Nonlinear and Nonseparable Structural Functions in Fuzzy Regression Discontinuity Designs

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Motivation

- Regression discontinuity (RD) design - one of the most important non-experimental methods for causal inference
- Treatment changes discontinuously as a function of some underlying index which we call the running variable
- Most theoretical results study binary treatment variable
  - either 0 or 1
- In practice, the treatment may well be continuous
  - taking values inside an interval

RD first stage: binary treatment

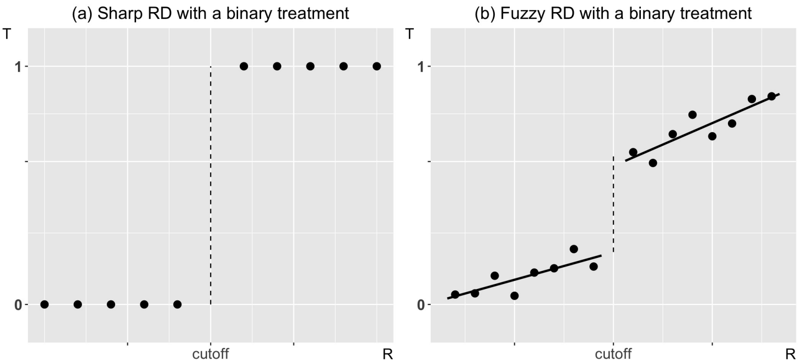


Figure: Demonstration of RD designs with a binary treatment.

RD first stage: continuous treatment

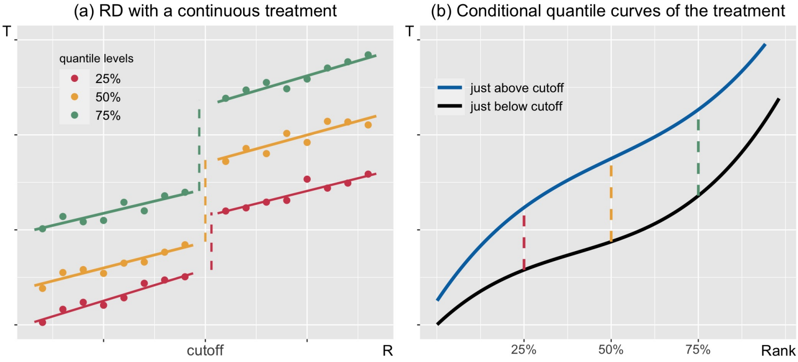


Figure: Demonstration of RD designs with a continuous treatment.

Causal object of interest

- The structural function (SF)

$$Y = g^*(T, R, \varepsilon)$$

- We use the RD to identify the SF at the cutoff $\bar{r}$

$$g^*(t, \bar{r}, \varepsilon), \text{ for all } t$$

- With a binary treatment, we care about the scalar treatment effect

$$g^*(1, \bar{r}, \varepsilon) - g^*(0, \bar{r}, \varepsilon)$$

Main contributions

- General specification of the structural function: nonlinear and nonseparable
- Nonparametric identification result under shape restrictions (monotonicity and smoothness)
- Semiparametric estimator
  - same convergence rate as with a binary treatment ($n^{-2/5}$)
  - asymptotic normality

Example: time zone boundaries

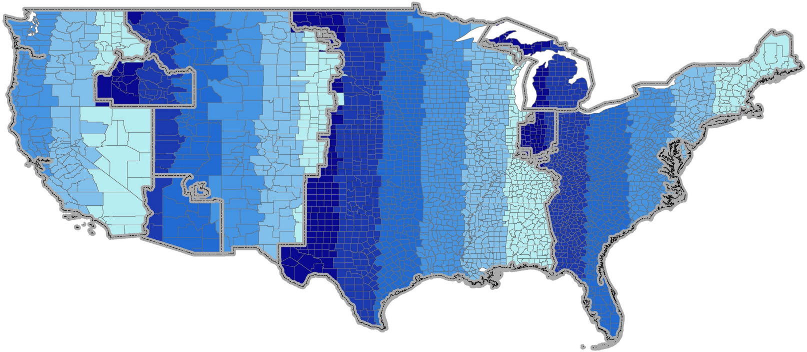


Figure: Time Zones and Average Sunset Time. The darker the color, the later the average sunset time.

