ENME665: Nonlinear Oscillations

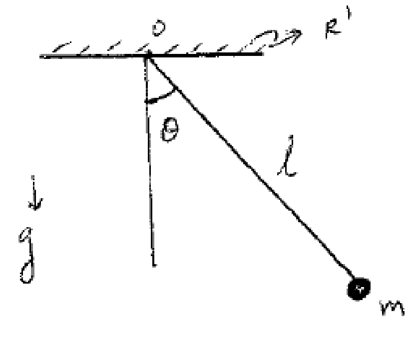
- Material covered last class
  - Introductory material
  - Examples of nonlinear systems
  - Examples of nonlinear phenomena
  - Animations of nonlinear motions

- Examples: In the last class, we covered some examples outside of engineering. While system models used in Aerospace and Mechanical sciences and engineering are governed by Newton's Laws of motion, the same is not true for system models in other fields (e.g., Chemistry and Biology).

- Today's class
  - ❖ Examples (to be continued)

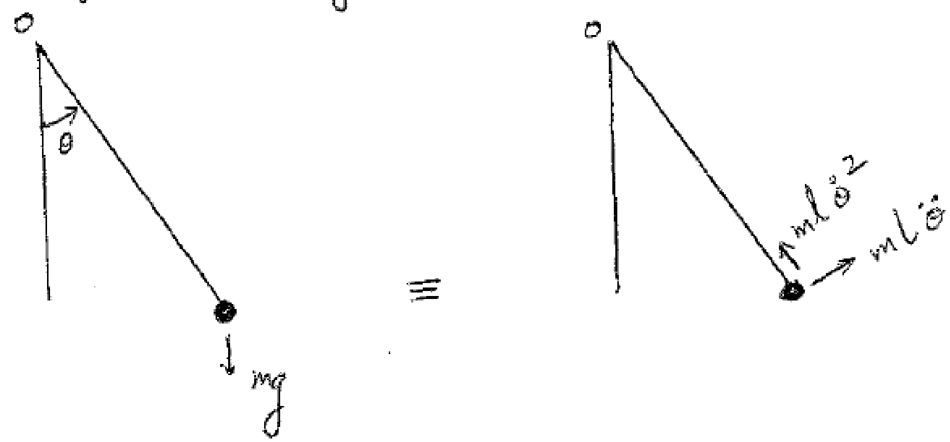
Example: Planar Pendulum Oscillations

Section 2.1.1, Nayfeh and Mook (1979)



- * Frictionless pivot
- * No air damping
- * Inextensional & massless string
- * Motions in the plane

Equations of Motion Using Newton's Second Law



Moment about Pivot O: $ml^2\ddot{\theta} = -mgl\sin\theta \Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$

Example: Planar Pendulum Oscillations (continued)

$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$ $\Rightarrow$ Differential equation describes how θ evolves with respect to time

$\rightarrow$ example of a dynamical system

$\Rightarrow$ No explicit time-dependent terms in the equation; called an autonomous system

Equilibrium Positions:

$\sin\theta = 0$; $\theta_e = 0$ or π

$\theta_e = 0$ $\rightarrow$ pendulum at the bottom

π $\rightarrow$ pendulum at the top

Example: Planar Pendulum Oscillations (continued)

<u>Oscillations about $\theta_e = 0$:</u>

$$\theta(t) = \theta_e + \hat{\theta}(t)$$

<u>"small" oscillations:</u> $\sin\theta(t) \approx \cancel{\sin\theta_e}^{\,0} + \cancel{\cos\theta_e}^{\,1}\,\hat{\theta}(t) - \cancel{\sin\theta_e}^{\,0}\,\frac{(\hat{\theta}(t))^2}{2} - \cos\theta_e\,\frac{(\hat{\theta}(t))^3}{6} + \cdots$

Taylor-Series Expansion

$$\Rightarrow \ddot{\hat{\theta}} + \frac{g}{\ell}\left[\hat{\theta} - \frac{\hat{\theta}^3}{6} + \cdots\right] = 0$$

, restoring force of "softening" type

↪ "approximate" solutions for nonlinear systems

Example: Planar Pendulum Oscillations (continued)

"Large" Oscillations: $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

Closed-form solution by using elliptic integrals for period of motion [Section 2.4 of Nayfeh and Mook (1979)]

