ENME665: Nonlinear Oscillations

- Material covered in previous week
  - Chapter 1, Nayfeh and Balachandran (1995, 2006)
  - MATLAB Program to do with Logistic Map is posted on Canvas under the same folder

- Dynamical System – One whose state evolves or changes with time *t*
  - Systems with continuously varying time – arise as solutions of differential equations – called flows
  - Systems with discrete changes in time – arise as solutions of algebraic equations – maps

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- Discrete-Time Systems

  - Logistic Map – Non-Invertible Map – A two-to-one Map

$$x_{k+1} = 4\alpha x_k(1 - x_k)$$

where $0 \leq x_k \leq 1$ and $0 < \alpha \leq 1$. For $\alpha = 0.50$, the orbit of the map initiated at $x_0 = 0.25$ is

- Orbit of x_o=0.25, $O(x_o=0.25)$= $\{0.25,\ 0.375,\ 0.46875,\ \cdots\}$

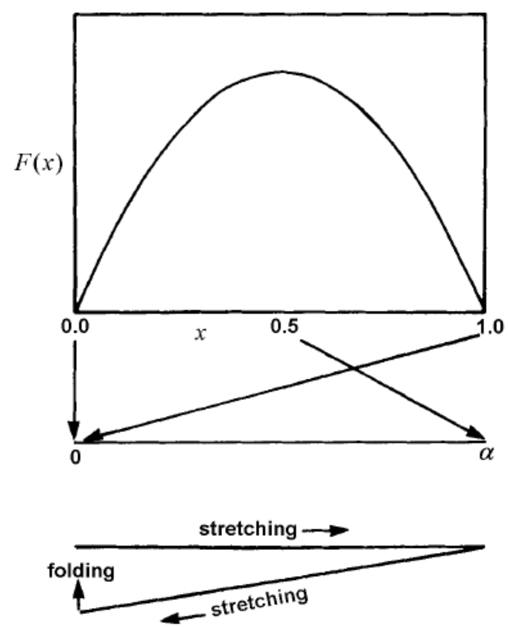

Fig. 5.1.4 Illustration of the action of the logistic map when $\alpha \simeq 1$.

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- Discrete-Time Systems

  - ❖ Hénon Map (Hénon, 1976) – Invertible Map if β not equal to 0.

$$x_{k+1} = 1 + y_k - \alpha x_k^2$$
$$y_{k+1} = \beta x_k$$

- Orbit of $\left\{ \begin{matrix} x_0 \\ y_0 \end{matrix} \right\} = \left\{ \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\}$ under the action of the Hénon map,

$$\left\{ \ldots, \left\{ \begin{matrix} -3.33 \\ 1.22 \end{matrix} \right\}, \left\{ \begin{matrix} 0.0 \\ -1.0 \end{matrix} \right\}, \left\{ \begin{matrix} 0.0 \\ 0.0 \end{matrix} \right\}, \left\{ \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\}, \left\{ \begin{matrix} 0.8 \\ 0.3 \end{matrix} \right\}, \left\{ \begin{matrix} 1.17 \\ 0.24 \end{matrix} \right\}, \left\{ \begin{matrix} 0.97 \\ 0.35 \end{matrix} \right\}, \ldots \right\}$$

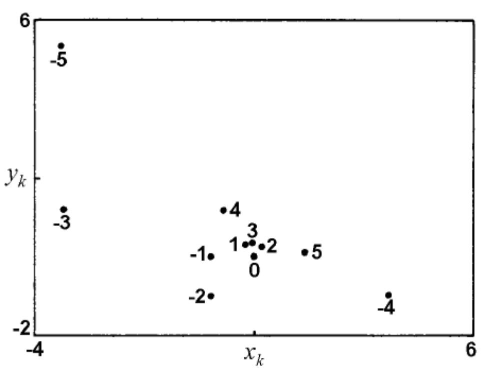

Fig. 1.1.1 Some of the discrete points that make up the orbit of (1, 0) of the Hénon map for $\alpha = 0.2$ and $\beta = 0.3$. The index k associated with each point is also shown.

