Homework I, Advanced Algorithms 2022

Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Michael Kapralov. You are, however, allowed to discuss problems in groups of up to three students; it is sufficient to hand in one solution per group.

1 Problem 1

Let $\vec{x}$ be an extreme point of the LP mentioned in the problem. First we claim that none of the following structures can found on the support of $\vec{x}$:

- **Structure 1 :** Even Cycles
- **Structure 2 :** A path P of length ≥ 3, where $P = (v_1, v_2, \ldots, v_n)$ and $x_{(v_1,v_2)} = x_{(v_{n-1},v_n)} = 1$.
- **Structure 3 :** Odd cycles C_1 and C_2 connected by a path of length ≥ 1, i.e. $C_1 = (v_1, v_2, \ldots, v_{2n+1})$ and $C_2 = (w_1, w_2, \ldots, w_{2m+1})$, and a path $P = (v_1 = u_1, u_2 \ldots, u_{k-1}, u_k = w_1)$ such that $\forall\, i \in [k], u_i \notin C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$.
- **Structure 4 :** Odd cycles sharing a single vertex. i.e. C_1 and C_2 are odd cycles such that $|C_1 \cap C_2| = 1$.
- **Structure 5 :** An odd cycle with a path incident on it, i.e. $C = (v_1, v_2, \ldots, v_{2n+1})$ is an odd cycle and $P = (v_1 = u_1, u_2 \ldots u_k)$ is an incident path where $x_{(u_{k-1},u_k)} = 1$.

Given these claims we will first show what the question asks, and then we'll give the individual proofs of the claims which are rather mechanical and follows the standard technique of changing x values in the support by $+$ or $-\ \varepsilon$.

Theorem. *Every extreme point solution to the linear program in the problem is supported on a disjoint union of odd length cycles and stars.*

Proof. Intuitively, we will prove the theorem by traversing the support of the extreme point starting from a vertex and reasoning about the potential paths and cycles we encounter along the way. This process terminates as our graph is finite and we will argue that if the support avoids all the 5 structures described previously, then only odd cycles and stars remain.
More formally pick any vertex v and start traversing any edge adjacent to v, repeating this process until either one of the two happen.

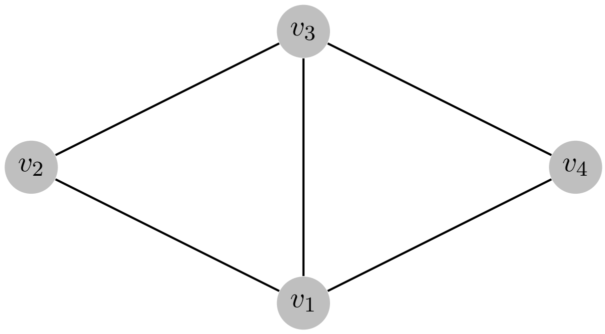

Figure 1: Cycles sharing more than one vertex. v_1, v_2, v_3, v_4 is an even cycle.

$(1a)$ You land on an already visited vertex.

$(1b)$ The vertex you are visiting doesn't have any other edge you can traverse.

$(2a)$ You land on an already visited vertex during either of the walks.

$(2b)$ The vertex you are visiting doesn't have any other edge you can traverse.

Note that by claim 1, all of the cycles formed by this walk have to be of odd length. Now let's examine all of the different cases.

Case $(1a) + (2a)$ **:** In this case, we have two connected odd cycles. Either

- The cycles are connected by a path, which is not possible by Claim 3.
- The cycles share a single vertex, which is not possible by Claim 4.
- The cycles share more than one vertex. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the set of edges in the two cycles. Both of them must be odd length cycles, i.e. $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$ are odd numbers. Then $(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus (\mathcal{C}_1 \cap \mathcal{C}_2)$ is a also cycle. It has length $|\mathcal{C}_1| + |\mathcal{C}_2| - 2|\mathcal{C}_1 \cap \mathcal{C}_2|$, which is an even number. Since there are no even length cycles, this case is not possible by Claim 1. Refer to Figure 1.

Case $(1a) + (2b)$ **or** $(1b) + (2a)$ **:** In this case, either

- We have an odd cycle connected to a path. Since we made sure that there are no other incident edge in the end vertex of the path, we know that the x value of the last edge of the path is at least 1. Hence we can use Claim 5 to show that this case is not possible.
- We have an odd cycle that is not connected to anything.

