

Nonlinear and Nonseparable Structural Functions in Fuzzy Regression Discontinuity Designs

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Motivation

- Regression discontinuity (RD) design - one of the most important non-experimental methods for causal inference
- Treatment changes discontinuously as a function of some underlying index which we call the running variable
- Most theoretical results study binary treatment variable
 - either 0 or 1
- In practice, the treatment may well be continuous
 - taking values inside an interval

RD first stage: binary treatment

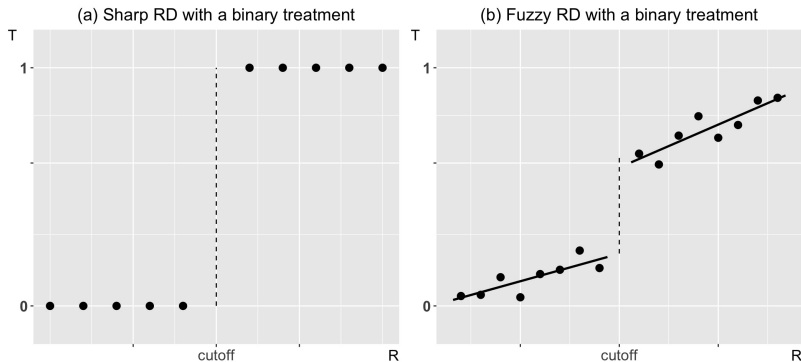


Figure: Demonstration of RD designs with a binary treatment.

RD first stage: continuous treatment

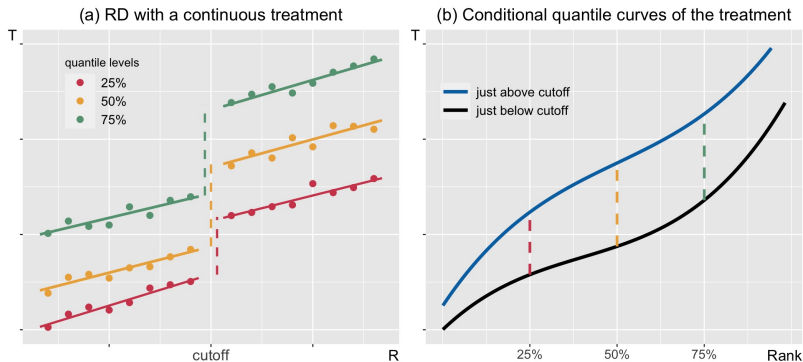


Figure: Demonstration of RD designs with a continuous treatment.

Causal object of interest

- The structural function (SF)

$$Y = g^*(T, R, \varepsilon)$$

- We use the RD to identify the SF at the cutoff \bar{r}

$$g^*(t, \bar{r}, \varepsilon), \text{ for all } t$$

- With a binary treatment, we care about the scalar treatment effect

$$g^*(1, \bar{r}, \varepsilon) - g^*(0, \bar{r}, \varepsilon)$$

Main contributions

- General specification of the structural function:
nonlinear and nonseparable
- Nonparametric identification result under shape
restrictions (monotonicity and smoothness)
- Semiparametric estimator
 - same convergence rate as with a binary treatment ($n^{-2/5}$)
 - asymptotic normality

Example: time zone boundaries

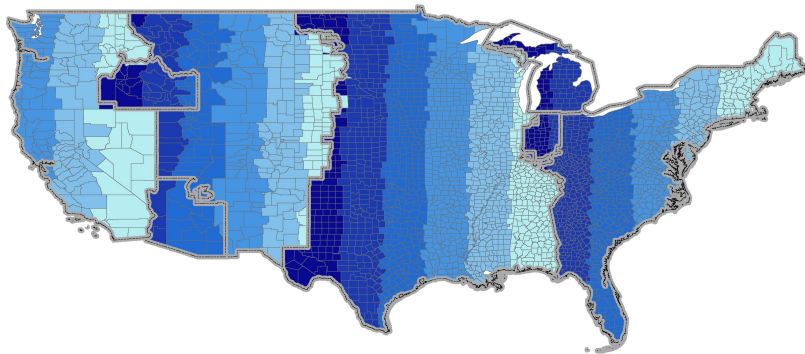


Figure: Time Zones and Average Sunset Time. The darker the color, the later the average sunset time.

