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Lyapunov Function Based Stability

This approach can be used to draw conclusion about the stability of the considered equilibrium solution of a system without explicitly determining the trajectories describing the solutions. Central to this approach, is what is called a Lyapunov function that is discussed below.

Theorem 1: Consider the dynamical system

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X},\mathbf{M}) ; \qquad \mathbf{X} \in \Re^n ; \quad \mathbf{M} \in \Re^m ; \quad \mathbf{F} : \Re^n \times \Re^m \to \Re^n$$

Let $\mathbf{X} = \mathbf{X}_0$ be an equilibrium solution of this system in $\Re^n$, and let there be a scalar function $V(\mathbf{X}) : D_l \to \Re$; that is a C^1 function on some neighborhood D_l of $\mathbf{X}_0$ such that

i) $V(\mathbf{X}_0) = 0$ and $V(\mathbf{X}) > 0$ for $\mathbf{X} \neq \mathbf{X}_0$;
ii) $\dot{V}(\mathbf{X}) \leq 0$ in $D_l \sim \{\mathbf{X}_0\}$

Then, $\mathbf{X}_0$ is stable in D_l.

Remarks:

a) $V(\mathbf{X})$ is called a Lyapunov function;
b) $D_l \sim \{\mathbf{X}_0\}$ refers to the region excluding $\{\mathbf{X}_0\}$;
c) Proof of this theorem can be found in the following textbooks:

i) Krasovskii, N. N. (1963). Stability of Motion, Stanford university Press, Stanford, CA
ii) Khalil, H. K. (1996). Nonlinear System Analysis, Prentice Hall, Upper Saddle River, NJ. ; Section 3.1
iii) Vidyasagar, M. (1993). Nonlinear System Analysis, Prentice Hall, Upper Saddle River, NJ. ; Chapter 5
iv) Bauel, F. and Nobel, J. A. (1969). The Qualitative Theory of Ordinary Differential Equations; An Introduction, Chapter 5

d) This approach is also called the direct approach.

Theorem 2: (Local Asymptotic Stability)

If condition i) of Theorem 1 is satisfied and if the following condition is satisfied

iii) $\dot{V}(\mathbf{X}) < 0$ in $D_l \sim \{\mathbf{X}_0\}$

Then, $\mathbf{X}_0$ is asymptotically stable in D_l.

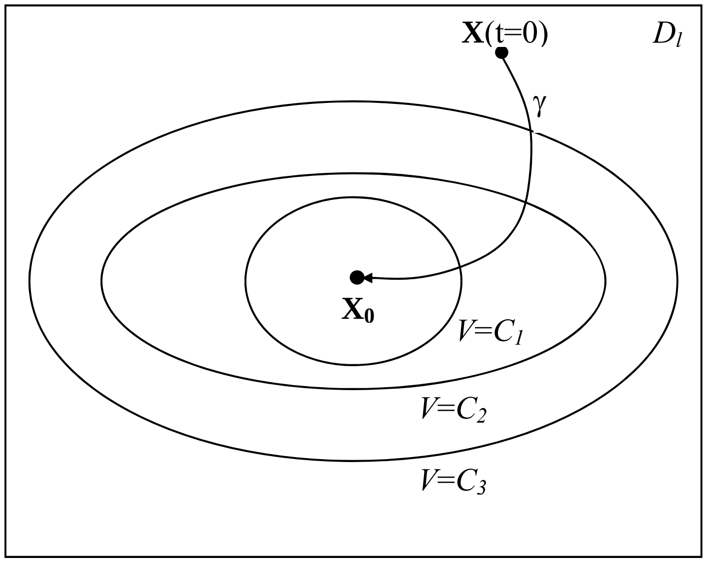

Figure 1. Trajectory γ evolving towards $\mathbf{X}_0$ which is asymptotic stable in D_l .

