Notes on Matrix Rigidity

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Chapter 1

Motivation and Definition

Definition 1.0.1. The **density** of a matrix A is the number of nonzero elements drawn from a field $\mathbb{F}$, denoted by $\text{dens}(A)$.

Definition 1.0.2. The **rigidity** of a matrix A is the function $\mathcal{R}_A^{\mathbb{F}}(r) : \{1, ..., N\} \to \{0, 1, ..., N^2\}$ defined by

$$\mathcal{R}_A^{\mathbb{F}}(r) := \min\{i | \exists\, B, \text{dens}(B) = i, \text{rank}(A + B) \leq r\} \tag{1.0.0.1}$$

Valiant motivated the definition of matrix rigidity from the analysis of circuit complexity and proved that if A is a rigid matrix, i.e. $\mathcal{R}_A^{\mathbb{F}}(\epsilon N) = N^{1+\delta}$ for some $\epsilon, \delta > 0$, then the linear program to compute Ax cannot be a circuit of $+$ gates of size $O(N)$ and depth $O(n = \log N)$. Moreover, Valiant gave a non-constructive proof for the following.

Theorem 1.0.3 ([Val77])**.**

1. *For an infinite field $\mathbb{F}$, for all N, there exists a $N \times N$ matrix A such that $\mathcal{R}_A^{\mathbb{F}}(r) = (N - r)^2$.*
2. *For a finite field $\mathbb{F}$ with c elements, for all N, there exists a $N \times N$ matrix A such that for all $r < N - \sqrt{2N \log_c 2 + \log_2 N}$,*

$$\mathcal{R}_A^{\mathbb{F}}(r) \geq \frac{(N - r)^2 - 2N \log_c 2 - \log_2 N}{2 \log_c N + 1} \tag{1.0.0.2}$$

Open Problem 1.0.4. Find *explicit* matrices A.

Chapter 2

Explicit Lower Bounds

For explicit lower bounds, most of the proofs consist of two steps: first, we show that most sub-matrices of the given matrix M has large or full rank r; second, if $\mathcal{R}_A^{\mathbb{F}}(r)$ is small, we are likely to get a sub-matrix that remains intact. The first step implies that it is plausible to find rigid matrices with high regularity.

Definition 2.0.1 ([BCS97])**.** A matrix A is called **totally regular** if and only if every minor of A is invertible.

A relaxation of this definition will suffice for our purpose.

Definition 2.0.2. A matrix A is called **almost totally regular** if and only if every $r \times r$ minor of A has rank $\Omega(r)$.

Another notion of regularity based on expectation is introduced by Pudlak, called densely regular.

Definition 2.0.3 ([Pud94])**.** Let A be an $N \times N$ matrix, $0 \leq \epsilon, \delta, \eta \leq 1$. We say that A is (ϵ, δ, η)**–densely regular**, if for every k with $\eta N \leq k \leq N$, there are nonempty sets of k elements subset $\mathcal{X}, \mathcal{Y} \in [1, n]^k$ such that for every $i, j = 1, ..., N$

$$\delta \mathbb{P}[i \in X] \leq k/N \quad \text{and} \quad \delta \mathbb{P}[j \in Y] \leq k/N \tag{2.0.0.1}$$

where $X \in \mathcal{X}, Y \in \mathcal{Y}$ are chosen with some probability distributions and such that for random $X \in \mathcal{X}, Y \in \mathcal{Y}$ the mean value of the rank of the matrix determined by X and Y is at least ϵk.

Again, a relaxation of this will also be sufficient to us.

Definition 2.0.4 ([Che05])**.** Let A be an $N \times N$ matrix. We say that A is ϵ**-densely regular**, if there is a constant $0 < \epsilon < 1$ such that for every k with $0 \leq k \leq N$, a $k \times k$ minor of A picked uniformly at random has an expected rank at least ϵk.

However, as pointed out by Lokam [Lok00], any proof relying on the second step cannot produce a lower bound better than $\Omega((n^2/r)\log(n/r))$. Moreover, due to the existence of linear size superconcentrators, the first step is far from sufficient to show a desirable rigidity.

Proposition 2.0.5 ([Val77])**.** *For each N there is an $N \times N$ totally regular matrix A such that*

$$\mathcal{R}_A^{\mathbb{F}}\left(\frac{N \log\log\log N}{\log\log N}\right) \leq N^{1+O(\frac{1}{\log\log N})} \tag{2.0.0.2}$$

Nevertheless, finding explicit matrices with high regularity is still a reasonable start.

2.1 Totally Regular Matrices ❖

2.1.1 A Combinatorial Lemma ❄

Lemma 2.1.1 ([SSS97])**.** *If fewer than*

$$\mu(N,r) = (N-r+1)(N-(r-1)^{1/r}N^{1-1/r}) \tag{2.1.1.1}$$

entries of an $N \times N$ matrix A is marked, then there is an $r \times r$ submatrix that remains intact.

Proof. Think of A as the adjacency matrix of a bipartite graph $G_{N,N}$ which contains an edge (i,j) if and only if $A_{i,j}$ has not been marked. Hence, having an $r \times r$ submatrix that remains intact is equivalent to having a $K_{r,r}$ complete bipartite subgraph in $G_{N,N}$. Hence, we have that $G_{N,N}$ cannot have more than $n^2 - \mu(n,r)$ edges, or it will contain a $K_{r,r}$ complete bipartite subgraph (see [Juk11], Theorem 2.10). □

Corollary 2.1.2. *Let $r \geq \log^2 n$ and let N be sufficiently large. If fewer than*

$$\frac{N(N-r+1)}{2r}\log\frac{N}{r-1} \tag{2.1.1.2}$$

changes are made to an $N \times N$ matrix A, then there exists an $r \times r$ submatrix that remains intact.

It is easy to see that if any $r \times r$ minor has rank $\Omega(r)$, we have that the rigidity of A is $\Omega(\frac{N^2}{r}\log\frac{N}{r})$.

2.1.2 Cauchy Matrix ❄

Definition 2.1.3. Let $x_1, ..., x_n, y_1, ..., y_n$ be elements of a field $\mathbb{F}_N$ with the property that

$$\prod_{i\neq j}(x_i - x_j) \neq 0, \quad \prod_{i\neq j}(y_i - y_j) \neq 0, \quad \prod_{i,j}(x_i + y_j) \neq 0 \tag{2.1.2.1}$$

we define the **Cauchy matrix** by

$$C := \Big(\frac{1}{x_i + y_j}\Big)_{1\leq i,j\leq n} \tag{2.1.2.2}$$

Hence, for every $1 \leq r \leq n$, each of its $r \times r$-submatrix has the determinant

$$\frac{\prod_{i\neq j}(x_i - x_j)\prod_{i\neq j}(y_i - y_j)}{\prod_{i,j}(x_i + y_j)} \tag{2.1.2.3}$$

which is nonzero. In other words, the Cauchy matrix is totally regular.

The following theorem is thus a direct result of Corollary 2.1.2.

Theorem 2.1.4 ([SSS97])**.** *Let $\mathbb{F}_N$ be a sequence of fields and let (C_N) be a sequence of Cauchy matrices where $C_N \in \mathbb{F}_N^{N\times N}$. Then if $\log^2 N \leq r \leq N/2$, we have*

$$\mathcal{R}_{C_N}^{\mathbb{F}_N} \geq \Big(\frac{N^2}{4r}\log\frac{N}{r-1}\Big) \tag{2.1.2.4}$$

2.1.3 Fourier Transform Matrix ❄

Another type of totally regular matrices is the **Discrete Fourier Transform Matrix**. Hence, the following result is also a direct consequence of Corollary 2.1.2. Also note that Discrete Fourier Transform Matrix is a type of Vandermonde Matrix.