Example: time zone boundaries (cont'd)

- Giuntella and Mazzonna (2019)
- Sleep time \rightarrow Health
- Treatment T : sleep time (continuous variable)
- Outcome of interest Y : health (measured by BMI)
- Running variable R : distance to the time zone boundary (boundary is the cutoff/threshold \bar{r})

Example: time zone boundaries (cont'd)

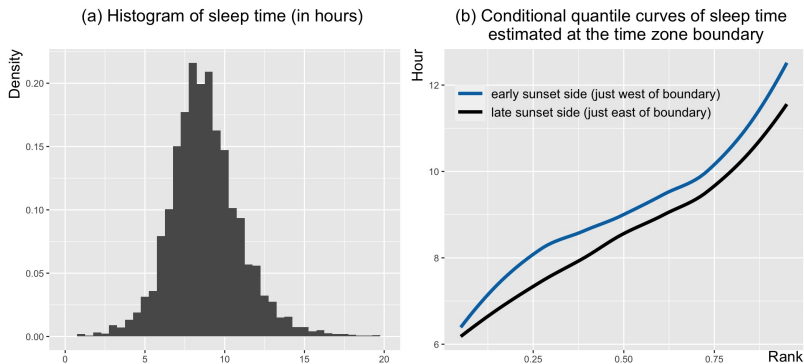


Figure: Empirical illustration of RD designs with a continuous treatment.

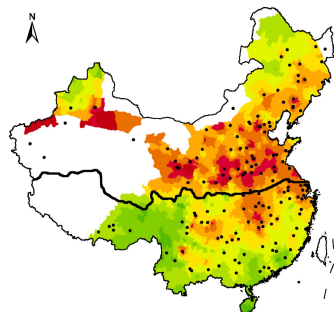
Example: time zone boundaries (cont'd)

Why general specification of SF?

- nonlinearity: undersleeping and oversleeping are both bad
- nonseparability: effect heterogeneity due to eating habits
- determining the optimal sleep time needs a nonlinear specification

More empirical examples

- China's Huai River winter heating policy
- Chen et al. (2013), Ebenstein et al. (2017)
- Y – life expectancy
- T – air pollution
- R – distance to Huai River



Huai River and PM₁₀ level

More empirical examples

- Very low birth weight threshold
- Almond et al. (2010),
Barreca et al. (2011),
Bharadwaj et al. (2013)
- Y – infants health
- T – medical spending,
days of hospitalization
- R – birth weight



More empirical examples

- Brazil tax redistribution
- Litschig and Morrison (2010)
- Y – incumbent's reelection
- T – local government spending
- R – district population



More empirical examples

- Child-related tax benefits
- Barr et al. (2021), Cole (2021)
- Y – academic achievements
- T – family income
- R – birth date



Roadmap

- 1 Introduction
- 2 The model and comparison with the literature
- 3 Nonparametric identification
- 4 Semiparametric estimation
- 5 Numerical results

Setup

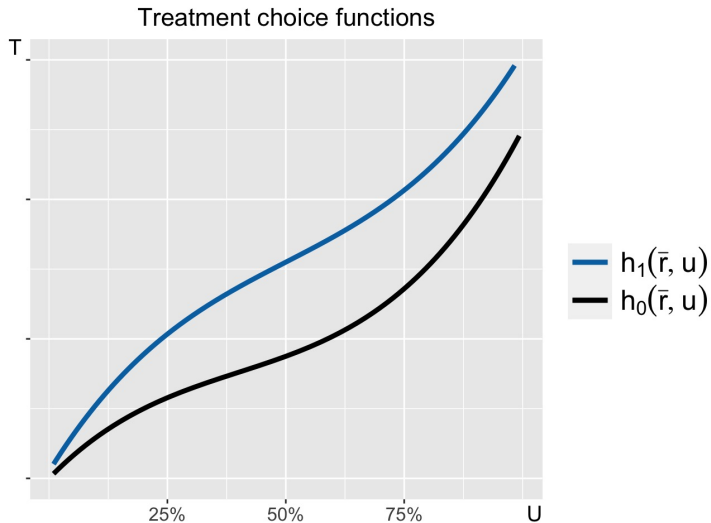
A triangular system

Outcome Equation: $Y = g^*(T, R, \varepsilon)$

$$\text{Treatment Choice: } T = h(R, U) = \begin{cases} h_0(R, U), & R < \bar{r} \\ h_1(R, U), & R \geq \bar{r} \end{cases}$$