Example: time zone boundaries (cont'd)

- Giuntella and Mazzonna (2019)
- Sleep time $\rightarrow$ Health
- Treatment T: sleep time (continuous variable)
- Outcome of interest Y: health (measured by BMI)
- Running variable R: distance to the time zone boundary (boundary is the cutoff/threshold $\bar{r}$)

Example: time zone boundaries (cont'd)

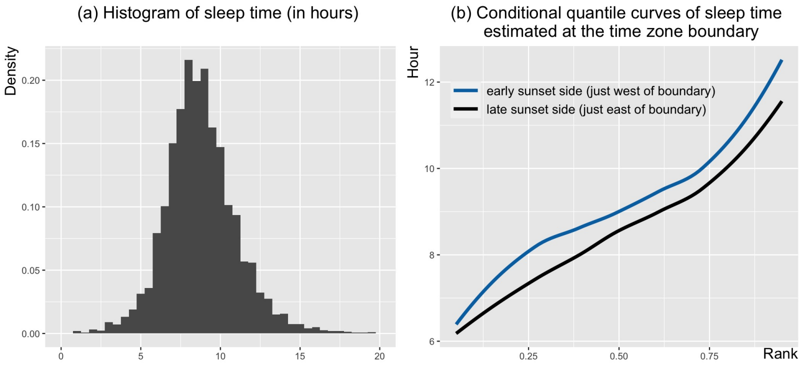


Figure: Empirical illustration of RD designs with a continuous treatment.

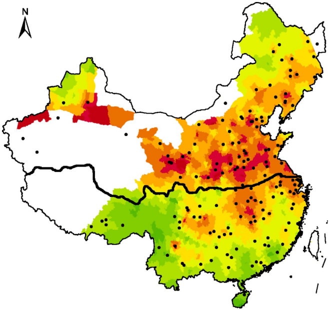
Example: time zone boundaries (cont'd)

Why general specification of SF?

- nonlinearity: undersleeping and oversleeping are both bad
- nonseparability: effect heterogeneity due to eating habits
- determining the optimal sleep time needs a nonlinear specification

More empirical examples

- China's Huai River winter heating policy
- Chen et al. (2013), Ebenstein et al. (2017)
- Y – life expectancy
- T – air pollution
- R – distance to Huai River



Huai River and PM_{10} level

More empirical examples

- Very low birth weight threshold
- Almond et al. (2010),
  Barreca et al. (2011),
  Bharadwaj et al. (2013)
- Y – infants health
- T – medical spending,
  days of hospitalization
- R – birth weight

More empirical examples

- Brazil tax redistribution
- Litschig and Morrison (2010)
- Y – incumbnet's reelection
- T – local government spending
- R – district population

More empirical examples

- Child-related tax benefits
- Barr et al. (2021), Cole (2021)
- Y – academic achievements
- T – family income
- R – birth date