Theorem 2.1.5 ([Lok00], [Lok09])**.** *Let $F = (\omega_i^{j-1})_{i,j=0}^{N-1}$, where ω is a primitive nth root of unity. Then, as N ranges over all prime numbers and $\log^2 N \leq r \leq N/2$,*

$$\mathcal{R}_F(r) \geq \frac{N^2}{4(r+1)}\log\frac{N}{r} \tag{2.1.3.1}$$

Theorem 2.1.6 ([DL19])**.** *Let F denote the $N \times N$ Fourier transform matrix. For any fixed $0 < \epsilon < 0.1$ and N sufficiently large,*

$$\mathcal{R}_F\Big(\frac{N}{2^{\epsilon^6 n^{0.35}}}\Big) \leq N^{15\epsilon} \tag{2.1.3.2}$$

where $n = \log N$.

2.1.4 Asymptotically Good Error Correcting Codes ❄

Lemma 2.1.7 (Tsfasman-Vladut-Zink Bound [TVZ82], as stated in [Lok09])**.** *Let q be the square of a prime. Then for every rate p, there exists an infinite sequence $[n_t, k_t, \mathcal{R}_t], t = 1, 2, ...$ of codes over $\mathbb{F}_q$ such that the asymptotic rate $p = \lim_{t\to\infty} k_t/n_t$ and the asymptotic relative distance $\delta = \lim_{t\to\infty} d_t/n_t$ such that*

$$p \geq 1 - \delta - \frac{1}{\sqrt{q} - 1} \tag{2.1.4.1}$$

Corollary 2.1.8. *For infinitely many N, there exists a $[2N, N, d]$-code with $d \geq (1 - \epsilon)N$ where $\epsilon = 2/(\sqrt{q} - 1)$.*

For a given N, let Γ be the $[2N, N, d]$-code as in Corollary 2.1.8, whose generator matrix has the form $(I_N|A)$ where I_N is the $N \times N$ identity matrix.

Theorem 2.1.9 ([SSS97])**.** *Let A be an $N \times N$ matrix as defined above. Then, for $\max(\log^2 N, \epsilon N) \leq r \leq N/4$,*

$$\mathcal{R}_A^{\mathbb{F}_q}(r) \geq \frac{N^2}{8r} \log \frac{N}{2r - 1} \tag{2.1.4.2}$$

Proof. We first show that for all $2r \times 2r$ submatrix of A, the rank must be at least r. Suppose on the contrary, then let B be a $2r \times 2r$ submatrix of A with r dependent rows. Then a linear combination of these r rows of the generator matrix gives a code word of weight

$$r + N - 2r = N - r \leq (1 - \epsilon)N - 1 \leq d \tag{2.1.4.3}$$

where the first r comes from the identity matrix and the $N - 2r$ comes from the fact that these r rows are dependent. Hence, we reach a contradiction.
By Corollary 2.1.2, we obtain the desired result. □

2.2 Densely Regular Matrices ❖

Theorem 2.2.1. *For every positive ϵ, δ, there exists a positive α such that for any field $\mathbb{F}$, every $\eta, 0 < \eta \leq 1$, and any $N \times N$ (ϵ, δ, η)–densely regular matrix A,*

$$\mathcal{R}_A^{\mathbb{F}}(r) \geq \alpha \frac{N^2}{r} \tag{2.2.0.1}$$

for $\epsilon\eta N/2 \leq r \leq \epsilon N/2$

Proof. Let $\epsilon\eta N/2 \leq r \leq \epsilon N/2$ and sets $\mathcal{X}, \mathcal{Y}$ be given and set $k = \lceil 2r/\epsilon \rceil$. For random variables $X \in \mathcal{X}, Y \in \mathcal{Y}$,, we have that for a coordinate (i, j),

$$\mathbb{P}[(i, j) \in X \times Y] \leq \frac{k^2}{\delta^2 N^2} \tag{2.2.0.2}$$

Let Z be the minimal set of coordinates changed to reduce the rank of A to r and let z be the number of coordinates in the set $Z \cap (X \times Y)$. Then, we have

$$\mathbb{E}[z] = \sum_{(i,j)\in Z} \mathbb{P}[(i, j) \in X \times Y] \leq |Z| \frac{k^2}{\delta^2 N^2} \tag{2.2.0.3}$$

Notice that on the other hand, the mean value of the rank of the matrix determined by X and Y is at least ϵk, which implies

$$\mathbb{E}[z] \geq \epsilon k - r \geq 2r - r = r \tag{2.2.0.4}$$

Therefore, we have

$$r \leq \mathbb{E}[z] \leq |Z| \frac{k^2}{\delta^2 N^2} \tag{2.2.0.5}$$

and thus by plugging $k = \lceil 2r/\epsilon \rceil$

$$|Z| \geq \frac{r\delta^2 N^2}{k^2} = \Omega(\frac{N^2}{r}) \tag{2.2.0.6}$$

□

The proof for ϵ-densely regular matrix is very similar.

2.2.1 Vandermonde Matrix ❄

Proposition 2.2.2. *Let $V = (x_i^{j-1})_{i,j=1}^N$ be a Vandermonde matrix with distinct x_i over some field. V is $(1, 1/2, 0)$-densely regular.*

By theorem 2.2.1, we prove the $\Omega(N^2/r)$ lower bound for Vandermonde matrices. An alternative proof is given by Shparlinsky.

2.2.2 Hadamard Matrix ❄

Definition 2.2.3. A matrix $H = (h_{i,j}) \in \mathbb{C}^{N\times N}$ is called a **(generalised) Hadamard matrix** if $|h_{i,j}| = 1$ for all $i, j \in [n]$ and $HH^* = NI_N$ where H^* is the conjugate transpose of H and I_N is the $N \times N$ identity matrix.

In other words, a matrix H is a (generalised) Hadamard matrix if $|h_{i,j}| = 1$ for all $i, j \in [n]$ and the rows of H are pairwise orthogonal.

Definition 2.2.4. The **Frobenius norm** of a matrix $A \in \mathbb{C}^{N\times N}$ is

$$||A||_F := \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{1/2} \tag{2.2.2.1}$$

Definition 2.2.5. The **trace** of a matrix $A \in \mathbb{C}^{N\times N}$ is the sum of its eigenvalues.

$$\text{Tr}(A) = \sum_{i=1}^{N} \lambda_(A) \tag{2.2.2.2}$$

Definition 2.2.6. The i**th singular value** $\sigma_i(A)$ is defined by

$$\sigma_i(A) := \sqrt{\lambda_i(AA^*)}, 1 \leq i \leq n \tag{2.2.2.3}$$

where λ_i denotes the ith largest eigenvalue of AA^*.

We recall some fact about singular value decomposition and Frobenius norm. The proof can be found in chapter 2.4 in [GVL12].

Proposition 2.2.7. *For any matrix $A \in \mathbb{C}^{N\times N}$,*

- *there exists unitary matrices $U, V \in \mathbb{C}^{N\times N}$ such that*

$$U^*AV = \text{diag}(\sigma_1(A), \sigma_2(A), ..., \sigma_N(A)) \tag{2.2.2.4}$$

- $||A||_F^2 = \sigma_1^2(A) + \sigma_2^2(A) + ... + \sigma_N^2(A)$.

