ENME665: Nonlinear Oscillations

- Material covered in previous week and this week
  - ❖ Chapter 1, Nayfeh and Balachandran (1995, 2006)
    - ➢ Dissipation, Attraction, Attracting Sets
    - ➢ Lyapunov function based stability analyses
  - ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006)
    - ➢ Local stability analyses- Linearization

- Material to be covered today and next week
  - ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006)
    - ❖ Fixed points of maps and differential equation systems
    - ❖ Local stability analyses

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❖ Fixed Points of Maps (Section 2.2, N&B)

Here, we consider fixed points of the map

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \mathbf{M}) \tag{2.2.1}$$

A fixed point $\mathbf{x}_0$ of this map satisfies the condition

$$\mathbf{x}_0 = \mathbf{F}^m(\mathbf{x}_0; \mathbf{M}_0) \quad \text{for all} \quad m \in \mathcal{Z} \tag{2.2.2}$$

where $\mathbf{M} = \mathbf{M}_0$ is the value of the vector of control parameters. We note that an orbit of a map initiated at a fixed point of the map is the fixed point itself. Moreover, the fixed points of a map are examples of invariant sets.

2.23. Consider the following three–dimensional map of Klein, Baier, and Rössler (1991):

$$\begin{aligned}
x_{n+1} &= \alpha - \alpha y_n^2 + d z_n \\
y_{n+1} &= x_n + \beta + \gamma z_n \\
z_{n+1} &= y_n
\end{aligned}$$

(a) Examine if this map is dissipative in each of the following cases: (i) $d < 1$, (ii) $d = 1$, and (iii) $d > 1$.

(b) Determine the fixed points of this map and discuss their stability.

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- Fixed Points of Continuous-Time Systems (Section 2.1, N&B)

In the case of the autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \mathbf{M}) \tag{2.1.1}$$

the fixed points are defined by the vanishing of the vector field; that is,

$$\mathbf{F}(\mathbf{x}; \mathbf{M}) = 0 \tag{2.1.2}$$

A location in the state space where this condition is satisfied is called a **singular point**. At such a point, the integral curve of the vector field $\mathbf{F}$ corresponds to the point itself. Also, an orbit of a fixed point is the fixed point itself. Fixed points are also called **stationary solutions**, **critical points**, **constant solutions**, and sometimes **steady–state solutions**. Physically, **a fixed point corresponds to an equilibrium position of a system**. Further, fixed points are examples of invariant sets of (2.1.1).

2.25. Consider the following speed–control system investigated by Fallside and Patel (1965):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = K_d x_2 - x_1 - G x_1^2 \left(-\frac{x_2}{K_d} + x_1 + 1 \right)$$

(a) For $K_d = -1$ and $G = 6$, determine the fixed points and their stability.

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❖ Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Linearization near an equilibrium solution

In the case of the autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \mathbf{M}) \tag{2.1.1}$$

the fixed points are defined by the vanishing of the vector field; that is,

$$\mathbf{F}(\mathbf{x}; \mathbf{M}) = \mathbf{0} \tag{2.1.2}$$

Let the solution of (2.1.2) for $\mathbf{M} = \mathbf{M}_0$ be $\mathbf{x}_0$, where $\mathbf{x}_0 \in \mathcal{R}^n$ and $\mathbf{M}_0 \in \mathcal{R}^m$. To determine the stability of this equilibrium solution, we superimpose on it a small disturbance $\mathbf{y}$ and obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{y}(t) \tag{2.1.3}$$

Substituting (2.1.3) into (2.1.1) yields

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{x}_0 + \mathbf{y}; \mathbf{M}_0) \tag{2.1.4}$$

We note that the fixed point $\mathbf{x} = \mathbf{x}_0$ of (2.1.1) has been transformed into the fixed point $\mathbf{y} = \mathbf{0}$ of (2.1.4). Assuming that $\mathbf{F}$ is at least $\mathcal{C}^2$, expanding (2.1.4) in a Taylor series about $\mathbf{x}_0$, and retaining only linear terms in the disturbance leads to

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{x}_0; \mathbf{M}_0) + D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y} + O(\| \mathbf{y} \|^2)$$

or

$$\dot{\mathbf{y}} \approx D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y} \equiv A\mathbf{y} \tag{2.1.5}$$

where A, the matrix of first partial derivatives, is called the **Jacobian matrix.** If the components of F are

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- Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Linearization near an equilibrium solution

