the governing equation and solving the system at different orders of ε, we obtain

$$
\begin{aligned}
v(t, \varepsilon) &= a_o \cos\left(t + \beta_o\right) \\
&\quad + \varepsilon \left[a_1 \cos\left(t + \beta_1\right) - \frac{3}{8} \alpha a_o^3 t \sin\left(t + \beta_o\right) + \frac{1}{32} \alpha a_o^3 \cos\left(3t + 3\beta_o\right) \right] \\
&\quad + \ldots.
\end{aligned}
$$

Here, the amplitudes a_i and phases β_i are determined by initial conditions. Again, the presence of a mixed secular term can be noted. The straightforward expansion is not valid for t $\geq O\left(1/\varepsilon\right)$. We need $\dfrac{v_1(t)}{v_o(t)}$ to be bounded as $t \to \infty$.

- Example iii) Nonlinear oscillator with weak nonlinearity and weak damping

$\ddot{v} + v + 2\varepsilon\mu\dot{v} + \varepsilon\alpha v^3 = 0;\ |\varepsilon| \ll 1, \mu = O(1),$ and $\alpha = O(1)$; Arbitrary initial conditions

Seek a straightforward Expansion: $v(t, \varepsilon) = v_o(t) + \varepsilon v_1(t) + O(\varepsilon^2)$

After substituting the expansion into the governing equation andsolving the system at different orders of ε, we obtain

$$v(t, \varepsilon) = a_o \cos\left(t + \beta_o\right)$$
$$+ \varepsilon\left[a_1 \cos\left(t + \beta_1\right) + \mu a t \cos\left(t + \beta_o\right) - \frac{3}{8}\alpha a_o^3 t \sin\left(t + \beta_o\right) + \frac{1}{32}\alpha a_o^3 \cos\left(3t + 3\beta_o\right)\right]$$
$$+ \ldots.$$

Here, the amplitudes a_i and phases β_i are determined by initial conditions. Again, the presence of mixed secular terms can be noted. The straightforward expansion is not valid for t $\geq O\left(1/\varepsilon\right)$. We need $\dfrac{v_1(t)}{v_o(t)}$ to be bounded as $t \to \infty$.

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- Example iv) Forced nonlinear oscillator with damping

$$\ddot{v} + \omega_o^2 v + 2\varepsilon\mu\dot{v} + \varepsilon\alpha v^3 = F\cos(\Omega t); \; |\varepsilon| \ll 1, \mu = O(1), \alpha = O(1), \text{and } F = O(1)$$

Arbitrary initial conditions

Use straightforward expansions to pick up resonances

$O(\varepsilon^0)$: Resonance at $\Omega = \omega_o$

$O(\varepsilon)$: Resonances at $\Omega = \frac{1}{3}\omega_o$ and $\Omega = 3\omega_o$

Resonances at $\Omega = \frac{1}{3}\omega_o$ and $\Omega = 3\omega_o$ particular to nonlinear system. For the present system, these are first-order resonances. At higher orders, additional resonances would be present.

Resonances cause straightforward expansions to breakdown.

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- Example v) Coupled, nonlinear oscillators

$$\ddot{v}_1 + \omega_1^2 v_1 = -\varepsilon\left[2\mu_1\dot{v}_1 + \delta_1 v_1 v_2\right]; \; |\varepsilon| \ll 1, \mu_1 = O(1), \text{ and } \delta_i = O(1)$$

$$\ddot{v}_2 + \omega_2^2 v_2 = -\varepsilon\left[2\mu_2\dot{v}_2 + \delta_2 v_1^2\right]$$

Arbitrary initial conditions

Seek straightforward expansions

$$v_i(t,\varepsilon) = v_{i0}(t) + \varepsilon v_{i1}(t) + \ldots; \; i = 1, 2$$

Expansions breakdown when $\omega_2 = 2\omega_1$, which is an example of aninternal resonance; in particular,a two-to-one internal resonance.

- Lindstedt-Poincaré Method (Section 2.3.2, Nayfeh and Mook, 1979)

$$\ddot{v} + \omega_0^2 v = -\varepsilon\left[f(v, \dot{v}) \right]; \ \ |\varepsilon| \ll 1$$

Introduce a new time scale

$$\tau = \omega t$$

where

$$\omega = \omega_0 + \varepsilon\omega_1 + \ldots$$

with the ω_i unknowns at this stage. Then,the governing equation is transformed to

$$\omega^2 \frac{d^2}{d\tau^2}(v) + \omega_0^2 v = -\varepsilon\left[f(v, \frac{dv}{d\tau}) \right]$$

Seek an expansion for the solution of the form

$$v(t, \varepsilon) = v_o(t) + \varepsilon v_1(t) + \ldots$$

and choose the ω_i to eliminate the sources of secular terms.

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- Lindstedt-Poincaré Method

  Rescale time as $\tau = \omega t$
  Expand $x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots$
  Expand $\omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$

  ω_0 is linear natural frequency

  Next, set source of secular terms equal to zero, by choosing ω_is (which are unknowns, for *i*>0).

- This method can be used to find frequency-amplitude relationships.

- This method cannot be used to find transient response (and hence cannot be applied to study damped systems).

