ENME665: Nonlinear Oscillations

•Material to be covered today and next class
- ❖ Lindstedt-Poincaré method (Nayfeh and Mook, 1979)
- ❖ Method of Multiple Scales (Nayfeh and Mook, 1979)

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- The Method of Multiple Scales is used to introduce different (time) scales into the analyses:

  fast-time scale⟶t

  slow-time scale ⟶ϵt

  and in general, $T_n = \epsilon^n t$

The analytical approximation for $x(t,\epsilon)$is written as a generalized asymptotic series; that is,

$$x(t,\epsilon) \\ = x_0(T_0, T_1, \dots) + \epsilon x_1(T_0, T_1, \dots) + \epsilon^2 x_2(T_0, T_1, \dots) + \cdots$$

- The new time scales necessitate care to be taken with the derivatives:

Let $D_i = \dfrac{\partial(\cdot)}{\partial T_i}$

$$\text{Then } \frac{d(\cdot)}{dt} = \frac{dT_0}{dt}\frac{\partial(\cdot)}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial(\cdot)}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial(\cdot)}{\partial T_2} + \ldots$$
$$= D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots$$

This method can be used to treat damped systems as well, since it can capture transient responses.

- Example: i) Damped Duffing oscillator

$$\ddot{u} + 2\mu\epsilon\dot{u} + u + \epsilon u^3 = 0$$

This equation is ordered in such a way that the damping and nonlinearity are "weak" due to the ϵ parameter. Here, the damping coefficient μ is positive and O(1).

Expanding u(t, ϵ) as a generalized asymptotic series, we arrive at

$$\left({D_0}^2 + 2\epsilon D_0 D_1 + \epsilon^2\left({D_1}^2 + 2D_1 D_2\right) + \cdots\right)(u_0 + \epsilon u_1 + \cdots) + 2\mu\epsilon(D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots)\,(u_0 + \epsilon u_1 + \cdots) + (u_0 + \epsilon u_1 + \cdots) + \epsilon(u_0 + \epsilon u_1 + \cdots)^3$$

- Collecting terms at same orders of ϵ, we have

$$O(\epsilon^0): {D_0}^2 u_0 + u_0 = 0,$$

which has solution

$$u_0 = a(T_1, T_2, \dots)\cos(T_0 + \beta(T_1, T_2, \dots))$$
$$= a(T_1, T_2, \dots)\left[\frac{e^{i(T_0 + \beta(T_1, T_2, \dots))} + c.c.}{2}\right]$$

If we only seek a one-term approximation, we can use a fast scale and a slow scale; that is, we keep only time scales T_0 and T_1. Here, and further, c.c. denotes the complex conjugate of the preceding term.

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- Setting $A(T_1) = \frac{a(T_1)e^{i\beta(T_0\ )}}{2}$, at the next order of ϵ , we obtain

$$
\begin{aligned}
&O(\epsilon^1)\colon\ {D_0}^2 u_1 + u_1 = {u_0}^3 - 2\mu D_0 u_0 - 2D_0 D_1 u_0 \\
&= -\left[A^3 e^{3iT_0} + 3A^2 \bar{A} e^{iT_0} + c.c\right] - 2\mu\left[iAe^{iT_0} + c.c\right] \\
&- 2\left[iD_1 A e^{iT_0} + c.c.\right]
\end{aligned}
$$

Recognizing that terms with e^{iT_0} (or e^{-iT_0}) would produce secular terms, we set these terms equal to zero. Note that, in general, the complex conjugate terms do not yield additional information.

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- Complex Modulation Equation describes the time variation of the *complex amplitude* A.

Complex Modulation Equation (CME):

$$-[2iD_1A + 2\mu iA + 3A^2\bar{A}]e^{iT_0} = 0$$

Since $D_1A = \frac{1}{2}(a'e^{i\beta} + ai\beta' e^{i\beta})$, the CME can be rewritten as shown next. Note that the prime superscript indicates a time derivative with respect to the slow scale T_1.

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- CME: $e^{i\beta}[ia' - a\beta' + \mu i a + \frac{3a^3}{8}]$=0
- Separating the real and imaginary parts of the above equation, we arrive at the modulation equations, presented below.

Modulation Equations (slow-scale equations):

$$\begin{cases} Im: a' + \mu a = 0 \\ Re: a\beta' = \dfrac{3a^3}{8} \end{cases}$$

- We see from the slow-scale equations that the amplitude is effected by the damping, and the phase is effected by the cubic nonlinearity.

- If there is no damping (i.e., $\mu = 0$), then

$$a' = 0.$$

So, a is a constant, a_0, and

$$\beta' = \frac{3{a_0}^2}{8}, \text{ if } a_0 \neq 0.$$

Hence, we have the analytical approximation,

$$u(t,\epsilon) = a_0 \cos\left(\left(1 + \epsilon\frac{3{a_0}^2}{8}\right)t + \beta_0\right) + O(\epsilon)$$

- The Method of Multiple Scales can be used to study many different phenomena. The complex modulation equations describe the behavior of the amplitude and phase variations on the slow scale, in the example discussed thus far.

- Other examples will be discussed in the classroom.

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- Material to be covered next class
  - ❖ Method of Multiple Scales (Nayfeh and Mook, 1979)