- h - treatment choice function (conditional quantile)
- U - conditional rank of T given R
- \bar{r} - cutoff/threshold
- ε can correlate with T, R

Back to the graph



Common practice: TSLS & Wald ratio

- Empirical studies use TSLS to estimate the Wald ratio

$$\text{WR} = \frac{\lim_{r \uparrow \bar{r}} \mathbb{E}[Y|R=r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[Y|R=r]}{\lim_{r \uparrow \bar{r}} \mathbb{E}[T|R=r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[T|R=r]}$$

- Binary treatment (Hahn et al., 2001):
 - $\text{WR} = (\text{local})$ average treatment effect
- Continuous treatment (Lee and Lemieux, 2010)
if g^* is linear and separable

$$g^*(T, R, \varepsilon) = \beta T + \delta R + \varepsilon$$

then $\text{WR} = \beta$

- General SF: $\text{WR} \neq \text{SF}$

Literature: RD with a continuous treatment

Dong et al. (2021)

- Quantile specific LATE

$$\frac{\lim_{r \uparrow \bar{r}} \mathbb{E}[Y|U = u, R = r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[Y|U = u, R = r]}{\lim_{r \uparrow \bar{r}} \mathbb{E}[T|U = u, R = r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[T|U = u, R = r]}$$

Literature: RD with a continuous treatment

Dong et al. (2021)

- Quantile specific LATE

$$\frac{\lim_{r \uparrow \bar{r}} \mathbb{E}[Y|U = u, R = r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[Y|U = u, R = r]}{\lim_{r \uparrow \bar{r}} \mathbb{E}[T|U = u, R = r] - \lim_{r \downarrow \bar{r}} \mathbb{E}[T|U = u, R = r]}$$

$$= \int_{h_0(\bar{r}, u)}^{h_1(\bar{r}, u)} \frac{\mathbb{E}[\frac{\partial}{\partial t} g^*(t, \bar{r}, \varepsilon) | U = u, R = \bar{r}]}{h_1(\bar{r}, u) - h_0(\bar{r}, u)} dt$$

- a weighted average derivative of g^*
- my paper focuses on how to directly identify the structural function $g^*(\cdot, \bar{r}, \cdot)$ at the cutoff

Literature: IV

- Instruments with small support
 - Torgovitsky (2015, 2016),
D'Haultfoeuille and Février (2015), Ishihara (2021)
 - Need independent instrument: $R \perp \varepsilon$
and exclusion restriction: R excluded from g^*, h_0, h_1
- Challenges we face in RD designs
 - R can violate independence and exclusion restrictions
 - estimation theory is more complicated (local to the cutoff)

Literature: nonseparable structural functions

- Matzkin (2003, 2013, 2016),
Hoderlein and Mammen (2007, 2009),
Sasaki (2015), Su et al. (2015, 2019)
- Need independent treatment: $T \perp \varepsilon | R$

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- 2 The model and comparison with the literature
- 3 Nonparametric identification**
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Identification assumptions

Dual monotonicity

- $g^*(T, R, \varepsilon)$ is strictly increasing in ε (ε is one-dimensional)
- $h(R, U)$ is strictly increasing in U
- one-to-one mapping $(Y, T) \leftrightarrow (\varepsilon, U)$ given R

Smoothness

- g^*, h_0, h_1 continuous
- $F_{\varepsilon|U,R}(e|u, r)$ is strictly increasing in e and

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u, r) = \lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u, r)$$

Identification assumptions (cont'd)

Fuzzy RD

- There are treatment levels that are taken both below and above the cutoff.
- Overlapping support:

$$\text{Supp}(h_0(\bar{r}, U)) \cap \text{Supp}(h_1(\bar{r}, U)) \neq \emptyset$$

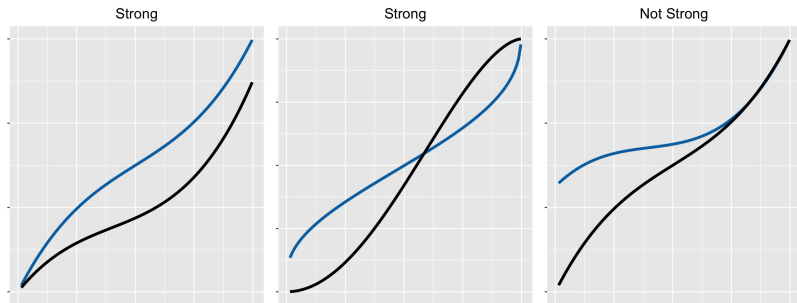
Support invariance

- $\text{Supp}(\varepsilon|U = u, R = r)$ does not depend on u or r
- still allows ε to be correlated with U or R arbitrarily

