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β -Robust scheduling for single-machine systems with uncertain processing times

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In scheduling environments with processing time uncertainty, system performance is determined by both the sequence in which jobs are ordered and the actual processing times of jobs. For these situations, the risk of achieving substandard system performance can be an important measure of scheduling effectiveness. To hedge this risk requires an explicit consideration of both the mean and the variance of system performance associated with alternative schedules, and motivates a β -robustness objective to capture the likelihood that a schedule yields actual performance no worse than a given target level. In this paper we focus on β -robust scheduling issues in single-stage production environments with uncertain processing times. We define a general β -robust scheduling objective, formulate the β -robust scheduling problem that results when job processing times are independent random variables and the performance measure of interest is the total flow time across all jobs, establish problem complexity, and develop exact and heuristic solution approaches. We then extend the β -robust scheduling model to consider situations where the uncertainty associated with individual job processing times can be selectively controlled through resource allocation. Computational results are reported to demonstrate the efficiency and effectiveness of the solution procedures.

1. Introduction

In this paper we consider a single-machine scheduling environment where the processing times of individual jobs are uncertain. An immediate consequence of processing time uncertainty in this environment is that system performance, as measured by the scheduling criteria of interest, can vary with both the sequence in which jobs are ordered and the actual processing times of jobs. Given probabilistic processing time information, the distribution of achievable system performance associated with any sequence can be determined, thus providing a means for evaluating alternative schedules. Stochastic scheduling models (see, for example, [1-4]) typically assess schedule quality by focusing on the mean of the outcome distribution, and seek to identify the schedule whose expected system performance is optimal. However, formulations that anchor on expected system performance do not explicitly address the concerns of decision makers who may reasonably be more interested in minimizing the risk of achieving unacceptably poor system performance than in optimizing average performance. These concerns are particularly relevant when there is a clear distinction between acceptable and substandard performance (e.g., when a promised delivery date must be met, or when there is a lead time threshold beyond which customers are lost), and the scheduling objective is to ensure as far as possible that actual system performance is acceptable. In

these situations, a decision maker may strongly prefer a schedule with suboptimal average performance, but low performance variability, over an alternative schedule with optimal average performance and significantly higher variance. More generally, the risk of achieving unacceptably poor system performance reflects a schedule's effectiveness in hedging against the underlying processing time uncertainty, and motivates a scheduling approach that considers both average system performance and performance variability in determining the optimal schedule. Such an approach is consistent with the robust decision-making formulations presented by Gupta and Rosenhead [5], Rosenhead *et al.* [6], Rosenblatt and Lee [7], Kouvelis *et al.* [8,9], and Daniels and Kouvelis [10].

In this paper we develop scheduling approaches for single-stage production environments with uncertain processing times. In Section 2 we define a general β -robustness measure that is based on the likelihood of achieving system performance no worse than a given target level, formulate the β -robust scheduling problem that results when the processing times of individual jobs are independent random variables and the performance measure of interest is the total flow time across all jobs, and establish problem complexity. A branch-and-bound algorithm and a heuristic approach for solving the β -robust scheduling problem are presented in Section 3. In Section 4 we extend the β -robust scheduling model to consider situations where the uncertainty associated with

individual job processing times can be selectively controlled through resource allocation. Section 5 reports the results of a set of experiments designed to test the computational performance of the solution approaches. Section 6 concludes with a summary and suggestions for further research.

2. Problem setting

Consider a set $\{1, 2, \dots, n\}$ of n independent jobs to be processed on a single machine. We assume that the processing times of individual jobs can be specified only imprecisely, due to, e.g., incomplete information on the processing requirements of the jobs. This processing time uncertainty is captured within the set of processing time scenarios, Λ . Each scenario $\lambda \in \Lambda$ represents a unique set of job processing times, which can be realized with some positive probability. Let p_i^λ denote the processing time of job i in scenario $\lambda \in \Lambda$, $P^\lambda = \{p_i^\lambda : i = 1, 2, \dots, n\}$ the vector of job processing times associated with scenario λ , and $\gamma(P^\lambda)$ the probability of realizing the processing times corresponding to scenario λ .

Let Ω represent the set of permutations sequences that can be constructed from the n jobs, and let $\pi = \{\pi(1), \pi(2), \dots, \pi(n)\}$ denote a given sequence in Ω , with $\pi(k)$ specifying the job that occupies position k in sequence π . Suppose that system performance is evaluated by using measure $\varphi(\pi, P^\lambda)$ (note that due to processing time uncertainty, $\varphi(\pi, P^\lambda)$ varies with both the job sequence and the actual processing times of the n jobs). Suppose also that alternative schedules are evaluated according to the corresponding likelihood that actual system performance will be no worse than a given target level T , which might represent, e.g., a promised delivery date or a limit on tolerable waiting time. Without loss of generality, assume that the elements of Λ can be numbered, $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{|\Lambda|}\}$, for each sequence π such that $\lambda_j < \lambda_{j'}$ if and only if $\varphi(\pi, P^{\lambda_j}) \leq \varphi(\pi, P^{\lambda_{j'}})$. Then $\beta(\pi, T)$, the probability that the actual system performance associated with sequence π will be no greater than the specified target level, can be calculated by identifying scenario j^* such that $j^* = \max\{j : \varphi(\pi, P^{\lambda_j}) \leq T\}$:

$$\beta(\pi, T) = \text{Prob}[\varphi(\pi, P^\lambda) \leq T] = \sum_{j=1}^{j^*} \gamma(P^{\lambda_j}). \quad (1)$$

The scheduling objective is to determine the sequence that maximizes the likelihood of achieving system performance no worse than target level T , with this β -robust schedule, π_β , defined such that $\beta(\pi_\beta, T) = \max_{\pi \in \Omega} \beta(\pi, T)$.

