



Exercise Set II, Advanced Algorithms 2022

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

- 1 For a bipartite graph, devise an efficient algorithm for finding an augmenting path P (if one exists). What is the total running time of the AUGMENTINGPATHALGORITHM explained in the second lecture?

Solution: Let the bipartition of the bipartite graph $G = (V, E)$ be A and B , and let M be the current bipartite matching maintained by AUGMENTINGPATHALGORITHM. From G , obtain the directed graph where each edge in M is directed from the vertex in B to the vertex in A . All other edges $E \setminus M$ are directed from A to B . Now start a breadth-first-search from the unmatched vertices in A . Note that if an unmatched vertex in B is reached by the breadth-first-search, then we have found an augmenting path. The runtime analysis is

- $O(|E|)$ for the construction of the directed graph from G .
- $O(|V| + |E|)$ for the run of the breadth-first-search.

So we can find an augmenting path in time $O(|V| + |E|)$. Since we increase the size of the matching each time we find an augmenting path, the total running time is $O(k(|V| + |E|)) = O(|V|(|V| + |E|))$ where k is the cardinality of a maximum matching.

- 2 You have just started your prestigious and important job as the Swiss Cheese Minister. As it turns out, different fondues and raclettes have different nutritional values and different prices:

Food	Fondue moitie moitie	Fondue a la tomate	Raclette	Requirement per week
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin B [mg/kg]	60	300	0.5	15 mg
Vitamin C [mg/kg]	30	20	70	4 mg
[price [CHF/kg]	50	75	60	—

Formulate the problem of finding the cheapest combination of the different fondues (moitie moitie & a la tomate) and Raclette so as to satisfy the weekly nutritional requirement as a linear program.

Solution: We have a variable x_1 for moitie moitie, a variable x_2 for a la tomate, and a variable x_3 for Raclette. The linear program becomes

$$\begin{aligned} \text{Minimize} \quad & 50x_1 + 75x_2 + 60x_3 \\ \text{Subject to} \quad & 35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5 \\ & 60x_1 + 300x_2 + 0.5x_3 \geq 15 \\ & 30x_1 + 20x_2 + 70x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3 Consider the following linear program for finding a maximum-weight matching:

$$\begin{aligned} \text{Maximize} \quad & \sum_{e \in E} x_e w_e \\ \text{Subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

(This is similar to the perfect matching problem seen in the lecture, except that we have inequality constraints instead of equality constraints.) Prove that, for bipartite graphs, any extreme point is integral.

Solution: We prove that all the extreme points are integral by contradiction. To that end, assume that there exists an extreme point x^* that is not integral. Let $G = (V_1, V_2, E)$ be the given bipartite graph and let $E_f = \{e \in E \mid 0 < x_e^* < 1\}$. If E_f contains a cycle, then the proof follows in the same way as the proof in the lecture notes. Therefore, we assume that E_f does not contain any cycles. Consider any maximal path in E_f ; let it have vertices v_1, \dots, v_k and edges e_1, \dots, e_{k-1} . Choose any ϵ such that $0 < \epsilon < \min(x_{e_i}^*, 1 - x_{e_i}^* : i = 1, \dots, k-1)$. Note that, since E_f only contains edges that are fractional, such an ϵ exists. Let y, z be the following two solutions to the linear program:

$$y = \begin{cases} x_e^* + \epsilon & \text{if } e \in \{e_1, e_3, e_5, e_7, \dots\} \\ x_e^* - \epsilon & \text{if } e \in \{e_2, e_4, e_6, e_8, \dots\} \\ x_e^* & \text{otherwise} \end{cases}$$

$$z = \begin{cases} x_e^* - \epsilon & \text{if } e \in \{e_1, e_3, e_5, e_7, \dots\} \\ x_e^* + \epsilon & \text{if } e \in \{e_2, e_4, e_6, e_8, \dots\} \\ x_e^* & \text{otherwise} \end{cases}$$

One can see that $x^* = \frac{y+z}{2}$.

