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Discrete Optimization

Robust scheduling of parallel machines with sequence-dependent set-up costs

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Abstract

In this paper we propose a robust approach for solving the scheduling problem of parallel machines with sequence-dependent set-up costs. In the literature, several mathematical models and solution methods have been proposed to solve such scheduling problems, but most of which are based on the strong assumption that input data are known in a deterministic way. In this paper, a fuzzy mathematical programming model is formulated by taking into account the uncertainty in processing times to provide the optimal solution as a trade-off between total set-up cost and robustness in demand satisfaction. The proposed approach requires the solution of a non-linear mixed integer programming (NLMIP), that can be formulated as an equivalent mixed integer linear programming (MILP) model. The resulting MILP model in real applications could be intractable due to its NP-hardness. Therefore, we propose a solution method technique, based on the solution of an approximated model, whose dimension is remarkably reduced with respect to the original counterpart. Numerical experiments conducted on the basis of data taken from a real application show that the average deviation of the reduced model solution over the optimum is less than 1.5%.

Keywords: Scheduling; Sequence-dependent set-up; Fuzzy programming; Goal programming

1. Introduction

In the planning of manufacturing production systems, generally, set-up includes the work to prepare machines or position works in the material-processing phase. The production problems that explicitly take into account set-up considerations may be grouped into two different classes. In the first one, set-up time (cost) depends only on the job to be processed and the problem is classified, from the set-up point of view, as sequence-independent. In the latter, set-up depends on both the job to be processed and the immediately preceding one. In this case, the problem is classified as sequence dependent.

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Scheduling problems requiring explicit treatment of set-up can be found in several shop environments. For a complete research review on scheduling involving set-up consideration see [1,17]. In some applications, the set-up costs are directly proportional to set-up times and, consequently, schedules optimal with respect to set-up times are also optimal with respect to set-up costs. This is true when set-up operations are related only to machine idle time. In many other cases, particularly when set-up operations require high-skilled labour, set-up costs are relatively high while set-up time is relatively less. The problem studied in this paper regards the scheduling of independent jobs on identical parallel machines with sequence-dependent set-up costs, in order to minimise the total set-up cost. In the scheduling literature, several authors have addressed such NPhard scheduling problems, investigating the use of heuristic solution able to give a good solution in reasonable computation time [20,21]. In particular, Sumischrast et al. [20,21] develop an efficient heuristic, realistic sized problems characterising the industrial applications motivating the study, by exploiting a mathematical model from Dearing and Henderson [4]. The proposed models are based on the assumption that input data are deterministically known. In this way, if some input data change with respect to those used in the computations, the solution given by the scheduler may be not optimal or even feasible anymore and consequently the application to real cases may be compromised. This is one of the reasons for which it is important to consider uncertainty in the modelling problem phase.

There are two major manners to represent uncertainty: random numbers and fuzzy numbers. In order to choice how to incorporate uncertainty in the optimisation problem some knowledge of the uncertainty is required. In the stochastic approach uncertain data are modelled by specifying the probability distributions, for example inferred from historical data. The literature of stochastic scheduling addresses the identical parallel machines problem in several research works, most of them focused on demonstrating the optimality of priority-index policies. In particular, the main performance objectives investigated has been total expected flow-time minimisation as well as expected makespan minimisation. In particular, the shortest expected processing time rule (SEPT) has been shown to be optimal for the flow-time objective in the following cases: all the jobs processing time distributions are exponential [10]; all the jobs have a common general processing time distribution with a no-decreasing hazard rate function [23]; job processing times are stochastically ordered [24]. For the expected makespan objective, the longer expected processing times policy (LEPT) has been shown to be optimal if the following conditions are verified: the processing times distributions are exponential [3]; jobs have a common processing time distribution with a non-increasing hazard rate function [23]. If historical data are not available (e.g. the plants often diversify their products or introduce in the production plans new product typologies), an alternative way to model imprecision is represented by the fuzzy approach. In these cases, the production data, on which the scheduling models are based, are characterised by imprecision and expressed in linguistic terms. Fuzzy sets theory [19] provides a conceptual framework which may efficiently be used for dealing with situations characterised by imprecision and provide a very efficient framework to reduce scheduling computational complexity with respect to the same problem formulated in a probabilistic way. We should make clear here that such an imprecision is due to subjective and qualitative evaluations, rather than the effect of uncontrollable events.

Zimmermann [26] provides an extensive coverage of the theoretical and applied approaches to fuzzy sets. Fuzzy sets concepts enrich traditional Operational Research in various applications. In particular, fuzzy sets theory has been exploited in the scheduling applications to model flexible constraints, and the uncertainty in the definition of time parameters in flow shop, job shop and project problems (see [10,11,13–15,18,25]).

In this paper an identical parallel-machine scheduling problem with sequence-dependent set-up costs under the hypothesis of fuzzy processing times knowledge is analysed. In the considered shop environment, since processing requirement of each different lot is expressed in terms of product demand, each lot can be split arbitrarily into sub-lots and processed independently on machines. Therefore, the uncertain processing times are modelled through uncertain machine production speeds. In Section 2, the scheduling problem is described and modelled through an integer linear programming (ILP) model. Section 3 illustrates the mathematical features of the corresponding fuzzy scheduling problem. In particular, the necessity degree measure is used in order to formulate a *minimum risk approach*, where the optimal solution is a trade-off between total set-up cost and robustness in demand satisfaction. Since the minimum risk approach requires to solve a non-linear mixed integer programming (NLMIP) model, in Section 4, we illustrate how to build an equivalent formulation in which we avoid the non-linearity presented in the preceding formulation. The resulting mixed integer linear programming (MILP) model in real applications could be intractable due to its NP-hardness. Therefore, we propose in Section 5 a specific technique, on the basis of which we are able to reduce the dimension of the model, and, consequently, to increase the number of real applications solvable. From the approximated model, a feasible solution of the original model can easily be found. Numerical experiments conducted on the basis of data taken from a real application show the effectiveness and efficiency of the proposed approach. The results are reported in Section 6. Section 7 is devoted to the conclusion of this work.

