Notes on Generating Functions

Yangxinyu Xie

Summer 2019

1 Background **2**

2 Definitions **2**

2.1 Ordinary Generating Functions 2

2.2 Probability Generating Functions 3

2.3 Moment Generating Functions 5

3 Recurrent Events **5**

Last updated January 11, 2020

1. Background

❖

Theorem 1.1. [l'Hôpital's Rule, 0/0 form]. *Let f and g be differentiable on some interval, with a in that interval, and assume they satisfy*

$$\lim_{x\to a} f(x) = 0, \text{ and } \lim_{x\to a} g(x) = 0.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. Definitions

❖

2.1 Ordinary Generating Functions ❖

Definition 2.1. For a sequence $a_0, a_1, a_2, ... \in \mathbb{R}$, if there exists $s_0 > 0$ such that

$$G(s) := \sum_{i=0}^{\infty} a_i s^i \tag{2.1.1}$$

converges when $|s| \le s_0$, then we call $G(s)$ the **ordinary generating function** of the sequence $\{a_i\}$. If the sum converges if $|s| \le s_0$ and diverges if $|s| > s_0$, we call s_0 the *radius of convergence*.

For example, the *exponential generating function* is given by

$$E_a(s) = \sum_{i=0}^{\infty} \frac{a_i s^i}{i!} \tag{2.1.2}$$

Specifically,

$$e^s = \sum_{i=0}^{\infty} \frac{s^i}{i!} \tag{2.1.3}$$

Note that a generating function uniquely determines its sequence. In fact,

$$a_i = \frac{G_a^{(i)}(0)}{i!} \tag{2.1.4}$$

Definition 2.2. The **convolution** of two sequences $\{a_n, n \ge 0\}$ and $\{b_n, n \ge 0\}$ is the new sequence $\{c_n, n \ge 0\}$ whose nth element is given by

$$c_n = \sum_{i=0}^{n} a_i b_{n-i} \tag{2.1.5}$$

we write $\{c_n\} = \{a_n\} * \{b_n\}$.

Example 2.3. Suppose X and Y are independent, non-negative, integer-valued random variables with

$$\mathbb{P}[X = k] = a_k, \quad \mathbb{P}[Y = k] = b_k, k = 1, 2, ... \tag{2.1.6}$$

then for $n \ge 0$

$$\mathbb{P}[X + Y = n] = \sum_{i=0}^{n} \mathbb{P}[X = i, Y = n - i] = \sum_{i=0}^{n} \mathbb{P}[X = i]\mathbb{P}[Y = n - i] = \sum_{i=0}^{n} a_i b_{n-i} \tag{2.1.7}$$

We obtain $\{\mathbb{P}[X + Y = n]\} = \{\mathbb{P}[X = n]\} * \{\mathbb{P}[Y = n]\}$.

Theorem 2.4. *If $\{a_n, n \geq 0\}$ and $\{b_n, n \geq 0\}$ have generating functions G_a and G_b, then the generating function of $\{c_n, n \geq 0\} = \{a_n\} * \{b_n\}$ is*

$$G_c(s) = \sum_{n=0}^{\infty} c_n s^n = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} a_i b_{n-i} \Big) s^n = \sum_{i=0}^{\infty} a_i s^i \sum_{n=i}^{\infty} b_{n-i} s^{n-i} = G_a(s) G_b(s) \tag{2.1.8}$$

Example 2.5. The *combinatorial identity*

$$\sum_i \binom{n}{i}^2 = \binom{2n}{n} \tag{2.1.9}$$

can be verified using generating functions. Let $a_i = \binom{n}{i}$, we have $a * a = \sum_i \binom{n}{i}\binom{n}{n-i} = \sum_i \binom{n}{i}^2$, $G_a(s) = \sum_i \binom{n}{i} s^i = (1+s)^n$. Hence,

$$G_{a*a}(s) = G_a(s)^2 = (1+s)^{2n} = \sum_i \binom{2n}{i} s^i \tag{2.1.10}$$

2.2 Probability Generating Functions ❖

Definition 2.6. The **probability generating function** of the random variable X is defined to be the generating function

$$G_X(s) = \mathbb{E}(s^X) = \sum_{i=0}^{\infty} s^i \mathbb{P}(X = i) \tag{2.2.1}$$

Using the result from Theorem 2.4, we have that in Example 2.3, if X and Y are independent, non-negative, integer-valued random variables

$$G_{X+Y}(s) = G_X(s) G_Y(s) \tag{2.2.2}$$

Example 2.7. Bernoulli Generating Function

$$G_X(s) = \mathbb{E}(s^X) = s^0(1-p) + sp = (1-p) + sp \tag{2.2.3}$$

Geometric Generating Function

$$G_X(s) = \mathbb{E}(s^X) = \sum_{i=1}^{\infty} (1-p)^{i-1} p s^i = ps \sum_{i=0}^{\infty} [s(1-p)]^i = \frac{ps}{1 - s(1-p)} \tag{2.2.4}$$

Poisson Generating Function

$$G_X(s) = \mathbb{E}(s^X) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} s^i = e^{-\lambda} e^{s\lambda} = e^{s(\lambda - 1)} \tag{2.2.5}$$

The following theorem allow us to conclude that $G(0) = \mathbb{P}(X = 0)$ and $G(1) = 1$ where we use $G(1)$ to denote $\lim_{s \uparrow 1} G(s)$.