Remarks:

a) If D_l can be chosen to be all of $\Re^n$, then $\mathbf{X}_0$ is said to be asymptotically stable in $\Re^n$; see end of Theorem 4, for when this may not be true.

b) From geometric considerations, referring to Figure 1, one can note that once $V(\mathbf{X})$ is a monotonically decreasing (positive definite) function, then the trajectory initiated from $\mathbf{X}(t=0)$ approaches the limit surface $V = C_o$ ($\lim_{t\to\infty} V(\mathbf{X}) = C_o$) from its outside. Note that $C_o \geq 0$, in general, and *o* in our case, which degenerates to a point in $\Re^n$. Also note that a trajectory initiated from within the level surface $V(\mathbf{X}) = C_i$ never moves outside this surface. For asymptotic stability according to Lyapunov, the following need to be taken notice of:

1) Level surfaces $V(\mathbf{X}) = C$ are closed in the vicinity of $\mathbf{X}_0$
2) lim $\lim_{t\to\infty} V(\mathbf{X}) = 0$ ensure that the trajectory converages to $\mathbf{X}_0$

c) For proof of Theorem 2, refer to the textbooks mentioned on page 1.

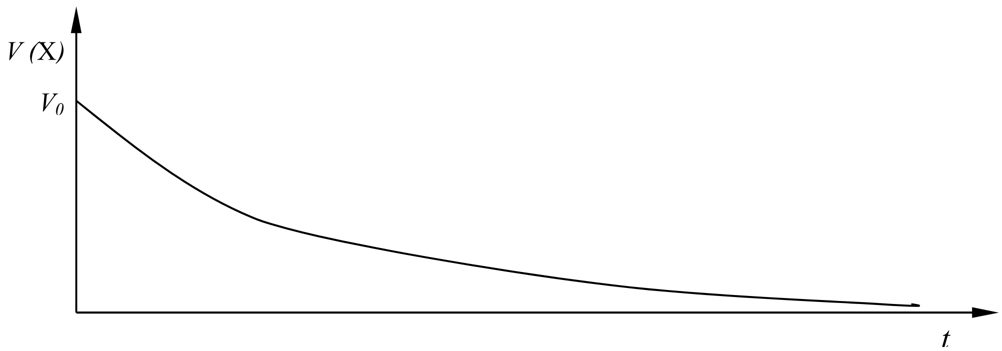

Figure 2. $V(\mathbf{X})$ decreases monotonically to zero.

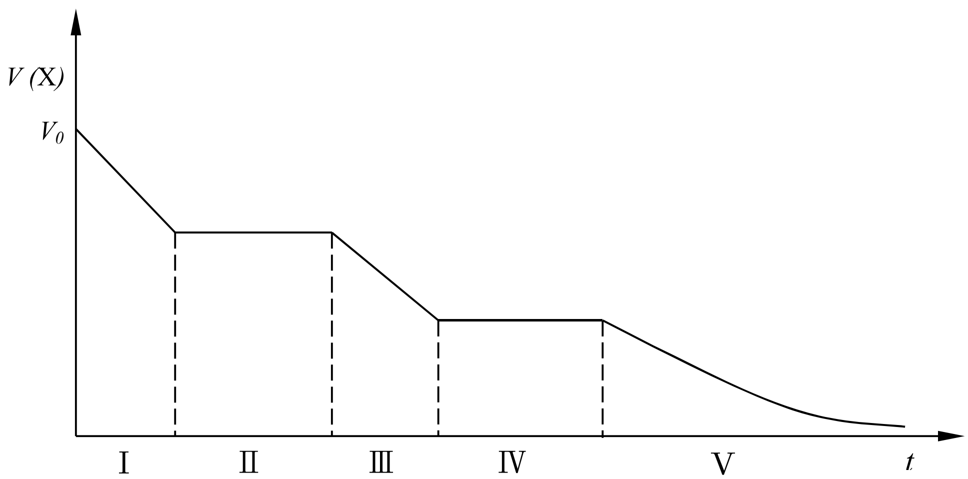

Figure 3. $V(\mathbf{X})$ not decreasing monotonically to zero.

