FRW

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April 13, 2019

Part I
FRW

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 sin^2\theta]d\phi^2$$

1 Close, open or flat

$$a^3(t)\int_0^\infty \frac{4\pi r^2}{\sqrt{1-Kr^2}}dr$$

- K=0

$$a^3(t)\int_0^\infty 4\pi r^2 dr = \infty$$

- K>0

if $1 - Kr^2 < 0$, FEW is meanless,

$$a^3(t)\int_0^1 \frac{4\pi r^2}{\sqrt{1-r^2}}dr = a^3(t)4\pi\int_0^{\frac{\pi}{2}} sin^2\theta d\theta$$

- K<0

$$a^3(t)\int_0^1 \frac{4\pi r^2}{\sqrt{1+r^2}}dr = a^3(t)4\pi\int_0^\infty sinh^2\theta d\theta = \infty$$

Comoving coordinate

2 Redshift

2.1 Classical method

For photons,

$$\frac{1}{a(t)}dt = \frac{1}{\sqrt{1-Kr^2}}dr = d\chi$$

Send $(t_1, t_1+\delta t_1)$, receive $(t_2, t_2+\delta t_2)$.

$$\int_t^{t+\delta t} \frac{1}{a(t)}dt = \frac{\delta t}{a(t)}$$

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_2}{a(t_2)}$$

$$\Rightarrow \nu_1 : \nu_2 = a(t_2) : a(t_1)$$

2.2 Quantum

$$ds^2 = a^2(t)[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 sin^2\theta]d\phi^2, dt^2 = a^2(t)d\eta^2$$

$$H = i\frac{\partial}{\partial\eta} = i\frac{1}{a(t)}\frac{\partial}{\partial t}$$

2.3 a,z

$$a = \frac{a_0}{1+z}$$

Assume t=0, $a = 0, z = \infty$ and $t = t_0, z = 0, a = 1$.
Because

$$H = \frac{\dot{a}}{a} \to dt = \frac{da}{aH} = -\frac{dz}{(1+z)H}$$

$$T = \int_0^t dt' = \int_0^\infty \frac{1}{(1+z')H(z')}dz'$$

3 Distance and volume

- Comoving distance $d_c = \chi$
- Proper distance $d_p = a(t)\chi$
- Angular diameter distance $d_A = \frac{s_p}{\theta} = a(t)r = \frac{r}{1+z}, r = f_K(\chi), d_A^c = \frac{d_A}{a} = r$

- Luminosity distance

$$d_L = \sqrt{\frac{L}{4\pi l}}$$

$$l = \frac{N_\gamma E_{obs}}{t_{obs} A} = \frac{N_\gamma E}{tA} \frac{1}{(1+z)^2}, L = \frac{N_\gamma E}{tA}$$

$$d_L = (1+z)r$$

4 FRW equation

$$\begin{aligned} H^2 + \frac{K}{a^2} &= \frac{8\pi G}{3}\rho \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \\ \dot{H} &= -4\pi G(\rho + P) + \frac{K}{a^2} \end{aligned}$$

Proof:

$$\begin{aligned} 2\dot{H}Ha &= \frac{8\pi G}{3}\dot{\rho}a + \frac{K}{a^2}\dot{a} \\ \dot{H} &= \frac{4\pi G}{3}\frac{\dot{\rho}}{H} + \frac{K}{a^2} \\ -4\pi G(\rho + P) + \frac{K}{a^2} &= \frac{4\pi G}{3}\frac{\dot{\rho}}{H} + \frac{K}{a^2} \end{aligned}$$

$$\Rightarrow \dot{\rho} = -3(\rho + P)H = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho = \rho_0 a^{-3(1+w)}$$

Matter w=0

Radiation w=1/3

vacuum_energy w=-1

More that one componets,

$$\rho = \sum \rho_i, p = \sum w_i \rho_i$$

and

$$\dot{\rho}_i/\rho_i = -3(1+w_i)\frac{\dot{a}}{a} \Rightarrow \rho = \rho_0 a^{-3(1+w)}$$

4.1 Critical density

$$\rho_c \equiv \frac{3H^2}{8\pi G}, \Omega_i = \frac{\rho_i}{\rho_c}$$

$$\rho = \rho_c + \frac{3K}{8\pi G a^2}$$

4.2 Dynamics

$$(\frac{\dot{a}}{a})^2 = \alpha^2 a^{-4} + \beta^2 a^{-3} - K a^{-2} + \frac{\Lambda}{3}$$

Suppose ($n \neq 0$)

$$\frac{\dot{a}}{a} = k a^{-\frac{n}{2}} \rightarrow \frac{2}{n} a^{\frac{n}{2}} = kt, \text{with a(0)=0}$$

if $n = 0$,

$$\frac{\dot{a}}{a} = \Lambda \Rightarrow a = (e^{\Lambda t} - 1)C$$

4.3 Cosmological parameters

$$\Omega_i = \frac{\rho_i}{\rho_{cri}}, \Omega_K = -\frac{3K}{8\pi G a^2}/\rho_{cri}$$

Then

$$\rho = \rho_{cri} + \frac{3K}{8\pi G a^2} \Rightarrow \sum_i \Omega_i + \Omega_K = 1$$

$$H^2 = H_o^2 \sqrt{\Omega_r (1+z)^4 + \Omega_m (1+z)^3 + \Omega_K (1+z)^2 + \Omega_\Lambda}$$