Roadmap

Setup

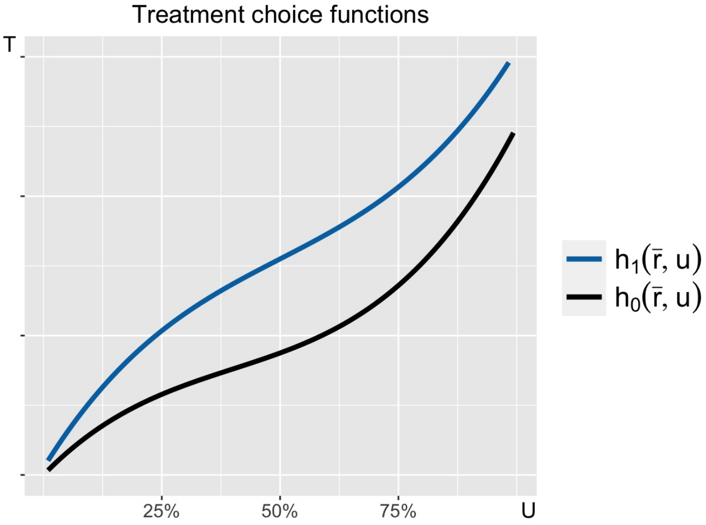
A triangular system

Outcome Equation: $Y = g^*(T, R, \varepsilon)$

Treatment Choice: $T = h(R, U) = \begin{cases} h_0(R, U), R < \bar{r} \\ h_1(R, U), R \geq \bar{r} \end{cases}$

- h - treatment choice function (conditional quantile)
- U - conditional rank of T given R
- $\bar{r}$ - cutoff/threshold
- ε can correlate with T, R

Back to the graph



Common practice: TSLS & Wald ratio

- Empirical studies use TSLS to estimate the Wald ratio

$$\mathrm{WR} = \frac{\lim_{r\uparrow\bar{r}} \mathbb{E}[Y|R=r] - \lim_{r\downarrow\bar{r}} \mathbb{E}[Y|R=r]}{\lim_{r\uparrow\bar{r}} \mathbb{E}[T|R=r] - \lim_{r\downarrow\bar{r}} \mathbb{E}[T|R=r]}$$

- Binary treatment (Hahn et al., 2001):
  - WR = (local) average treatment effect

- Continuous treatment (Lee and Lemieux, 2010) if g^* is linear and separable

$$g^*(T, R, \varepsilon) = \beta T + \delta R + \varepsilon$$

  then $\mathrm{WR} = \beta$

- General SF: $\mathrm{WR} \neq \mathrm{SF}$

Literature: RD with a continuous treatment

Dong, Lee, and Gou (2021)

- Quantile specific LATE

$$\frac{\lim_{r\uparrow\bar{r}} \mathbb{E}[Y|U=u, R=r] - \lim_{r\downarrow\bar{r}} \mathbb{E}[Y|U=u, R=r]}{\lim_{r\uparrow\bar{r}} \mathbb{E}[T|U=u, R=r] - \lim_{r\downarrow\bar{r}} \mathbb{E}[T|U=u, R=r]}$$

Literature: RD with a continuous treatment

Dong, Lee, and Gou (2021)

- Quantile specific LATE

$$\frac{\lim_{r\uparrow\bar{r}}\mathbb{E}[Y|U=u,R=r]-\lim_{r\downarrow\bar{r}}\mathbb{E}[Y|U=u,R=r]}{\lim_{r\uparrow\bar{r}}\mathbb{E}[T|U=u,R=r]-\lim_{r\downarrow\bar{r}}\mathbb{E}[T|U=u,R=r]}$$

$$=\int_{h_0(\bar{r},u)}^{h_1(\bar{r},u)}\frac{\mathbb{E}[\frac{\partial}{\partial t}g^*(t,\bar{r},\varepsilon)|U=u,R=\bar{r}]}{h_1(\bar{r},u)-h_0(\bar{r},u)}dt$$

- a weighted average derivative of g^*
- my paper focuses on how to directly identify the structural function $g^*(\cdot,\bar{r},\cdot)$ at the cutoff

Literature: IV

- Instruments with small support
  - Torgovitsky (2015, 2016), D'Haultfoeuille and Février (2015), Ishihara (2021)
  - Need independent instrument: $R \perp \varepsilon$ and exclusion restriction: R excluded from g^*, h_0, h_1
- Challenges we face in RD designs
  - R can violate independence and exclusion restrictions
  - estimation theory is more complicated (local to the cutoff)

Literature: nonseparable structural functions

- Identification and estimation:
  Matzkin (2003, 2013, 2016),
  Hoderlein and Mammen (2007, 2009),
  Sasaki (2015), Su, Ura, and Zhang (2019)
- Testing: Su, Tu, and Ullah (2015)
- Need (conditional) independence: $T \perp \varepsilon$