As noted earlier, for asymptotic stability, the Lyapunov function has to decrease monotically to *o* according to Theorem 2. For instance, if one had a Lyapunov function as show in Figure 3, where $V(\mathbf{X})$ decreases montonically in regions Ⅰ, Ⅲ and Ⅴ and stays constant in other regions such as Ⅱ and Ⅳ, then although $V(\mathbf{X})$ decreases in a piecewise-monotonic fashion and approaches *o* in the limit. Theorem 2 will not allow us to draw the conclusion of asymptotic stability in this case. This is addressed by the Krasovskii-Lasalle theorem (Vidyasagar, 1993).

<u>Theorem 3</u>: (Local Asymptotic Stability)

Consider the dynamical system

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X},\mathbf{M})\,; \qquad \mathbf{X} \in \Re^n\,; \quad \mathbf{M} \in \Re^m\,; \quad \mathbf{F} : \Re^n \times \Re^m \to \Re^n$$

Let $\mathbf{X} = \mathbf{X}_0$ be an equilibrium position of this system in $\Re^n$ and $V(\mathbf{X}) : D_l \to \Re$ be a C^1 function on some neighborhood D_l of $\mathbf{X}_0$ such that

i) $V(\mathbf{X}_0) = 0$ and $V(\mathbf{X}) > 0$ for $\mathbf{X} \neq \mathbf{X}_0$;

ii) $\dot{V}(\mathbf{X}) \leq 0$ in D_l; $D_l = \{\mathbf{X} \in \Re^n : V(\mathbf{X}) \leq l\}$

iii) $E = \{\mathbf{X} \in \Re^n \mid \dot{V}(\mathbf{X}) = 0\}$ does not contain any solutions $\mathbf{X}^*(E)$ other than $\mathbf{X}_0$ that can remain forever in E (i.e., for $0 \leq t < \infty$)

Then, $\mathbf{X}_0$ is asymptotically stable in D_l.

Remarks:

a) A geometric interpretation of the theorem can be provided with the aid of the Figure 4. In this figure, the trajectory initiated from $\mathbf{X}(t=0)$ gets onto the manifold (or surface) E and stays on the level surface $V(\mathbf{X}) = C_3$ because $\dot{V}(\mathbf{X}) = 0$ in E. Since E does not contain any other solution other than $\mathbf{X}_0$ that can remain in it forever (i.e., $0 \leq t < \infty$) or no positive invariant sets other than $\mathbf{X}_0$, the trajectory γ has to leave E. When γ leaves E, it enters the inside $V(\mathbf{X}) = C_3$ and evolves towards $\mathbf{X}_0$ before it gets onto E again and stays on $V(\mathbf{X}) = C_1$ for some time. Eventually, in the limit, γ is attracted to $\mathbf{X}_0$.

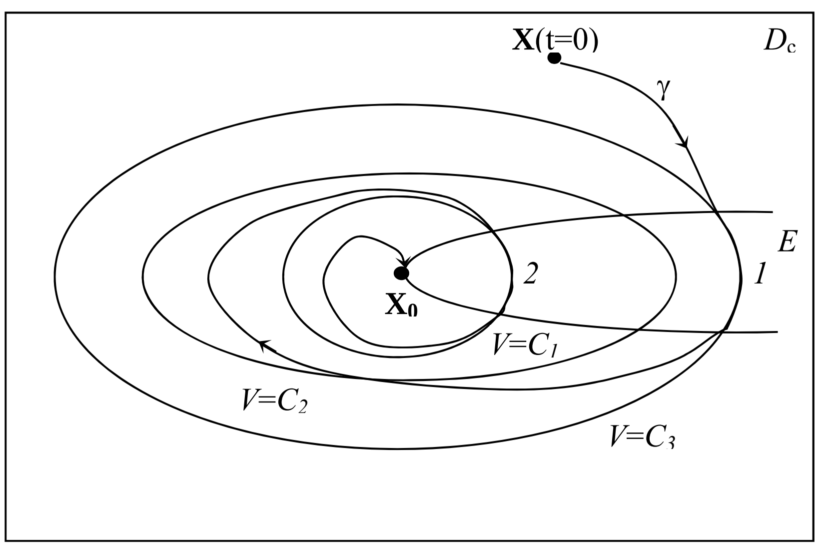

Figure 4. Evolution of γ towards the equilibrium point where the Lyapunov function's time derivative is zero in a region.