Proposition 2.2.8. *If $A \in \mathbb{R}^{N\times N}$ is symmetric, then*

$$\frac{\text{Tr}(A)^2}{||A||_F^2} \leq \text{rank}(A) \tag{2.2.2.5}$$

Proof. Let $B = AA^*$. Notice that

$$\mathrm{Tr}(B) = \sum_{i=1}^{N} \lambda_i(B) = ||A||_F^2 \tag{2.2.2.6}$$

Because A is symmetric,

$$\sum_{i=1}^{N} \lambda_i(B) = \mathrm{Tr}(B) = \sum_{i=1}^{N} \lambda_i(A^2) = \sum_{i=1}^{N} \lambda_i^2(A) \tag{2.2.2.7}$$

Moreover, B has only rank(B) =rank(A) non-zero eigenvalues, which are all positive. Assume without loss of generality $\lambda_1^2(A) \geq \lambda_2^2(A) \geq ... \geq \lambda_N^2(A)$. Then, $\sum_{i=1}^{N} \lambda_i^2(A) = \sum_{i=1}^{\mathrm{rank}(A)} \lambda_i^2(A)$. Hence, by Cauchy-Schwarz inequality, we have

$$||A||_F^2 = \sum_{i=1}^{\mathrm{rank}(A)} \lambda_i^2(A) \geq \frac{\left(\sum_{i=1}^{\mathrm{rank}(A)} \lambda_i(A)\right)^2}{\mathrm{rank}(A)} \geq \frac{\mathrm{Tr}(A)^2}{\mathrm{rank}(A)} \tag{2.2.2.8}$$

□

Proposition 2.2.9 ([KR98])**.** *Let H be an $N \times N$ generalised Hadamard matrix. Let G be a random $q \times N$ submatrix of H and let A be a random $q \times q$ submatrix of G. Then* $\mathbb{E}[\mathrm{rank}(A)] \geq r/8$.

Proof. Let $B = AA^*$. Then B is a positive definite symmetric matrix in $\mathbb{R}^{q \times q}$. Recall that $h_{i,j} = 1$, which implies all entries of B on the main diagonal equals to q. Thus $\mathrm{Tr}(B) = q^2$. By proposition 2.2.8, we obtain that rank$(A) \leq r$ for some positive integer r implies

$$||B||_F^2 \geq \frac{\mathrm{Tr}(B)^2}{\mathrm{rank}(B)} \geq \frac{q^4}{r} \tag{2.2.2.9}$$

Let

$$\epsilon_j = \begin{cases} 1 & \text{if the } j\text{th column of } H \text{ is in } H_0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2.2.10}$$

we have

$$\mathbb{E}[\epsilon_{j_1}\epsilon_{j_2}] = \begin{cases} \frac{q}{N} & \text{if } j_1 = j_2 \\ \frac{q(q-1)}{N(N-1)} & \text{if } j_1 \neq j_2 \end{cases} \tag{2.2.2.11}$$

Now, notice that $b_{i,j} = \sum_{k=1}^{q} a_{i,k}a_{j,k}^* = \sum_{k=1}^{q} g_{i,k}g_{j,k}^*\epsilon_k$ and $b_{i,j}^* = \sum_{l=1}^{q} a_{i,l}^* a_{j,l} = \sum_{k=1}^{q} g_{i,k}g_{j,k}^*\epsilon_l$

$$\begin{aligned} ||B||_F^2 &= \sum_{1\leq i,j\leq q} |b_{i,j}|^2 = \sum_{1\leq i,j\leq q} b_{i,j}b_{i,j}^* = \sum_{1\leq i\leq q, 1\leq j\leq n} \sum_{1\leq k,l\leq q} g_{i,k}g_{j,k}g_{i,l}^*g_{j,l}^*\epsilon_k\epsilon_l \\ &= \sum_{1\leq k,l\leq q} \left(\epsilon_k\epsilon_l \sum_{1\leq i\leq q, 1\leq j\leq n} g_{i,k}g_{j,k}g_{i,l}^*g_{j,l}^*\right) \end{aligned} \tag{2.2.2.12}$$

Thus,

$$\begin{aligned} \mathbb{E}[||B||_F^2] &= \sum_{1\leq k,l\leq q} \left(\mathbb{E}[\epsilon_k\epsilon_l] \sum_{1\leq i\leq q, 1\leq j\leq n} g_{i,k}g_{j,k}g_{i,l}^*g_{j,l}^*\right) \\ &= \frac{q(q-1)}{N(N-1)} \sum_{1\leq k,l\leq q} \sum_{1\leq i\leq q, 1\leq j\leq n} g_{i,k}g_{j,k}g_{i,l}^*g_{j,l}^* + \left(\frac{q}{N} - \frac{q(q-1)}{N(N-1)}\right) \sum_{1\leq i\leq q, 1\leq j\leq n} g_{i,k}g_{j,k}g_{i,l}^*g_{j,l}^* \\ &= \frac{q(q-1)}{N(N-1)}||GG^*||_F^2 + \left(\frac{q}{N} - \frac{q(q-1)}{N(N-1)}\right) \sum_{k=1}^{q} \sum_{1\leq i\leq q, 1\leq j\leq n} g_{i,k}g_{i,k}^*g_{j,k}g_{j,k}^* \end{aligned} \tag{2.2.2.13}$$

Notice that $GG^* = NI_q$ where I_q is the $q \times q$ identity matrix.

$$\begin{aligned}\mathbb{E}[||B||_F^2] &= \frac{q(q-1)}{N(N-1)}||GG^*||_F^2 + \Big(\frac{q}{N} - \frac{q(q-1)}{N(N-1)}\Big)\sum_{k=1}^{q}\sum_{1\le i\le q, 1\le j\le n} g_{i,k}g_{i,k}^*g_{j,k}g_{j,k}^* \\ &= \frac{q(q-1)}{N(N-1)}N^2q^2 + \Big(\frac{q}{N} - \frac{q(q-1)}{N(N-1)}\Big)Nq^2 \\ &= q^2(q + (N-q)\frac{q-1}{N-1}) \le 2q^3\end{aligned} \tag{2.2.2.14}$$

By Chebyshev's inequality, we have

$$\mathbb{P}[||B||_F^2 \ge \frac{q^4}{r}] \le \frac{r}{q^4}\mathbb{E}[||B||_F^2] \le \frac{2r}{q} \tag{2.2.2.15}$$

Thus, we have $\mathbb{P}[\text{rank}(A) \le r] \le \mathbb{P}[||B||_F^2 \ge \frac{q^4}{r}] \le 2r/q$. Choosing $r = q/4$, we have$\mathbb{P}[\text{rank}(A) \le q/4] \le 1/2$. Again, using Chebyshev's inequality,

$$\mathbb{E}[\text{rank}(A)] \ge \frac{q}{4}\mathbb{P}[\text{rank}(A) \ge \frac{q}{4}] = \frac{q}{4}(1 - \mathbb{P}[\text{rank}(A) \ge \frac{q}{4}]) \ge \frac{q}{8} \tag{2.2.2.16}$$

□

Corollary 2.2.10. *Let H be an $N \times N$ generalised Hadamard matrix, then H is $1/8$-densely regular.*

2.3 Averaging Argument ❖

Another set of proofs utilises averaging argument to select some number of rows that has small changes and then show that the remaining part of these rows has high rank.