Identification assumptions (cont'd)

Strong discontinuity

- $h_0(\bar{r}, \cdot)$ and $h_1(\bar{r}, \cdot)$ intersect but do not overlap



Identification equation

Smoothness

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u, r)$$

$$||$$

$$\lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u, r)$$

Identification equation

Smoothness

Dual Monotonicity

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u, r) = F_{Y|T,R}^{-}(g^*(h_0(u), e)|h_0(u))$$

||

$$\lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u, r)$$

Identification equation

Smoothness

Dual Monotonicity

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u, r) = F_{Y|T,R}^{-}(g^*(h_0(u), e)|h_0(u))$$

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Identification equation

Smoothness

Dual Monotonicity

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||

$$\lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u, r) = F_{Y|T,R}^{+}(g^*(h_1(u), e)|h_1(u))$$

- $F_{Y|T,R}$ and (h_0, h_1) are identified
 \implies restrictions on the structural function
- These restrictions identify a unique structural function

Proof

Nonparametric identification

Theorem 1

If a qualifying g satisfies the following condition

$$F_{Y|T,R}^{-}(g(h_0(u), e)|h_0(u)) = F_{Y|T,R}^{+}(g(h_1(u), e)|h_1(u)),$$

then there exists a continuous and strictly increasing function λ^g such that

$$g^*(T, \bar{r}, \varepsilon) = g(T, \bar{r}, \lambda^g(\varepsilon)).$$

- identification at the cutoff (\bar{r} suppressed in the notation)
- identification up to a monotone transformation in ε

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Parametrize the structural function

Nonlinear: $g_\gamma(T, \bar{r}, \varepsilon) = \gamma_1 T + \gamma_2 T^2 + \varepsilon$

Nonseparable: $g_\gamma(T, \bar{r}, \varepsilon) = \gamma_1 T + \gamma_3 T \varepsilon + \varepsilon$

Both: $g_\gamma(T, \bar{r}, \varepsilon) = \gamma_1 T + \gamma_2 T^2 + \gamma_3 T \varepsilon + \varepsilon$

More general: in the paper

Criterion function

Rearranging identification equation + integral smoothing

$$D_{\gamma,h}(e, u) = \int_0^u (F_{Y|T,R}^-(g_\gamma(h_0(v), e)|h_0(v)) - F_{Y|T,R}^+(g_\gamma(h_1(v), e)|h_1(v)))dv$$

Take L^2 -norm

$$\|D_{\gamma,h}\|_w = \left(\int_0^1 \int_{\mathbb{R}} |D_{\gamma,h}(e, u)|^2 w(e, u) dedu \right)^{1/2},$$

where $w(e, u)$ is a positive weighting function on $\mathbb{R} \times [0, 1]$.

Semiparametric identification

$$\|D_{\gamma,h}\|_w \geq 0, \text{ and } \|D_{\gamma,h}\|_w = 0 \text{ iff } \gamma = \gamma^*$$

Semiparametric estimation

Three-step estimation procedure

- 1 estimate conditional quantiles of $T|R$: $\hat{h}_0(\bar{r}, u)$, $\hat{h}_1(\bar{r}, u)$
- 2 local linear regression estimates $\hat{F}_{Y|T,R}^{\pm}(y|t, \bar{r})$
- 3 $\hat{\gamma}$ minimizes the empirical criterion function $\|\hat{D}_{\gamma, \hat{h}}\|_w$

Asymptotic normality

Theorem 2

Under regularity conditions, the following term is asymptotically standard normal

$$n^{2/5}(\Sigma_- + \Sigma_+)^{-1/2}\Delta(\hat{\gamma} - \gamma^*) - (B_- - B_+)$$

Regularity conditions

Proof

- same convergence rate as in the binary case
- second step estimators $\hat{F}_{Y|T,R}^\pm(y|t, \bar{r})$ converges slower
two nonparametric dimensions $\implies n^{-1/3}$
- analogy: a \sqrt{n} -semiparametric estimator with a nonparametric first step being $o(n^{-1/4})$