b) Because of the nature of $V(\mathbf{X})$, $\dot{V}(\mathbf{X}) < 0$ outside of *E*, and hence, a trajectory such as γ shown in Figure 4 cannot intersect $V(\mathbf{X})$=constant from the inner side.

c) Although condition i) and ii) can be easily examined, the same is not true for condition iii). Before addressing this, let us recall that condition i) requires for one to show that $V(\mathbf{X})$ is a positive definite function and condition iii) require for one to show that $\dot{V}(\mathbf{X}) = 0$ in *E* and negative definite outside *E*. To show that *E* does not contain any other solutions that remain in it forever, an indirect check can be carried out. To explain this, consider the figure next.

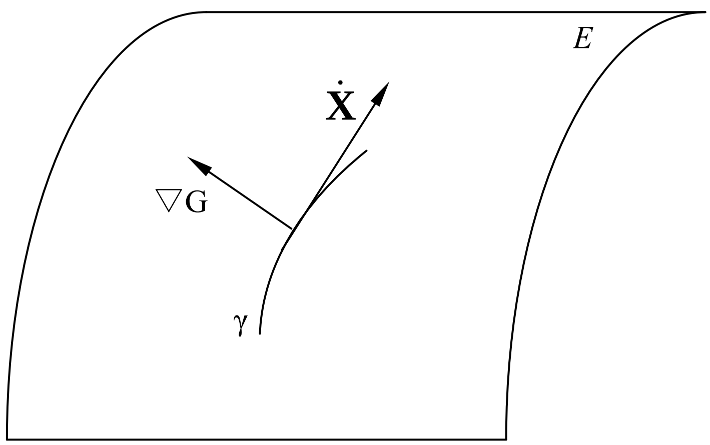

Figure 5. Trajectory along with gradient of G.

Let us suppose $G(x_1, x_2, \cdots, x_n) = 0$ in $\Re^n$ that defines the smooth surface *E*. Then, in order to ensure that the trajectory γ does not lie on this surface, one needs to ensure that

$$\dot{\mathbf{X}} \cdot \nabla G \neq 0$$

That is, the velocity field is not normal to the gradient.

Example: Consider the stability of $(x_1 = 0, x_2 = 0)$ of the system

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 + 5x_2^2 \\ \dot{x}_2 &= -x_1 x_2 - 4x_2^3 \end{aligned}\right\} \tag{1}$$

Let

$$V(\mathbf{X}) = \frac{1}{2}(x_1^2 + x_2^2) \tag{2}$$

which satisfies condition i) of Theorem 3. The time derivative of this function is

$$\dot{V}(\mathbf{X}) = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2 \quad (3)$$

Making use of Eq. (1) in Eq. (3), we obtain

$$\begin{aligned}\dot{V}(\mathbf{X}) &= x_1(-x_1 + 5x_2^2) + x_2(-x_1x_2 - 4x_2^3) \\ &= -(x_1 - 2x_2^2)^2\end{aligned} \quad (4)$$

On examining $\dot{V}(\mathbf{X})$ given by (4), it is clear that $\dot{V}(\mathbf{X})$ is a negative semi-definite function and it satisfies condition ii) of theorem 3.

Noting that

$$E = \{(x_1, x_2) \in \Re^2 : \dot{V}(\mathbf{X}) = 0\} \qquad (5)$$

leads to

$$G(x_1, x_2) = 0 \qquad \Rightarrow \qquad x_1 - 2x_2^2 = 0 \quad (6)$$

To indirectly verify that condition iii) is satisfied, let us examine

$$\begin{aligned}\dot{\mathbf{X}} \cdot \nabla G &= \dot{x}_1 \frac{\partial G}{\partial x_1} + \dot{x}_2 \frac{\partial G}{\partial x_2} \\ &= \dot{x}_1 - 4\dot{x}_2 x_2 \\ &= (-x_1 + 5x_2^2) - 4(-x_1x_2 - 4x_2^3)x_2 \\ &= -x_1 + 5x_2^2 + 4x_1x_2^2 + 16x_2^4 \\ &= -2x_2^2 + 5x_2^2 + 8x_2^4 + 16x_2^4 \\ &= 3x_2^2 + 24x_2^4\end{aligned} \qquad (7)$$

From equation (7), it is clear that

$$\dot{\mathbf{X}} \cdot \nabla G = 0$$

only at $x_1 = x_2 = 0$ and not equal to zero elsewhere. Hence, *E* does not contain any trajectories that will remain in it forever other that $(0,0)$.