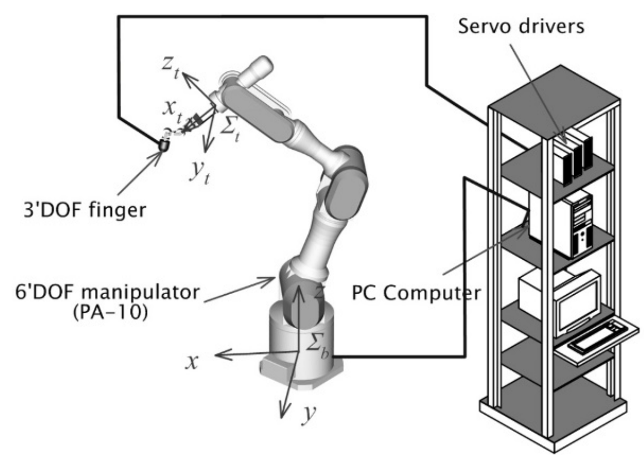
Lagrangian Equations:

$$\mathcal{L} = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

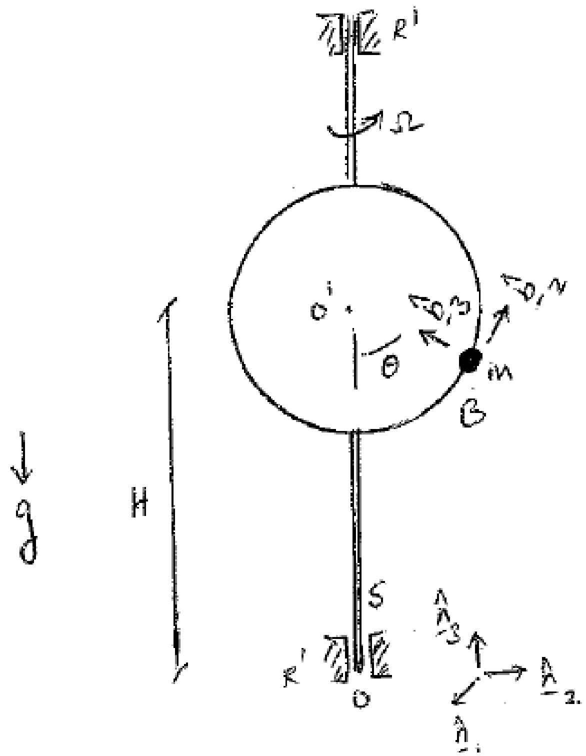
•Nonlinear terms arise from potential energy

•Robotics, manipulator arm motions – Nonlinearities arise due to large motions



(source: Huang, J., Hara, M., and Yabuta, T. (2010). Controlling a Finger-arm Robot to Emulate the Motion of the Human Upper Limb by Regulating Finger Manipulability, in Motion Control, Federico Casolo, Editor; www.intechopen.com)

Example: Bead on a Smooth and Circular Hoop



* Hoop assumed to be smooth and massless
* Angular ~~velocity~~ speed Ω constant
* $\hat{\underline{n}}_1, \hat{\underline{n}}_2, \& \hat{\underline{n}}_3$: fixed in inertial frame
* $\hat{\underline{b}}_1, \hat{\underline{b}}_2, \& \hat{\underline{b}}_3$ fixed to mass (or bead) m

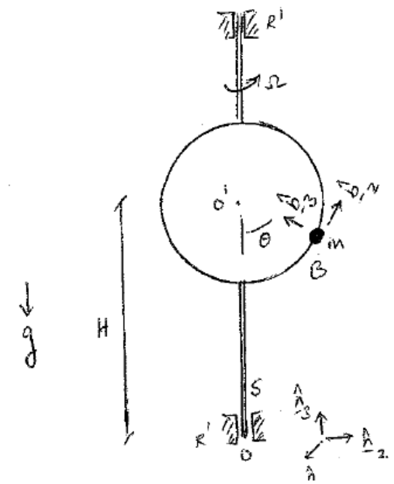
$$\underline{r}^{m/O} = H\,\hat{\underline{n}}_3 - R\,\hat{\underline{b}}_3$$

$$\dot{\underline{r}}^{m/O} = {}^{R'}\underline{\omega}^{B} \times (-R\,\hat{\underline{b}}_3)$$

$$= -\left[\Omega s\theta\,\hat{\underline{b}}_2 + \Omega c\theta\,\hat{\underline{b}}_3 + \dot{\theta}\,\hat{\underline{b}}_1\right] \times R\,\hat{\underline{b}}_3$$

$$= (R\dot{\theta}\,\hat{\underline{b}}_2 - R\Omega s\theta\,\hat{\underline{b}}_1)$$