Case $(1b) + (2b)$ **:** In this case, we have a path such that both end edges have value at least 1. By Claim 2, we know that this path is of length at most 2.

Conclusion : We have shown that the only acceptable structure in an extreme point are odd length cycles that are their own connected components, or paths of length at most 2. But the connected components of paths of length less than 2 are exactly stars. We can now conclude that an extreme point is supported on a disjoint union of odd length cycles and stars.

Proving the claims

Proof of Non-Existence of Structure 1. Let x be a feasible solution of the linear problem, and let $C = \{e_1, \dots, e_{2n}\}$ be a cycle of even length supporting x. Let $\varepsilon = \frac{1}{2}\min_{e\in[2n]} x_e$. We define the following two feasible solutions y and z :

$$y_{e_k} = \begin{cases} x_{e_k} + \varepsilon & \text{if } k \le 2n \text{ and } k \text{ is even,} \\ x_{e_k} - \varepsilon & \text{if } k \le 2n \text{ and } k \text{ is odd,} \\ x_{e_k} & \text{otherwise.} \end{cases} \qquad z_{e_k} = \begin{cases} x_{e_k} - \varepsilon & \text{if } k \le 2n \text{ and } k \text{ is even,} \\ x_{e_k} + \varepsilon & \text{if } k \le 2n \text{ and } k \text{ is odd,} \\ x_{e_k} & \text{otherwise.} \end{cases}$$

We have $x = \frac{1}{2}(y+z)$, so we only need to prove that y and z are indeed feasible, and are different from x. Note that by definition, $\varepsilon > 0$, which guarantees that $y \neq x$, and $\varepsilon < x_e$ for all $e \in \mathcal{C}$ which guarantees that $y, z \geq 0$. (Note that this is valid for all the proofs, so for the subsequent proofs, we will only prove that $\sum_{e\in\delta(v)} y_e \geq 1, \forall v$
Finally, for $u \in V$, we have $\sum_{e\in\delta(v)} z_e = \sum_{e\in\delta(v)} y_e = \sum_{e\in\delta(v)} x_e \geq 1$ by construction.
This ensures that y and z are feasible, and that x is not an extreme point. □

Non-Existence of Structure 2. Let $\varepsilon = \frac{1}{2}\min\limits_{e\in\mathcal{P}} x_e$ and define the following points :

$$y_e = \begin{cases} x_e + \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}), e \notin \{(v_1, v_2), (v_{n-1}, v_n)\}, \\ x_e - \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), e \notin \{(v_1, v_2), (v_{n-1}, v_n)\}, \\ x_e & \text{otherwise.} \end{cases}$$
$$z_e = \begin{cases} x_e - \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}), e \notin \{(v_1, v_2), (v_{n-1}, v_n), \\ x_e + \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), e \notin \{(v_1, v_2), (v_{n-1}, v_n), \\ x_e & \text{otherwise.} \end{cases}$$

Because the two end edges have value at least1, the end vertices are not affected by the change of value, and still verify the constraints. On the other hand, the vertices that are in the interior of the path also verify the constraints, because they are all connected to one edge whose value reduced by ε, and one edge whose value increased by ε. □

Proof of Non-Existence of Structure 3. Suppose that $\mathcal{P}$ is a path of even length. Let $\varepsilon = \frac{1}{4}\min\limits_{e\in\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{P}} x_e$ and define the following points :

$$y_e = \begin{cases} x_e + \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e - \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e - 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \\ x_e + 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \\ x_e - \varepsilon & \text{if } e = (w_{2j-1}, w_{2j}) \text{ or } e = (w_1, w_{2m+1}), \\ x_e + \varepsilon & \text{if } e = (w_{2j}, w_{2j+1}), \\ x_e & \text{otherwise.} \end{cases}$$
$$z_k = \begin{cases} x_e - \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e + \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e + 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \\ x_e - 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \\ x_e + \varepsilon & \text{if } e = (w_{2j-1}, w_{2j}) \text{ or } e = (w_1, w_{2m+1}), \\ x_e - \varepsilon & \text{if } e = (w_{2j}, w_{2j+1}), \\ x_e & \text{otherwise.} \end{cases}$$

Refer to Figure 2. One can check that y is indeed a feasible solution. With similar ideas to the