Theorem 4: (Barbarshin-Krasovskii Theorem) (Global Asymptotic Stability)

Consider the dynamical system

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X},\mathbf{M})\,; \qquad \mathbf{X} \in \Re^n\,; \quad \mathbf{M} \in \Re^m\,; \quad \mathbf{F} : \Re^n \times \Re^m \to \Re^n$$

and let $\mathbf{X} = \mathbf{X}_0$ be an equilibrium position of this system in $\Re^n$. Let $V(\mathbf{X}) : \Re^n \to \Re$ be a C^1 function such that

i) $V(\mathbf{X}_0) = 0$ and $V(\mathbf{X}) > 0$ for $\mathbf{X} \neq \mathbf{X}_0$;
ii) $\dot{V}(\mathbf{X}) \leq 0$ in $\Re^n$
iii) $E = \{\mathbf{X} \in \Re^n \mid \dot{V}(\mathbf{X}) = 0\}$ does not contain any solutions $\mathbf{X}^*(E)$ other than $\mathbf{X}_0$ that remain forever in E (i.e., for $0 \leq t < \infty$)
IV) $\lim_{\|\mathbf{X}\| \to \infty} V(\mathbf{X}) = \infty$

Then, $\mathbf{X}_0$ is globally asymptotically stable in $\Re^n$.

Remarks:

a) Condition (IV) requires that $V(\mathbf{X})$ be a radially unbounded function. If this condition is not satisfied, then $V(\mathbf{X})$=constant may correspond to closed surfaces for "small" $\|\mathbf{X}\|$ and open surfaces for "large" $\|\mathbf{X}\|$. In this case, a trajectory initiated for from $\mathbf{X}_0$ will evolve towards to $\mathbf{X}_0$ with $V(\mathbf{X})$ decreasing in value, but in the limit may not approach to $\mathbf{X}_0$.

Example: Consider the stability of $(x_1 = 0, x_2 = 0)$ of the system (Khalil, 1996; Exercise 3.7)

$$\left.\begin{aligned} \dot{x}_1 &= \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 &= \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{aligned}\right\} \qquad (1)$$

Let

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2 \qquad (2)$$

$V(\mathbf{X})$ is a positive function, satisfies condition i) of Theorem 4.

$$\begin{aligned}\dot{V}(x_1,x_2) &= \frac{2x_1\dot{x}_1}{1+x_1^2} - \frac{x_1^2}{(1+x_1^2)^2}(2x_1\dot{x}_1) + 2x_2\dot{x}_2 \\ &= \frac{2x_1\dot{x}_1}{(1+x_1^2)^2} + 2x_2\dot{x}_2 \\ &= \frac{2x_1(-6x_1)}{(1+x_1^2)^4} + \frac{4x_1x_2}{(1+x_1^2)^2} + \frac{2x_2(-2x_1-2x_2)}{(1+x_1^2)^2} \\ &= \frac{-12x_1^4}{(1+x_1^2)^4} - \frac{4x_2^2}{(1+x_1^2)^2} \qquad (3)\end{aligned}$$

which is negative definite in $\Re^2 \sim \{\mathbf{0}\}$

On the basis of Theorem 2, the local asymptotic stability of $(0,0)$ follows. Now, in order to know global asymptotic stability, it needs to be checked if

$$\lim_{\|\mathbf{X}\|\to\infty} V(\mathbf{X}) = \infty \qquad (4)$$

Since, from equation (2)

$$\lim_{\substack{x_1\to\infty \\ x_2=0}} V(x_1,x_2) = 1+c^2 \qquad (5)$$

the requirement (4) is not satisfied; that is, the radial unbounded condition is not satisfied.

To understand the situation, a graph of $V(\mathbf{X})$ is shown in Figure 6.