To illustrate the development of a detailed solution approach for determining the β -robust schedule, assume that the actual processing time of each job i , p_i , is an independent random variable with mean μ_i and variance σ_i^2 , and that the performance criterion of interest is the

total flow time over all jobs. Define x_{ik} to indicate the position of job i in sequence π , i.e., $x_{ik} = 1$ if $\pi(k) = i$ and $x_{ik} = 0$ otherwise, with $X = \{x_{ik} : i = 1, 2, \dots, n \text{ and } k = 1, 2, \dots, n\}$. Then the total flow time of sequence π given processing time scenario λ can be expressed as

$$\varphi(X, P^\lambda) = \sum_{i=1}^n \sum_{k=1}^n (n - k + 1) p_i^\lambda x_{ik}. \quad (2)$$

Because the total flow time of sequence π is calculated as the weighted sum of a set of independent random variables (with the weights determined by associated assignment vector X), by the Central Limit Theorem $\varphi(X, P^\lambda)$ has an approximate normal distribution as $n \rightarrow \infty$. The mean and variance of this distribution can be expressed as

$$\bar{\varphi}(X) = \sum_{i=1}^n \sum_{k=1}^n (n - k + 1) \mu_i x_{ik}, \quad (3)$$

$$\sigma^2[\varphi(X)] = \sum_{i=1}^n \sum_{k=1}^n (n - k + 1)^2 \sigma_i^2 x_{ik}. \quad (4)$$

The associated standard-normal statistic, $z(X, T) = [T - \bar{\varphi}(X)] / \sqrt{\sigma^2[\varphi(X)]}$, then yields the likelihood of achieving actual flow time performance no worse than the specified target level.

The quality of the normal approximation of $\varphi(X, P^\lambda)$ depends both on the number of jobs and on the characteristics of the individual job processing time distributions. Hines and Montgomery [11] indicate that close approximations should be obtained for even small values of n ($n \approx 4$) when processing time distributions are unimodal and nearly symmetric, whereas a moderate number of jobs ($n \approx 12$) is required to validate the approximation for processing time distributions with no prominent mode and nearly uniform density.

The β -Robust Scheduling Problem (β -RSP), i.e., identifying the schedule with the maximum likelihood of achieving flow time performance no greater than target level T , can be formulated as:

$$(\beta\text{-RSP}) \quad \max \frac{T - \sum_{i=1}^n \sum_{k=1}^n (n - k + 1) \mu_i x_{ik}}{\sqrt{\sum_{i=1}^n \sum_{k=1}^n (n - k + 1)^2 \sigma_i^2 x_{ik}}} \quad (5)$$

$$\text{s.t.} \quad \sum_{k=1}^n x_{ik} = 1, \quad i = 1, 2, \dots, n; \quad (6)$$

$$\sum_{i=1}^n x_{ik} = 1, \quad k = 1, 2, \dots, n; \quad (7)$$

$$x_{ik} \in \{0, 1\}, \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, n. \quad (8)$$

Problem (β -RSP) thus recognizes the effect of sequencing on both expected flow time performance and flow time

variability across processing time scenarios, and determines the best schedule by considering performance across both dimensions. Note that the β -robust schedule also minimizes total flow time subject to $\text{Prob}[\varphi(\pi, P^i)] = \beta(\pi_\beta, T)$, i.e., when the sequencing decision is based on the performance of alternative schedules at the $\beta(\pi_\beta, T)$ -confidence level of the associated flow time distribution. Thus, the β -robust scheduling approach can be tailored to reflect the level of risk that an individual decision maker is willing to bear in hedging against processing time uncertainty. This differs significantly from an approach that exclusively seeks to minimize expected total flow time by sequencing jobs in nondecreasing order of the μ_i to form the shortest expected processing time (SEPT) schedule (see, for example, [12]). Because the SEPT schedule may also exhibit high flow time variance, an alternative schedule with higher expected flow time but lower variance may be preferred by a risk-averse decision maker.

Whereas the SEPT schedule to minimize expected flow time can be constructed in $O(n \log n)$ time, the β -robust scheduling problem is NP-hard, as established by the following result.