We continue by showing that y is a feasible solution to the linear program. One can see that for any vertex $v \in G$ except v_1 and v_k we have $\sum_{e \in \delta(v)} y_e = \sum_{e \in \delta(v)} x_e^*$. So, we only need to show that the linear program constraint holds for v_1 and v_k . Let us first state two observations. First, by the definition of ϵ , we have that $0 \leq x_{e_1}^* + \epsilon \leq 1$, $0 \leq x_{e_1}^* - \epsilon \leq 1$, $0 \leq x_{e_{k-1}}^* + \epsilon \leq 1$, and $0 \leq x_{e_{k-1}}^* - \epsilon \leq 1$. Second, since the path is maximal and E_f does not contain any cycles, the degrees of v_1 and v_k in E_f are both one. Therefore $\sum_{e \in \delta(v_1)} y_e = y_{e_1}$ and $\sum_{e \in \delta(v_k)} y_e = y_{e_{k-1}}$.

Putting together the previous two observations, we get that the linear program constraint also holds for v_1 and v_k , so y is a feasible solution.

We can similarly show that z is also a feasible solution. This shows that we can write x^* as a convex combination of y and z , which contradicts the fact that x^* is an extreme point.

- 4 (half a *) Use the integrality of the bipartite perfect matching polytope (as proved in class) to show the following classical result:

The edge set of a k -regular bipartite graph $G = (A \cup B, E)$ can in polynomial time be partitioned into k disjoint perfect matchings.

A graph is k -regular if the degree of each vertex equals k . Two matchings are disjoint if they do not share any edges.

Solution: We show how to find such k disjoint perfect matchings in a k -regular bipartite graph in polynomial time.

Let $G_0 = (A \cup B, E)$ be a k -regular bipartite graph. Consider the LP for bipartite perfect matching on G_0 . The LP is feasible because setting $x_e = 1/k$ for all $e \in E$ satisfies all the constraints (recall that each vertex of a k -regular graph is incident to exactly k edges). Now we find an extreme point solution to the LP in polynomial time, and due to the integrality of such solutions, we get a valid perfect matching M_1 .

Notice that M_1 , being a perfect matching, forms a 1-regular sub-graph of G . Therefore, if we remove the matching M_1 from the original graph G_0 , we get a new $(k - 1)$ -regular graph $G_1 = (A \cup B, E \setminus M_1)$.

Now we repeat the process k times. Formally, at each iteration $i = 1, \dots, k$, we start with $(k - i + 1)$ -regular graph G_{i-1} . By solving the bipartite perfect matching LP for G_{i-1} to get an extreme point solution, we obtain a perfect matching M_i . We remove M_i from G_{i-1} to obtain a $(k - i)$ -regular graph G_i , which is a sub-graph of G_{i-1} .

Since we remove the already found perfect matchings at each iteration, the k -perfect matchings M_1, \dots, M_k are disjoint. Furthermore, since all graphs G_1, \dots, G_{k-1} are sub-graphs of the original graph G_0 , the matchings M_1, \dots, M_k are all valid perfect matchings of G_0 .

- 5 (*) Consider an undirected graph $G = (V, E)$ and let $s \neq t \in V$. Recall that in the min s, t -cut problem, we wish to find a set $S \subseteq V$ such that $s \in S$, $t \notin S$ and the number of edges crossing the cut is minimized. Show that the optimal value of the following linear program equals the number of edges crossed by a min s, t -cut:

$$\begin{array}{ll}
 \text{minimize} & \sum_{e \in E} y_e \\
 \text{subject to} & y_{\{u,v\}} \geq x_u - x_v \quad \text{for every } \{u,v\} \in E \\
 & y_{\{u,v\}} \geq x_v - x_u \quad \text{for every } \{u,v\} \in E \\
 & x_s = 0 \\
 & x_t = 1 \\
 & x_v \in [0, 1] \quad \text{for every } v \in V
 \end{array}$$

The above linear program has a variable x_v for every vertex $v \in V$ and a variable y_e for every edge $e \in E$.

Hint: Show that the expected value of the following randomized rounding equals the value of the linear program. Select θ uniformly at random from $[0, 1]$ and output the cut $S = \{v \in V : x_v \leq \theta\}$.

Solution: Let OPT be the number of edges that cross a minimum s, t -cut, and let OPT_{LP} be the value of the given LP. To show that $OPT = OPT_{LP}$, we show that $OPT_{LP} \leq OPT$ and $OPT_{LP} \geq OPT$.