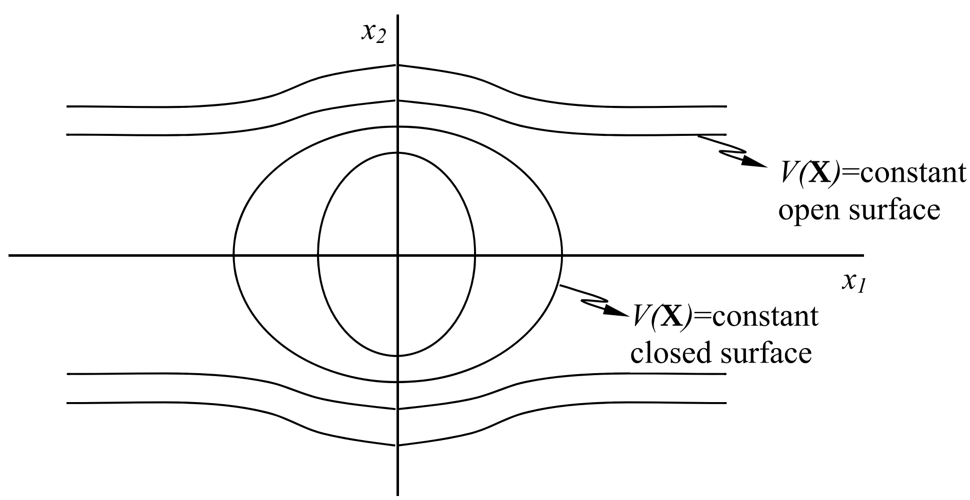

Figure 6.

The set $D_l = \{\mathbf{X} \in \Re^n \mid V(\mathbf{X}) \leq l\}$ is closed for "small" values of l and it becomes open after a certain 'l'. Considering

$$V(\mathbf{X}) = l = \frac{x_1^2}{1+x_1^2} + x_2^2 \text{, we have}$$

$$x_2 = \pm\sqrt{l - 1 + \frac{1}{1+x_1^2}} \qquad (6)$$

so, we can get

$$\begin{aligned} &0 < l < 1; \text{ closed surfaces} \\ &l \geq 1; \quad \text{open surfaces} \end{aligned}$$

Consider the hyperbola

$$x_2 = \frac{2}{x_1 - \sqrt{2}} \qquad (7)$$

The slope of the tangent to this curve is

$$k_1 = \frac{dx_2}{dx_1} = \frac{-2}{(x_1 - \sqrt{2})^2} \qquad (8)$$

On the hyperbola given by equation (7), equations (1) take the form

$$\left.\begin{aligned} \dot{x}_1 &= \frac{-6x_1}{(1+x_1^2)^2} + \frac{4}{x_1 - \sqrt{2}} \\ \dot{x}_2 &= \frac{-2(x_1 + \frac{2}{x_1 - \sqrt{2}})}{(1+x_1^2)^2} \end{aligned}\right\} \qquad (9)$$

From equations (9), we have

$$k_2 = \frac{dx_2}{dx_1} = \frac{-2(x_1^2 - \sqrt{2}x_1 + 2)}{-6x_1^2 + 6\sqrt{2}x_1 + 4(1+x_1^2)^2} \qquad (10)$$

Then, $\lim_{|x_1| \to \infty} \frac{k_2}{k_1} = \frac{1}{2}$; that is less than one. Therefore, one can find a large positive number for x_1, say d, such that for all $x_1 \geq d$, the inequality

$$|k_2| < |k_1| \qquad (11)$$

will be valid. Considering a domain ($x_1 \geq d$; $x_2 > \frac{2}{x_1 - \sqrt{2}}$) to the right of the hyperbola, a trajectory initiated in domain can intersect (7) from the inside to the outside if $|k_2| > |k_1|$ at the point of intersection. However, for $x_1 > d$, since $|k_2| < |k_1|$, this is not possible here. This point along with the other property that the trajectory initiated in this domain moves away from $x_1 = d$ to right indicate that the trajectory can move outside this domain; hence, one will not reach $(0,0)$ - solution not globally asymptotically stable.