Example: Bead on a Smooth and Circular Hoop (continued)



$$T = \frac{1}{2} m \left(\dot{\underline{r}}^{m/o} \cdot \dot{\underline{r}}^{m/o} \right) = \frac{1}{2} m \left[R^2 \Omega^2 s^2_\theta + R^2 \dot{\theta}^2 \right]$$

$$V = mg \left(H + R - R\cos\theta \right)$$

Nonlinear terms in kinetic energy and potential energy

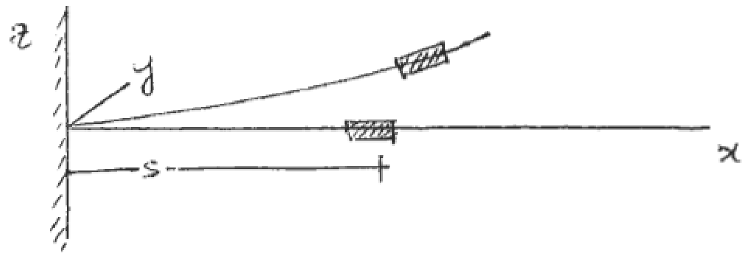
$$\mathcal{L} = T - V = \frac{1}{2} m \left[R^2 \Omega^2 s^2_\theta + R^2 \dot{\theta}^2 \right] - mg \left[H + R - R\cos\theta \right]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \boxed{m R^2 \ddot{\theta} - m R^2 \Omega^2 s\theta\, c\theta + mgR\sin\theta = 0}$$

Nonlinear terms due to inertia and large oscillations

Example: Thin Elastica

Governing Equations with Cubic Nonlinearities

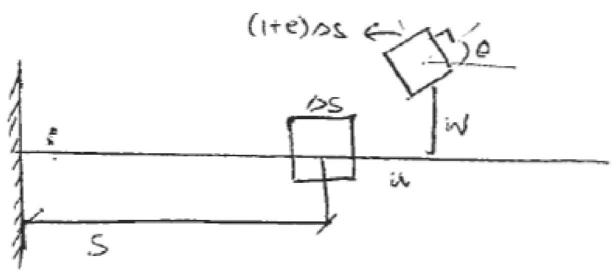


ρ: mass density
$A(s)$: Area of cross-section.

* Euler-Bernoulli beam theory

* Undeformed case: co-ordinates of point located in x-y-z space (before deformation) at $(s, 0, 0)$

* Deformed case: co-ordinates of point (after deformation) are $(s+u, 0, w)$



* $\sin\theta = \dfrac{\Delta w}{(1+e)\Delta s}$

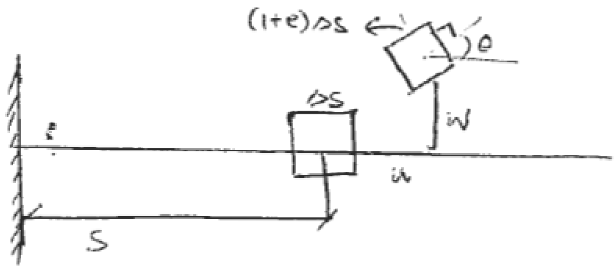
* Radius of Curvature
$$R = \frac{(1+e)\Delta s}{\Delta \theta}$$

Example: Thin Elastica (continued)

Thin Elastica (Governing Equations with Cubic Nonlinearities)

* Undeformed case: co-ordinates of point located in x-y-z space (before deformation) at $(s, 0, 0)$