2. Problem description

The problem under consideration deals with the scheduling of independent lots on parallel identical machines over an assigned planning horizon in order to minimise the total set-up cost. In the production system, it is supposed that different product typologies may be processed. Each production type requires a single machining operation and pre-emption is not allowed. The changeover required between any two distinct product types results in sequence-dependent set-up cost with no machine idle time. During the fixed planning horizon, for each product typology, the relative due dates may not be unique. In the following, it is supposed that the total number of machines is referred to as M, the number of production periods as T and the total number of different product typologies as N. Under the stated previous assumptions, the problem can modelled as an integer linear programming as stated below.

Indexes, constants, variables

t, ℓ	planning period indexes ($t, \ell = 1, \ldots, T$)
i, j	product type indexes ($i, j = 1, \ldots, N$)
X_{it}	number of machines assigned to the production of part type i in period t
Y_{ijt}	number of machines switched from part type i to part type j in period t
S_{ij}	cost to switch from product i to j with $S_{ii} = 0$
a_i	production rate of product i in number of machined units per period
Ω_i	number of machines initially set up to produce product i with $\sum_{i=1,\ldots,N} \Omega_i = M$
$d_{i\ell}$	demand for product i—in machine periods—in period ℓ

$$\text{Min} \quad \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} S_{ij} \sum_{t=1}^{T} Y_{ijt} \tag{1}$$

$$\text{Subject to :} \quad X_{i1} - \sum_{\substack{j=1 \\ j \neq i}}^{N} Y_{ij1} + \sum_{\substack{j=1 \\ j \neq i}}^{N} Y_{ji1} = \Omega_i \quad i = 1, \ldots, N, \tag{2}$$

$$X_{it-1} - \sum_{\substack{j=1 \\ j \neq i}}^{N} Y_{ijt} + \sum_{\substack{j=1 \\ j \neq i}}^{N} Y_{jit} = X_{it} \quad i = 1, \ldots, N; \; t = 2, \ldots, T, \tag{3}$$

$$a_i \sum_{\ell=1}^{t} X_{i\ell} \geqslant \sum_{\ell=1}^{t} d_{i\ell} \quad i = 1, \ldots, N;\ t = 1, \ldots, T, \tag{4}$$

$$X_{it} \geqslant 0,\ \text{integer} \quad i = 1, \ldots, N;\ t = 1, \ldots, T, \tag{5}$$

$$Y_{ijt} \geqslant 0,\ \text{integer} \quad i = 1, \ldots, N;\ j = 1, \ldots, N;\ t = 1, \ldots, T. \tag{6}$$

The constraints (2) state the relationships between the first planning period decision variables and the initial assignment (i.e. Ω_i with $i = 1, \ldots, N$). The constraints (3) represent the relationship between the number of machines assigned to product type i and the number of changeovers. The constraints (4) state that for each period the demand of product type i has to be satisfied without backlogging. Finally, (5) and (6) are the integer constraints.

Since all input data are deterministic, the model (1)–(6) is useful in a production environment where there are no sources of uncertainty. However, in the actual industrial applications, the input data are estimated and they often have a degree of uncertainty such that the solution given by a deterministic scheduler may not be optimal or even feasible any more. For this reason, it is important to take into account uncertainty in the modelling problem phase. In general, in the solution of industrial scheduling problems one of the main sources of uncertainty is related to the definition of processing times. Often, it is difficult to estimate a priori such time parameters since they might vary according to a very large number of factors. For these reasons, for example if the production of new product type must be scheduled in the system, it is necessary to take into account uncertainty about processing times in the modelling phase. In the mathematical model (1)–(6) each lot is expressed in terms of product demand. Since each lot can be split arbitrarily and processed independently on the machines, the uncertainty of processing times is modelled considering the machine production rates a_i as uncertain data. Since the assumed hypothesis (e.g. the plants often diversify their products and various new product types are repeatedly required to the manufacturer) the machine production rates can be modelled through a set of possible values. For each product type, the set lower and upper bounds are, respectively, the minimum acceptable value and the maximum feasible machine production rate. On the basis of the above considerations, the scheduling model (1)–(6) has been reformulated in order to take into account of uncertainty and machine production speeds have been expressed in terms of fuzzy numbers. Orders has been considered as already negotiated and consequently demands and due dates are well-defined parameters. In the next section it will be described how the previous scheduling problem has been reformulated as fuzzy mathematical model.

3. The fuzzy mathematical programming problem

The notion of fuzzy set is a generalisation of the classical set notion. If in the classical notion, the membership of an element to a set is true or false, in fuzzy sets theory the membership of an element to the set is mapped by the definition of a membership function. Formally, a fuzzy set is characterised by a membership function, valued in the range $[0, 1]$, mapping the degree with which each considered element belongs to the set. For reading on fuzzy sets theory, refer also to [26].

The possibility offered by fuzzy sets theory have been exploited to represent uncertainty in the mathematical model formulated in the previous section. The ambiguity about the estimated machine production rates can be modelled representing uncertain data values in terms of fuzzy numbers. In the considered case, the machine production rate of product i is supposed to be described by the linguist expression '*the production rate of product i is equal to about m_i product units per period*'. In particular, it is supposed that the decision-maker is able to define three values of the machine production rate for each product. The three values are the minimum acceptable value, the most possible value and the maximum feasible one, respectively referred to as h, m and u. By this definition, it is possible to represent the machine production

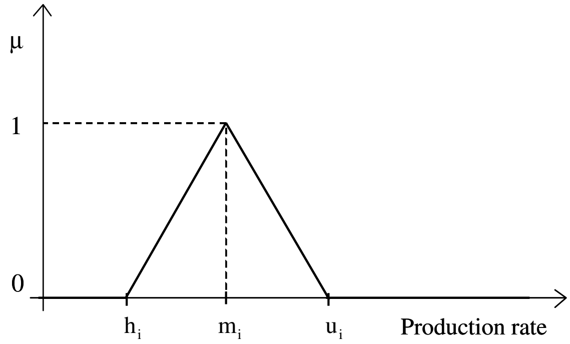

Fig. 1. The triangular fuzzy number $\langle h_i, m_i, u_i \rangle$.