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{x}_0; \mathbf{M}_0) + D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y} + O(\| \mathbf{y} \|^2)$$

or

$$\dot{\mathbf{y}} \approx D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0; \mathbf{M}_0)\mathbf{y} \equiv A\mathbf{y} \tag{2.1.5}$$

where A, the matrix of first partial derivatives, is called the **Jacobian matrix.** If the components of F are

$$F_1(x_1, x_2, \cdots, x_n), F_2(x_1, x_2, \cdots, x_n), \cdots, F_n(x_1, x_2, \cdots, x_n),$$

then

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & . & . & . & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & . & . & . & \frac{\partial F_2}{\partial x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & . & . & . & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

Next, we show that the eigenvalues of the constant matrix A provide information about the **local stability** of the fixed point $\mathbf{x}_0$. We say **local** because we have considered a small disturbance and linearized the vector field.

The solution of (2.1.5) that passes through the initial condition $\mathbf{y}_0 \in \mathcal{R}^n$ at time $t_0 \in \mathcal{R}$ can be expressed as

$$\mathbf{y}(t) = e^{(t-t_0)A}\mathbf{y}_0 \tag{2.1.6}$$

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- Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Linearization near an equilibrium solution

If the eigenvalues λ_i of the matrix A are distinct, then there exists a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix with entries $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$; that is,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix}$$

If the eigenvalues are complex, then the matrix P will also be complex. The columns of the matrix P are the right eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ of the matrix A corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$; that is, $P = [\mathbf{p}_1\ \mathbf{p}_2\ \cdots\ \mathbf{p}_n]$. Hence,

$$AP = [A\mathbf{p}_1\ A\mathbf{p}_2\ \cdots\ A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1\ \lambda_2\mathbf{p}_2\ \cdots\ \lambda_n\mathbf{p}_n] = PD$$

Consequently,

$$D = P^{-1}AP$$

Introducing the transformation $\mathbf{y} = P\mathbf{v}$ into (2.1.5), we obtain

$$P\dot{\mathbf{v}} = AP\mathbf{v} \quad \text{or} \quad \dot{\mathbf{v}} = D\mathbf{v}$$

Hence,

$$\mathbf{v} = e^{(t-t_0)D}\mathbf{v}_0$$

where $\mathbf{v}_0 = \mathbf{v}(t_0) = P^{-1}\mathbf{y}_0$. In terms of $\mathbf{y}$, this solution becomes

$$\mathbf{y}(t) = Pe^{(t-t_0)D}P^{-1}\mathbf{y}_0 \tag{2.1.7}$$

The matrix $e^{(t-t_0)D}$ is a diagonal matrix with entries $e^{(t-t_0)\lambda_i}$. Hence, the eigenvalues of A are also known as the **characteristic exponents** associated with $\mathbf{F}$ at $(\mathbf{x}_0, \mathbf{M}_0)$.

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❖ Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Linearization near an equilibrium solution

If the eigenvalues of A are not distinct, then there exists a matrix P such that $P^{-1}AP = J$ is a **Jordan canonical form** with off–diagonal entries; that is,

$$J = \begin{bmatrix} J_1 & \phi & \cdot & \cdot & \cdot & \phi \\ \phi & J_2 & \cdot & \cdot & \cdot & \phi \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi & \phi & \cdot & \cdot & \cdot & J_k \end{bmatrix}$$

where ϕ represents a matrix with zero entries and

$$J_m = \begin{bmatrix} \lambda_m & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_m & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_m \end{bmatrix}$$

In writing the matrix J, we have assumed that A has k distinct eigenvalues. Further, let the (algebraic) multiplicity of the mth eigenvalue λ_m be n_m. Then, the matrix J_m corresponding to the eigenvalue λ_m differs from the diagonal matrix D due to the presence of the elements 1 above the diagonal elements. In this case, the columns $\mathbf{p}_i$ of the matrix P are the **generalized eigenvectors** corresponding to the eigenvalues λ_i of the matrix A. There are n_m generalized eigenvectors corresponding to the eigenvalue λ_m. These vectors are the nonzero solutions of

$$(A - \lambda_m \mathbf{I})\,\mathbf{p} = \mathbf{0},\ (A - \lambda_m \mathbf{I})^2\,\mathbf{p} = \mathbf{0},\ \cdots,\ (A - \lambda_m \mathbf{I})^{n_m}\,\mathbf{p} = \mathbf{0}$$

For an $n \times n$ matrix with n distinct eigenvalues, the generalized eigenvectors are also the eigenvectors of the matrix. The components of $\mathbf{v}$ have terms of the form $t^k e^{(t-t_0)\lambda_i}$, where the integer k depends on the multiplicity n_i of the eigenvalue λ_i.