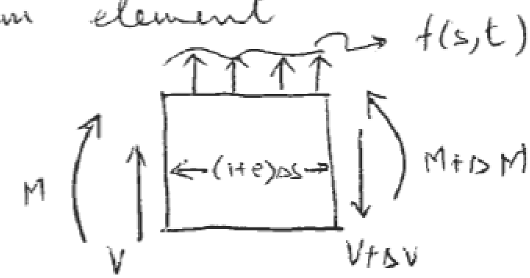
* Deformed case: co-ordinates of point (after deformation) are $(s+u, 0, w)$



* $\sin\theta = \dfrac{\Delta w}{(1+e)\Delta s}$

* Radius of Curvature
$$R = \frac{(1+e)\Delta s}{\Delta \theta}$$

* FBD for beam element



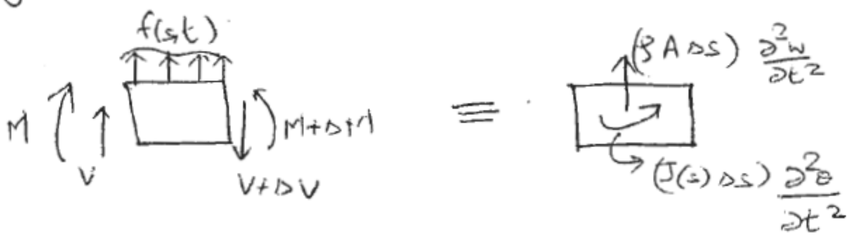
* Inextensional case, $e = 0$

Example: Thin Elastica (continued)

Thin Elastica (Governing Equations with Cubic Nonlinearities)

* $J(s)$: polar mass moment of inertia per unit length
* $f(s,t)$: load per unit length.

* Apply Newton's Second Law of Motion to an Infinitesimal Element



$$\Sigma F_z = m a_z : \quad V + f(s,t)\,\Delta s - (V + \Delta V) = (\rho A \Delta s)\,\frac{\partial^2 w}{\partial t^2}$$

+ve

$$\Sigma M_B = I_M \omega : \quad -M + (M + \Delta M) - V \Delta s - (f \Delta s)(\alpha_1 \Delta s)$$

$$= \left(\rho A \,\Delta s\, \frac{\partial^2 w}{\partial t^2}\right)(\alpha_2 \,\Delta s)$$

$$+ \left(J(s)\,\Delta s\right)\frac{\partial^2 \theta}{\partial t^2}$$

$0 < \alpha_1 < 1$
$0 < \alpha_2 < 1$

α_1 & α_2 : to take into account non-uniform distribution

Example: Thin Elastica (continued)

Thin Elastica (Governing Equations with Cubic Nonlinearities)

$$\Sigma F_z = m a_z : \quad V + f(s,t)\Delta s - (V + \Delta V) = (\rho A \Delta s)\frac{\partial^2 w}{\partial t^2}$$

+ve

$$\left(\Sigma M_B = I_M \omega : \quad - M + (M + \Delta M) - V\Delta s - (f\Delta s)(\alpha_1 \Delta s)\right.$$

$$= \left(\rho A \Delta s \frac{\partial^2 w}{\partial t^2}\right)(\alpha_2 \Delta s)$$

$$+ (J(s)\Delta s)\frac{\partial^2 \theta}{\partial t^2}$$

$0 < \alpha_1 < 1$

$0 < \alpha_2 < 1$

α_1 & α_2: to take into account non-uniform distribution

* Dividing throughout by Δs and considering the limit as $\Delta s \to 0$

$$\Rightarrow \boxed{\begin{aligned} f(s,t) - \frac{\partial V}{\partial s} &= \rho A(s)\frac{\partial^2 w}{\partial t^2} \\ \frac{\partial M}{\partial s} - V &= J(s)\frac{\partial^2 \theta}{\partial t^2} \end{aligned}}$$

Equations of dynamic equilibrium for a beam element

Example: Thin Elastica (continued)

Thin Elastica (Governing Equations with Cubic Nonlinearities)

$$\Rightarrow \boxed{\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M}{\partial s^2} - \frac{\partial}{\partial s}\left(J(s) \frac{\partial^2 \theta}{\partial t^2} \right) = f(s,t)}$$