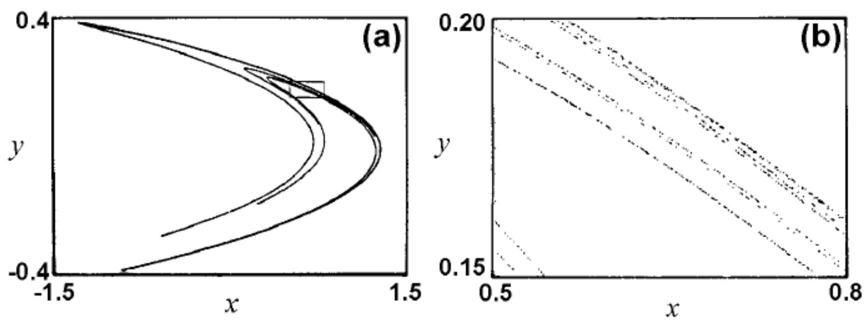

Fig. 5.1.5 (a) Attractor of the Hénon map for $\alpha = 1.4$ and $\beta = 0.3$ and (b) enlargement of box shown in a.

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• Continuous-Time Systems

❖ Nonautonomous Systems: $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t; \mathbf{M})$

$$\ddot{x} + 2\mu\dot{x} + \omega^2 x = F\cos(\Omega t)$$

$$\dot{x}_1 = x_2 \tag{1.2.2}$$

$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2 + F\cos(\Omega t) \tag{1.2.3}$$

For $\omega^2 = 8$, $\mu = 2$, $F = 10$, and $\Omega = 2$, the solution of (1.2.2) and (1.2.3) is

$$x_1 = e^{-2t}\left[a\cos(2t) + b\sin(2t)\right] + 0.5\cos(2t) + \sin(2t)$$

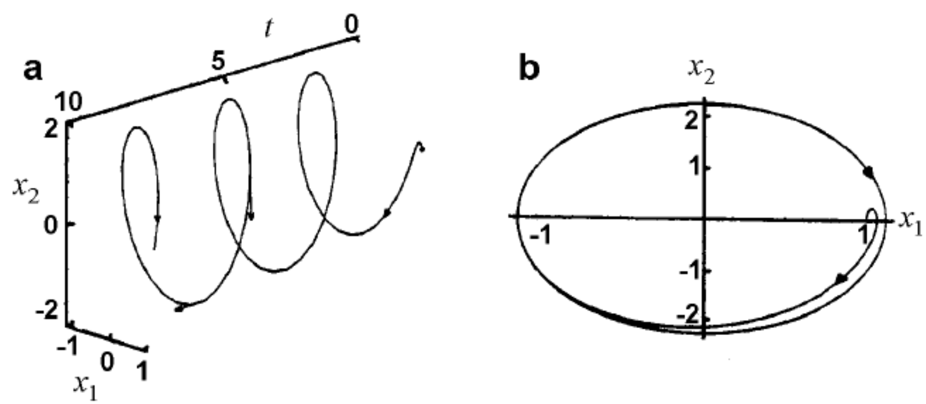

Fig. 1.2.1 Solution of (1.2.2) and (1.2.3) initiated from (1, 0) at $t = 0$ for $\omega^2 = 8$, $\mu = 2$, $F = 10$, and $\Omega = 2$: (a) integral curve and (b) positive orbit.

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- Continuous-Time Systems

  - ❖ Autonomous Systems: $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \mathbf{M})$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2$$