rate of each product by the triangular fuzzy number A_i (Fig. 1). As shown in the figure, the support range of A_i is the interval $[h_i, u_i]$ and the membership function assumes value equal to one in correspondence of the value m_i. The membership value $\mu_{Ai}(r)$ represents the possibility degree of the event 'the machine production rate of product type i is equal to r'. By these hypotheses, it is possible to represent the coefficients a_i of the previous mathematical model as a possibilistic variable and $\mu_{Ai}(r)$ as the relative possibility distribution. By this definition, the function $f_{it}(X_{i1}, \ldots, X_{it}) = \sum_{\ell=1}^{t} a_i X_{i\ell}$ has an ambiguous value too. By applying the Zadeh's extension principle to the possibilistic linear functions of constraints (4), it is possible to define the fuzzy number $F_{it}(X_{i1}, \ldots, X_{it})$, for each function $f_{it}(X_{i1}, \ldots, X_{it})$, as reported in (7).

$$F_{it}(X_{i1}, \ldots, X_{it}) = \left\langle \sum_{\ell=1}^{t} h_i X_{i\ell}, \sum_{\ell=1}^{t} m_i X_{i\ell}, \sum_{\ell=1}^{t} u_i X_{i\ell} \right\rangle. \tag{7}$$

The representation of the coefficients a_i as possibilistic variables implies the non-correct definition, in the mathematical traditional sense, of constraints (4) and make ill-posed the overall mathematical model (1)–(6). In the fuzzy programming approach [12], such ill-posed problem is transformed to a conventional mathematical programming model. The model uncertainty is managed by means of various well-known interpretations of the fuzzy inequality relations. In the proposed approach, two well-known relations based on the Dubois and Prade possibility and necessity measures have been used. Let be μ_A the possibility distribution μ_A of a possibilistic variable α. Possibility and necessity measures of the event that α is in a fuzzy sets B are defined as follows:

$$\Pi_A(B) = \sup_r \min(\mu_A(r), \mu_B(r)), \tag{8}$$

$$N_A(B) = \inf_r \max(1 - \mu_A(r), \mu_B(r)). \tag{9}$$

The measures $\Pi_A(B)$ and $N_A(B)$ respectively denote to what extent it is possible and certain that the possibilistic variable α is in the fuzzy set B. If the fuzzy set A has to be compared with a crisp value g, it is possible to exploit the defined measures, by considering B as a crisp set of real numbers not lower than g. The membership function μ_B is equal to 1 for values greater or equal to g, equal to 0 otherwise. In this case, the possibility and certainty degrees, that the possibilistic variable α is not smaller than a given real number g, are

$$\mathrm{Pos}(\alpha \geqslant g) = \prod_A([g, +\infty)) = \sup\{\mu_A(r) | r \geqslant g\}, \tag{10}$$

$$\mathrm{Nes}(\alpha \geqslant g) = N_A([g, +\infty)) = 1 - \sup\{\mu_A(r) | r < g\}. \tag{11}$$

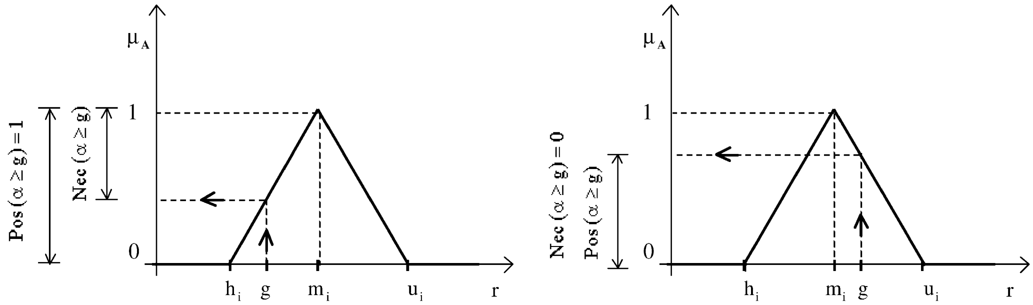

Fig. 2. Possibility and necessity degrees of $\alpha \geqslant g$.

A graphical representation of the relation $\mathrm{Pos}(\alpha \geqslant g)$ ($\mathrm{Nes}(\alpha \geqslant g)$) is showed in Fig. 2. Since the possibilistic linear function value $f_{it}(X_{i1},\ldots,X_{it})$ is a possibilistic variable restricted by $F_{it}(X_{i1},\ldots,X_{it})$, it is possible to substitute $f_{it}(X_{i1},\ldots,X_{it})$ for α and $F_{it}(X_{i1},\ldots,X_{it})$ for A in (10) and (11). By this assumption, if R_{it} represents the minimum certainty degree with which the decision-maker states that the demand of product type i for period t has to be satisfied with no backlogging, the constraints (4) can be treated as follows:

$$\mathrm{Nes}\left(\sum_{\ell=1}^{t} a_i X_{i\ell} \geqslant D_{it}\right) \geqslant R_{it}, \quad \text{with } D_{it} \text{ equal to } \sum_{\ell=1}^{t} d_{i\ell}. \tag{12}$$

From (7) the fuzzy number $F_{it}(X_{i1},\ldots,X_{it})$ restricting $f_{it}(X_{i1},\ldots,X_{it}) = a_i \sum_{\ell=1}^{t} X_{i\ell}$ is a triangular fuzzy number $\langle \sum_{\ell=1}^{t} h_i X_{i\ell}, \sum_{\ell=1}^{t} m_i X_{i\ell}, \sum_{\ell=1}^{t} u_i X_{i\ell} \rangle$. For this fuzzy number, the necessity degree $\mathrm{Nes}(\sum_{\ell=1}^{t} a_i X_{i\ell} \geqslant D_{it})$ is depicted in Fig. 3. As shown in figure, in order to satisfy the constraint (12) the value q has to be greater than D_{it}. Since the triangles $\triangle$ECH and $\triangle$EFG are similar we obtain:

$$q = \sum_{\ell=1}^{t} m_i X_{i\ell} - R_{it}\left(\sum_{\ell=1}^{t} m_i X_{i\ell} - \sum_{\ell=1}^{t} h_i X_{i\ell}\right). \tag{13}$$

The equivalent conditions to (12) are:

$$\sum_{\ell=1}^{t} m_i X_{i\ell} - R_{it}\left(\sum_{\ell=1}^{t} m_i X_{i\ell} - \sum_{\ell=1}^{t} h_i X_{i\ell}\right) \geqslant D_{it}, \quad i = 1,\ldots N,\ t = 1,\ldots,T. \tag{14}$$