Theorem 1. *Problem (β -RSP) is NP-hard.*

Proof: Consider optimal solution X^β to problem (β -RSP), and let $x_{ik}^\beta = 1$ if job i is assigned to position k in this solution (with $x_{ik}^\beta = 0$ otherwise) for $i = 1, 2, \dots, n$ and $k = 1, 2, \dots, n$. Suppose that the variance in total flow time for the schedule associated with solution X^β is given by $\sigma^2[\varphi(X^\beta)] = \sum_{i=1}^n \sum_{k=1}^n (n-k+1)^2 \sigma_i^2 x_{ik}^\beta = V$. Then, solution X^β could be determined by solving the following equivalent formulation of problem (β -RSP).

$$\begin{aligned} \min \quad & \sum_{i=1}^n \sum_{k=1}^n (n-k+1) \mu_i x_{ik} \\ \text{s.t.} \quad & \sum_{i=1}^n \sum_{k=1}^n (n-k+1)^2 \sigma_i^2 x_{ik} \leq V, \\ & (6), (7), \text{ and } (8). \end{aligned}$$

Setting $c_{ik} = (n-k+1) \mu_i$, $r_{ik} = (n-k+1)^2 \sigma_i^2$, and $b = V$, this problem is immediately recognized as an assignment problem with a single side constraint, which has been shown to be NP-hard (see, for example, [13,14]). ■

3. Solution approaches for the β -robust scheduling problem

Given processing time information (μ_i and σ_i^2) for each job i and a specified target level of flow time performance T , problem (β -RSP) can be solved by using a standard branch-and-bound approach that systematically assigns jobs to alternative sequence positions. Each node generated at the l th level of the tree corresponds to the job assigned to position l in the associated partial

sequence. One branch emanating from this node is generated for each of the $n-l$ jobs not occupying the first l positions in the associated partial schedule. Considering in this manner all n jobs for assignment to each sequence position implies a total of $n!$ possible schedules. However, the number of schedules requiring explicit evaluation can be reduced considerably by using the following result.

Theorem 2. *Let $\bar{\varphi}^*$ represent the expected total flow time associated with the SEPT schedule.*

- (a) *If $\mu_i \leq \mu_{i'}$ and $\sigma_i^2 \leq \sigma_{i'}^2$, then there exists a β -robust schedule in which job i precedes job i' for any $T \geq \bar{\varphi}^*$.*
- (b) *If $\mu_i \leq \mu_{i'}$ and $\sigma_i^2 \geq \sigma_{i'}^2$, then there exists a β -robust schedule in which job i precedes job i' for any $T < \bar{\varphi}^*$.*

Theorem 2 indicates that both mean and variance information must be considered to determine job precedence unambiguously. When $T \geq \bar{\varphi}^*$, the β -robust schedule should reflect risk aversion, with shorter and more certain jobs favored for positions early in the schedule. At the extreme, the SEPT schedule yields an optimal hedge against processing time uncertainty for any $T \geq \bar{\varphi}^*$ when job processing time variance is a nondecreasing function of the mean processing time, i.e., when longer jobs are also uniformly more uncertain. In contrast, when $T < \bar{\varphi}^*$, the β -robust schedule should reflect risk seeking, and jobs with low mean processing times but higher variances should be scheduled earlier. The proof of Theorem 2 is easily obtained using a straightforward interchange argument (see [15]).

The search for the β -robust schedule can be further enhanced by computing an upper bound on the objective value for a given partial schedule. To illustrate, consider an assignment of jobs to the first l positions in sequence, represented by partial schedule π_p . A lower bound on the expected flow time of any schedule that starts with partial sequence π_p can be determined by constructing schedule π'_p , consisting of partial sequence π_p followed by the remaining $n-l$ jobs sequenced in nondecreasing order of μ_i :

$$LB_{\bar{\varphi}}(\pi_p) = \sum_{k=1}^n (n-k+1) \mu_{\pi_p(k)}. \quad (9)$$

Similarly, if $T \geq \bar{\varphi}^*$, a lower bound on the flow time variance associated with any schedule that starts with partial schedule π_p can be determined by constructing schedule π''_p , in which π_p is followed by the $n-l$ unscheduled jobs sequenced in nondecreasing order of σ_i^2 :

$$LB_{\sigma^2[\varphi]}(\pi_p) = \sum_{k=1}^n (n-k+1)^2 \sigma_{\pi_p''(k)}^2. \quad (10)$$

If $T < \bar{\varphi}^*$, then the $n-l$ unscheduled jobs in π''_p should be sequenced in nonincreasing order of σ_i^2 , and expression (10) represents an upper bound on the flow time variance

of any schedule starting with partial sequence π_p . In either case, an upper bound on the value of objective function (5) associated with partial sequence π_p is obtained by combining $LB_{\bar{\varphi}}(\pi_p)$ and $LB_{\sigma^2[\varphi]}(\pi_p)$:

$$UB(\pi_p) = [T - LB_{\bar{\varphi}}(\pi_p)] / \sqrt{LB_{\sigma^2[\varphi]}(\pi_p)}. \quad (11)$$

Since π'_p and π''_p are feasible schedules, they also provide lower bounds on the optimal solution.

The dominance property and bounding approach are easily incorporated into the branch-and-bound algorithm for the β -robust scheduling problem. Partial sequences are fathomed when Theorem 2 is violated, or when the associated upper bound is exceeded by the best lower bound encountered thus far. Complete sequences are evaluated by computing objective function (5) for that schedule, with the best sequence and objective value consistently retained. At termination, the algorithm yields the β -robust schedule, π_β , and the associated maximum probability of achieving flow time performance no worse than target level T , $\beta(\pi_\beta, T)$.