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Empirical study: sleep time

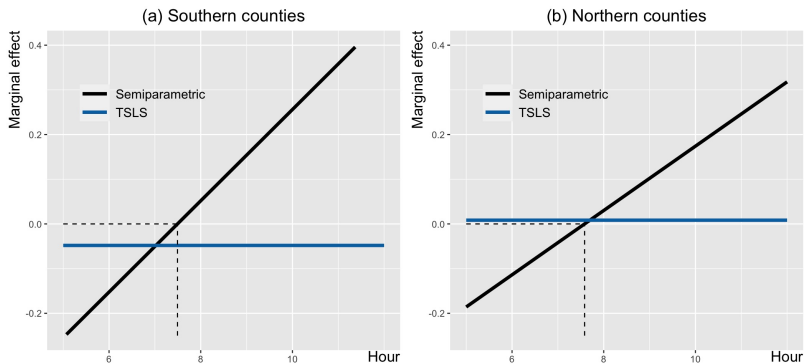


Figure: Estimated marginal effects (structural derivative) of sleep time on BMI (kg/m^2).

Empirical findings

- Marginal effects are increasing, indicating a nonlinear (U-shaped) structural function.
- Optimal sleep time is between 7 and 8 hours.
- The TSLS is not capable of demonstrating such results.

Simulations

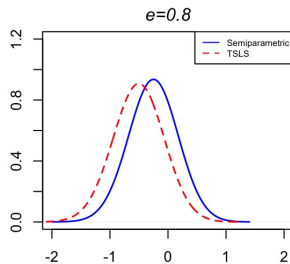
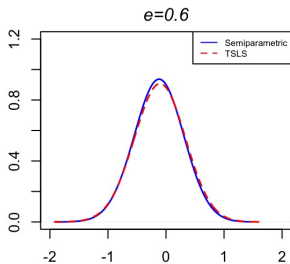
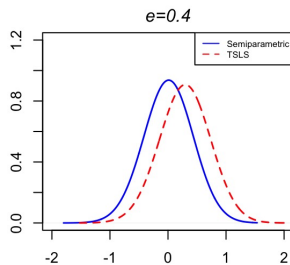
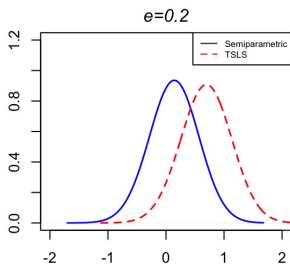
- Semiparametric estimator vs TSLS
- Estimate the marginal effect

$$\text{SP: } \frac{\partial}{\partial t} g_{\hat{\gamma}}(t, \bar{r}, \varepsilon)$$

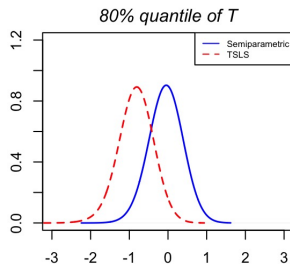
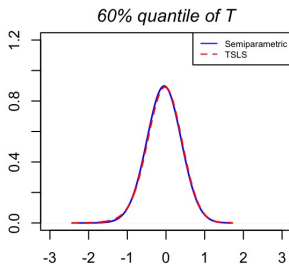
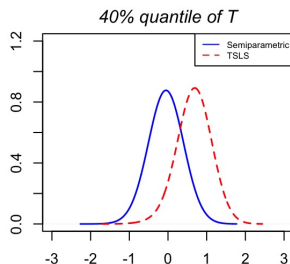
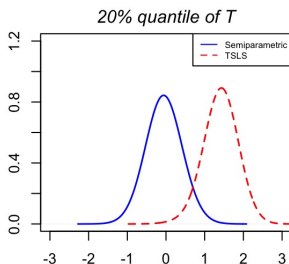
$$\text{TSLS: } \widehat{WR}$$

- Simulate distribution of the estimator centered at the true marginal effect
- Different values of T, ε