Roadmap

Identification assumptions

Dual monotonicity

- $g^*(T, R, \varepsilon)$ is strictly increasing in ε (ε is one-dimensional)
- $h(R, U)$ is strictly increasing in U
- one-to-one mapping $(Y, T) \leftrightarrow (\varepsilon, U)$ given R

Smoothness

- g^*, h_0, h_1 continuous
- $F_{\varepsilon|U,R}(e|u, r)$ is strictly increasing in e and

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u, r) = \lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u, r)$$

Identification assumptions (cont'd)

Fuzzy RD

- There are treatment levels that are taken both below and above the cutoff.
- Overlapping support:
  $Supp(h_0(\bar{r}, U)) \cap Supp(h_1(\bar{r}, U)) \neq \varnothing$

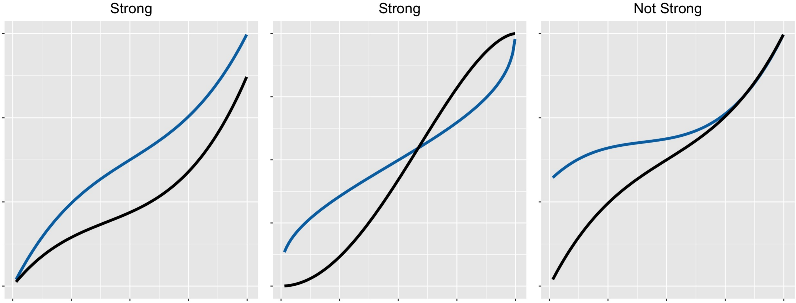
Support invariance

- $\mathrm{Supp}(\varepsilon | U = u, R = r)$ does not depend on u or r
- still allows ε to be correlated with U or R arbitrarily

Identification assumptions (cont'd)

Strong discontinuity

- $h_0(\bar{r}, \cdot)$ and $h_1(\bar{r}, \cdot)$ intersect but do not overlap



Identification equation

Smoothness

$$\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u,r)$$

$$||$$

$$\lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u,r)$$

Identification equation

Smoothness **Dual Monotonicity**

$$\begin{array}{ll} \lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u,r) & = F^{-}_{Y|T,R}(g^*(h_0(u),e)|h_0(u)) \\ \| & \\ \lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u,r) & \end{array}$$

Identification equation

$$
\begin{array}{ccc}
\textbf{Smoothness} & & \textbf{Dual Monotonicity} \\
\lim_{r \uparrow \bar{r}} F_{\varepsilon|U,R}(e|u,r) & = & F^{-}_{Y|T,R}(g^*(h_0(u),e)|h_0(u)) \\
\| & & \\
\lim_{r \downarrow \bar{r}} F_{\varepsilon|U,R}(e|u,r) & = & F^{+}_{Y|T,R}(g^*(h_1(u),e)|h_1(u))
\end{array}
$$

Identification equation

Smoothness		Dual Monotonicity
$\lim_{r \uparrow \bar{r}} F_{\varepsilon \mid U,R}(e \mid u, r)$	$=$	$F^{-}_{Y \mid T,R}(g^*(h_0(u), e) \mid h_0(u))$
$\vert\vert$		
$\lim_{r \downarrow \bar{r}} F_{\varepsilon \mid U,R}(e \mid u, r)$	$=$	$F^{+}_{Y \mid T,R}(g^*(h_1(u), e) \mid h_1(u))$

- $F_{Y \mid T,R}$ and (h_0, h_1) are identified
  $\Longrightarrow$ restrictions on the structural function
- These restrictions identify a unique structural function