The rationale behind the heuristic for the β -robust scheduling problem is to generate discrete processing time scenarios that are likely to yield flow time performance near the β -confidence level for any sequence. This is achieved by defining $z_\beta = [T - \bar{\varphi}^*] / \sqrt{\sigma^2[\varphi(\text{SEPT})]}$ (where $\sigma^2[\varphi(\text{SEPT})]$ represents the flow time variance of the SEPT schedule), and setting $p_i = \mu_i + (z_\beta/\alpha)\sqrt{(\sigma_i^2)}$, $i = 1, 2, \dots, n$, for several values of control parameter α . Sequencing the jobs in shortest processing time (SPT) order yields the corresponding optimal schedule. This approach ensures that Theorem 2 is not violated, e.g., $\mu_i \leq \mu_r$ and $\sigma_i^2 \leq \sigma_r^2$ implies that $\mu_i + (z_\beta/\alpha)\sqrt{(\sigma_i^2)} \leq \mu_r + (z_\beta/\alpha)\sqrt{(\sigma_r^2)}$ for $z_\beta \geq 0$ ($T \geq \bar{\varphi}^*$), and hence job i will precede job r in the associated heuristic schedule since $p_i \leq p_r$. The SPT schedule is then evaluated by using objective function (5), and the best schedule encountered in the search over alternative values of α is consistently retained.

Procedure β -Heuristic

Input. Mean processing time, μ_i , and processing time variance, σ_i^2 , for each job $i = 1, 2, \dots, n$, target level of flow time performance, T , mean flow time, $\bar{\varphi}^*$, and flow time variance, $\sigma^2[\varphi(\text{SEPT})]$, associated with the SEPT schedule, and selected parameter values, α_j , $j = 1, 2, \dots, I$.

Output. Heuristic sequence, π_h , that provides a lower bound, LB , on the probability of achieving flow time performance no worse than target level T .

Step 1. Set $j = 0$, $LB = 0$, and $z_\beta = [T - \bar{\varphi}^*] / \sqrt{\sigma^2[\varphi(\text{SEPT})]}$.

Step 2. Set $j = j + 1$. If $j > I$, stop. Otherwise, set $p_i = \mu_i + (z_\beta/\alpha_j)\sqrt{(\sigma_i^2)}$ for $i = 1, 2, \dots, n$.

Step 3. Construct schedule π_j by sequencing the jobs in nondecreasing order of the p_i , and set:

$$LB_j = \left[T - \sum_{k=1}^n (n-k+1)\mu_{\pi_j(k)} \right] \div \sqrt{\sum_{k=1}^n (n-k+1)^2 \sigma_{\pi_j(k)}^2}.$$

If $LB_j > LB$, set $LB = LB_j$ and $\pi_h = \pi_j$. Return to Step 2.

Procedure β -Heuristic requires a total of I iterations, with Steps 2 and 3 of each iteration performed in $O(n)$ and $O(n \log n)$ time, respectively. The complexity of the heuristic is therefore $O[I(n \log n)]$. Clearly, the number of values of parameter α considered by the procedure also affects solution quality. The computational results reported in Section 5 indicate that close approximations of the β -robust schedule are obtained in very few iterations.

4. Variance reduction and β -robust scheduling

The formulation of problem (β -RSP) can be extended to include resource-allocation decisions that reduce processing time uncertainty, e.g., managerial time and effort spent interacting with customers to better understand the detailed requirements of a particular job. We model the selective reduction of processing time uncertainty by assuming that there exists a single resource, available in limited supply (R), that can be applied to individual jobs to linearly decrease the associated processing time variance. The processing time of job i is again represented as an independent random variable with mean μ_i and variance $\sigma_i^2 = \bar{\sigma}_i^2 - a_i y_i$, where $\bar{\sigma}_i^2$ denotes the normal variance in the processing time of job i , a_i denotes the rate at which the application of resource reduces the processing time variance of job i , and y_i represents the amount of resource allocated to job i . We also assume a minimum processing time variance for job i , $\underline{\sigma}_i^2$, so that $\underline{\sigma}_i^2 \leq \sigma_i^2 \leq \bar{\sigma}_i^2$.

Assigning resource among the n jobs thus reduces flow time variance without affecting the expected performance of any given schedule, thereby allowing investigation of the impact of variance reduction on the schedule-selection process. Note that opportunities to decrease processing time uncertainty would not be considered by decision makers focused solely on optimizing expected system performance, who simply use the SEPT priority rule to directly determine the optimal schedule. However, variance reduction is an important concern for decision makers interested in hedging the risk of unacceptably poor system performance, because these resource allocation decisions ultimately affect any schedule's performance variability. If T again represents a given target level of flow time performance, then the β -Robust Scheduling Problem with Variance Reduction (β -RSPVR) can be formulated to simultaneously determine the

optimal job sequence and allocation of resource among the n jobs:

$$(\beta\text{-RSPVR}) \max \frac{T - \sum_{i=1}^n \sum_{k=1}^n (n-k+1) \mu_i x_{ik}}{\sqrt{\sum_{i=1}^n \sum_{k=1}^n (n-k+1)^2 (\bar{\sigma}_i^2 - a_i y_i) x_{ik}}} \quad (12)$$

$$\text{s.t. } \sum_{i=1}^n y_i \leq R, \quad (13)$$

$$0 \leq y_i \leq \frac{\bar{\sigma}_i^2 - \underline{\sigma}_i^2}{a_i}, \quad i = 1, 2, \dots, n, \quad (14)$$

(6), (7), and (8).