Nonseparable SF

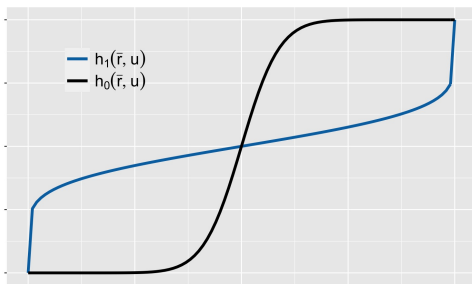


Nonlinear SF

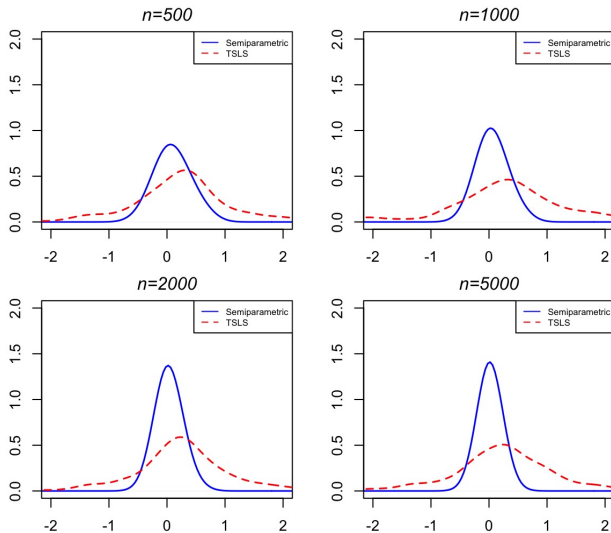


Linear SF: weakly identified TSLS

- $\hat{\gamma}$ can outperform TSLS even with a linear g^*
- weak identification in the difference in means
- $h_0(\bar{r}, \cdot)$ – quantile of $Beta(0.1, 0.1)$,
 $h_1(\bar{r}, \cdot)$ – quantile of $Beta(10, 10)$



Linear SF: weakly identified TSLS



Conclusion

- RD designs with a continuous treatment
- Nonlinear and nonseparable structural function
- Nonparametric identification
- Semiparametric estimation
- Future work
 - partial identification when some assumptions fail
 - extrapolation away from the cutoff
 - Regression kink designs

Thank You!

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Sequencing approach

Prove that the identification equation pins down a unique structural function. [Back](#)

- If g satisfies the identification equation, then

$$g^{-1}(h_0(u), g^*(h_0(u), e)) = g^{-1}(h_1(u), g^*(h_1(u), e))$$

- Define $\pi(t) = h_1(h_0^{-1}(t))$, then

$$\begin{aligned} g^{-1}(t, g^*(t, e)) &= g^{-1}(\pi(t), g^*(\pi(t), e)) \\ &= g^{-1}(\pi(\pi(t)), g^*(\pi(\pi(t)), e)) \\ &= g^{-1}(\pi^m(t), g^*(\pi^m(t), e)) \end{aligned}$$

- The sequence $t, \pi(t), \pi(\pi(t)), \dots, \pi^m(t), \dots$ is convergent.
- $g^{-1}(t, g^*(t, e))$ does not depend on t .

Regularity conditions

- Distributions of Y, T , and R [detail](#)
- Complexity of the parametric model [detail](#)
- Kernel functions [detail](#)
- Bandwidth [detail](#)
- First-step conditional quantile estimators [detail](#)

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Proof

- 1 uniform convergence results on the LLR estimators
- 2 consistency of $\hat{\gamma}$
- 3 initial bound on the convergence rate of $\hat{\gamma}$
- 4 stochastic equicontinuity of the criterion function
- 5 linearization of the criterion function
- 6 asymptotic normality of the minimizer of the linearized criterion function
- 7 asymptotic normality of $\hat{\gamma}$

Back

- ① The support of T does not vary with R except when crossing the cutoff \bar{r} , i.e., $\text{Supp}(T|R = r) = [t'_0, t''_0]$ for $r < \bar{r}$ and $\text{Supp}(T|R = r) = [t'_1, t''_1]$ for $r > \bar{r}$. The density functions $f_{T,R}^-$ and $f_{T,R}^+$ are bound away from zero.
- ② The density functions $f_{T,R}^-$ and $f_{T,R}^+$ are twice continuously differentiable, and $\frac{\partial^2}{\partial T^2} f_{T,R}^-(t, \bar{r})$ and $\frac{\partial^2}{\partial T^2} f_{T,R}^+(t, \bar{r})$ are Lipschitz continuous with respect to t .
- ③ The support of Y , \mathcal{Y} , is compact. The conditional distribution functions $F_{Y|T,R}^-$ and $F_{Y|T,R}^+$ are three-times continuously differentiable over $\mathcal{Y} \times [t'_0, t''_0] \times [r_0, \bar{r}]$ and $\mathcal{Y} \times [t'_1, t''_1] \times [\bar{r}, r_1]$, respectively.