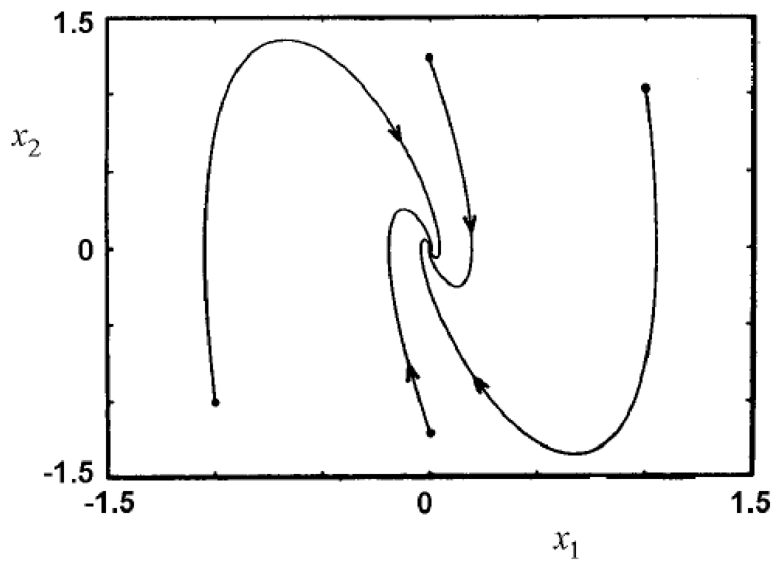

Fig. 1.2.4 Positive orbits of (1.2.6) and (1.2.7) initiated at $t = 0$ from $(1.0, 1.0)$, $(0.0, -1.2)$, $(-1.0, -1.0)$, and $(0.0, 1.2)$ for $\omega^2 = 8$ and $\mu = 2$. All four orbits approach the origin as $t \to \infty$.

•Dissipation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2$$

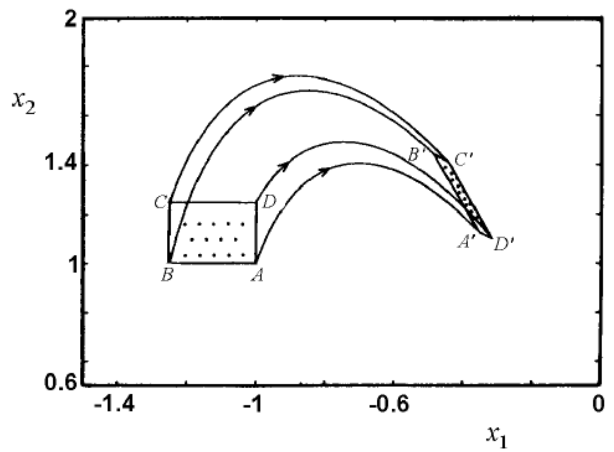

Fig. 1.2.5 Four positive orbits of (1.2.6) and (1.2.7) initiated at $t = 0$ when $\mu > 0$. The orbits are shown for $0 \le t \le 0.5$. The area $A'B'C'D'$ occupied by the final states is less than the area $ABCD$ occupied by the initial states. Furthermore, the orientation of the corners of the initial area is preserved.

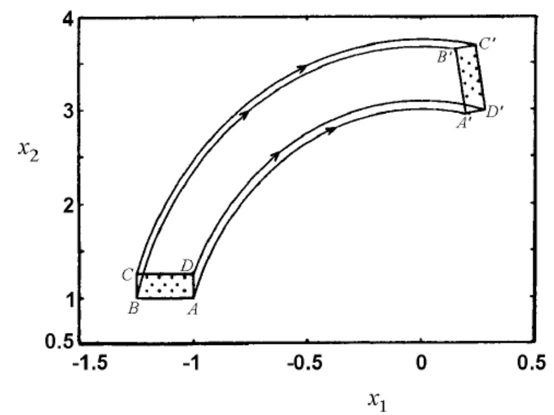

Fig. 1.3.1 Four positive orbits of (1.2.6) and (1.2.7) initiated at $t = 0$ when $\mu = 0$. The orbits are shown for $0 \le t \le 0.5$. The area $A'B'C'D'$ occupied by the final states is equal to the area $ABCD$ occupied by the initial states. Furthermore, the orientation of the corners of the initial area is preserved.