Problem (β -RSPVR) thus maximizes the probability of achieving a given target level of system performance while limiting the amount of resource allocated among the n jobs to the available supply, and ensuring that $\underline{\sigma}_i^2 \leq \sigma_i^2 \leq \bar{\sigma}_i^2$ for each job i . Since problem (β -RSPVR) generalizes problem (β -RSP) (with $a_i = 0$ for $i = 1, 2, \dots, n$), problem (β -RSPVR) is clearly NP-hard.

As in Section 3, a branch-and-bound algorithm can be developed for solving problem (β -RSPVR) given processing time (μ_i and $\bar{\sigma}_i^2$) and variance-reduction (a_i and $\underline{\sigma}_i^2$) information for each job i , and given a target level of flow time performance (T). Nodes in the search tree again represent the assignment of jobs to sequence positions, and branches in the tree correspond to feasible partial sequences. Complete schedules are evaluated by determining the allocation of resource that maximizes the associated likelihood of achieving actual performance no worse than the target level. Sequence π is thus evaluated by solving:

$$(\pi\text{-EVAL}) \max \frac{T - \sum_{k=1}^n (n-k+1) \mu_{\pi(k)}}{\sqrt{\sum_{k=1}^n (n-k+1)^2 (\bar{\sigma}_{\pi(k)}^2 - a_{\pi(k)} y_{\pi(k)})}} \quad (15)$$

s.t. (13) and (14).

Let $s_\pi(i)$ denote the position of job i in sequence π (i.e., $s_\pi(i) = k$ if and only if $\pi(k) = i$), and suppose without loss of generality that the jobs in sequence π are renumbered such that if $(n - s_\pi(i) + 1)^2 a_i \geq (n - s_\pi(i') + 1)^2 a_{i'}$, then $i < i'$ (with ties broken arbitrarily). Let critical (renumbered) job i^* be identified such that $\sum_{i=1}^{i^*-1} [(\bar{\sigma}_i^2 - \underline{\sigma}_i^2)/a_i] \leq R$ and $\sum_{i=1}^{i^*} [(\bar{\sigma}_i^2 - \underline{\sigma}_i^2)/a_i] > R$. Then the optimal allocation of resource across the renumbered jobs in sequence π is given by:

$$y_i^* = \begin{cases} (\bar{\sigma}_i^2 - \underline{\sigma}_i^2)/a_i, & \text{for } i < i^*; \\ R - \sum_{i=1}^{i^*-1} [(\bar{\sigma}_i^2 - \underline{\sigma}_i^2)/a_i], & \text{for } i = i^*; \\ 0, & \text{for } i > i^*. \end{cases} \quad (16)$$

So long as a consistent mapping is maintained between the renumbered jobs in expression (16) and the original jobs in sequence π , the complexity of the process for evaluating complete sequence π is $O(n \log n)$. Note also that a similar marginal analysis yields the optimal allocation of resource across jobs for a given sequence whenever σ_i^2 is a convex, nonincreasing function of y_i for $i = 1, 2, \dots, n$. An extension of the dominance result in Section 3 can be used to reduce the actual number of sequences requiring explicit evaluation by using expression (16).

Theorem 3. Let $\bar{\varphi}^*$ represent the expected total flow time associated with the SEPT schedule.

- (a) If $\mu_i \leq \mu_{i'}$, $\bar{\sigma}_i^2 \leq \bar{\sigma}_{i'}^2$, $\underline{\sigma}_i^2 \leq \underline{\sigma}_{i'}^2$, and $a_i \geq a_{i'}$, then there exists an optimal solution to problem (β -RSPVR) in which job i precedes job i' for any $T \geq \bar{\varphi}^*$.
- (b) If $\mu_i \leq \mu_{i'}$ and $\sigma_i^2 \geq \sigma_{i'}^2$, then there exists an optimal solution to problem (β -RSPVR) in which job i precedes job i' for any $T < \bar{\varphi}^*$.

The proof of Theorem 3 again utilizes a straightforward interchange argument (see [15]).

We employ a bounding process similar to that developed in Section 3. To illustrate, consider partial schedule π_p in which jobs have been assigned to the first l positions in sequence, and note that a lower bound on the expected flow time of any schedule starting with π_p , $LB_{\bar{\varphi}}(\pi_p)$, is again given by expression (9). If $T \geq \bar{\varphi}^*$, a lower bound on the flow time variance of π_p can be determined by constructing schedule π_p'' , in which partial sequence π_p is followed by $n - l$ artificial jobs. The makeup and sequence of the artificial unscheduled jobs is such that $\bar{\sigma}_{\pi_p''(k)}^2 \leq \bar{\sigma}_{\pi_p''(k')}^2$, $a_{\pi_p''(k)} \geq a_{\pi_p''(k')}$, and $\underline{\sigma}_{\pi_p''(k)}^2 \leq \underline{\sigma}_{\pi_p''(k')}^2 \forall l + 1 \leq k < k' \leq n$. Expression (16) can then be used to determine the corresponding optimal allocation of resource across jobs in artificial schedule π_p'' , from which a lower bound on flow time variance can be calculated as

$$LB_{\sigma^2[\varphi]}(\pi_p) = \sum_{k=1}^n (n-k+1)^2 (\bar{\sigma}_{\pi_p''(k)}^2 - a_{\pi_p''(k)} y_{\pi_p''(k)}^*). \quad (17)$$