The constraints (14) state that the request of a not null certainty degree is equivalent to estimating a crisp machine production speed value lower than m_i which is equal to $m_i - R_{it}(m_i - h_i)$. This condition implies

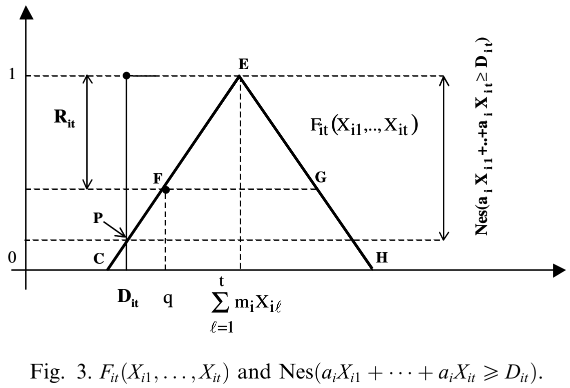

Fig. 3. $F_{it}(X_{i1},\ldots,X_{it})$ and $\mathrm{Nes}(a_i X_{i1} + \cdots + a_i X_{it} \geqslant D_{it})$.

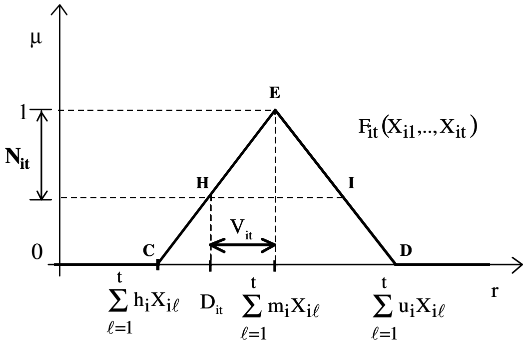

Fig. 4. The triangles △EHI and △ECD.

that the total set-up cost is a non-decreasing function of the certainty degrees R_{it}. In fact, in order to satisfy the demand of the product i without backlogging and with a machine production rate lower than m_i, the number of machines assigned to the product i cannot decrease with respect to the deterministic optimal solution. Consequently, the total set-up cost cannot decrease either. The constraints (14) restrict the variables X_{it} and the overall feasible region stronger than the constraints (4). Since not all the possible combinations of R_{it} values could be feasible, the calculation of the certainty degrees R_{it} could result in a further combinatorial optimisation problem. On the basis of the above considerations, in the proposed method a minimum risk approach to the scheduling problem is taken. The optimal solution is obtained as a trade-off between the minimum total set-up cost and the maximum global satisfaction level for due date constraints. In this formulation, each necessity degree measure (N_{it}) has to be considered as a decision variable. The necessity degree measures (N_{it}) are determined as reported in (15), by considering the similarity of the triangles △EHI and △ECD (Fig. 4). The maximisation of the global satisfaction level for due-date constraints is a multiple-criteria satisfaction problem. In this sense, fuzzy optimisation literature suggests the max–min framework, in which the global satisfaction level for a solution to a set of fuzzy constraints is the level of satisfaction of the least satisfied constraints [2,9]. Generally, the max–min approach is limited by a lack of discrimination respect to the satisfaction levels greatest than the least one. Indeed, the min based ordering relation tends to equalise the satisfaction level of solutions and the Pareto-optimality is not ensured [4–8]. In the considered case this drawback is overcome by the total set-up cost optimisation, which represents the discrimination rule among solutions characterised by the same minimum necessity degree $N_{\min}$

$$N_{it} = \frac{m_i \sum_{\ell=1}^{t} X_{i\ell} - D_{it}}{m_i \sum_{\ell=1}^{t} X_{i\ell} - h_i \sum_{\ell=1}^{t} X_{i\ell}} \quad i = 1, \dots N,\ t = 1, \dots, T. \tag{15}$$

The minimisation of the total set-up cost and the maximisation of the minimum necessity degree ($N_{\min}$) are conflicting objectives; in this way, the minimum risk approach results in the formulation of the following multi-objective programming mathematical model.

$$\text{Max} \qquad z_1(X) = N_{\min} \tag{16}$$

$$\text{Max} \qquad z_2(Y) = \left(-\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} S_{ij} \sum_{t=1}^{T} Y_{ijt} \right) \tag{17}$$

$$\text{Subject to:} \quad N_{\min} \leqslant N_{it} \quad i = 1, \ldots, N; \; t = 1, \ldots, T, \tag{18}$$

$$m_i \sum_{\ell=1}^{t} X_{i\ell} - N_{it}\left(m_i \sum_{\ell=1}^{t} X_{i\ell} - h_i \sum_{\ell=1}^{t} X_{i\ell}\right) = D_{it} \quad i = 1, \ldots, N; \; t = 1, \ldots, T, \tag{19}$$

$$0 \leqslant N_{\min} \leqslant 1, \tag{20}$$

$$(X, Y) \in C_D. \tag{21}$$

The symbol C_D denotes the sets of constraints (2), (3), (5) and (6) of the deterministic mathematical programming model. The first step in the solution of the model (16)–(20), is the normalisation of the two goals. Since the necessity degree measures are valued in the interval $[0, 1]$, a zero–one normalisation technique has been applied to the total set-up cost objective function, leading to the following revised algebraic formulation:

$$f_S(Y) = 1 - \frac{\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} S_{ij} \sum_{t=1}^{T} Y_{ijt} - S_{\min}}{S_{\min} - S_{\max}}. \tag{22}$$

The $S_{\min}(S_{\max})$ value denotes the best (the worst) value of the total set-up cost obtained by the minimisation (maximisation) of the objective function (1) subject to the constraints (2)–(6). It is assumed that for each product typology i, the relative machine production rate is equal to the most possible value m_i. Therefore, the objective function is defined as $z = (f_S(Y), N_{\min})$ and the target values are equal to one for both the two objectives. Accordingly to the weighted goal-programming approach [22], the following mathematical problem is formulated.

$$\min \quad \sum_{k=1}^{2} (b_k d_k^+ + c_k d_k^-) \tag{23}$$

$$\text{Subject to:} \quad f_S(Y) + d_1^- - d_1^+ = 1, \tag{24}$$

$$N_{\min} + d_2^- - d_2^+ = 1, \tag{25}$$

$$(X, Y) \in C_D \cup C_F. \tag{26}$$

The symbol C_F denotes the set of constraints (18)–(20), $d_k^+(d_k^-)$ represents the negative (positive) deviation from target value and $b_k(c_k)$ is the relative positive weight. In particular, since both $N_{\min}$ and $f_S(Y)$ values belong to the $[0, 1]$ interval, in all the feasible solutions of the problem (23)–(26) it results $d_1^+ = d_2^+ = 0$.