- Derivatives on new time scale:

$$\frac{d(\cdot)}{dt} = \frac{d(\cdot)}{d\tau}\frac{d\tau}{dt} = \frac{d(\cdot)}{d\tau}\omega = \omega\frac{d(\cdot)}{d\tau}$$

$$\Rightarrow \frac{d^2}{dt^2}(\cdot) = \omega^2 \frac{d^2}{d\tau^2}(\cdot)$$

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- Examples: i) unforced Duffing oscillator and ii) unforced pendulum

- i) unforced Duffing oscillator

  $\ddot{u} + {\omega_0}^2\, u = -\epsilon f(u, \dot{u}) = -\epsilon\alpha u^3$

  Applying the Lindstedt-Poincaré method substitutions,

  $$\omega^2 \frac{d^2u}{d\tau^2} + {\omega_0}^2 u(\tau) = \epsilon\alpha u^3,$$

  which in turn leads to

  $$(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2 (u_0'' + \epsilon u_1'' + \epsilon^2 u_2'' + \cdots)$$
  $$+ {\omega_0}^2 (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)$$
  $$= -\epsilon\alpha (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)^3$$

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- Collecting terms, we have

$\mathcal{O}(1)$: $u_0'' + u_0 = 0$

which has solution

$u_0(\tau) = a_0 \cos(\tau + \beta_0)$

$\mathcal{O}(\epsilon)$: ${\omega_0}^2 u_1'' + {\omega_0}^2 u_1 = -2\omega_0\omega_1 u_0'' - \alpha(u_0)^{\,3}$

$=2\omega_0\omega_1 a_0 \cos(\tau + \beta_0)$

$-\alpha(a_0)^3 * {}^1\!/_4 \left[\cos(3\tau + 3\beta_0) + 3\cos(\tau + \beta_0)\right]$

which has solution

$u_1(\tau) = a_1 \cos(\tau + \beta_1) + A\left[2\omega_0\omega_1 a_0 - \alpha(a_0)^3 * {}^3\!/_4\right] \cos(\tau + \beta_0) * \tau - B\alpha(a_0)^3 \cos(3\tau + 3\beta_0)/4$

- Setting source of secular terms equal to zero,

$$\omega_1 = \frac{3\alpha(a_0)^2}{8\omega_0}$$

and so the frequency expansion is

$$\omega = \omega_0 + \epsilon\frac{3\alpha(a_0)^2}{8\omega_0} + \ldots$$

Therefore,

$$u = a_0 \cos\left(\left(\omega_0 + \epsilon\frac{3\alpha {a_0}^2}{8\omega_0} + \ldots\right)t + \beta_0\right) + O(\epsilon)$$

- ii) unforced pendulum

$$\ddot{\theta} + {\omega_0}^2 \sin(\theta) = 0$$
$$\rightarrow \ddot{\theta} + {\omega_0}^2 \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = 0$$

Since even terms don't appear in the Taylor expansion of sine, and by a symmetry argument, assume the following expansions:
θ=$\epsilon\theta_1 + \epsilon^3\theta_3 + \cdots$
and
$\omega = \omega_0 + \epsilon^2\omega_2 + \cdots$

Here, ϵ is a measure of the amplitude of motion and is used to indicate that the motions have small amplitudes.

- Expanding the equation of motion,

$$(\omega_0+\epsilon^2\omega_2+\cdots)^2(\epsilon\theta_1''+\epsilon^3\theta_3''+\cdots)+\omega_0{}^2\left([\epsilon\theta_1+\epsilon^3\theta_3+\cdots]-\frac{1}{6}[\epsilon\theta_1+\epsilon^3\theta_3+\cdots]^3+\cdots\right)=0$$

Collecting terms,

$$\mathcal{O}(\epsilon): \theta_1''+\theta_1=0$$

which has solution

$$\theta_1=a_1\cos(\tau+\beta_1)$$

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- $\mathcal{O}(\epsilon^3)$: $\omega_0{}^2(\theta_3'' + \theta_3) = \left[2\omega_0\omega_1 a_1 + \frac{\omega_0{}^2(a_1)^3}{8}\right]\cos(\tau + \beta_1) + \frac{\omega_0(a_1)^3}{24}\cos(3\tau + 3\beta_1)$

- So, the solution is

$$\theta_3 = a_3\cos(\tau + \beta_3) + A(\theta_3'' + \theta_3)$$
$$= \left[2\omega_0\omega_1 a_1 + \frac{\omega_0{}^2(a_1)^3}{8}\right]\cos(\tau + \beta_1)\tau$$
$$+ B\frac{\omega_0(a_1)^3}{24}\cos(3\tau + 3\beta_1)$$

- Setting source of secular terms equal to zero, it is found that

$$\omega_2 = -\frac{\omega_0 {a_1}^2}{16}$$

Thus,

$$\theta(t) = a_1 \cos\left(\left[\omega_0 - \frac{\omega_0 {a_1}^2}{16} + \cdots\right] t + \beta_1\right) + \cdots$$

where the bookkeeping parameter ϵ has been set equal to one.

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- Material to be covered next several classes
  - ❖ Lindstedt-Poincaré Method (Nayfeh and Mook, 1979)
  - ❖ Method of Multiple Scales (Nayfeh and Mook, 1979)