ENME665: Nonlinear Oscillations

- Material covered last class
  - Examples of systems with nonlinear damping

  - Qualitative Analyses (Chapter 2, Nayfeh and Mook): Seek information about all solutions, would like to know whether a certain property of these solutions remain unchanged if the system is subjected to various types of changes
    - ❖ First integral of motion
    - ❖ Phase portraits
    - ❖ Examples: Undamped and damped pendulum systems

$$\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0; \quad \text{undamped case}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0; \quad \text{undamped case}$$

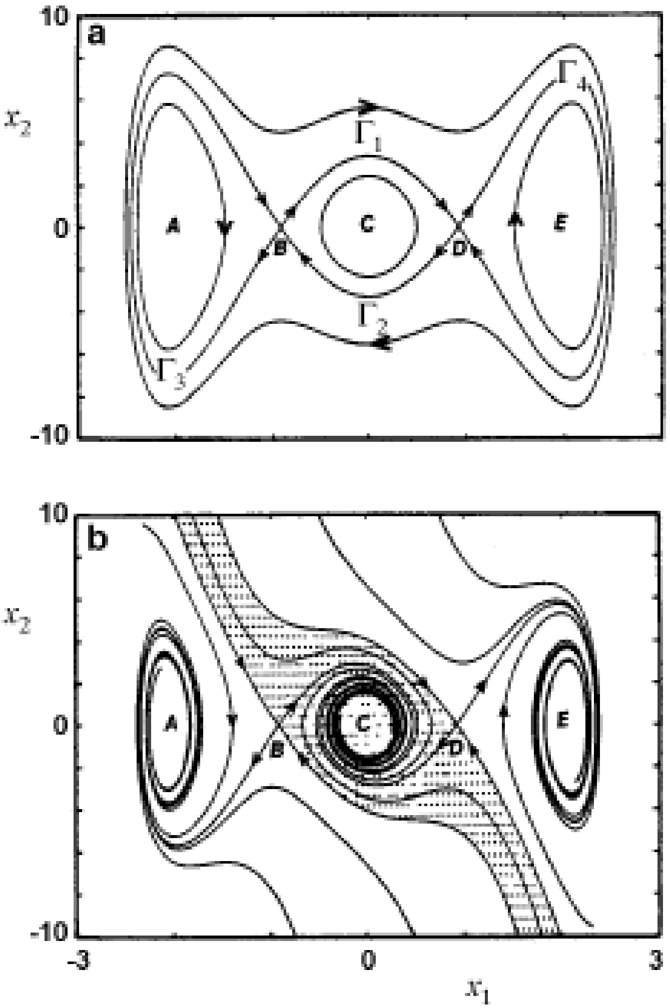
$$\ddot{\theta} + 2\mu\dot{\theta} + \frac{g}{l}\sin\theta = 0; \quad \text{Damped case}$$

ENME665: Nonlinear Oscillations

- Today's class
  - Qualitative Analyses
    - Ship-roll motions
    - Duffing oscillator
  - Quantitative Analyses
    - Landau symbols and ordering
    - Straightforward expansions
    - Lindstedt-Poincaré technique
    - Method of multiple scales

ENME665: Nonlinear Oscillations

•Example 2.8 (Nayfeh and Balachandran, 1995): Ship-Roll Motions



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\left(\omega_0^2 x_1 + \alpha_3 x_1^3 + \alpha_5 x_1^5\right) - \left(2\mu_1 x_2 + \mu_3 x_2^3\right)$$

$\omega_0 = 5.278;\ \alpha_3 = -1.402\omega_0^2;\ \alpha_5 = 0.271\omega_0^2$

undamped case: $\mu_1 = \mu_3 = 0$

damped case: $\mu_1 = 0.086$ and $\mu_3 = 0.108$

ENME665: Nonlinear Oscillations

•Duffing oscillator $\ddot{x} + ax + bx^3 = 0$

•Georg Duffing (1861-1944)

•Experimental prototype: Dynamics and Control Laboratory, University of Maryland

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•Quantitative Analyses

•Landau symbols, O(ε) and o(ε): Used to represent the asymptotic order of a quantity

•Two scalar functions *f*(ε) and *g*(ε), near ε=0, say |ε| ≤ ε_1; |ε| is a "small" positive quantity, |ε|≪ 1

•*f*(ε)=O(*g*(ε)) if there exists a positive number *K* independent of ε and an ε_0 such that

$$\boxed{\left|f(\varepsilon)\right| \le K\left|g(\varepsilon)\right|} \text{ for all } |\varepsilon| \le |\varepsilon_0|$$

which is equivalent to

$$\boxed{\lim_{\varepsilon \to 0} \frac{\left|f(\varepsilon)\right|}{\left|g(\varepsilon)\right|} \le \infty}$$

Function *f*(ε) has been ordered by using function *g*(ε).