2.3.1 Vandermonde Matrix ❄

Theorem 2.3.1 (Shparlinsky, see [Lok00])**.** *Let $V = (x_i^{j-1})_{i,j=1}^N$ be a Vandermonde matrix with distinct x_i over some field. Then*

$$\mathcal{R}_V(r) \ge \frac{(N-r)^2}{r+1} \tag{2.3.1.1}$$

Proof. Let r be given and let $s = \mathcal{R}_V(r)$. By averaging argument, we can select $r+1$ consecutive columns such that the total number of changes within these columns are at most $s(r+1)/(N-r)$. Then we select the rows that do not contain any changes in these these columns, which gives us at least $N - s(r+1)/(N-r)$ rows. Hence, we constructed a submatrix S of size $(r+1) \times (N - s(r+1)/(N-r))$. Because the rank of this submatrix is at most r, we have that there exists a nonzero vector g such that $S\mathbf{g} = 0$. In other words, we obtain a polynomial $\sum_{t=0}^r g_t x^t = 0$ with at least $(N - s(r+1)/(N-r))$ roots. On the other hand, this polynomial can have at most r roots. Therefore,

$$r \ge (N - s(r+1)/(N-r)) \tag{2.3.1.2}$$

which gives $s \ge (N-r)^2)/(r+1)$. □

2.3.2 Hadamard Matrix ❄

Proposition 2.3.2 (Alon, see [Juk11])**.** *Every non-trivial linear combination of any k rows of a Hadamard matrix $H = (h_{i,j}) \in \mathbb{C}^{N\times N}$ has at least N/k nonzero entries.*

Proof. Let A be a $k \times n$ submatrix of H and let $y = x^T A$ for some nonzero vector $x \in \mathbb{R}^k$. Let S be the set of the coordinates of the non-zero entries in y and let $s = |S|$. We need to show that $s \geq n/k$.
Assume without loss of generality that $x_1 = \max_{i\in[k]} |x_i|$. Let a^i denote the ith row of A. Because the rows of A are mutually orthogonal, we have

$$kx_1^2 N \geq \sum_{i=1}^{k} x_i^2 N = \sum_{i=1}^{k} \langle x_i a^i, x_i a^i \rangle = \langle \sum_{i=1}^{k} x_i a^i, \sum_{i=1}^{k} x_i a^i \rangle \tag{2.3.2.1}$$

Notice that $\sum_{i=1}^{k} x_i a^i = x^T A = y$, we have

$$\langle \sum_{i=1}^{k} x_i a^i, \sum_{i=1}^{k} x_i a^i \rangle = \langle y, y \rangle = \sum_{j=1}^{N} y_j^2 = \sum_{j=1}^{N} y_j^2 = \sum_{j\in S} y_j^2 = \sum_{j\in S} |y_j|^2 \tag{2.3.2.2}$$

Using Cauchy-Schwarz inequality, we have

$$\sum_{j\in S} |y_j|^2 \geq \frac{1}{s} (\sum_{j\in S} |y_j|)^2 = \frac{1}{s} (\sum_{j=1}^{N} |y_j|)^2 \tag{2.3.2.3}$$

On the other hand, because $|a_{i,j}| = 1$, we have

$$\begin{aligned} \sum_{j=1}^{N} |y_j| &\geq \sum_{j=1}^{N} y_j a_{1,j} = \sum_{j=1}^{N} \langle x, a^j \rangle a_{1,j} = \sum_{j=1}^{N} \sum_{i=1}^{k} x_i a_{i,j} a_{1,j} \\ &= \sum_{i=1}^{k} x_i \sum_{j=1}^{N} a_{i,j} a_{1,j} = \sum_{i=1}^{k} x_i \langle a^i, a^1 \rangle = x_1 \langle a^1, a^1 \rangle = x_1 N \end{aligned} \tag{2.3.2.4}$$

This gives us

$$kx_1^2 N \geq \frac{1}{s} (x_1 N)^2 \tag{2.3.2.5}$$

Thus, $s \geq N/k$. □

Notice that the following two corollaries holds for real-valued Hadamard matrices.

Corollary 2.3.3 (Alon, see [Juk01])**.** *If $t > (1 - 1/r)N$, then every $r \times t$ sub-matrix H' of an $N \times N$ Hadamard matrix $H \in \mathbb{R}^{N\times N}$ has rank r.*

Proof. For the sake of contradiction, we assume the opposite that $\text{rank}(H') < r$. Hence, there exists a nonzero vector $x \in \mathbb{R}^r$ such that $x^t H' = 0$. Because $t > (1 - 1/r)N$, this contradicts with proposition 2.3.2 that any nonzero linear combination of these r rows of H has at least N/r nonzero entries. □

Corollary 2.3.4 (Alon, see [Juk01])**.** *If fewer than $(n/r)^2$ entries of an $N \times N$ Hadamard matrix $H \in \mathbb{R}^{N\times N}$ are changed), then the rank of the resulting matrix remains at least r.*

Proof. By averaging argument, we can choose (n/r) rows that has fewer than (n/r) changes in total. Therefore, the number of columns that remain intact in these (n/r) rows is greater than $(1 - 1/r)N$. By 2.3.3, we complete the proof. □

Lemma 2.3.5 ([Lok95])**.** *For any $u \times v$ submatrix H_0 if an $N \times N$ generalised Hadamard matrix H,* $\text{rank}(H_0) \geq uv/N$.

Proof. Let $A \in \mathbb{C}^{k\times k}$ for some $k > 0$. Let $\lambda_1(A)$ be the largest eigenvalue of AA^*. We thus have

$$\frac{||A||_F^2}{\lambda_1(A)} = \frac{\sum_{i=1}^{N} \lambda_i(A)}{\lambda_1(A)} \tag{2.3.2.6}$$

Notice that AA^* has exactly rank(AA^*) =rank(A) nonzero entries, all of which are positive, which implies

$$\frac{||A||_F^2}{\lambda_1(A)} = \frac{\sum_{i\in[N],\lambda_i(A)>0}\lambda_i(A)}{\lambda_1(A)} \leq \text{rank}(A) \tag{2.3.2.7}$$

On the other hand, H_0 is a submatrix of H, we have $\lambda_1(H_0) \leq \lambda_1(H)$. Thus

$$\frac{||H_0||_F^2}{\lambda_1(H_0)} \geq \frac{||H_0||_F^2}{\lambda_1(H)} = \frac{uv}{N} \tag{2.3.2.8}$$

Notice that the last equality follows from the fact that $H_0H_0^* = vI_u$. Therefore, we have rank$(H_0) \geq ||H_0||_F^2/\lambda_1(H_0) \geq uv/N$. □

Theorem 2.3.6 ([dW06])**.** *If $r \leq N/2$, then $\mathcal{R}_H(r) \geq N^2/4r$.*

Proof. Let r be given and let $s = \mathcal{R}_H(r)$. By averaging argument, we can select $2r$ rows that has fewer than $2rs/N$ changes. If $2rs/N \geq N$, we have $s \geq N^2/(2r)$ and we are done. If $2rs/N < N$, we then have that by lemma 2.3.5, for the submatrix H_0 that contains the $N - 2rs/N$ intact columns of these $2r$ rows,

$$r \geq \text{rank}(H_0) \geq \frac{2r(N-2rs/N)}{N} \tag{2.3.2.9}$$

which implies $s \geq N^2/4r$. □

The same bound can be proved with a much simpler argument for a special type of Hadamard matrix called the **Sylvester matrix**, which is recursively defined as follows:

- $S_1 := (1)$.
- $S_{2n} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H_n = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$

Theorem 2.3.7 ([Mid05])**.** *If $S(N)$ is a Sylvester matrix and $r \leq N/2$ is a power of 2, then*

$$\mathcal{R}_{S(N)}(r) \geq \frac{N^2}{4r} \tag{2.3.2.10}$$

Proof. Let r be given and let $s = \mathcal{R}_S(r)$. Assume on the contrary that $s < N^2/4r$. If we divide S into $(N/2r)^2$ grids of size $2r \times 2r$, then by averaging argument, there exists a grid that has fewer than

$$s \cdot \frac{(2r)^2}{N^2} < \frac{N^2}{4r} \cdot \frac{(2r)^2}{N^2} = r \tag{2.3.2.11}$$

changes. Notice that each grid has full rank because it is exactly a Sylvester matrix of size $2r \times 2r$. Then this grid still has rank more than $2r - r = r$ after these r changes. Hence, the rank of S after these s changes will be more than r, which gives us a contradiction. □

Remark 2.3.8. Notice that this simple proof can be applied to any totally regular matrix, for example, the Discrete Fourier Transform Matrix.