Proof

Nonparametric identification

Theorem 1

If a qualifying g satisfies the following condition

$$F^{-}_{Y|T,R}(g(h_0(u),e)|h_0(u)) = F^{+}_{Y|T,R}(g(h_1(u),e)|h_1(u)),$$

then there exists a continuous and strictly increasing function λ^g such that

$$g^*(T,\bar{r},\varepsilon) = g(T,\bar{r},\lambda^g(\varepsilon)).$$

- identification at the cutoff ($\bar{r}$ suppressed in the notation)
- identification up to a monotone transformation in ε

Roadmap

Parametrize the structural function

Parametric class

$$
\begin{aligned}
\text{Nonlinear:} &\quad g_\gamma(T,\bar{r},\varepsilon) = \gamma_1 T + \gamma_2 T^2 + \varepsilon \\
\text{Nonseparable:} &\quad g_\gamma(T,\bar{r},\varepsilon) = \gamma_1 T + \gamma_3 T\varepsilon + \varepsilon \\
\text{Both:} &\quad g_\gamma(T,\bar{r},\varepsilon) = \gamma_1 T + \gamma_2 T^2 + \gamma_3 T\varepsilon + \varepsilon \\
\text{More general:} &\quad \text{any parametric class compatible with} \\
&\quad \text{previous identification assumptions}
\end{aligned}
$$

These parametrizations normalize the scale of ε.

Criterion function

Rearranging identification equation + integral smoothing

$D_{\gamma,h}(e,u) =$
$\int_0^u (F^-_{Y|T,R}(g_\gamma(h_0(v),e)|h_0(v)) - F^+_{Y|T,R}(g_\gamma(h_1(v),e)|h_1(v)))dv$

Take L^2-norm

$||D_{\gamma,h}||_w = \left(\int_0^1 \int_{\mathbb{R}} |D_{\gamma,h}(e,u)|^2 w(e,u) dedu \right)^{1/2},$
where $w(e,u)$ is a positive weighting function on $\mathbb{R} \times [0,1]$.

Semiparametric identification

$||D_{\gamma,h}||_w \geq 0$, and $||D_{\gamma,h}||_w = 0$ iff $\gamma = \gamma^*$

Semiparametric estimation

Three-step estimation procedure

1. estimate conditional quantiles of $T|R$: $\hat{h}_0(\bar{r}, u)$, $\hat{h}_1(\bar{r}, u)$
2. local linear regression estimates $\hat{F}^{\pm}_{Y|T,R}(y|t, \bar{r})$
3. $\hat{\gamma}$ minimizes the empirical criterion function $||\hat{D}_{\gamma,\hat{h}}||_w$

Asymptotic normality

Theorem 2

Under regularity conditions, the following term is asymptotically standard normal

$$n^{2/5}(\Sigma_- + \Sigma_+)^{-1/2}\Delta(\hat{\gamma} - \gamma^*) - (B_- - B_+)$$

Regularity conditions   Proof

- same convergence rate as in the binary case
- second step estimators $\hat{F}^{\pm}_{Y|T,R}(y|t,\bar{r})$ converges slower
  two nonparametric dimensions $\implies n^{-1/3}$
- analogy: a $\sqrt{n}$-semiparametric estimator with a nonparametric first step being $o(n^{-1/4})$

Roadmap

Empirical study: sleep time

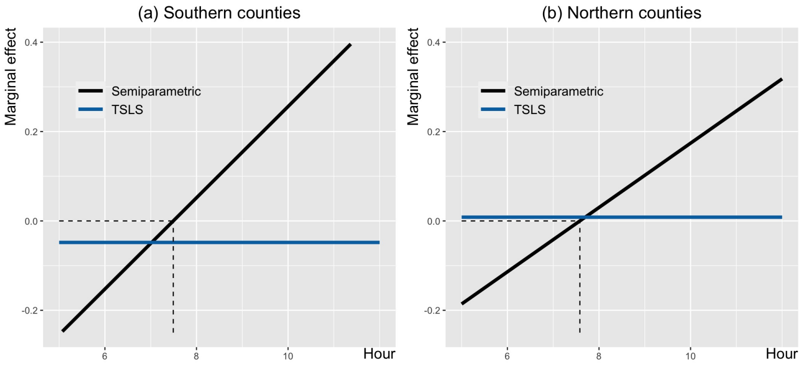


Figure: Estimated marginal effects (structural derivative) of sleep time on BMI (kg/m^2).