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❖ Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Classification of equilibrium solutions

When all of the eigenvalues of A have nonzero real parts, the corresponding fixed point is called a **hyperbolic fixed point**, irrespective of the values of the imaginary parts; otherwise, it is called a **nonhyperbolic fixed point**.

There are three types of hyperbolic fixed points: **sinks**, **sources**, and **saddle points**. If all of the eigenvalues of A have negative real parts, then all of the components of the disturbance $\mathbf{y}$ decay in time, and hence $\mathbf{x}$ approaches the fixed point $\mathbf{x}_0$ of (2.1.1) as $t \to \infty$. Therefore, the fixed point $\mathbf{x}_0$ of (2.1.1) is asymptotically stable according to Section 1.4.2. An asymptotically stable fixed point is called a **sink**. If the matrix A associated with a sink has complex eigenvalues, the sink is also called a **stable focus**. On the other hand, if all of the eigenvalues of the matrix A associated with a sink are real, the sink is also called a **stable node**. A sink is stable in forward time (i.e., $t \to \infty$) but unstable in reverse time (i.e., $t \to -\infty$). Further, all sinks qualify as attractors.

If one or more of the eigenvalues of A have positive real parts, some of the components of $\mathbf{y}$ grow in time, and $\mathbf{x}$ moves away from the fixed point $\mathbf{x}_0$ of (2.1.1) as t increases. In this case, the fixed point $\mathbf{x}_0$ is said to be unstable. When all of the eigenvalues of A have positive real parts, $\mathbf{x}_0$ is said to be a **source**. If the matrix A associated with a source has complex eigenvalues, the source is also called an **unstable focus**. On the other hand, if all of the eigenvalues of the matrix A associated with a source are real, the source is also called an **unstable node.** A source is unstable in forward time but stable in reverse time. Because trajectories move away from a source in forward time, the source is an example of a repellor.

When some, but not all, of the eigenvalues have positive real parts while the rest of the eigenvalues have negative real parts, the associated fixed point is called a **saddle point**. Because a saddle point is unstable in both forward and reverse times, some authors call it a **nonstable fixed point** (e.g., Parker and Chua, 1989).

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❖ Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Classification of equilibrium solutions

Next, we address nonhyperbolic fixed points. A nonhyperbolic fixed point is unstable if one or more of the eigenvalues of A have positive real parts. If some of the eigenvalues of A have negative real parts while the rest of the eigenvalues have zero real parts, the fixed point $\mathbf{x} = \mathbf{x}_0$ of (2.1.1) is said to be **neutrally** or **marginally stable**. If all of the eigenvalues of A are purely imaginary and nonzero, the corresponding fixed point is called a **center**.

❖ Example: Duffing oscillator revisited

$$\dot{x}_1 = x_2 \tag{1.4.1}$$

$$\dot{x}_2 = -x_1 + x_1^3 - 2\mu x_2 \tag{1.4.2}$$

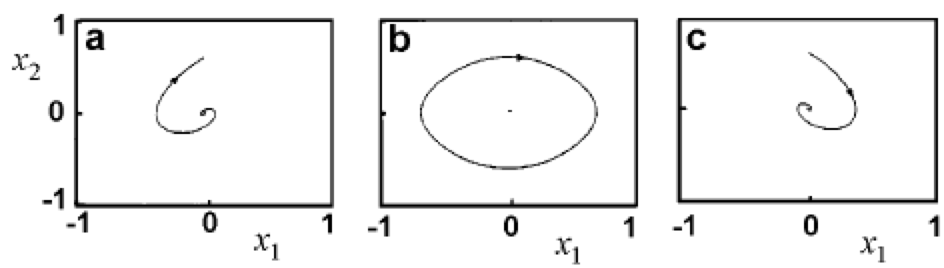

Fig. 2.1.1 Phase portraits in the vicinity of the origin of (1.4.1) and (1.4.2): (a) $\mu = -0.4$, (b) $\mu = 0$, and (c) $\mu = 0.4$.

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❖ Example: Duffing oscillator revisited

$$\dot{x}_1 = x_2 \tag{1.4.1}$$

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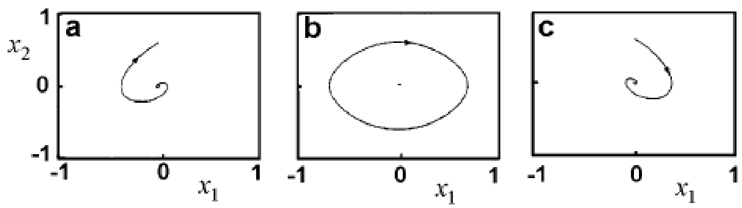

Fig. 2.1.1 Phase portraits in the vicinity of the origin of (1.4.1) and (1.4.2): (a) $\mu = -0.4$, (b) $\mu = 0$, and (c) $\mu = 0.4$.