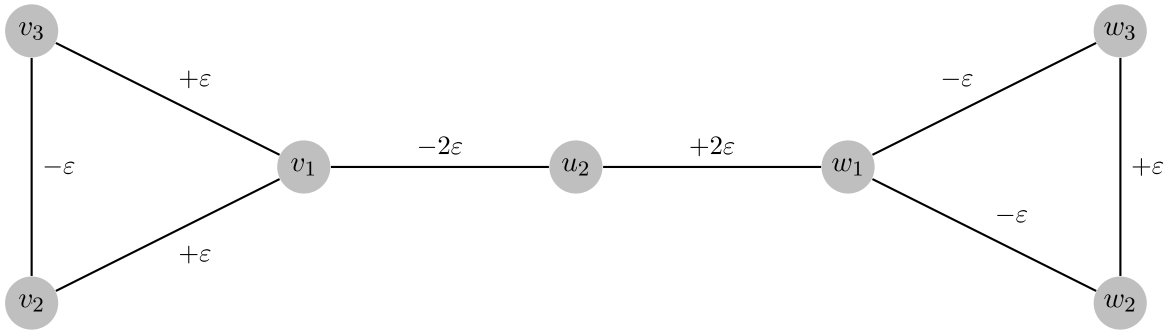

Figure 2: Odd cycles with even path in between

previous proofs, one can prove that the vertices that are in the middle of the path or in the cycles still verify the constraints of the LP. The only new vertices to consider are v_1 and w_1 (which are the vertices where the cycles and the path connect). We have

$$\sum_{e\in\delta(v_1)} y_e = \sum_{e\in\delta(v_1)\setminus(\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{P})} y_e + (x_{(v_1,v_2)} + \varepsilon) + (x_{(v_1,v_{2n+1})} + \varepsilon) + (x_{(v_1,u2)} - 2\varepsilon) = \sum_{e\in\delta(v_1)} x_e \geq 1$$

So u_1 and w_1 verify the constraints of the LP, and we showed that y is feasible.

If $\mathcal{P}$ is a path of odd length then the following points still work, refer to Figure 3.:

$$y_e = \begin{cases} x_e + \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e - \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e - 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \\ x_e + 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \\ x_e + \varepsilon & \text{if } e = (w_{2j-1}, w_{2j}) \text{ or } e = (w_1, w_{2m+1}), \\ x_e - \varepsilon & \text{if } e = (w_{2j}, w_{2j+1}), \\ x_e & \text{otherwise.} \end{cases}$$

$$z_k = \begin{cases} x_e - \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e + \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e + 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \\ x_e - 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \\ x_e - \varepsilon & \text{if } e = (w_{2j-1}, w_{2j}) \text{ or } e = (w_1, w_{2m+1}), \\ x_e + \varepsilon & \text{if } e = (w_{2j}, w_{2j+1}), \\ x_e & \text{otherwise.} \end{cases}$$

□

Proof of Non-Existence of Structure 4. The proof of this claim is completely analogous to the proof of claim 3, with a path of length 0 □

Proof of Non-Existence of Structure 5. Let $\varepsilon = \frac{1}{4} \min_{e\in\mathcal{C}\cup\mathcal{P}} x_e$ and define the following points :

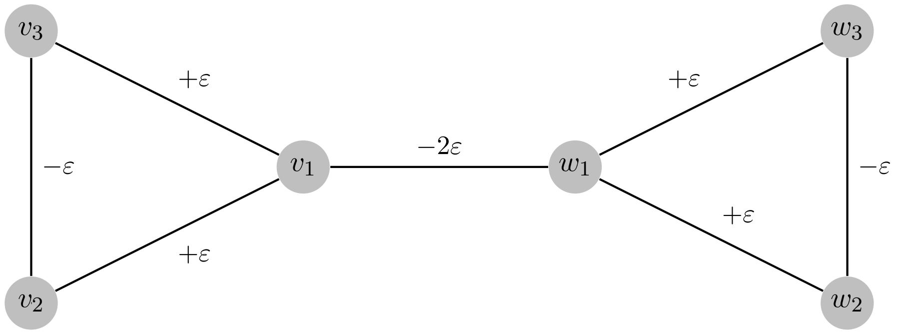

Figure 3: Odd cycles with odd path in between

$$y_e = \begin{cases} x_e + \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e - \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e - 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \text{ and } e \neq (u_{k-1}, u_k) \\ x_e + 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \text{ and } e \neq (u_{k-1}, u_k) \\ x_e & \text{otherwise.} \end{cases}$$

$$, \quad z_e = \begin{cases} x_e - \varepsilon & \text{if } e = (v_{2j-1}, v_{2j}) \text{ or } e = (v_1, v_{2n+1}), \\ x_e + \varepsilon & \text{if } e = (v_{2j}, v_{2j+1}), \\ x_e + 2\varepsilon & \text{if } e = (u_{2j-1}, u_{2j}), \text{ and } e \neq (u_{k-1}, u_k) \\ x_e - 2\varepsilon & \text{if } e = (u_{2j}, u_{2j+1}), \text{ and } e \neq (u_{k-1}, u_k) \\ x_e & \text{otherwise.} \end{cases}$$