Empirical findings

- Marginal effects are increasing, indicating a nonlinear (U-shaped) structural function.
- Optimal sleep time is between 7 and 8 hours.
- The TSLS is not capable of demonstrating such results.

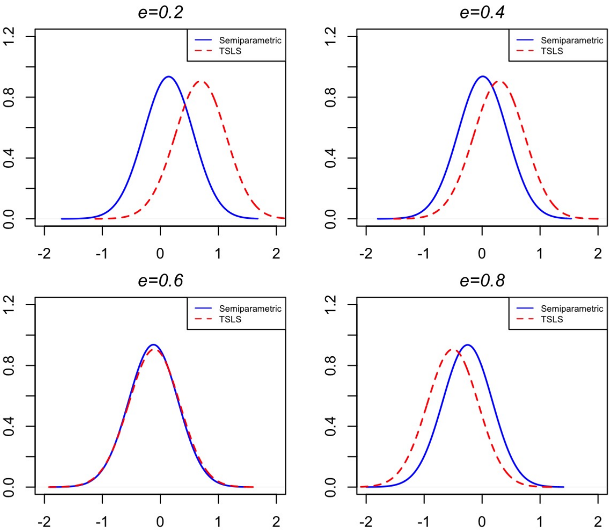
Simulations

- Semiparametric estimator vs TSLS
- Estimate the marginal effect

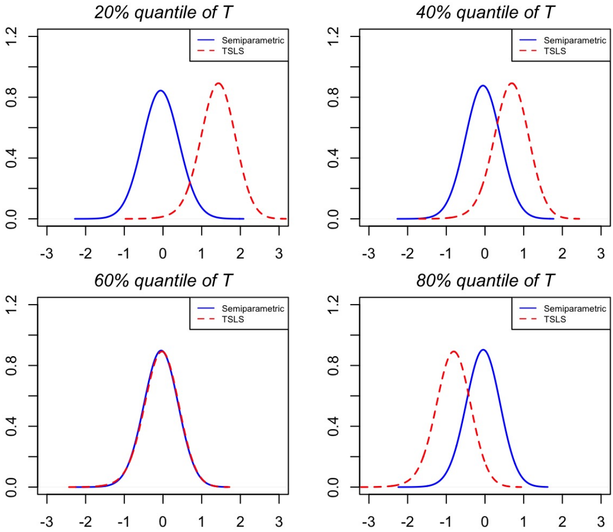
$$\text{SP: } \frac{\partial}{\partial t} g_{\hat{\gamma}}(t, \bar{r}, \varepsilon)$$
$$\text{TSLS: } \widehat{\text{WR}}$$

- Simulate distribution of the estimator centered at the true marginal effect
- Different values of T, ε