Back

The parametrization $\{g_\gamma(\cdot, \bar{r}, \cdot) : \gamma \in \Gamma\}$ satisfies the following conditions.

- 1 The parameter space Γ is compact.
- 2 The class of functions $\{T \mapsto g_\gamma(T + v, \bar{r}, e) : \gamma \in \Gamma, v \in (-1, 1), e \in \mathcal{E}\}$ is finite-dimensional.
- 3 The function $g_\gamma(t, \bar{r}, e)$ is twice continuously differentiable over $\gamma \in \Gamma$, $t \in [t'_0, t''_0] \cup [t'_1, t''_1]$, and $e \in \mathcal{E}$.
- 4 The gradient $\nabla_\gamma D_{\gamma^*, h^*}(e, u)$ is a vector of linearly independent functions of (e, u) .

Back

- ① The kernel functions k_T and k_R are (1) supported on $[-1, 1]$, (2) strictly greater than zero in the interior of the support, (3) of bounded variation, (4) continuously differentiable on \mathbb{R} .
- ② The kernel function k_Y is (1) nonnegative and (2) integrable on \mathbb{R} with $\int k_Y(y)dy = 1$ and satisfies (3) $\int y k_Y(y)dy = 0$.

Back

Bandwidth

The bandwidth b_1 and b_2 satisfy the following conditions:

- ① $b_1 \asymp b_2$.
- ② $(n \log n) b_1^6 = o(1)$.
- ③ $n b_1^{\frac{13}{3} + \epsilon} \rightarrow \infty$, for some sufficiently small $\epsilon > 0$.

In particular, the optimal bandwidth $n^{-1/5}$ is allowed.

Back

First-step conditional quantile estimators (1)

Monotonicity and smoothness: for every n sufficiently large, there exist $C > 0$ and deterministic and finite partitions \mathcal{P}_0^n and \mathcal{P}_1^n on $(0, 1)$ such that

$$\mathbb{P}\left(\hat{h}_0(\bar{r}, \cdot) \notin \mathcal{H}_0(\mathcal{P}_0^n)\right), \mathbb{P}\left(\hat{h}_1(\bar{r}, \cdot) \notin \mathcal{H}_1(\mathcal{P}_1^n)\right) = O(\sqrt{b_1}),$$

where $\mathcal{H}_0(\mathcal{P}_0^n)$ includes functions that are strictly increasing with a three-times continuously differentiable inverse on each element of \mathcal{P}_0^n . $\mathcal{H}_1(\mathcal{P}_1^n)$ is defined analogously by replacing \mathcal{P}_0^n with \mathcal{P}_1^n .

Back

First-step conditional quantile estimators (2)

Uniform Bahadur representation:

$$\begin{aligned}\hat{h}_0(\bar{r}, u) - h_0(\bar{r}, u) &= b_1^2 \nu_0(u) + O_p(b_1^3) \\ &+ \frac{1}{nb_1} \sum_{i=1}^n q_0(T_i, R_i; u) k_{Q,0} \left(\frac{R_i - \bar{r}}{b_1} \right) \mathbf{1}\{R_i < \bar{r}\} + o_p(1/\sqrt{nb_1}),\end{aligned}$$

uniformly over $u \in (0, 1)$. The same for \hat{h}_1 . The functions ν_0 and ν_1 are bounded. The functions q_0 and q_1 are (1) bounded, (2) centered, that is,

$\mathbb{E}[q_0(T, R; u)|T, R] = \mathbb{E}[q_1(T, R; u)|T, R] = 0$, and (3) does not vary with n . The functions $k_{Q,0}$ and $k_{Q,1}$ are bounded.

Back

First-step conditional quantile estimators (3)

Uniform convergence rate:

$$\begin{aligned}\|\hat{h} - h\|_{\infty} &= \sup_{u \in (0,1)} |\hat{h}_0(\bar{r}, u) - h_0(\bar{r}, u)| \vee |\hat{h}_1(\bar{r}, u) - h_1(\bar{r}, u)| \\ &= O_p \left(\sqrt{\log n / (nb_1)} + b_1^2 \right).\end{aligned}$$

Back