Example 2.1. For illustration, we consider the classification of the fixed points $(0,0)$, $(-1,0)$, and $(1,0)$ of (1.4.1) and (1.4.2). In the vicinity of a fixed point, we obtain the following system after linearization:

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1+3x_1^2 & -2\mu \end{bmatrix} \mathbf{y} \tag{2.1.8}$$

Hence, the eigenvalues of the Jacobian matrix are

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 1 + 3x_1^2} \quad \text{and} \quad \lambda_2 = -\mu + \sqrt{\mu^2 - 1 + 3x_1^2} \tag{2.1.9}$$

For all three fixed points, both of the eigenvalues have nonzero real parts when $\mu \neq 0$. Hence, all three fixed points are hyperbolic fixed points.

In the vicinity of the fixed point $(0,0)$, (2.1.8) and (2.1.9) become

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1 & -2\mu \end{bmatrix} \mathbf{y} \tag{2.1.10}$$

and

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 1} \quad \text{and} \quad \lambda_2 = -\mu + \sqrt{\mu^2 - 1} \tag{2.1.11}$$

respectively. We conclude from (2.1.11) that the fixed point $(0,0)$ is a center when $\mu = 0$, an unstable node when $\mu \leq -1$, an unstable focus when $-1 < \mu < 0$, a stable focus when $0 < \mu < 1$, and a stable node when $\mu \geq 1$. In Figures 2.1.1a–c, we show phase portraits in the vicinity of the origin of the $x_2 - x_1$ space when the origin is an unstable focus, a center, and a stable focus, respectively. A positive orbit spirals away from a neighborhood of the unstable focus in Figure 2.1.1a, and a positive orbit spirals into the stable focus in Figure 2.1.1c. The orbit of Figure 2.1.1b, which corresponds to a periodic solution, closes on itself.

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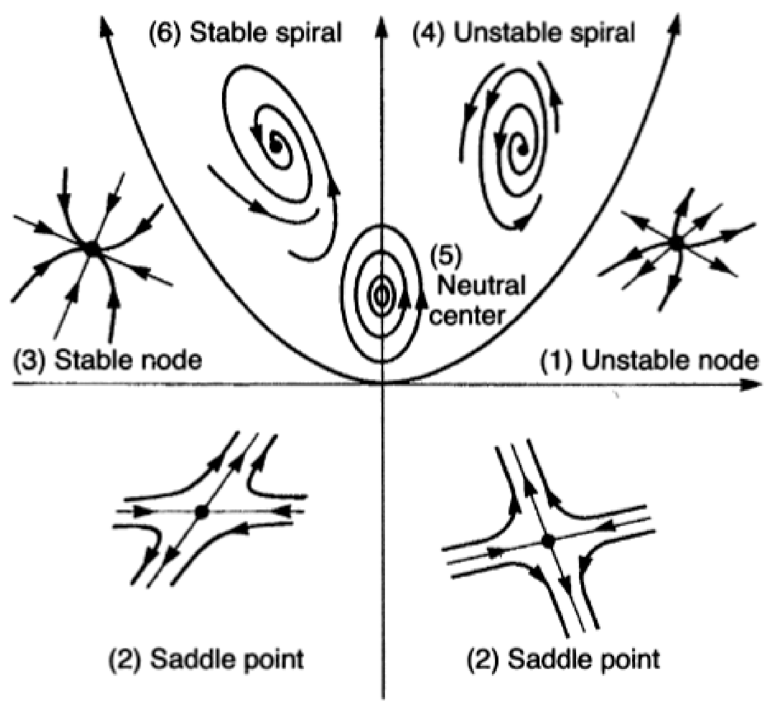
- ❖ Fixed Points of Continuous-Time Systems (Section 2.1, N&B): Classification of equilibrium solutions – Planar Systems
- ❖ Different types of solutions of a planar linear system of differential equations are shown below. The trace of the Jacobian Matrix is plotted along the x-axis and the determinant of the Jacobian matrix is plotted on the y-axis. Along the parabolic boundary, the two eigenvalues have zero imaginary parts, since the square of the trace = four times the determinant.



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- Material to be covered next week
  - ❖ Chapter 2, Nayfeh and Balachandran (1995, 2006)