In the proposed approach, the robust scheduling model with sequence-dependent set-up costs may be formulated as represented by (23)–(26). Due to the non-linearity of constraints (19), the overall model is a NLMIP model. In the next section, we show how to transform it in an equivalent MILP formulation.

4. An equivalent formulation

The necessity degree functions N_{it} are non-linear in the Xs variables. They perform a mapping between the production level of product i in period t and real values in $[0, 1]$ interval. The non-linearity can be dealt with by approximating the non-linear functions by piecewise linear functions and then by modelling these in a mixed integer framework by introducing new 0–1 variables [16]. The particular mathematical feature of the model (23)–(26) makes possible to formulate a piecewise linear representation for the functions N_{it}. Such representation is exploited in the following theorem in order to demonstrate that the NLMIP model (23)–(26) is equivalent to a MILP model.

Theorem 1. *The MNPL problem* (23)–(26) *and the following problem are equivalent:*

$$\min \quad \sum_{k=1}^{2}(b_k d_k^+ + c_k d_k^-) \tag{27}$$

$$\text{Subject to:} \quad f_S(Y) + d_1^- - d_1^+ = 1, \tag{28}$$

$$N_{\min} + d_2^- - d_2^+ = 1, \tag{29}$$

$$n_{it}^k + \gamma_{it}^k \sum_{\ell=1}^{t} X_{i\ell} = \delta_{it}^k \quad i = 1,\dots,N;\ t = 1,\dots,T;\ k = 1,\dots,Q_{it}, \tag{30}$$

$$N_{\min} \leqslant n_{it}^k \quad i = 1,\dots,N;\ t = 1,\dots,T;\ k = 1,\dots,Q_{it}, \tag{31}$$

$$0 \leqslant n_{it}^k \leqslant 1 \quad i = 1,\dots,N;\ t = 1,\dots,T;\ k = 1,\dots,Q_{it}, \tag{32}$$

$$(X, Y) \in C_D. \tag{33}$$

Proof. Given the triangular fuzzy number $A_i = \langle h_i, m_i, u_i \rangle$, let w_i^{L} denote its left spread, whilst let V_{it} represent the most possible value of the inventory level, for product i, at the end of the period t, once assigned a feasible production planning.

$$V_{it} = m_i \sum_{\ell=1}^{t} X_{i\ell} - D_{it}, \quad i = 1,\dots,N;\ t = 1,\dots,T, \tag{34}$$

$$w_i^{\mathrm{L}} = m_i - h_i \quad i = 1,\dots,N;\ t = 1,\dots,T. \tag{35}$$

The necessity degree N_{it} can be expressed as a function of V_{it}:

$$N_{it} = \begin{cases} 0 & \text{if } V_{it} < 0, \\ \frac{m_i}{w_i^{\mathrm{L}}}\left(\frac{V_{it}}{V_{it}+D_{it}}\right) & \text{if } 0 \leqslant V_{it} \leqslant VM_{it}, \\ 1 & V_{it} > VM_{it}, \end{cases} \tag{36}$$

where,

$$\frac{m_i}{w_i^{\mathrm{L}}}\left(\frac{VM_{it}}{VM_{it}+D_{it}}\right) = 1 \iff VM_{it} = \frac{w_i^{\mathrm{L}} D_{it}}{(m_i - w_i^{\mathrm{L}})} \quad i = 1,\dots,N;\ t = 1,\dots,T. \tag{37}$$

Since the Xs are integer, $V_{it}(X_{i1},\dots,X_{it})$ is a linear function from $\aleph^t$ into the discrete set $S \subset \Re$. Consequently, given i and t, the domain of $N_{it}(V_{it})$ in $[0, VM_{it}]$ is a discrete and finite set consisting of Q_{it} elements, i.e. $(V_{it}^1, V_{it}^2, \dots, V_{it}^{Q_{it}})$, as well as,

$$N_{it}^k = N_{it}(V_{it}^k) \quad \forall k, \text{ with } 1 \leqslant k \leqslant Q_{it} \tag{38}$$

are the Q_{it} function values corresponding to the $V_{it}^k (k = 1,\dots,Q_{it})$ elements in the domain (see Fig. 5). The domain $N_{it}(V_{it})$ can be represented as a piecewise linear function of V_{it} as follows,

$$N_{it} = \begin{cases} 0 & \text{if } V_{it} < 0, \\ \min_k \{n_{it}^k | 2 \leqslant k \leqslant Q_{it}\} & \text{if } 0 \leqslant V_{it} \leqslant VM_{it}, \quad i = 1,\dots,N;\ t = 1,\dots,T, \\ 1 & \text{if } V_{it} > VM_{it}, \end{cases} \tag{39}$$

where n_{it}^k is defined as the line passing through the points (V_{it}^k, N_{it}^k) and $(V_{it}^{k-1}, N_{it}^{k-1})$.

$$n_{it}^k = \beta_{it}^k V_{it} + \alpha_{it}^k \quad k = 2,\dots,Q_{it};\ i = 1,\dots,N;\ t = 1,\dots,T, \tag{40}$$

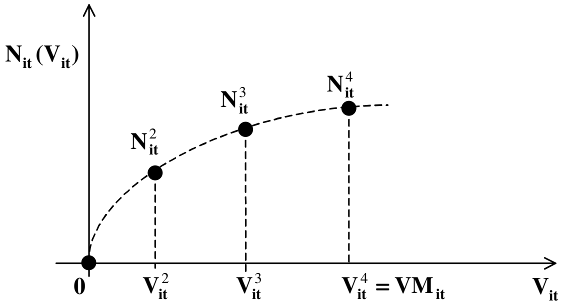

Fig. 5. The $N_{it}(V_{it})$ function.