Chapter 3

Somewhat-Explicit Lower Bounds

3.1 The Shoup–Smolensky Dimensions ❖

Definition 3.1.1 ([Mor96])**.** Let K be a field extension of $\mathbb{F}$, and let $t_1, ..., t_n \in K$. The set $\{t_1, ..., t_n\}$ is **algebraically independent** over $\mathbb{F}$ if $f(t_1, ..., t_n) \neq 0$ for all nonzero polynomials $f \in \mathbb{F}[x_1, ..., x_n]$.

Theorem 3.1.2 ([Lok00])**.** *Let $V = (x_i^{j-1})_{i,j=1}^N$ be a Vandermonde matrix where x_i are algebraically independent over $\mathbb{Q}$. Then*

$$\mathcal{R}_V(r) \geq \frac{N(N - cr^2)}{2} \tag{3.1.0.1}$$

where $c > 0$ is an absolute constant.

Theorem 3.1.3 ([Lok06])**.** *Let A be an $N \times N$ matrix over C and $0 \leq r \leq n$. Suppose, $D_{Nr}(A) = \binom{N^2}{Nr}$,i. e., all products of Nr distinct entries of A are linearly independent over $\mathbb{Q}$. Then,*

$$\mathcal{R}_A(r) \geq N(N - 16r) \tag{3.1.0.2}$$

3.2 Random Toeplitz/Hankel Matrices ❖

Let $m, k \in \mathbb{N}, 16 \leq k \leq m$. Let $A \in \mathbb{F}_2^{m \times m}$ be the random matrix

$$\begin{bmatrix} a_1 & a_2 & ... & a_m \\ a_{k+1} & a_{k+2} & ... & a_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(m-1)k+1} & a_{(m-1)k+2} & ... & a_{(m-1)k+m} \end{bmatrix}$$

where $a_1, a_2, ..., a_{(m-1)k+m}$ are uniform independent random bits, and let $S \in \mathbb{F}_2^{m \times m}$ be some fixed matrix. We aim to show the following lemma

Lemma 3.2.1 ([GT18])**.** $\mathbb{P}_A[\text{rank}(S + A) \leq m/2] \leq 2^{-km/16}$.

Let $B = S + A$ and let B_i denote the ith row of B. If rank$(B) \leq m/2$, then we can find a basis $B_{i_1}, B_{i_2}, ..., B_{i_{\text{rank}(B)}}$ of the row space spanned by B in the following constructive fashion:

1. Let i_1 be the index of the first nonzero row of B.
2. For each t, let i_t be the index of the first row of B that cannot be spanned by $B_{i_1}, ..., B_{i_{t-1}}$.

Let an index set $\mathcal{I} = \{i_1, ..., i_r\}, r \leq m/2$ be given and set $\mathcal{J} = [m] \setminus \mathcal{I}$. We have that

$$\forall j \in J, B_j \in \text{span}\{B_i : i \in I, i < j\} \tag{3.2.0.1}$$

Let an arbitrary $j \in J$ be given. Notice that if we fix the random bits $a_1, ..., a_{j-1}k$, the jth row is completely undetermined because the first entry of the jth row is $a_{(j-1)k+1}$.

Claim 3.2.2. Let $\mathcal{I}' = \mathcal{I} \cap [j-1]$ and $p = |\mathcal{I}'|$ and fix a vector $\mathbf{c} \in \{0,1\}^p$. We have that

$$\mathbb{P}[B_j = \sum_{i \in \mathcal{I}'} c_i B_i] = 2^{-m}. \tag{3.2.0.2}$$

Proof. Notice that for all $h \in [m], B_{j,h} = S_{j,h} + a_{(j-1)k+h}$ where $S_{j,h}$ is fixed but $a_{(j-1)k+h}$ is not. Hence, since $\sum_{i \in \mathcal{I}'} c_i B_i$ is fixed, we have that the hth bit of this linear combination will be equal to the hth bit of B_j with probability exactly $1/2$ since $a_{(j-1)k+h}$ is uniformly chosen from $\{0,1\}$. Also notice that each of these probabilities are independent since $a_{(j-1)k+h}$ are independently chosen. □

Let $\Delta = \lceil m/k \rceil$, then we can select an increasing sequence of $|J|/\Delta$ indices in J such that each two indices differ by at least Δ. Let $j_1, j_2, ..., j_t$ be such a sequence of indices where $t \geq |J|/\Delta$. For each $l \in [t]$, let E_l be the event that j_lth row is spanned by the rows indexed by $\mathcal{I} \cap [j_l - 1]$.

Claim 3.2.3. For all $l \in [t], \mathbb{P}[E_l | E_1, E_2, ..., E_{l-1}] \leq 2^{-m/2}$.

Proof. Notice that $j_l \geq j_{l-1} + \Delta =\geq j_{l-1} + \lceil m/k \rceil$. That is, $(j_l - 1)k \geq (j_{l-1} - 1)k + m$. On the other hand, given fixed bits $a_1, ..., a_{j_l - 1}$, we can determine the rows $B_{j_1}, ..., B_{j_{l-1}}$ but $B_{j_l} = (a_{(j_l-1)k+1}, ..., a_{(j_l-1)k+m})$. Hence, we have

$$\mathbb{P}[E_l | E_1, E_2, ..., E_{l-1}] \leq \mathbb{P}[E_l | a_1, ..., a_{j_l - 1}] \tag{3.2.0.3}$$

By claim 3.2.2, for a fixed linear combination, we have $\mathbb{P}[E_l | a_1, ..., a_{j_l-1}, \mathbf{c}] = 2^{-m}$. Let $\mathcal{I}' = \mathcal{I} \cap [j-1]$ and $p = |\mathcal{I}'|$. Because there are 2^p different values for $\mathbf{c}$, and recall that $p \leq r \leq \text{rank}(B) \leq m/2$, by union bound, $\mathbb{P}[E_l | a_1, ..., a_{j_l - 1}] \leq 2^p 2^{-m} \leq 2^{-m/2}$. □

We are now in a good shape to prove the following lemma.

Lemma 3.2.4. *Let E be the event that*

$$\forall j \in \mathcal{J}, B_j \in \textit{span}\{B_i : i \in \mathcal{I}, i < j\} \tag{3.2.0.4}$$

for a given index set $\mathcal{I}$. Then $\mathbb{P}[E] \leq 2^{-mk/8}$.

Proof. Recall that $(j_1, j_2, ..., j_t)$ is a sequence of indices where $t \geq |J|/\Delta$ and each two indices are Δ apart. For each $l \in [t]$, let E_l be the event that j_lth row is spanned by the rows indexed by $\mathcal{I} \cap [j_l - 1]$. Hence,

$$\mathbb{P}[E] \leq \mathbb{P}[E_1, E_2, ..., E_{t-1}, E_t] = \mathbb{P}[E_1]\mathbb{P}[E_2|E_1]...\mathbb{P}[E_t | E_1, E_2, ..., E_{t-1}] \leq \left(2^{-m/2}\right)^t \tag{3.2.0.5}$$

Notice that

$$t \geq |J|/\Delta \geq \frac{m/2}{\lceil m/k \rceil} \geq k/4 \tag{3.2.0.6}$$

Therefore, $\mathbb{P}[E] \leq \left(2^{-m/2}\right)^t \leq 2^{-mk/8}$. □

proof of lemma 3.2.1. We can simply apply union bound among all possible choices of $\mathcal{I}$, which is less than 2^m. Hence, because we chose $k \geq 16, \mathbb{P}_A[\text{rank}(S + A) \leq m/2] \leq 2^m \mathbb{P}[E] \leq 2^{-km/16}$. □

In this section we show a proof for Hankel matrices in the field $\mathbb{F}_2$ of the following theorem. The main idea of this proof is based on an averaging argument very similar to the one used to prove theorem 2.3.7.

