Some Remarks on Stability

One may pose two forms of questions regarding the stability of a system. These are as follows.

i) Given a system and a solution of it, would one return to this solution when disturbed or perturbed from it ? This disturbance can due to a change in the initial conditions. *This question is in the sense of Lyapunov, and the associated stability is sometimes referred to as being in the sense of Lyapunov*.

ii) Given a system, how would a "small" change to the system influence the stability of solutions of the system ? *This question addresses the coarseness or robustness issue, and the associated stability is sometimes referred to as structural stability.*

Example: Behavior of a perfect beam under axial loading that can buckle the beam compared with the behavior of an imperfect beam under axial loading (see Figures 2.3.15 and 2.3.16 of Nayfeh and Balachandran (1995)].

How does one define structural stability ? To answer this question, let us consider the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x};\boldsymbol{M})$$

Then, the system $\boldsymbol{F}(\boldsymbol{x};\boldsymbol{M})$ is called *structurally stable*, if for a sufficiently "small" perturbation of the vector field $\boldsymbol{F}$, the perturbed system is topologically equivalent to the unperturbed system [e.g., Stewart, 1981].

Stewart, I. (1981). Applications of the Catastrophe Theory to the Physical Sciences, *Physica 2D*, pp. 245—305.

Example: The system $\ddot{x} + x = 0$ is structurally unstable to the perturbation $2\mu\dot{x}$ ($\mu \neq 0$), since the phase portraits in the two cases are qualitatively different.

In Nayfeh and Balachandran (1995), structural stability is addressed in Chapters 2 through 5 through different examples.

Additional Reading for Structural Stability:

1. Thom, R. (1969). Topological Models in Biology, *Topology*, Vol. 8, pp. 313—335.
2. Arnold, V. I. (1983). Geometrical Methods in Ordinary Differential Equations, Springer Verlag, NY.

Both of the above mentioned references contain material on structural stability for two-dimensional or planar systems.

Stability in the Sense of Lyapunov in Finite and Infinite-Dimensional Systems

Throughout Nayfeh and Balachandran (1995), and in many other books, the stability discussions are limited to systems that are described by finite number of states. So, this would mean that, if we took a spatially continuous model of a beam or a plate and discretize this model in terms of a finite number of vibration modes, the discussion in the textbook would apply to these finite-dimensional systems. A fundamental result that is important to take note of is the principle of equivalence of norms.

When one looks at the stability in the sense of Lyapunov, a metric (or a norm) to measure how "close" or "far" two solutions are is needed. Let us suppose that we have two norms available, one the Euclidean norm $\|\boldsymbol{x}\|$ given by Eq. (1.1.2) and another norm, say $\rho(\boldsymbol{x})$, that can be defined according to $\rho(\boldsymbol{x}) = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$ $1 \le p < \infty$. Then, according to the principle of equivalence of norms, in finite-dimensional systems there exists the relation

$$\alpha\|\boldsymbol{x}\| \le \rho(\boldsymbol{x}) \le \beta\|\boldsymbol{x}\|$$

where a and b are real-valued positive constants (e.g., Hirsch and Smale, 1974; page 77). This means that if a solution is shown to be stable by using one metric (or norm), then it will also be stable if another metric is used. The same goes for instability as well. However, this convenient situation for finite-dimensional systems does not exist for infinite-dimensional systems or distributed-parameter systems.

Some Related Material

Brauchli, H. (1976). On the norm-dependence of the concept of stability, Vol. 53 of Lecture Notes in Mathematics, Springer Verlag, pp. 235—238.

Shield, R. T. and Green, A. E. (1963). On certain methods in the stability theory of continuous systems, Archives of Rational Mechanics and Analysis, Vol. 12, pp. 354—360.