where

$$\beta_{it}^k = \frac{N_{it}^k - N_{it}^{k-1}}{V_{it}^k - V_{it}^{k-1}}, \tag{41}$$

$$\alpha_{it}^k = \frac{N_{it}^{k-1} V_{it}^k - N_{it}^k V_{it}^{k-1}}{V_{it}^k - V_{it}^{k-1}}. \tag{42}$$

On the basis of the piecewise linear representations of the N_{it} functions ($i = 1, \ldots, N;\ t = 1, \ldots, T$) given in (39), $N_{\min}$ can be redefined as follows:

$$N_{\min} = \min \left\{ n_{it}^k | i = 1, \ldots, N;\ t = 1, \ldots, T;\ k = 2, \ldots, Q_{it} \right\}. \tag{43}$$

By substituting the expression (34) in (40) results in

$$n_{it}^k = \beta_{it}^k \left(\sum_{\ell=1}^{t} m_i X_{it} - D_{it} \right) + \alpha_{it}^k \quad i = 1, \ldots, N;\ t = 1, \ldots, T;\ k = 2, \ldots, Q_{it}, \tag{44}$$

$$\gamma_{it}^k = -m_i \beta_{it}^k \quad i = 1, \ldots, N;\ t = 1, \ldots, T;\ k = 1, \ldots, Q_{it}, \tag{45}$$

$$\delta_{it}^k = \alpha_{it}^k - \beta_{it}^k D_{it} \quad i = 1, \ldots, N;\ t = 1, \ldots, T;\ k = 1, \ldots, Q_{it}. \tag{46}$$

On the basis of the $N_{\min}$ definition (43) and by setting the coefficients in constraints (30) as reported in (45) and (46), the equivalence between the MILP problem (27)–(33) and the NLMIP problem (23)–(26) is demonstrated. □

5. An approximated formulation

The scheduling problem described in Section 1 is typically encountered in the textile and fibreglass industries. These industries have a rather limited set of major products, typically four to ten. By considering the structure of the problem (1)–(6), we can apply the approach [20] to obtain a significant reduction of the number of integer variables. This approach makes possible to solve realistic size problems. In the model (27)–(33), the problem size is a function of the number of products N, the planning horizon length T as well as the sum Q of Q_{it}

$$Q = \sum_{i=1}^{N} \sum_{t=1}^{T} Q_{it}. \tag{47}$$

By adopting the robust approach (27)–(33), if Q is a significant number, then realistic sized problems (i.e., when $N=4$; $T=6$) might not be solved in a reasonable amount of time. For this reason, we propose in the sequel a relaxation of the model (27)–(33), in order to reduce the number of constraints with the aim to increase the size of problems tractable with this approach. The idea is to consider a piecewise linear approximation, say G_{it}, of N_{it}. Let Ω_{it} denote a sub-set of $\{V_{it}^1, V_{it}^2, \ldots, V_{it}^{Q_{it}}\}$, the G_{it} functions are defined as follows:

$$G_{it} = \begin{cases} 0 & \text{if } V_{it} < 0, \\ \min_k \{g_{it}^h | 2 \leqslant h \leqslant |\Omega_{it}|\} & \text{if } 0 \leqslant V_{it} \leqslant VM_{it}, \\ 1 & \text{if } V_{it} > VM_{it}, \end{cases} \tag{48}$$

$$\Omega_{it} = \{V_{it}^1, V_{it}^{Q_{it}}\} \cup S_{it} \quad \text{with } S_{it} \subseteq \{V_{it}^2, \ldots, V_{it}^{Q_{it}-1}\}, \tag{49}$$

where g_{it}^h is defined as the line passing through two consecutive points belonging to Ω_{it}. Choosing $|\Omega_{it}|$ large enough, the G_{it} function approximates the corresponding N_{it} function arbitrarily closely (Figs. 6 and 7). On the other hand, a significant large value of Ω (see (50)) might result in the formulation of a problem computationally intractable.

$$\Omega = \sum_{i=1}^{N} \sum_{t=1}^{T} |\Omega_{it}|. \tag{50}$$

It is worth evaluating the G_{it} functions in terms of approximation errors as well as computational efficiency, through the Ω value. Let the propositions PG and PN be defined as follows:

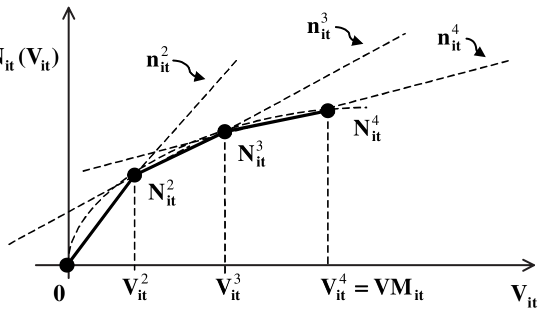

Fig. 6. The $n_{it}(V_{it})$ linear functions, $Q_{it}=4$.

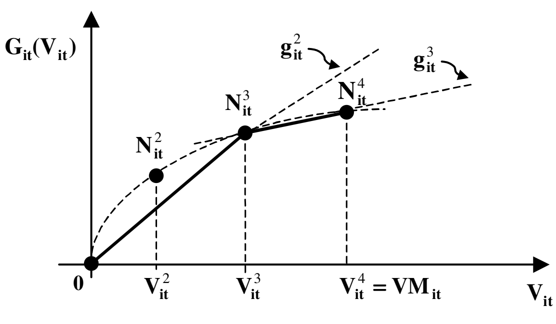

Fig. 7. The $G_{it}(V_{it})$ function with $\Omega_{it} = \{V_{it}^1, V_{it}^3, V_{it}^4\}$.

PG = 'maximizing the minimum value of G_{it}'
PN = 'maximizing the minimum of N_{it}'.

In the following, we illustrate how *to measure* the equivalence between PG and PN. In particular, the goal is to determine the maximum percentage error made by taking PG instead of PN. For the sake of simplicity, in the remainder, each product i is considered as having only one due date over the planning horizon T. Consequently, the notation can be simplified by eliminating the subscript t in all occurring cases.