Theorem 3.2.5 ([GT18])**.** *Let T be a random Toeplitz/Hankel matrix of size $N \times N$. Then, for every $r \in [\sqrt{n}, n/32]$, with probability $1 - o(1)$, the matrix T has rigidity $\Omega(\frac{n^3}{r^2 \log n})$.*

Chapter 4

Paturi-Pudlák Dimensions

We first introduce the definition of sparsity.

Definition 4.0.1 (Sparsity)**.** A vector $v \in \mathbb{F}^n$ is s**-sparse** if the number of non-zero coordinates in v is at most s. A matrix $A \in \mathbb{F}^{M\times N}$ is s**-sparse** if it has s nonzero entires. A is s**-row sparse** if each of its row is s-sparse. A subspace $V \in \mathbb{F}^M$ is s**-sparse** if it is the *column space* of a s-row sparse matrix B.

4.1 Friedman's result ❖

In [Fri93], Friedman defined that an $N \times N$ matrix is A is (s,t)-rigid if for any s-sparse matrix B, we have that rank$(A+B) \geq t$.

Theorem 4.1.1 ([Fri93])**.** *For any constant $C_1 > 0$ there is a constant $C_2 > 0$ such that the following holds. Let $\mathbb{F}$ be a finite field of q elements. Let A be an $N \times N$ matrix such that the first $N/2$ rows are the basis of a linear error-correcting code in $\mathbb{F}^N$ of minimum distance $\geq C_1 N$. If B is any $N \times N$ matrix over $\mathbb{F}$ with at most k non-zero entries in each row, where $k \leq N/C_2$, then we have*

$$\text{rank}(A+B) \geq \frac{N}{C_2 k}(\log_q k + \log_q(q-1)) \tag{4.1.0.1}$$

Let r denote the rank of $A+B$, notice that this theorem implies

$$k \geq \frac{N}{r}(\log_q k + \log_q(q-1)) \geq \frac{N}{r}\log_q \frac{N}{r} \tag{4.1.0.2}$$

for small finite fields and $k \geq q$. Many people claim that this implies an $\Omega(\frac{N^2}{r}\log_q \frac{N}{r})$ lower bound of matrix rigidity. However, it is not very obvious from this theorem as the converse of the statement only ensures that some rows of B will have more than k entries.

proof of 4.1.1. Let $A_{N/2}$ denote the first $N/2$ rows of A and $B_{N/2}$ the first $N/2$ rows of B. We set $D_{N/2} \in \mathbb{F}_q^{N/2\times N}$ by $D_{N/2} = A+B$ and let r denote the rank of $_{N/2}$. Let S denote the linear space spanned by all vectors $w \in \mathbb{F}_q^{N/2}$ such that

$$w \cdot D_{N/2} = 0 \tag{4.1.0.3}$$

We see that S is a subspace of $\mathbb{F}_q^{N/2}$ with dimension $n/2 - r$.

Claim 4.1.2. Suppose t is an integer such that the size of a Hamming sphere of radius $t/2$ in $\mathbb{F}_q^{N/2}$ is at least q^r. Then there is a vector $w \in S$ with weight at most t.

Proof. Suppose on the contrary that the weight of w is greater than t for all $w \in S$. Let l denote the size of a Hamming sphere of radius $t/2$ in $\mathbb{F}_q^{N/2}$. Then

$$|S| \cdot l > |S| \cdot q^r = q^{N/2-r} \cdot q^r = q^{N/2} \tag{4.1.0.4}$$

However, we know that there are at most $q^{N/2}$ points in $\mathbb{F}_q^{N/2}$. That is

$$q^{N/2} \geq |S| \cdot l \tag{4.1.0.5}$$

Hence, we complete the proof by contradiction. ■

Now, let $w \in S$ be a vector of weight t as defined in the claim above. We then have

$$0 = w \cdot D_{N/2} = w \cdot A_{N/2} + w \cdot B_{N/2} \tag{4.1.0.6}$$

Because all the rows in $A_{N/2}$ are independent, we have $w \cdot A_{N/2} \neq 0$ and thus $w \cdot B_{N/2} \neq 0$. Since each row of B has at most k non-zero entries and w has weight t, we have that the weight of $w \cdot B_{N/2}$ is at most tk. On the other hand, since the code represented by $A_{N/2}$ has minimum distance $C_1 N$, we have that the weight of $w \cdot A_{N/2}$ is at least $C_1 N$. Therefore, we must have

$$tk \geq C_1 N \tag{4.1.0.7}$$

Take $t_0 = \lceil C_1 N/k \rceil$, we then have the size of a Hamming sphere of radius $t/2$ in $\mathbb{F}_q^{N/2}$ is at most q^t because the weight of w must be greater than t_0 to achieve $0 = w \cdot D_{N/2}$. Then

$$q^r \geq \binom{n/2}{t_0/2}(q-1)^{t_0/2} \tag{4.1.0.8}$$

which is

$$r \geq \log_q \left[\binom{N/2}{t_0/2}(q-1)^{t_0/2}\right] \geq \log_q \binom{N/2}{t_0/2} + \frac{t_0}{2}\log_q(q-1) \geq \frac{t_0}{2}\frac{N}{k}\log_q k \tag{4.1.0.9}$$

Choosing $C_2 \sim 1/C_1$, we have

$$r \geq \log_q \left[\binom{N/2}{t_0/2}(q-1)^{t_0/2}\right] \geq \frac{n}{C_2 k}(\log_q k + \log_q(q-1)) \tag{4.1.0.10}$$

□

4.2 Strong Rigidity and Paturi-Pudlák Dimensions ❖

In [PP06], Paturi and Pudlák introduced two dimensions that refine the notion of rigidity studied by Friedman.

In 4.1.1, Friedman also gave the following notion of strong rigidity.

Definition 4.2.1 (Strong Rigidity)**.** Let $V \subseteq \mathbb{F}^M$ be a subspace. V is (s,t)-**strongly rigid** if for any s-sparse subspace $U \in \mathbb{F}^M$ with $\dim(U) \leq \dim(V)$,

$$\dim(V \cap U) \leq \dim(V) - t \tag{4.2.0.1}$$

Definition 4.2.2 (Inner Dimension)**.** Let $V \subseteq \mathbb{F}^M$ be a subspace, and s be a positive integer less than M. We defined the **inner dimension** $d_V(s)$ **of V** by

$$d_V(s) := \max\{\dim(V \cap U) | U \in \mathbb{F}^M, \dim(U) \leq \dim(V), U \text{ is } s\text{-sparse}\} \tag{4.2.0.2}$$

Notice that the above definition is based on strong rigidity. To see the relation, we define $\rho_A(s)$ for a matrix $A \in \mathbb{F}^{M \times N}$.

$$\rho_A(s) := \min_B\{\text{rank}(A-B) : B \text{ is } s\text{-row sparse}\} \tag{4.2.0.3}$$

Proposition 4.2.3. *Let $A \in \mathbb{F}^{M\times N}$ with $M > N$ and $0 < s \leq N$ be given. Let V be the row space of A. Then*

$$\text{rank}(A) - d_V(s) \leq \rho_A(s) \tag{4.2.0.4}$$

Proof. Let B be the matrix that matches $\rho_A(s)$, i.e., $\text{rank}(A-B) = \rho_A(s)$. Let U be the row space of B and let W be the row space of $A - B$. Then we have $\dim(W) = \text{rank}(A-B) = \rho_A(s)$ and thus

$$\dim(V \cup U) \leq \dim(U) + \dim(W) = \dim(U) + \rho_A(s) \tag{4.2.0.5}$$

Hence, we obtain

$$\begin{aligned} \text{rank}(A) - d_V(s) &= \dim(V) - d_V(s) = \dim(V \cap U) + \dim(V \cup U) - \dim(U) - d_V(s) \\ &\leq \dim(V \cap U) + \dim(U) + \rho_A(s) - \dim(U) - d_V(s) \\ &= \dim(V \cap U) - d_V(s) + \rho_A(s) \end{aligned} \tag{4.2.0.6}$$