Nonseparable SF

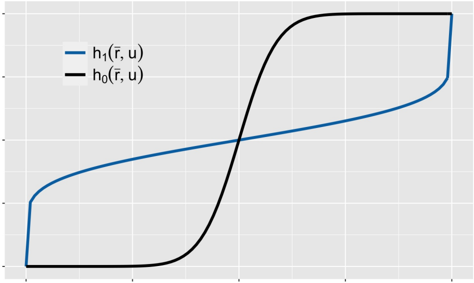


Nonlinear SF

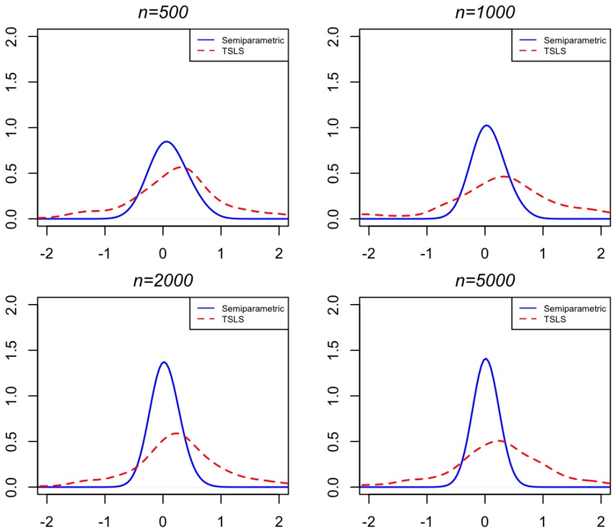


Linear SF: weakly identified TSLS

- $\hat{\gamma}$ can outperform TSLS even with a linear g^*
- weak identification in the difference in means
- $h_0(\bar{r}, \cdot)$ – quantile of $Beta(0.1, 0.1)$,
  $h_1(\bar{r}, \cdot)$ – quantile of $Beta(10, 10)$



Linear SF: weakly identified TSLS



Conclusion

- RD designs with a continuous treatment
- Nonlinear and nonseparable structural function
- Nonparametric identification
- Semiparametric estimation
- Future work
    - testing the identification assumptions
    - partial identification when some assumptions fail
    - extrapolation away from the cutoff
    - regression kink designs

Thank You!

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Sequencing approach

Prove that the identification equation pins down a unique structural function. Back

- If g satisfies the identification equation, then

$$g^{-1}(h_0(u), g^*(h_0(u), e)) = g^{-1}(h_1(u), g^*(h_1(u), e))$$

- Define $\pi(t) = h_1(h_0^{-1}(t))$, then

$$\begin{aligned} g^{-1}(t, g^*(t, e)) &= g^{-1}(\pi(t), g^*(\pi(t), e)) \\ &= g^{-1}(\pi(\pi(t)), g^*(\pi(\pi(t)), e)) \\ &= g^{-1}(\pi^m(t), g^*(\pi^m(t), e)) \end{aligned}$$

- The sequence $t, \pi(t), \pi(\pi(t)), \cdots, \pi^m(t), \cdots$ is convergent.
- $g^{-1}(t, g^*(t, e))$ does not depend on t.

Regularity conditions

- Distributions of Y, T, and R detail
- Complexity of the parametric model detail
- Kernel functions detail
- Bandwidth detail
- First-step conditional quantile estimators detail

Back

Proof

1. uniform convergence results on the LLR estimators
2. consistency of $\hat{\gamma}$
3. initial bound on the convergence rate of $\hat{\gamma}$
4. stochastic equicontinuity of the criterion function
5. linearization of the criterion function
6. asymptotic normality of the minimizer of the linearized criterion function
7. asymptotic normality of $\hat{\gamma}$

Back

1. The support of T does not vary with R except when crossing the cutoff $\bar{r}$, i.e., $\text{Supp}(T|R=r) = [t_0', t_0'']$ for $r < \bar{r}$ and $\text{Supp}(T|R=r) = [t_1', t_1'']$ for $r > \bar{r}$. The density functions $f_{T,R}^-$ and $f_{T,R}^+$ are bound away from zero.

2. The density functions $f_{T,R}^-$ and $f_{T,R}^+$ are twice continuously differentiable, and $\frac{\partial^2}{\partial T^2} f_{T,R}^-(t, \bar{r})$ and $\frac{\partial^2}{\partial T^2} f_{T,R}^+(t, \bar{r})$ are Lipschitz continuous with respect to t.

3. The support of Y, $\mathcal{Y}$, is compact. The conditional distribution functions $F_{Y|T,R}^-$ and $F_{Y|T,R}^+$ are three-times continuously differentiable over $\mathcal{Y} \times [t_0', t_0''] \times [r_0, \bar{r}]$ and $\mathcal{Y} \times [t_1', t_1''] \times [\bar{r}, r_1]$, respectively.

Back

The parametrization $\{g_\gamma(\cdot, \bar{r}, \cdot) : \gamma \in \Gamma\}$ satisfies the following conditions.