•Dissipation in continuous-time systems

For dynamical systems governed by (1.2.4) or (1.2.5), we appeal to concepts of fluid mechanics (e.g., Karamcheti, 1976) to determine whether a flow is conservative or dissipative. At an instant in time t, we consider a set of points occupying a small region with volume V and surface S in the associated n–dimensional state space. (As the set of points are transported in the state space under the considered flow, V and S change with time.) Considering a small elemental area ΔS, the associated change in the volume V over a time interval Δt is given by

$$\mathbf{v}\Delta t \cdot \mathbf{n}\Delta S \tag{1.3.1}$$

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•Dissipation

where $\mathbf{v} = \dot{\mathbf{x}} = \mathbf{F}$ is the velocity vector and $\mathbf{n}$ is the outward unit normal on S. To determine the total change in V, we let $\Delta S \to 0$ and integrate (1.3.1) over the surface S. The result is

$$\Delta V = \Delta t \int\int_S \mathbf{F} \cdot \mathbf{n} dS \tag{1.3.2}$$

From the divergence theorem (e.g., Karamcheti, 1976), we have

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS = \int\int\int_V (\bigtriangledown \cdot \mathbf{F}) dV \tag{1.3.3}$$

But, for an infinitesimally small V, the right-hand side of (1.3.3) can be approximated by $(\bigtriangledown \cdot \mathbf{F})V$. Therefore, (1.3.2) and (1.3.3) lead to

$$\frac{1}{V}\frac{\Delta V}{\Delta t} = \bigtriangledown \cdot \mathbf{F} \tag{1.3.4}$$

Consequently, a flow is conservative or dissipative, depending on whether the divergence of its vector field is zero or negative. In other words, in conservative systems

$$\sum_{i=1}^{n} \frac{\partial F_i(\mathbf{x})}{\partial x_i} = 0 \tag{1.3.5}$$

and in dissipative systems (e.g., systems with damping)

$$\sum_{i=1}^{n} \frac{\partial F_i(\mathbf{x})}{\partial x_i} < 0 \tag{1.3.6}$$

where the F_i and x_i are the scalar components of $\mathbf{F}$ and $\mathbf{x}$ in (1.2.5), respectively. In this book, we shall mainly be interested in dissipative systems.

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•Dissipation

1.2. Consider the Rössler system (Rössler, 1976a):

$$\begin{aligned}\dot{x} &= -(y+z)\\ \dot{y} &= x + ay\\ \dot{z} &= b + (x-c)z\end{aligned}$$

Determine when this system is dissipative and when it is conservative.

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•Hamiltonian Systems and Dissipation

In mechanics, there are systems called **Hamiltonian systems**, which are governed by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.3.7}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.3.8}$$

for $i = 1, 2, \cdots, n$ and $H = H(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n, t)$. (The function H is called the **Hamiltonian**.) The divergence of the vector field of the system governed by (1.3.7) and (1.3.8) is

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right)\right] = 0 \tag{1.3.9}$$

if H is twice continuously differentiable. Therefore, volumes in state space are conserved in Hamiltonian systems and hence they form a subset of the set of conservative systems. The statement about the preservation of volumes in state space of Hamiltonian systems is called the **Liouville theorem** (e.g., Arnold, 1973, Chapter 3; Lichtenberg and Lieberman, 1992, Chapter 1).

•Attracting Sets

In summary, the **the flow in conservative systems is said to preserve volume (locally) in the state space.** Furthermore, as $t \to \infty$, the motion takes place in the full n–dimensional space. For dissipative systems, $V_f < V_0$ and $V_f \to 0$ as $t = t_f \to \infty$. This means that trajectories initiated from different conditions are attracted to a subspace of the state space. This phenomenon is called **attraction**, and the set to which the trajectories are attracted as $t \to \infty$ is called an **attracting set.**

Before we consider the notion of an attracting set in detail, we first explain what is meant by an **invariant set**. A set $\mathbf{P} \subset \mathcal{R}^n$ is called an **invariant set** if for any initial condition $\mathbf{x}(t = t_0) \in \mathbf{P}$ we have $\mathbf{x}(t) \in \mathbf{P}$ for $-\infty < t < \infty$. If this condition is satisfied only for $t \geq 0$ or $t \leq 0$, $\mathbf{P}$ is called a **positive** or **negative invariant set**, respectively. An **attracting set** is an invariant set. Further, it has an open neighborhood such that positive orbits initiated in this neighborhood are attracted to this set. As explained in Section 1.5, a special type of an attracting set is called an **attractor**.