Because B is s-row sparse, we have that U is s-sparse. Thus, $d_V(s) \geq \dim(V \cap U)$. Hence, we obtain $\text{rank}(A) - d_V(s) \leq \rho_A(s)$. □

Definition 4.2.4 (Outer Dimension)**.** Let $V \subseteq \mathbb{F}^M$ be a subspace, and s be a positive integer less than M. We defined the **outer dimension $D_V(s)$ of V** by

$$D_V(s) := \max\{\dim(U) | U \in \mathbb{F}^M, V \subseteq U, U \text{ is } s\text{-sparse}\} \tag{4.2.0.7}$$

4.2.1 A Simple Bound ❄

Proposition 4.2.5. *Let $V \subseteq \mathbb{F}^M$ be a subspace and s be a positive integer less than M. Then,*

$$d_V(s) + D_V(s) \geq 2\dim(V) \tag{4.2.1.1}$$

Proof. Let $V \subseteq \mathbb{F}^M$ be a subspace such that $V \subseteq U, U$ is s-sparse and $\dim(U) = D_V(s)$. Let $m = \dim(V)$ and W be an m-dimensional subspace of $\mathbb{F}^M$ such that $W \subseteq U$. Hence, $\dim(V \cap W) \leq d_V(s)$ and thus

$$\begin{aligned} 2\dim(V) &= \dim(V) + \dim(W) \\ &= \dim(V \cap W) + \dim(V \cup W) \\ &\leq d_V(s) + \dim(U) = d_V(s) + D_V(s) \end{aligned} \tag{4.2.1.2}$$

□

Theorem 4.2.6. *Let C be an $[n,k,d]$ linear code over $\mathbb{F}_2$. Then for $s \leq d/2$,*

$$\begin{aligned} D_C(s) &\geq k + \frac{d}{2s}\log(\frac{2sk}{d}) \\ d_C(s) &\leq k - \frac{d}{2s}\log(\frac{2sk}{d}) \end{aligned} \tag{4.2.1.3}$$

The proof is essentially the same as that of theorem 4.1.1. We use the following auxiliary lemma.

Lemma 4.2.7. *Let C be an $[n,k,d]$ linear code over $\mathbb{F}_2$. Then for $s \leq d/2$, then there exists a $[D_C(s), k, d/s]$-code.*

Proof. Consider the subspace $W \subseteq \mathbb{F}_2^n$ with $\dim(W) = D_C(s), C \subseteq W$ and W is s-sparse. Let $D = D_C(s)$ and $\{w_1, ..., w_D\}$ be a basis of W where each $w_i, i \in [D]$ is s-sparse. Hence, for any $x \in C$, we have that there exists a $y \in \mathbb{F}_2^D$ such that

$$x = \sum_{i=1}^{D} y_i w_i \tag{4.2.1.4}$$

Let E be the set of all such y for all x. Then, we have $\dim(E) = \dim(C) = k$. Let $y' \in E$ be a nonzero vector with minimum weight. Because $x' = \sum_{i=1}^{D} y'_i w_i$ has weight at least d and each w_i is s-sparse, we have that at least d/s coordinates of y' is nonzero. Hence, we obtain that E is a $[D_C(s), k, d/s]$-code. □

Proof of theorem 4.2.6. Using the sphere packing bound on the $[D = D_C(s), k, d/s]$-code E we just constructed. We have that the Hamming balls of radius at most $d/2s$ at each vector in B do not intersect with each other. Hence,

$$\sum_{j=1}^{d/2s} \binom{D}{j} \leq 2^{D-k} \tag{4.2.1.5}$$

Notice that

$$\sum_{j=1}^{d/2s} \binom{D}{j} \geq \binom{D}{d/2s} \geq \binom{k}{d/2s} \geq (2sk/d)^{d/2s} \tag{4.2.1.6}$$

Hence,

$$D - k \geq \frac{d}{2s} \log(\frac{2sk}{d}) \tag{4.2.1.7}$$

Let $U \subseteq \mathbb{F}_2^n$ be a subspace with $\dim(U) = k$, U is s-sparse and $\dim(C \cap U) = d_C(s)$. Then $F = C \cap U$ is simply a $[n, d_C(s), d]$ code. Applying lemma 4.2.7 again, we obtain a $[D_F(s), d_C(s), d/s]$-code. Hence, because $D_F(s) \leq \dim(U) = k$, equation 4.2.1.7 implies

$$k - d_C(s) \geq D_F(s) - d_C(s) \geq \frac{d}{2s} \log(\frac{2sk}{d}) \tag{4.2.1.8}$$

which means $d_C(s) \leq k - (d/2s)\log(2sk/d)$. □

Remark 4.2.8. There was a typo about this result in the original [PP06] paper.

4.2.2 Connection Between Outer and Inner Dimensions ❄

Theorem 4.2.9 ([DGW19])**.** *Let t and k be positive integers and let $0 < \epsilon < 1$. If $A \in \mathbb{F}^{M \times N}$ is a matrix whose columns space $V \subseteq \mathbb{F}^M$ has an outer dimension*

$$D_V(tk + N\epsilon^k) \geq \frac{N}{1-\epsilon} \tag{4.2.2.1}$$

then for some $N' \geq N\epsilon^k$, A contains a submatrix $B \in \mathbb{F}^{M \times N'}$ whose columns space $U \subseteq \mathbb{F}^M$ has an inner dimension

$$d_U \leq \text{rank}(B) - \epsilon N' \tag{4.2.2.2}$$

4.3 Row Rigidity ❖

In this section, we introduce the definition of *row rigidity* from [DGW19], specifically for rectangular matrices. In the next section, we will discuss the link between rigidity and data structure lower bounds.

Definition 4.3.1 (Rigidity for Rectangular Matrices)**.** A matrix $A \in \mathbb{F}^{M \times N}$ is said to be (r, s)-**rigid** if for any matrix $B \in \mathbb{F}^{M \times N}$ with dens$(B) \leq s$, we have $A + B$ has rank at least r. A matrix $A \in \mathbb{F}^{M \times N}$ is said to be (r, s)-**strongly rigid** if for any invertible matrix $C \in \mathbb{F}^{N \times N}$, we have $A \times C$ is (r, s) rigid.

Definition 4.3.2 (Row Rigidity)**.** A matrix $A \in \mathbb{F}^{M \times N}$ is said to be (r, s)-**row rigid** if for any matrix s-sparse $B \in \mathbb{F}^{M \times N}$, we have $A + B$ has rank at least r.

Definition 4.3.3 (Strong Row Rigidity)**.** A matrix $A \in \mathbb{F}^{M \times N}$ is said to be (r, s)-**strongly row rigid** if for any invertible matrix $C \in \mathbb{F}^{N \times N}$, we have $A \times C$ is (r, s)-row rigid.

4.3.1 Strong rigidity is equivalent to small inner dimension ❄

The following lemma shows that the definition of strong row rigidity of rectangular matrices is equivalent to the strong rigidity of subspaces as defined by Friedman. In particular, we limit our attention to matrices with more rows than columns, i.e. $M > N$.