1. The parameter space Γ is compact.
2. The class of functions $\{T \mapsto g_\gamma(T + v, \bar{r}, e) : \gamma \in \Gamma, v \in (-1, 1), e \in \mathcal{E}\}$ is finite-dimensional.
3. The function $g_\gamma(t, \bar{r}, e)$ is twice continuously differentiable over $\gamma \in \Gamma$, $t \in [t_0', t_0''] \cup [t_1', t_1'']$, and $e \in \mathcal{E}$.
4. The gradient $\nabla_\gamma D_{\gamma^*, h^*}(e, u)$ is a vector of linearly independent functions of (e, u).

Back

1. The kernel functions k_T and k_R are (1) supported on $[-1, 1]$, (2) strictly greater than zero in the interior of the support, (3) of bounded variation, (4) continuously differentiable on $\mathbb{R}$.

2. The kernel function k_Y is (1) nonnegative and (2) integrable on $\mathbb{R}$ with $\int k_Y(y)dy = 1$ and satisfies (3) $\int yk_Y(y)dy = 0$.

Back

Bandwidth

The bandwidth b_1 and b_2 satisfy the following conditions:

1. $b_1 \asymp b_2$.
2. $(n \log n) b_1^6 = o(1)$.
3. $n b_1^{\frac{13}{3}+\epsilon} \to \infty$, for some sufficiently small $\epsilon > 0$.

In particular, the optimal bandwidth $n^{-1/5}$ is allowed.

Back

First-step conditional quantile estimators (1)

Monotonicity and smoothness: for every n sufficiently large, there exist $C > 0$ and deterministic and finite partitions $\mathcal{P}_0^n$ and $\mathcal{P}_1^n$ on $(0,1)$ such that

$$\mathbb{P}\left(\hat{h}_0(\bar{r},\cdot) \notin \mathcal{H}_0(\mathcal{P}_0^n)\right), \mathbb{P}\left(\hat{h}_1(\bar{r},\cdot) \notin \mathcal{H}_1(\mathcal{P}_1^n)\right) = O(\sqrt{b_1}),$$

where $\mathcal{H}_0(\mathcal{P}_0^n)$ includes functions that are strictly increasing with a three-times continuously differentiable inverse on each element of $\mathcal{P}_0^n$. $\mathcal{H}_1(\mathcal{P}_1^n)$ is defined analogously by replacing $\mathcal{P}_0^n$ with $\mathcal{P}_1^n$.

Back

First-step conditional quantile estimators (2)

Uniform Bahadur representation:

$$\hat{h}_0(\bar{r}, u) - h_0(\bar{r}, u) = b_1^2 \nu_0(u) + O_p(b_1^3)$$
$$+ \frac{1}{nb_1} \sum_{i=1}^{n} q_0(T_i, R_i; u) k_{Q,0} \left(\frac{R_i - \bar{r}}{b_1} \right) \mathbf{1}\{R_i < \bar{r}\} + o_p\left(1/\sqrt{nb_1}\right),$$

uniformly over $u \in (0, 1)$. The same for $\hat{h}_1$. The functions ν_0 and ν_1 are bounded. The functions q_0 and q_1 are (1) bounded, (2) centered, that is, $\mathbb{E}[q_0(T, R; u)|T, R] = \mathbb{E}[q_1(T, R; u)|T, R] = 0$, and (3) does not vary with n. The functions $k_{Q,0}$ and $k_{Q,1}$ are bounded.

Back

First-step conditional quantile estimators (3)

Uniform convergence rate:

$$
\begin{aligned}
\|\hat{h} - h\|_\infty &= \sup_{u \in (0,1)} |\hat{h}_0(\bar{r}, u) - h_0(\bar{r}, u)| \vee |\hat{h}_1(\bar{r}, u) - h_1(\bar{r}, u)| \\
&= O_p\left(\sqrt{\log n/(nb_1)} + b_1^2\right).
\end{aligned}
$$

Back