Firstly let's prove that $OPT_{LP} \leq OPT$. Suppose that S is an optimal cut s, t -cut. We have $s \in S$ and $t \notin S$. We will create a solution for the LP problem whose value equals cut size defined by S and $E \setminus S$. Set $x_u = 0$ for all $u \in S$, and $x_v = 1$ for all $v \notin S$. Furthermore define

$$y_e = \begin{cases} 1 & \text{if } e \in \delta(S) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_e y_e = |\delta(S)| = OPT$. It remains to prove that the assignment to the variables $\{x_v\}_{v \in V}$, $\{y_e\}_{e \in E}$ is feasible:

- Consider any edge $\{u, v\}$. We need to verify that $y_{\{u, v\}} \geq x_u - x_v$ and $y_{\{u, v\}} \geq x_v - x_u$. In other words, that $y_{\{u, v\}} \geq |x_u - x_v|$.

- If $\{u, v\} \in \delta(S)$ then one of the vertices are in S and one is outside. Say $u \in S$ and $v \notin S$. Then

$$1 = y_{\{u, v\}} = |0 - 1| = |x_u - x_v|.$$

- If $\{u, v\} \notin \delta(S)$ then either $x_u = x_v = 0$ (both are in S) or $x_u = x_v = 1$ (both are outside S). In either case $|x_u - x_v| = 0$ and so the constraint $y_{\{u, v\}} = 0 = |x_u - x_v|$ is again verified (with equality).

- $x_s = 0$ and $x_t = 1$. Moreover, we have $x_v \in \{0, 1\} \subseteq [0, 1]$ for every $v \in V$.

This finishes one part of the proof - there is an assignment to the variables such that the LP outputs OPT . This means that OPT_{LP} is at most OPT , in other words $OPT_{LP} \leq OPT$.

Now let's prove that $OPT_{LP} \geq OPT$. Suppose that $(\{x_v^*\}_{v \in V}, \{y_e^*\}_{e \in E})$ is an optimal solution to the LP. Consider the following randomized rounding: select $\theta \in (0, 1)$ uniformly at random and let $S = S_\theta = \{v \in V : x_v^* \leq \theta\}$. Let's analyze this rounding algorithm.

It is clear that we always output a feasible cut since $x_s^* = 0$ and $x_t^* = 1$. This tells us that for every $\theta \in (0, 1)$ the associated S_θ is a valid solution and so $OPT \leq \delta(S_\theta)$. We thus have

$$OPT \leq \mathbb{E}_{\theta \in [0, 1]} [|\delta(\{v : x_v^* \leq \theta\})|].$$

We will now complete the proof by showing that the above expectation is at most OPT_{LP} . Let's introduce a new random variable $X_{e, \theta}$ that indicates if an edge is cut:

$$X_{e, \theta} = \begin{cases} 1 & \text{if } e \in \delta(S_\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Then the expectation above equals

$$\mathbb{E}_{\theta \in [0, 1]} \left[\sum_{e \in E} X_{e, \theta} \right] = \sum_{e \in E} \mathbb{E}_{\theta \in [0, 1]} [X_{e, \theta}]$$

Let's analyze $\mathbb{E}_{\theta \in [0,1]} [X_e] = \Pr_{\theta \in [0,1]} [e \text{ is cut in } S_\theta]$ for a specific edge $e = \{u, v\} \in E$. In the case when $x_u^* \leq x_v^*$, the edge e is cut if and only if $x_u^* \leq \theta \leq x_v^*$. The other case is analogous. It follows that

$$\Pr_{\theta \in [0,1]} [X_{\{u,v\},\theta}] = \begin{cases} \Pr_{\theta \in [0,1]} [\theta \in [x_u^*, x_v^*]] & \text{if } x_u^* \leq x_v^* \\ \Pr_{\theta \in [0,1]} [\theta \in [x_v^*, x_u^*]] & \text{if } x_u^* > x_v^* \end{cases} = |x_u^* - x_v^*|.$$

Now since the LP guarantees that $y_{\{u,v\}}^* \geq |x_u^* - x_v^*|$, we have

$$\sum_{\{u,v\} \in E} \mathbb{E}_{\theta \in [0,1]} [X_{\{u,v\},\theta}] = \sum_{\{u,v\} \in E} |x_u^* - x_v^*| \leq \sum_{\{u,v\} \in E} y_{\{u,v\}}^* = OPT_{LP}.$$

It follows that

$$OPT \leq \mathbb{E}_{\theta \in [0,1]} [|\delta(\{v : x_v^* \leq \theta\})|] \leq OPT_{LP}$$

and this finishes the proof.