Definition 1. Two distinct products i and j $(i = 1, \ldots, N; j = 1, \ldots, N)$ are defined as similar products if

$$\forall V : N_i(V) = N_j(V). \tag{51}$$

Theorem 2. *If product i is not similar to product j, it results in*

$$(VM_i \leqslant VM_j) \iff N_j(V) < N_i(V) \quad \forall V \in]0; VM_i[. \tag{52}$$

Proof. On the basis of expression (36) it results in

$$N_i(V) = \frac{m_i}{w_i}\left(\frac{V}{V + D_i}\right) \quad \forall V \in]0; VM_i[, \tag{53}$$

$$N_j(V) = \frac{m_j}{w_j}\left(\frac{V}{V + D_j}\right) \quad \forall V \in]0; VM_i[. \tag{54}$$

The necessity degrees difference $[N_i(V) - N_j(V)]$ assumes negative values for V values greater than $Q_{\max}$ (see (55)).

$$N_i(V) - N_j(V) = \frac{m_i}{w_i}\left(\frac{V}{V + D_i}\right) - \frac{m_j}{w_j}\left(\frac{V}{V + D_j}\right) \leqslant 0 \iff V \geqslant \frac{m_i w_j D_j - m_j w_i D_i}{m_j w_i - m_i w_j} = Q_{\max}. \tag{55}$$

However as reported in (56) $Q_{\max}$ is not lower than VM_{it}.

$$Q_{\max} - VM_i \geqslant 0 \iff \frac{m_i w_j D_j - m_j w_i D_i}{m_j w_i - m_i w_j} - \frac{w_i D_i}{m_i - w_i} \geqslant 0 \iff \frac{w_i D_i}{m_i - w_i} \leqslant \frac{w_j D_j}{m_j - w_j} \iff VM_i \leqslant VM_j. \tag{56}$$

Therefore: $VM_i \leqslant VM_j \iff N_j(V) < N_i(V)\ \forall V \in]0; VM_i[$. $\square$

The equivalence between PG and PN is valid if the definition of $G_i (i = 1, \ldots, N)$ functions fulfills the following two properties.

Property 1. *PG and PN are equivalent if the following conditions are satisfied*:

$$VM_j \leqslant VM_i : \forall V \wedge \overline{V} N_i(V) < N_j(\overline{V}) \iff G_i(V) < G_j(\overline{V}) \forall i \wedge j \text{ not similar}. \tag{57}$$

Property 2. *If product i is not similar to product j, it results in*

$$(VM_j \leqslant VM_i) \iff G_i(V) < G_j(V) \quad \forall V \in]0; VM_i[. \tag{58}$$

Since in $(-\infty, 0] \cup [VM_i, +\infty)$ the N_i function coincides with the corresponding G_i, thus we have that V and $\overline{V}$ belong to $]0, VM_j[$ (i.e. $VM_j < VM_i$). In particular the G_i functions have to be defined so that the following property has to be satisfied. There are two cases to be considered.

Case (a) $V \leqslant \overline{V}$.

On the basis of Theorem 2 and the definition of $N_i(V)$ functions, we obtain:

$$(VM_j \leqslant VM_i) \iff N_i(\overline{V}) < N_j(\overline{V}) \iff N_i(V) < N_i(\overline{V}) < N_j(\overline{V}). \tag{59}$$

Moreover on the basis of the definition of the G_i functions (see (48) and (58)), it results in

$$(VM_j \leqslant VM_i) \iff G_i(V) < G_j(V) < G_j(\overline{V}). \tag{60}$$

Therefore the conditions (51) are verified.

Case (b) $V > \overline{V}$.

Let define V_1 and V_2 as follows:

$$G_i(V_1) = G_j(\overline{V}), \tag{61}$$

$$N_i(V_2) = N_j(\overline{V}), \tag{62}$$

In the interval $[V_1, V_2]$ conditions (57) are not satisfied

$$\forall V \in [V_{\min}, V_{\max}] : (N_i(V) < N_j(\overline{V})) \quad \text{and} \quad (G_i(V) > G_j(\overline{V})), \tag{63}$$

$$V_{\min} = \min(V_1, V_2), \tag{64}$$

$$V_{\max} = \max(V_1, V_2). \tag{65}$$

Fig. 8 reports the definition of V_1 and V_2, if G_i functions are defined with $\Omega_i = 1 (i = 1, \ldots, N)$. In general, it is possible to define the parameter ε_{ij} as the maximum percentage error made by comparing G_i with G_j:

$$\varepsilon_{ij} = \max_{\overline{V} \in]0, VM_j[} (\varepsilon_{ij}(\overline{V})), \tag{66}$$

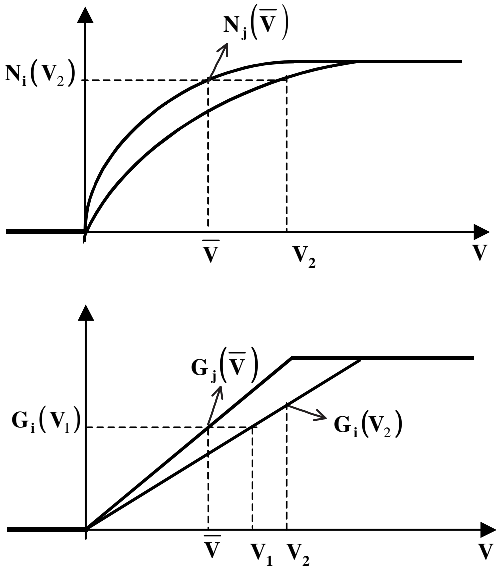

Fig. 8. The $[V_1, V_2]$ interval.

$$\varepsilon_{ij}(\overline{V}) = \max_{V\in[V_{\mathrm{INF}},V_{\mathrm{MAX}}]} \left(\frac{N_i(V) - N_j(\overline{V})}{N_j(\overline{V})} \right). \tag{67}$$

In conclusion, when each G_i $(i =, 1 \ldots, N)$ is defined, the maximum possible percentage error made by considering *PG* instead of *PN* is:

$$\varepsilon = \max_{i,j} (\varepsilon_{ij}). \tag{68}$$

The expressions (66)–(68) can be used in order to define an algorithm that determines the approximated piecewise linear representation of N_i functions characterised by a maximum percentage error ε lower than a fixed threshold. The determined G_i functions are used to formulate the approximated fuzzy mathematical model where the objective 'maximise $N_{\min}$' is substituted by objective 'maximise $G_{\min}$', where $G_{\min}$ is defined as follows.

$$G_{\min} = \left\{ g_{it}^{k} | i = 1, \ldots, N;\ t = 1, \ldots, T;\ k = 1, \ldots, \Omega_{it} \right\}. \tag{69}$$

The Xs and Ys variable values characterising the optimal solution of the approximated fuzzy model are feasible for the mathematical model (27)–(33); consequently, the corresponding value of (27) is a lower bound, say LB, on the optimal objective function of the fuzzy mathematical model.