ENME665: Nonlinear Oscillations

•Two scalar functions $f(\varepsilon)$ and $g(\varepsilon)$, near $\varepsilon=0$, say $|\varepsilon| \leq \varepsilon_1$; $|\varepsilon|$ is a "small" positive quantity, $|\varepsilon| \ll 1$

•$f(\varepsilon)=o(g(\varepsilon))$ as $\varepsilon \to 0$ if $\boxed{\lim_{\varepsilon\to 0} \frac{|f(\varepsilon)|}{|g(\varepsilon)|} = 0}$

- Similar notions also apply to vector functions
- Throughout this course, ε and powers of ε are to be used as gauge functions.
- Quantity denoted as O(1) is a bounded quantity.

ENME665: Nonlinear Oscillations

• If f is a function of another variable t in addition to ε, and g(t, ε) is a gauge function, one can also say that

$$f(t, \varepsilon)=O(g(t, \varepsilon)) \text{ as } \varepsilon \to 0$$

if there exists a positive number K independent of ε and an $\varepsilon_0>0$ so that

$$\boxed{\left|f(t,\varepsilon)\right| \le K\left|g(t,\varepsilon)\right|} \text{ for all } |\varepsilon| \le \varepsilon_0$$

If K and ε_0 are independent of t, the above relationship is said to hold uniformly.

ENME665: Nonlinear Oscillations

•Examples

- $\sin(\varepsilon) = O(\varepsilon)$
- $\sin(10\varepsilon) = O(\varepsilon)$
- $\sin(\varepsilon) = o(1)$
- $\sin(\varepsilon^2) = O(\varepsilon^2)$
- $\sin(\varepsilon^2) = o(\varepsilon)$
- $\cos(\varepsilon) = O(1)$
- $\cos(\varepsilon) = o(\varepsilon^{-\frac{1}{2}})$
- $1 - \cos(\varepsilon) = O(\varepsilon^2)$
- $\sinh(\varepsilon) = O(\varepsilon)$
- $\sin(2\varepsilon) - \sin(\varepsilon) = O(\varepsilon^3)$
- $\sin(t + \varepsilon) = O(1)$
- $\sin(t + \varepsilon) = O(\sin t)$ uniformly as $\varepsilon \to 0$
- $e^{-\varepsilon t} - 1 = O(\varepsilon)$ non-uniformly as $\varepsilon \to 0$

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- Asymptotic Expansion

$$\boxed{f(\varepsilon)=\sum_{n=0}^{N} c_n f_n(\varepsilon)}$$

- Sequence $\{f_n(\varepsilon)\},\ n=0,\ldots,N+1$ is such that

$$\boxed{f_n(\varepsilon)=o\left(f_{n-1}(\varepsilon)\right)} \text{ as } \varepsilon\to 0 \ \ for\ n=1,\ldots,N+1$$

- $$\lim_{\varepsilon\to 0}\left[f(\varepsilon)-\sum_{n=0}^{N} c_n f_n(\varepsilon)\right]=O\left(f_{n+1}(\varepsilon)\right)$$

- Asymptotic expansions need not converge as $N\to\infty$

- Asymptotic expansions do not uniquely determine functions

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- Examples of sequences: $\varepsilon^n,\ \varepsilon^{n/3},\ (\sin(\varepsilon))^n$

•Asymptotic expansion can be defined in terms of an asymptotic sequence; for instance, $\sum_{n=0}^{N} c_n \delta_n(\varepsilon)$ where C_n is independent of ε and $\delta_n(\varepsilon)$ is an asymptotic sequence.

•So,

$$v(\varepsilon) \approx \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon) \quad as\ \varepsilon \to 0$$

$$v(\varepsilon) = \sum_{n=0}^{N-1} a_n \delta_n(\varepsilon) + O\left(\delta_N(\varepsilon)\right) as\ \varepsilon \to 0$$

•In our case,

$$v(t,\varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n v_n(t) + O\left(\varepsilon^{N+1}\right)$$

•Expansion with n terms is called an nth order approximation and the expansion is called a Poincaré asymptotic expansion. An expansion is said to be uniformly valid upto $O(\varepsilon^k)$ if the error is $O(\varepsilon^{k+1})$ for all times t.

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- Illustration of straightforward expansion
  - Example:

    Damped, Linear Oscillator

    $$\ddot{x} + 2\varepsilon\dot{x} + x = 0; \;\; \varepsilon > 0; \;\; x(t=0) = 0; \; \dot{x}(t=0) = 1$$

    Expansion : $x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \ldots$

- Two problems for exploring straightforward expansions further

  Undamped, *Duffing* Oscillator

  $$\ddot{u} + u + \varepsilon\alpha u^3 = 0; \;\; \varepsilon > 0; \;\; u(t=0) = 0; \; \dot{u}(t=0) = 1$$

  Damped, *Duffing* Oscillator

  $$\ddot{u} + 2\varepsilon\mu\dot{u} + u + \varepsilon\alpha u^3 = 0; \;\; \varepsilon > 0; \;\; \mu > 0; \; u(t=0) = 0; \; \dot{u}(t=0) = 1$$

  Expansion: $u(t,\varepsilon) = u_0(t) + \varepsilon u_1(t) + \ldots$

ENME665: Nonlinear Oscillations

- Material to be covered next class

  - Quantitative Analyses (to be continued)
    - ❖ Straightforward expansions
    - ❖ Lindstedt-Poincaré technique
    - ❖ Method of multiple scales