Proving that y is feasible is completely analogous to Claims 2 and 3. □

2 Problem 2

We are given a matrix $A \in \mathbb{R}^{n \times kn}$ and we want to find out whether its columns can be partitioned into k bases of $\mathbb{R}^n$. Consider the following bipartite graph $G = (L \cup R, E)$. $L = [nk]$ and $R = [k]$. Here E will denote both the edges of this bipartite graph and the ground set of the two matroids. The edge set $E = \{(i, j) \mid i \in [nk], j \in [k]\}$. Thus this is the complete bipartite graph on $[nk] \times [k]$. L denotes the columns of the matrix and R denotes the partitions. An edge $(i.j)$ is picked in the solution if column i is assigned to partition j. Thus on a high level we want the following constraints to be met:

1. Each column is assigned to exactly one such partition
2. Each partition has linearly independent columns assigned to it.

In total if there are nk edges satisfying these two conditions then we have a partition that we require. Now let's formalise these notions.
Formal Solution:

Let G be the graph as defined above, and E is its edge set. Let's model the constraints as matroids. Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$. Here $\mathcal{I}_1 = \{F \subseteq E \mid |F \cap \delta(v)| \leq 1 \ \forall v \in L\}$. The edges incident on vertices in L form a partition $\{\delta(v)\}_{v \in L}$, thus $\mathcal{M}_1$ is a partition matroid.
Let $\mathcal{M}_2 = (E, \mathcal{I}_2)$. Here $\mathcal{I}_2 = \{F \subseteq E \mid rank(A_{N_F(v)}) = |N_F(v)| \ \forall v \in R\}$. Here $N_F(v) = \{u \in V \mid \{u, v\} \in F\}$ and A_X for any $X \subseteq [nk]$ denotes the matrix of columns indexed by X. Thus

all columns assigned to a vertex $v \in R$ are linearly independent. However a vertex $v \in L$ might have multiple edges incident to it but that's fine. We need to prove that $\mathcal{M}_2$ is indeed a matroid.

1. Property 1 - Let $Y \in \mathcal{I}_2$ and suppose $X \subseteq Y$. Let $v \in R$. Thus $rank(A_{N_Y(v)} = |N_Y(v)|)$. Thus the columns indexed by $N_Y(v)$ are linearly independent which also implies all the columns in X are linearly independent. Thus $rank(A_{N_X(v)}) = |N_X(v)|$ and $X \in \mathcal{I}_2$.

2. Property 2 - Let $X, Y \in \mathcal{I}_2$ such that $|Y| > |X|$. Thus there must exist atleast one vertex $v \in R$ such that $|N_Y(v)| > |N_X(v)|$. (Otherwise $|Y| = \sum_{v \in R} |N_Y(v)| \leq \sum_{v \in R} |N_X(v)| = |X|$). Since $X, Y \in \mathcal{I}_2$, $rank(A_{N_X(v)}) < rank(A_{N_Y(v)})$. Thus there must exist $u \in L$, such that $\{u, v\} \in Y$ and $u \notin span(A_{N_X(v)})$. (Otherwise there would be more linearly independent vectors than the dimension of the subspace spanned by columns of $N_X(v)$). Thus $\{u, v\} \notin X$ and add the edge $\{u, v\}$ to X. Since u didn't lie in the span of $A_{N_X(v)}$, columns in $N_X(v) \cup \{u\}$ are linearly independent. All other edges remain same, so no other vertex in R has its incident columns changed. Thus $X \cup \{u, v\} \in \mathcal{I}_2$.

Consider $\mathcal{M}_1 \cap \mathcal{M}_2$ and put weight function of $e \in E$ as 1 $\forall e \in E$. Now we can run the matroid intersection algorithm to check if there exists a base of weight nk. Clearly a base of weight nk exists if and only if there is a partition of the nk columns into k bases of $\mathbb{R}^n$.