If $T < \bar{\varphi}^*$, then the $n - l$ unscheduled jobs should simply be sequenced in nonincreasing order of $\bar{\sigma}_i^2$, and expression (17) (with $y_{\pi_p''(k)}^* = 0$ for $1 \leq k \leq n$) represents an upper bound on the flow time variance of π_p . In either case, an upper bound on the value of objective function (12) is obtained by combining $LB_{\bar{\varphi}}(\pi_p)$ and $LB_{\sigma^2[\varphi]}(\pi_p)$, i.e., $UB(\pi_p) = [T - LB_{\bar{\varphi}}(\pi_p)] / \sqrt{LB_{\sigma^2[\varphi]}(\pi_p)}$.

The evaluation process, dominance property, and bounding approach are easily incorporated into the branch-and-bound algorithm for problem (β -RSPVR). At termination, the algorithm yields the β -robust schedule, π_β , the associated assignment of resource across jobs, $Y_{\pi_\beta(k)} = \{y_{\pi_\beta(k)}^* : k = 1, 2, \dots, n\}$, and the resulting maximum probability of achieving flow time performance no worse than target level T , $\beta(\pi_\beta, T)$.

The heuristic developed for problem (β -RSPVR) is a decomposition approach that separately determines an appropriate job sequence given an estimate of each job's optimal processing time variance, and the optimal allocation of resource across jobs given a sequence. An initial job sequence, π_1 , is constructed using procedure β -Heuristic with $\sigma_i^2 = \bar{\sigma}_i^2$ for $i = 1, 2, \dots, n$. Sequence π_1 is evaluated by using expression (16) to determine the associated optimal allocation of resource, $Y_{\pi_1} = \{y_{\pi_1(k)}^* : k = 1, 2, \dots, n\}$; the resulting probability of achieving flow time performance no worse than target level T is then computed as $\beta(\pi_1, T) = [T - \sum_{k=1}^n (n-k+1)\mu_{\pi_1(k)}] \div \sqrt{[\sum_{k=1}^n (n-k+1)^2 (\bar{\sigma}_{\pi_1(k)}^2 - a_{\pi_1(k)} y_{\pi_1(k)}^*)]}$. Because this optimal allocation of resource implies a set of processing time variances that may be different from those used to construct sequence π_1 , we update the estimate of the optimal variance for each job, $\sigma_{\pi_1(k)}^2 = \bar{\sigma}_{\pi_1(k)}^2 - a_{\pi_1(k)} y_{\pi_1(k)}^*$ for $k = 1, 2, \dots, n$, and generate a new candidate job sequence, π_2 , using procedure β -Heuristic. This new sequence is evaluated as described above, with the best encountered solution consistently retained. The process continues until a previously encountered sequence is generated, as further search would simply result in cycling.

Decomposition Heuristic

Input. Processing time, μ_i and σ_i^2 , and variance-reduction, a_i and $\bar{\sigma}_i^2$, information for each job $i = 1, 2, \dots, n$, R units of available resource, and target level of flow time performance, T .

Output. Heuristic sequence, π_h , and resource-allocation, $y_{\pi_h(k)}^*$, $k = 1, 2, \dots, n$, that provide a lower bound, LB , on the probability of achieving flow time performance no worse than target level T .

Step 1. Set $\sigma_i^2 = \bar{\sigma}_i^2$ for $i = 1, 2, \dots, n$, $S = \emptyset$, $j = 0$, and $LB = 0$.

Step 2. Set $j = j + 1$. Construct schedule π_j by using Procedure β -Heuristic applied to μ_i and σ_i^2 for $i = 1, 2, \dots, n$. If $\pi_j \in S$, stop. Otherwise, set $S = S + \{\pi_j\}$ and go to Step 3.

Step 3. Evaluate schedule π_j by determining the associated optimal resource-allocation vector, $\{y_{\pi_j(k)}^* : k = 1, 2, \dots, n\}$, by using expression (16), and set:

$$LB_j = \left[T - \sum_{k=1}^n (n-k+1)\mu_{\pi_j(k)} \right] \div \sqrt{\sum_{k=1}^n (n-k+1)^2 (\bar{\sigma}_{\pi_j(k)}^2 - a_{\pi_j(k)} y_{\pi_j(k)}^*)}.$$

If $LB_j > LB$, set $LB = LB_j$, $\pi_h = \pi_j$, and $y_{\pi_h(k)}^* = y_{\pi_j(k)}^*$ for $k = 1, 2, \dots, n$. Set $\sigma_{\pi_j(k)}^2 = \bar{\sigma}_{\pi_j(k)}^2 - a_{\pi_j(k)} y_{\pi_j(k)}^*$ for $k = 1, 2, \dots, n$ and return to Step 2.