6. Computational experiments

In order to evaluate the performance of the solution method proposed in Section 5, ten test problems were considered. The test problems have been solved by CPLEX 7.0, running on a personal computer clocked at 800 Mhz and equipped with 256 MB of core memory and 512 KB of cache memory. In particular, we considered product demands and due dates characterising 60 working days of the weaving department in a stocking factory situated in Southern Italy. The shop environment consists of 300 parallel weaving machines. Each test problem is characterised by four product typologies, the production planning is weekly for an overall horizon of six weeks and the set-up cost is reported in Table 1. The machine production rate of each product type is characterised by a symmetric triangular fuzzy number represented as $A_i =< w_i,\ m_i,\ w_i >$, with m_i and w_i denoting the most possible value and the fuzzy number spread, respectively (see Table 2). For each test we have formulated the deterministic programming problem as well as the fuzzy problem and the approximated fuzzy model. Table 3 reports for each formulation the features of the mathematical problems. Moreover, the approximated model formulations adopt as maximum acceptable percentage error $\varepsilon = 3\%$ (see (68)). The expressions (70) and (71) give the total number of variables and constraints characterising the deterministic problem formulation

$$\text{Number of variables} = N *\ T + N *\ N *\ T, \tag{70}$$

$$\text{Number of constraints} = N *\ T + D. \tag{71}$$

Table 1
Set-up cost

From product	To product			
	1	2	3	4
1	0	120	40	30
2	30	0	40	30
3	30	120	0	30
4	30	90	40	0

Table 2
The fuzzy numbers for part type production rate

	Products			
	1	2	3	4
m_i	180	108	234	234
w_i	18	36	36	18

Table 3
Test problem sizes

Test number	Problem formulation type								
	Deterministic			Fuzzy			Approximated fuzzy		
	IV	TV	NC	IV	TV	NC	IV	TV	NC
1	120	120	34	120	1151	2094	120	161	114
2	120	120	33	120	1065	1921	120	158	107
3	120	120	33	120	1035	1861	120	157	105
4	120	120	34	120	1082	1956	120	161	114
5	120	120	34	120	1247	2288	120	161	114
6	120	120	35	120	1078	1949	120	164	121
7	120	120	34	120	1196	2184	120	164	120
8	120	120	34	120	1003	2038	120	161	114
9	120	120	35	120	1344	2481	120	165	123
10	120	120	34	120	1049	1890	120	161	114

The number of variables is the sum of the number of Xs variables and the number of Ys variables. The number of constraints is obtained as the sum of the constraints (2) and (3) and the number of due-dates D, that is number of not null product demands. As explained in Section 4, in the fuzzy formulation the problem dimension grows proportionally to Q (i.e. the total number of lines characterising the piecewise linear representation of N_{it} functions).

$$\text{Number of variables} = N * T + N * N * T + Q + 1, \tag{72}$$

$$\text{Number of constraints} = N * T + D + 2 * Q, \quad \text{with } Q = \sum_{i=1}^{N} \sum_{t=1}^{T} Q_{it}. \tag{73}$$

In the same way, in the approximated fuzzy formulation the problem dimension grows proportionally to Ω (see (50)), that is, the total number of lines characterising the approximated piecewise linear representation of N_{it} functions. We observe that the deterministic problems are small-sized and the execution time is 30 seconds on average. The problem size significantly grows in case of the fuzzy formulation. In particular, it is worth observing that CPLEX 7.0 solver has not been able to determine the optimal solution of any test problem within 1 day of computing time. In order to improve the efficiency of the solver we have used all the options provided by CPLEX and we have set the initial lower bound equal to solution found by the approximated fuzzy model. However, besides that, for any test the enhanced solver was not able to determine the optimal solution with a reasonable computing time. Anyhow, we have considered the upper bound, provided by the enhanced solver with a time limit of 5000 seconds, as reference solution in the evaluation of the quality provided by approximated fuzzy model. The results obtained are summarised in Table 4. Some remarks can be made about the data of Table 4. The quality of the lower bounds is always good. The relative gap exceeds 3% only in one case and it is equal to 1.4% on average. On the other hand

Table 4
Computational results

Test number	Variables	Time (seconds)	$\frac{UB-LB}{UB}\%$
1	131	1157	1.07
2	130	3.54	1.05
3	130	0.73	1.05
4	131	0.87	1.07
5	131	0.66	0.81
6	132	700	1.08
7	131	9.93	3.27
8	131	1000	1.06
9	132	480	2.40
10	131	1	1.05

the execution time never exceeds 20 minutes. These results cannot be taken as a conclusive evidence that all problems will be solved in similar times, but they are encouraging, especially with respect to the quality of obtained solutions.

7. Summary and conclusions

In this paper we have studied a robust approach for solving the scheduling problem of parallel machines with sequence-dependent set-up costs, typically encountered in the textile and fibreglass industries. We have provided the formulation of a fuzzy mathematical model by taking into account the uncertainty regarding the processing times. The proposed approach requires solving a non-linear mixed integer programming model. Based on a piecewise linear representation of the non-linear functions presented in the NLMIP model, an equivalent mixed integer linear programming model has been proposed. Since the resulting MILP model could not be solved in a reasonable time, a solution technique has been proposed, consisting in formulating a reduced MIP model where the piecewise linear functions are approximated in turn. Such approximated formulation of the fuzzy model has been tested on data taken from a real application. The computational results attest the efficiency and the effectiveness of the approach.

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