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- Attracting Sets

$$\dot{x}_1 = x_2 \tag{1.2.2}$$
$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2 + F\cos(\Omega t) \tag{1.2.3}$$

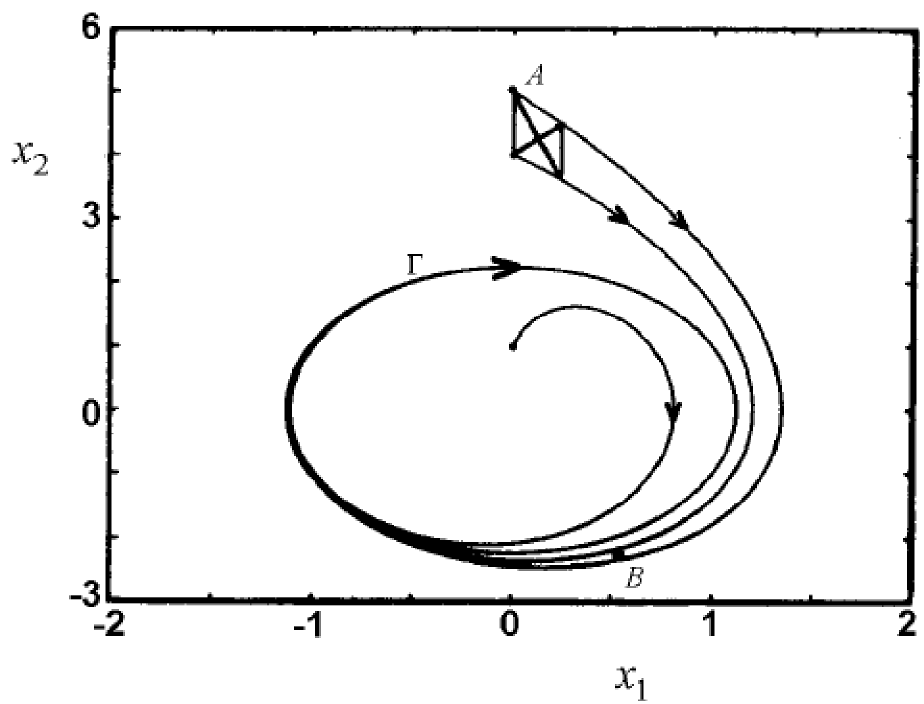

Fig. 1.3.2 Three positive orbits of (1.2.2) and (1.2.3) initiated at $t = 0$.

•Dissipation in maps

We note that the concepts of dissipation, invariant sets, attracting sets, and attractors also apply to the maps discussed in Section 1.1. The map (1.1.3) is said to be dissipative at $\mathbf{x}_k = \mathbf{x}_0$ if

$$|\det D_{\mathbf{x}_k}\mathbf{F}| < 1 \text{ at } \mathbf{x}_k = \mathbf{x}_0 \tag{1.3.10}$$

where $\det D_{\mathbf{x}_k}\mathbf{F}$ is the determinant of the $n \times n$ matrix of first partial derivatives of the scalar components of $\mathbf{F}$ with respect to the scalar components of $\mathbf{x}_k$.

Example 1.9. In the case of the Hénon map (1.1.5) and (1.1.6), we have

$$\det D_{\mathbf{x}_k}\mathbf{F} = \det \begin{bmatrix} -2\alpha x_k & 1 \\ \beta & 0 \end{bmatrix} = -\beta$$

Hence, when $|\beta| < 1$, the Hénon map is dissipative at all $\mathbf{x}_k$. Consequently, any area is contracted by the factor $|\beta|$ after each iterate.

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- Material to be covered in the week after Spring Break
  - ❖ Chapter 1, Nayfeh and Balachandran (1995, 2006)
  - ❖ Stability Concepts – Lyapunov Stability