The effectiveness of the decomposition heuristic is clearly a function of the number of job sequences generated by the process. The computational results reported in the next section indicate that the heuristic yields close approximations to the optimal solution while considering very few distinct sequences and resource-allocation policies.

5. Computational results

To explore the computational performance of the solution approaches, procedures were coded in FORTRAN and implemented on a 486 personal computer. The first experiment involved test problems consisting of $n = 10, 15$, and 20 jobs. The mean processing time for each job i was first randomly drawn from a uniform distribution of integers on the interval $\mu_i \in [10, 50\delta_1]$, where parameter δ_1 controls the variability in the average processing times across jobs in a given test problem. The processing time variance of each job i was then randomly drawn from a uniform distribution of integers on the interval $\sigma_i^2 \in [0, \frac{1}{9}\mu_i^2\delta_2]$, where parameter δ_2 controls the variance in the ratio $\mu_i/\sqrt{\sigma_i^2}$ across jobs in a given test problem, and $\delta_2 \leq 1$ ensures that virtually all of the mass of the processing time distribution for each job i lies to the right of $p_i = 0$. Three values of δ_1 (0.4, 0.7, and 1.0), three values of δ_2 (0.4, 0.7, and 1.0), and three values of z_β (1.04, 1.65, and 2.33, corresponding to target levels where the SEPT schedule yields acceptable performance with probability approx. 0.85, 0.95, and 0.99) were included in the experimental design to simulate a wide range of scheduling environments. Ten replications were generated for each combination of n , δ_1 , δ_2 , and z_β , resulting in a total of 810 test problems.

Each problem was solved using the branch-and-bound algorithm and the β -Heuristic procedure (with parameter α set to 1.0, 2.0, 4.0, and ∞) discussed in Section 3. Note that $\alpha = \infty$ corresponds to $p_i = \mu_i$ for $i = 1, 2, \dots, n$, so that the implications of ignoring processing time variance information and scheduling according to the SEPT priority rule can be investigated. The performance of the procedures is reported in Table 1, which provides information on the average CPU time (in seconds) for the branch-and-bound algorithm, the expected flow time of the β -robust schedule, $\bar{\varphi}(\pi_\beta)$, compared with the optimal expected flow time, $\bar{\varphi}^*$, and the approximation errors incurred by the SEPT and β -Heuristic solution approaches. Note that approximation errors are calculated to represent the additional risk of achieving substandard system performance if a heuristic schedule is selected instead of β -robust schedule π_β . The results in Table 1 are aggregated over the three values of δ_1 and the three values of δ_2 included in the experimental design.

The results in Table 1 show that over the 810 test problems, the expected flow time of the β -robust schedule

Table 1. Computational performance of β -robust solution procedures

n	z_β	Branch-and-bound			Percentage above optimal solution			
		CPU	$[\bar{\varphi}(\pi_\beta) - \bar{\varphi}^*]/\bar{\varphi}^*$		SEPT		β -Heuristic	
			Mean	Max.	Mean	Max.	Mean	Max.
10	1.04	0.2	0.1	0.8	4.0	24.0	0.6	6.0
	1.65	0.2	0.3	1.9	11.1	54.8	0.9	13.2
	2.33	0.3	0.6	2.5	32.0	135.4	1.2	14.5
	Avg. [Max.]	0.2	0.3	[2.5]	15.7	[135.4]	0.9	[14.5]
15	1.04	1.0	0.1	0.5	3.4	14.8	1.2	9.3
	1.65	1.7	0.2	1.0	10.9	33.7	1.4	11.9
	2.33	2.1	0.4	1.9	26.2	83.1	2.4	16.0
	Avg. [Max.]	1.6	0.2	[1.9]	13.5	[83.1]	1.6	[16.0]
20	1.04	12.6	0.1	0.3	3.6	11.7	1.5	6.7
	1.65	33.6	0.1	0.7	10.2	38.2	2.3	9.9
	2.33	38.2	0.2	1.1	25.1	90.2	3.1	22.6
	Avg. [Max.]	28.1	0.1	[1.1]	13.0	[90.2]	2.3	[22.6]

averaged 0.2% above the optimal expected flow time, with a maximum deviation of 2.5%. This indicates that the β -robust schedule maximizes the probability of achieving flow time performance no worse than the specified target level while maintaining excellent performance with respect to expected flow time. The results in Table 1 also show that close approximations to the β -robust schedule were obtained using the β -Heuristic procedure. Over all 810 test problems, the risk level associated with the best heuristic solution averaged 1.6% above the optimal solution, with a maximum deviation of 22.6%. Because only four iterations of the heuristic were required to identify consistently good schedules, we note that further improvement in solution quality could be realized with a moderate increase in computational effort by judiciously including additional values of α . In contrast, the poor performance (14.1% average error; 135.4% maximum error) of the SEPT schedule illustrates the substantial additional risk associated with a scheduling approach that ignores processing time variance information.

