Exercise Set III, Advanced Algorithms 2022

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. **This exercise set contains many problems.** So solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 *(*)* Consider the linear programming relaxation for minimum-weight vertex cover:

$$\begin{aligned} \text{Minimize} \quad & \sum_{v \in V} x_v w(v) \\ \text{Subject to} \quad & x_u + x_v \geq 1 \quad \forall \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \qquad \forall v \in V \end{aligned}$$

In class, we saw that any extreme point is integral when considering bipartite graphs. For general graphs, this is not true, as can be seen by considering the graph consisting of a single triangle. However, we have the following statement for general graphs:

Any extreme point x^* satisfies $x^*_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Prove the above statement.

2 Write the dual of the following linear program:

$$\begin{aligned} \text{Maximize} \quad & 6x_1 + 14x_2 + 13x_3 \\ \text{Subject to} \quad & x_1 + 3x_2 + x_3 \leq 24 \\ & x_1 + 2x_2 + 4x_3 \leq 60 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Hint: How can you convince your friend that the above linear program has optimum value at most z?

3 Consider the min-cost perfect matching problem on a bipartite graph $G = (A \cup B, E)$ with costs $c : E \rightarrow \mathbb{R}$. Recall from the lecture that the dual linear program is

$$\begin{aligned} \text{Maximize} \quad & \sum_{a \in A} u_a + \sum_{b \in B} v_b \\ \text{Subject to} \quad & u_a + v_b \leq c(\{a, b\}) \qquad \text{for every edge } \{a, b\} \in E. \end{aligned}$$

Show that the dual linear program is unbounded if there is a set $S \subseteq A$ such that $|S| > |N(S)|$, where $N(S) = \{v \in B : \{u, v\} \in E \text{ for some } u \in S\}$ denotes the neighborhood of S. This proves (as expected) that the primal is infeasible in this case.

4 *(half a *)* Prove Hall's Theorem:

> "An n-by-n bipartite graph $G = (A \cup B, E)$ has a perfect matching if and only if $|S| \leq |N(S)|$ for all $S \subseteq A$."

(Hint: use the properties of the augmenting path algorithm for the hard direction.)

5 Consider the Maximum Disjoint Paths problem: given an undirected graph $G = (V, E)$ with designated source $s \in V$ and sink $t \in V \setminus \{s\}$ vertices, find the maximum number of edge-disjoint paths from s to t. To formulate it as a linear program, we have a variable x_p for each possible path p that starts at the source s and ends at the sink t. The intuitive meaning of x_p is that it should take value 1 if the path p is used and 0 otherwise[1]. Let P be the set of all such paths from s to t. The linear programming relaxation of this problem now becomes

$$\begin{aligned} \text{Maximize} \quad & \sum_{p \in P} x_p \\ \text{subject to} \quad & \sum_{p \in P : e \in p} x_p \leq 1, \qquad \forall e \in E, \\ & x_p \geq 0, \qquad \forall p \in P. \end{aligned}$$

What is the dual of this linear program? What famous combinatorial problem do binary solutions to the dual solve?

[1] I know that the number of variables may be exponential, but let us not worry about that.