Theorem 2.8 (Abel's theorem). *If $a_i \geq 0$ for all i and $G_a(s)$ is finite for $|s| < 1$, then*

$$\lim_{s \uparrow 1} G_a(s) = \sum_{i=1}^{\infty} a_i \tag{2.2.6}$$

whether the sum if finite or equals to ∞.

Theorem 2.9. *If X has a generating function $G(s)$, then*

i . $\mathbb{E}(X) = G'_X(1)$.

ii . $\mathbb{E}[X(X-1)...(X-k+1)] = G_X^{(k)}(1)$.

Proof. Taking the derivatives,

$$G_X^{(k)}(s) = \sum_i s^{i-k} i(i-1)...(i-k+1)\mathbb{P}[X=i] \tag{2.2.7}$$

Using Abel's theorem 2.8, we have

$$G_X^{(k)}(1) = \sum_i i(i-1)...(i-k+1)\mathbb{P}[X=i] = \mathbb{E}[X(X-1)...(X-k+1)] \tag{2.2.8}$$

□

Using this result, we can calculate the variance of X by

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2 = G''_X(1) + G'_X(1) - [G'_X(1)]^2 \tag{2.2.9}$$

Theorem 2.10. *If $X_1, X_2, ...$ is a sequence of independent identically distributed random variables with a common generating function G_X, and N is a nonnegative random variable which is independent of the X_is and has the generating function G_N, then $S = X_1 + X_2 + ... + X_n$ has the generating function given by*

$$G_s(s) = G_N(G_X(s)) \tag{2.2.10}$$

Proof.

$$\begin{aligned} G_S(s) &= \mathbb{E}(s^S) = \mathbb{E}[\mathbb{E}(s^S|N)] \\ &= \sum_{n=0}^{\infty} \mathbb{E}(s^S|N)\mathbb{P}(N=n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(s^{X_1+X_2+...+X_n})\mathbb{P}(N=n) \end{aligned} \tag{2.2.11}$$

Using a generalisation of Equation 2.2.2, we obtain

$$\begin{aligned} G_S(s) &= \sum_{n=0}^{\infty} \mathbb{E}(s^{X_1+X_2+...+X_n})\mathbb{P}(N=n) \\ &= \sum_{n=0}^{\infty} G_{X_1}(s)G_{X_2}(s)...G_{X_n}(s)\mathbb{P}(N=n) \\ &= \sum_{n=0}^{\infty} [G_X(s)]^n \mathbb{P}(N=n) \\ &= G_N(G_X(s)) \end{aligned} \tag{2.2.12}$$

□

Definition 2.11. The **joint probability generating function** of variables X_1 and X_2 taking values in nonnegative integers is defined by

$$G_{X_1,X_2}(s_1, s_2) = \mathbb{E}(s_1^{X_1} s_2^{X_2}) \tag{2.2.13}$$

Theorem 2.12. *Random variables X_1 and X_2 are independent if and only if*

$$G_{X_1,X_2}(s_1, s_2) = G_{X_1}(s_1)G_{X_2}(s_2) \tag{2.2.14}$$

for all s_1 and s_2.

Proof. To prove the converse, we have

$$G_{X_1,X_2}(s_1,s_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} s_1^i s_2^j \mathbb{P}[X_1 = i, X_2 = j] \tag{2.2.15}$$

and

$$\begin{aligned} G_{X_1}(s_1)G_{X_2}(s_2) &= \sum_{i=0}^{\infty} s_1^i \mathbb{P}[X_1 = i] \sum_{j=0}^{\infty} s_2^j \mathbb{P}[X_2 = j] \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} s_1^i s_2^j \mathbb{P}[X_1 = i]\mathbb{P}[X_2 = j] \end{aligned} \tag{2.2.16}$$

Hence, we conclude the independence by $\mathbb{P}[X_1 = i, X_2 = j] = \mathbb{P}[X_1 = i]\mathbb{P}[X_2 = j]$. □

2.3 Moment Generating Functions

Definition 2.13. The **moment generating function** of the random variable X is defined to be the generating function

$$\begin{aligned} M_X(t) = G_X(e^t) &= \sum_{i=0}^{\infty} e^{ti}\mathbb{P}(X = i) \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(ti)^k}{k!}\mathbb{P}(X = i) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{\infty} i^k \mathbb{P}(X = i) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}[X^k] \end{aligned} \tag{2.3.1}$$