Isotropic + Elastic Beams : $M = \frac{EI}{R} = EI\,k$

$$M = EI\,k = EI \frac{\partial \theta}{\partial s}\,; \qquad \sin\theta = \frac{\partial w}{\partial s}$$

$$\cos\theta = \frac{\partial^2 w}{\partial s^2} \frac{\partial s}{\partial \theta}$$

$$\Rightarrow k = \frac{\dfrac{\partial^2 w}{\partial s^2}}{\cos\theta} = \frac{\dfrac{\partial^2 w}{\partial s^2}}{\left[1 - \left(\dfrac{\partial w}{\partial s}\right)^2\right]^{1/2}}$$

Example: Thin Elastica (continued)

Thin Elastica (Governing Equations with Cubic Nonlinearities)

$$\Rightarrow \kappa = \frac{\frac{\partial^2 w}{\partial s^2}}{\cos\theta} = \frac{\frac{\partial^2 w}{\partial s^2}}{\left[1-\left(\frac{\partial w}{\partial s}\right)^2\right]^{1/2}}$$

For $\left|\frac{\partial w}{\partial s}\right| \ll 1$; $\left(1-\left(\frac{\partial w}{\partial s}\right)^2\right)^{-1/2} = \left[1+\frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2+\frac{3}{8}\left(\frac{\partial w}{\partial s}\right)^4+\cdots\right]$

Binomial Expansion

$$M = EI\,\frac{\partial^2 w}{\partial s^2}\left[1+\frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2+\frac{3}{8}\left(\frac{\partial w}{\partial s}\right)^4+\cdots\right]$$

Curvature nonlinearities

$$\rho A\,\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial s^2}\left(EI\,\frac{\partial^2 w}{\partial s^2}\right) + \underbrace{\frac{\partial^2}{\partial s^2}\left(EI\,\frac{\partial^2 w}{\partial s^2}\left[\frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2+\frac{3}{8}\left(\frac{\partial w}{\partial s}\right)^4+\cdots\right]\right)}_{\text{Nonlinear Terms from Curvature}}$$

$$-\frac{\partial}{\partial s}\underbrace{\left(J(s)\,\frac{\partial^2 \theta}{\partial t^2}\right)}_{\text{Rotary inertia term}} = f(s,t)$$

Examples (continued)

- Physical configurations such as boundary conditions can also give rise to nonlinear terms in the system model
- Nonlinear springs
- Systems with backlash and damping (Chapter 3, Nayfeh and Mook)
- Coloumb damping

ENME665: Project Information

What constitutes a project?

Any combination of analytical, numerical, and experimental investigations into a nonlinear system can be selected to be a part of the project. The investigations can be tailored to carry out one or more of the following: a) uncover nonlinear phenomena and/or explain observed nonlinear phenomena, b) analyze stability of motions in detail (for example, stability analyses of equilibrium and/or periodic motions of the considered nonlinear system, c) catalog nonlinear characteristics and nonlinear behavior of the considered nonlinear system for different parameter ranges, d) go beyond what is covered in the classroom and understand a topic in more depth (for example, global analyses, dimension calculations, quasiperiodic motions, time series analyses), and e) develop a simplified reduced-order model (a map or a set of ODEs) of a complex system that captures same physics as that observed in experiments.

What are some possible project topics?

A partial list of examples includes the following: i) experimental investigations into nonlinear oscillations of mechanical, electrical, or structural systems, ii) study of instabilities in micro-scale and nano-scale systems, iii) study of bifurcations of solutions of nonsmooth systems (such as systems with friction and impact), and iv) study of phenomena such as solitons, chatter, and so forth.

What should the project report have in it?

The essential elements of the project report include an introduction to the problem studied, problem description, study undertaken, results obtained, conclusions/inferences from the study, and a list of references.

Project Description Due Date: March 28, 2017
Final Project Report Due Date: May 9, 2017

Representative final reports and project descriptions from the previous years are provided in CANVAS.

ENME665: Nonlinear Oscillations

- Check out the nonlinear dynamics and chaos laboratory demonstrations at the Cornell University site
  http://ecommons.library.cornell.edu/handle/1813/97
  (http://ecommons.library.cornell.edu/bitstream/1813/97/7/streaming_97_3.html)

- Material to be covered next class

  - Qualitative Analyses (Chapter 2, Nayfeh and Mook)
    - ❖ Seek information about all solutions, would like to know whether a certain property of these solutions remain unchanged if the system is subjected to various types of changes