A second experiment was designed to investigate the performance of the solution approaches developed for problem (β -RSPVR). Problem sizes ($n = 10, 15$, and 20 jobs) and mean processing times ($\mu_i \in [10, 50\delta_1]$) were determined as in the first experiment. Processing time variance data were next randomly generated from uniform distributions of integers such that $\bar{\sigma}_i^2 \in [0, \frac{1}{9}\mu_i^2]$ and $a_i \in [0, \bar{\sigma}_i^2]$, with the minimum variance of each job set to 0. Resource levels were then randomly generated from a uniform distribution of integers on the interval $R \in [0, \delta_2 \sum_{i=1}^n \bar{\sigma}_i^2/a_i]$, with parameter δ_2 included to control resource availability. Three values of δ_1 (0.4, 0.7, and 1.0), three values of δ_2 (0.4, 0.7, and 1.0), and three values of z_β (1.04, 1.65, and 2.33) were again included in the experimental design, and ten replications for each

combination of parameters resulted in a total of 810 test problems.

Each problem instance was solved using the branch-and-bound algorithm and the decomposition heuristic (with parameter α set to 1.0, 2.0, 4.0, and ∞ in the β -Heuristic procedure used to generate alternative sequences) discussed in Section 4. The optimal allocation of resource across jobs in the SEPT schedule was also determined to highlight the added risk that results from a sequencing approach that ignores processing time variance information, even when available opportunities to reduce processing time uncertainty are exploited through a separate resource allocation decision. The performance of the solution procedures is presented in Table 2, whose structure is identical to that of Table 1 with the exception that information on the average number of iterations required before termination of the decomposition heuristic is also reported.

The results in Table 2 emphasize the excellent expected flow time performance of the β -robust schedule with variance reduction, which averaged 0.3% above the optimal expected flow time (with a maximum deviation of 3.1%) over the 810 test problems. Table 2 also indicates that good approximations to the β -robust schedule with variance reduction were obtained in very few iterations using the decomposition heuristic. Over all 810 test problems, the risk level associated with heuristic solutions averaged 4.1% above the optimal solution, with a maximum deviation of 31.9%. In contrast, allocating resource optimally to jobs in the SEPT schedule yielded solutions whose risk level averaged 33.0% above the optimal solution, with a maximum deviation of 220.3%, illustrating again that considerable additional risk is incurred by a scheduling approach that ignores processing time variance information.

Table 2. Computational performance of solution procedures for problem (β -RSPVR)

<i>n</i>	<i>z_β</i>	Percentage above optimal solution							
		Branch-and-bound			Decomposition Heuristic			SEPT with resource allocation	
		CPU	$[\bar{\varphi}(\pi_{\beta}) - \bar{\varphi}^*]/\bar{\varphi}^*$		Number of Iterations			Mean	Max.
			Mean	Max.		Mean	Max.		
10	1.04	5.5	0.1	0.6	3.5	2.8	19.8	22.2	64.6
	1.65	6.8	0.4	1.3	3.2	3.7	22.5	36.1	129.6
	2.33	7.2	0.5	3.1	4.1	5.2	26.1	52.9	220.3
	Avg. [Max.]	6.5	0.3	[3.1]	3.6	3.9	[26.1]	37.1	[220.3]
15	1.04	35.4	0.1	0.7	4.6	3.1	22.8	19.9	78.6
	1.65	54.5	0.3	1.3	4.8	4.2	29.4	31.4	105.0
	2.33	70.0	0.4	1.7	4.7	5.4	31.9	46.8	169.4
	Avg. [Max.]	53.3	0.3	[1.7]	4.7	4.2	[31.9]	32.7	[169.4]
20	1.04	724.5	0.1	0.8	5.5	3.0	9.5	19.6	94.0
	1.65	1344	0.3	1.6	6.8	4.4	22.7	24.6	115.3
	2.33	1685	0.5	2.9	6.3	5.5	28.2	43.2	151.0
	Avg. [Max.]	1251	0.3	[2.9]	6.2	4.3	[28.2]	29.1	[151.0]

6. Conclusions

This paper has focused on scheduling environments with processing time uncertainty, and explored the scheduling implications of hedging the risk of substandard system performance by considering both the mean and variance of system performance in evaluating alternative schedules. We defined a measure of schedule β -robustness that represents the likelihood of achieving system performance no worse than a given target level. For single-stage production environments where job processing times are independent random variables and the performance measure of interest is the total flow time over all jobs, we formulated the β -robust scheduling problem, established its complexity, and developed exact and heuristic solution approaches. We then considered β -robust scheduling issues that arise when the uncertainty associated with individual job processing times can be selectively controlled through resource allocation. Computational results indicated that β -robust schedules provide effective hedges against processing time uncertainty while maintaining near-optimal performance with respect to expected total flow time. Close approximations to the optimal solution were consistently obtained using the heuristic procedures.

β -Robust scheduling can be an important approach for controlling the impact of manufacturing uncertainty on system performance. Further research should focus on providing additional linkages between sources of manufacturing variability, operational decisions that affect or are affected by these uncertainties, and the manner in which managers should incorporate risk into the decision-making process. Examples in the scheduling domain include development of β -robust models for other

scheduling environments where the variability in system performance associated with a schedule can be determined, with consideration of alternative performance measures that are sensitive to manufacturing variability and thus exposed to the risk of substandard performance, and detailed investigation of other sources of and modelling frameworks for manufacturing uncertainty.

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