3. Recurrent Events

We denote an event by H and suppose that at discrete time point $t = 1, 2, ...$, either H occurs or does not occur. Additionally, we use X_1 to denote the time point when H first occur, i.e. $X_1 = \min\{n : H \text{ occurs at time } n\}$; also, we let X_m be the time interval between the mth occurrence and $m-1$th occurrence of H. Thus, the time for the mth occurrence of H is $T_m = X_1 + X_2 + ... + X_m$. We assume that $X_1, X_2, ...$ are independent and $X_2, X_3, ...$ are identically distributed. Note that X_1 may be of some special distribution. Our goal is to obtain the probability that H occurs at some given time point. That is, we let H_n be the event that H occurs at time n and define

$$u_n := \sum_{i=1}^{n} \mathbb{P}(H_n|X_1 = i)\mathbb{P}(X_1 = i) = \sum_{i=1}^{n} \mathbb{P}(H_{n-i+1}|H_1)\mathbb{P}(X_1 = i) \tag{3.0.1}$$

Moreover, for $m \geq 2$,

$$\mathbb{P}(H_m|H_1) = \sum_{i=1}^{m-1} \mathbb{P}(H_m|H_1, X_2 = i)\mathbb{P}(X_2 = i) = \sum_{i=1}^{m-1} \mathbb{P}(H_{m-i}|H_1)\mathbb{P}(X_2 = i) \tag{3.0.2}$$

To get the conditional generating function $G_H(x) = \sum_{m=1}^{\infty} x^{m-1}\mathbb{P}(H_{m-i}|H_1)$, we have

$$\begin{aligned} G_H(x) - 1 = \sum_{m=2}^{\infty} x^{m-1}\mathbb{P}(H_m|H_1) &= \sum_{m=2}^{\infty} x^{m-1} \sum_{i=1}^{m-1} \mathbb{P}(H_{m-i}|H_1)\mathbb{P}(X_2 = i) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(X_2 = i)x^i \sum_{n=i}^{\infty} \mathbb{P}(H_{n-i+1}|H_1)x^{n-i} \quad \text{by convolution} \\ &= F(x)G_H(x) \end{aligned} \tag{3.0.3}$$

where $F(x) = \mathbb{E}(x^{X_2})$. Hence, $G_H(x) = 1/(1 - F(x))$. Let $U(x) = \sum_{n=0}^{\infty} x^n u_n$. Then

$$\begin{aligned} U(x) &= \sum_{n=0}^{\infty} x^n \sum_{i=1}^{n} \mathbb{P}(H_{n-i+1}|H_1)\mathbb{P}(X_1 = i) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(X_1 = i)x^i \sum_{n=i}^{\infty} \mathbb{P}(H_{n-i+1}|H_1)x^{n-i} \\ &= D(x)G_H(X) = \frac{D(X)}{1 - F(x)} \end{aligned} \tag{3.0.4}$$

where $D(x) = \mathbb{E}(x^{X_1})$. One choice of the generating function $D(x)$ is

$$D^* = \frac{1 - F(x)}{\mu(1 - x)} \tag{3.0.5}$$

for $|x| < 1$. We can verify by Theorem 1.1 that

$$D^*(1) = \lim_{x \uparrow 1} \frac{1 - F(x)}{\mu(1 - x)} = \lim_{x \uparrow 1} \frac{-F'(x)}{-\mu} = 1 \tag{3.0.6}$$

where $\mu = \mathbb{E}(X_2)$. With this choice of $D(x)$, we have

$$U^*(x) = \frac{D^*}{1 - F(x)} = \frac{1}{\mu(1 - x)} = \sum_{n=0}^{\infty} x^n \mu^{-1}. \tag{3.0.7}$$

In this case, the choice of μ_n is *constant*, so that the density of occurrence of H is constant as time passes. Such a process is called a *stationary* recurrent-event process. We conclude this section with the following theorem, whose proof is not provided in this note.

Theorem 3.1 (Renewal theorem). *If the mean inter-occurrence time μ is finite and the process is non-arithmetic, then $u_n = \mathbb{P}(H_n)$ satisfies $\mu_n \to \mu^{-1}$ as $n \to \infty$.*

where a process if *non-arithmetic* if $\gcd\{n : \mathbb{P}(X_2 = n) > 0\} = 1$.

References

[1] Grimmett, Geoffrey., and Stirzaker, David. Probability and Random Processes / Geoffrey R. Grimmett and David R. Stirzaker. 3rd ed., Oxford University Press, 2001.

[2] Resnick, Sidney I. Adventures in Stochastic Processes / Sidney Resnick. BirkhÃ¤user, 1992.

[3] Ross, Sheldon M. Introduction to Probability Models Sheldon M. Ross. 9th ed., Academic Press, 2007.