Lemma 4.3.4. *Let matrix $A \in \mathbb{F}^{M\times N}$ have rank N and let $V \subseteq \mathbb{F}^M$ be its columns space. Then the following are equivalent:*

1. *A is (r,s)-strongly row rigid.*
2. *$d_V(s) \leq \text{ rank}(A) - r$.*
3. *V is not contained in a subspace of the form $E \cup F$ where $E, F \subseteq \mathbb{F}^M$ are subspaces with $\dim(E) \leq N, \dim(F) < r$ and E is s-sparse.*

Proof. $(1 \Rightarrow 2)$: Suppose $d_V(t) > \text{ rank}(A) - r$. By the definition of inner dimension 4.2.2, there exists a subspace $U \in \mathbb{F}^M, \dim(U) \leq \dim(V) = \text{rank}(A) = N, U$ is s-sparse and $\dim(U \cap V) > \text{ rank}(A) - r$. In other words, there exists a subspace $W \in \mathbb{F}^M$ with $\dim(W) < r$ such that $V = U \cup W$. Let $C \in \mathbb{F}^{M\times N}$ be a s-row sparse basis matrix of U and $B \in \mathbb{F}^{M\times N}$ a basis matrix of W. There exists an invertible matrix $T \in \mathbb{F}^{N\times N}$ such that

$$A = C \times T + B \tag{4.3.1.1}$$

This implies $A \times T^{-1} = C + B \times T^{-1}$ is not (r,s)-row rigid because $\text{rank}(A \times T^{-1} - C) = \text{rank}(B \times T^{-1}) < r$. This means that A is not (r,s)-strongly row rigid and leads to a contradiction.
$(2 \Rightarrow 3)$: Because $d_V(s) \leq \text{ rank}(A) - r$, for all subspaces $E \in \mathbb{F}^M$ such that $\dim(E) \leq N = \dim(V)$ and E is s-sparse, we have $\dim(V \cap E) \leq N - r$. That is, for any subspace $F \in \mathbb{F}^M$ with $V \subseteq E \cup F$, we have

$$\begin{aligned} N - r \geq \dim(V \cap E) &= \dim(V) + \dim(E) - \dim(V \cup E) \\ &\geq \dim(V) + \dim(E) - \dim(E \cup F) \\ &= \dim(V) + \dim(E) - \dim(E) - \dim(F) + \dim(E \cap F) \\ &= N - \dim(F) + \dim(E \cap F) \end{aligned} \tag{4.3.1.2}$$

This implies $\dim(F) \geq r$.
$(3 \Rightarrow 1)$: Take any invertible $T \in \mathbb{F}^{N\times N}$, if we write

$$A \times T = C + B \tag{4.3.1.3}$$

where $C \in \mathbb{F}^{M\times N}$ be a s-row sparse and $B \in \mathbb{F}^{M\times N}$. Let E be the columns space of C and F the columns space of B. Because T is invertible and, $\text{rank}(A \times T) = \text{rank}(A) = \dim(V)$. As we have just seen, $\dim(F) \geq r$, hence $\text{rank}(B) > r$ and thus A must be (r,s)-strongly row rigid. □

Remark 4.3.5. Notice that by proposition 4.2.3, A is $(\rho_A(s), s)$-strongly row rigid if and only if $d_V(s) = \text{ rank}(A) - \rho_A(s)$.

4.3.2 From Row Rigidity to General Rigidity ❄

Theorem 4.3.6 ([DGW19])**.** *Let $A \in \mathbb{F}^{M\times N}$ be a rectangular matrix, $E \in \mathbb{F}^{L\times M}$ a $(t, \delta, 3/4)$-linear locally decodable code and set $B := EA$. Then,*

1. *If $A \in \mathbb{F}^{M\times N}$ is $(r, s+1)$-row rigid, then $B \in \mathbb{F}^{L\times N}$ is $(r, (\delta sL)/t)$-rigid.*
2. *If $A \in \mathbb{F}^{M\times N}$ is $(r, s+1)$-strongly row rigid, then $B \in \mathbb{F}^{L\times N}$ is $(r, (\delta sL)/t)$-strongly rigid.*

We need a few tools from locally decodable code to prove this theorem.

Notation 4.3.7. We use $\text{dist}(u, v)$ to denote the Hamming distance between two vectors u, v.

Definition 4.3.8. A linear code $C : \mathbb{F}^M \to \mathbb{F}^N$ is said to be (t, δ, ϵ)-**locally decodable** if there exists a randomised decoding algorithm $\mathcal{A}$ such that for all $m \in \mathbb{F}^M$ and all $w \in \mathbb{F}^N$ such that dist$(C(m), w) \leq \delta$:

1. For every index $i \in [M]$
$$\mathbb{P}[\mathcal{A}(w, i) = m_i] \geq 1 - \epsilon, \tag{4.3.2.1}$$
where the probability is taken over the random coin tosses of the algorithm $\mathcal{A}$.

2. $\mathcal{A}$ makes at most t queries to w.

We abuse the notation and write $C \in \mathbb{F}^{M \times N}$ as its generating matrix.

Lemma 4.3.9 ([GKST02], [DS07])**.** *Let $C \in \mathbb{F}^{M \times N}$ be a $(t, \delta, 3/4)$-linear locally decodable code and let R be a set of rows of C with $|R| \geq (1 - \delta)M$. For any $i \in [N]$, there exists a set of t rows in R which spans the ith standard basis vector e_i.*

proof of theorem 4.3.6. Let $A \in \mathbb{F}^{M \times N}$ be $(r, s + 1)$-row rigid and suppose that B is not $(r, (\delta sL)/t)$-rigid. Then we have $B = D + S$ where $D \in \mathbb{F}^{L \times N}$ has rank at most r and $S \in \mathbb{F}^{L \times N}$ has density dens$(S) \leq (\delta sL)/t$. Let S' be the set of row of S that are s/t-sparse. By averaging argument, we have that $|S'| \geq (1 - \delta)L$. Let D' be the corresponding rows in D. Because A is $(r, s + 1)$-row rigid, some rows A_i has Hamming distance at least $(s + 1)$ from the space generated by D.
On the other hand, by lemma 4.3.9, there exist t rows in D' and S' which spans A_i. This means that A_i has a Hamming distance at most $t \cdot (s/t) = s$ from the row space of L'.
Therefore, by contradiction, we must have B is $(r, (\delta sL)/t)$-rigid.
Let $A \in \mathbb{F}^{M \times N}$ be $(r, s + 1)$-strongly row rigid, then for all invertible matrix $T \in \mathbb{F}^{N \times N}$ such that AT is $(r, s + 1)$-row rigid. Notice that
$$E \times (A \times T) = (E \times A) \times T = B \times T \tag{4.3.2.2}$$
where $B \in (r, (\delta sL)/t)$. As we have just shown, $B \times T$ is $(r, (\delta sL)/t)$-rigid. Since T is arbitrary, we have that B is $(r, (\delta sL)/t)$-strongly rigid. □

The following corollary shows that we can obtain rigid square matrices from rigid rectangular matrices.

Corollary 4.3.10 (Rectangular Matrices to Square Matrices, [DGW19])**.** *For every constant $\alpha > 0$, and an $(r, s + 1)$-row rigid matrix $A \in \mathbb{F}^{M \times N}$, we can construct a square matrix $B \in \mathbb{F}^{L \times L}$, $L = M^{O(1/\alpha)}$, which is*

$$(r, \frac{L}{N} \cdot \frac{s}{(\log M)^{1+\alpha}})\text{-row rigid and } (r, \frac{L^2}{N} \cdot \frac{s}{(\log M)^{1+\alpha}})\text{-rigid.}$$

in polynomial time.

To prove this theorem we use the following result from locally decodable code without proof. This lemma allows us to construct linear locally decodable codes in polynomial time.

Lemma 4.3.11 ([Dvi11])**.** *For every $\alpha, \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ and an explicit family of $((\log N)^{1+\alpha}, \delta, \epsilon)$-linear locally decodable codes $C \in \mathbb{F}^{\mathbb{M} \times \mathbb{N}}$ for $M = N^{O(1/\alpha)}$.*

proof of corollary 4.3.10. Let $L = M^{O(1/\alpha)}$ be a multiple of N and δ some constant. We also let $C \in \mathbb{F}^{L \times M}$ be a $((\log M)^{1+\alpha}, \delta, 3/4)$-linear locally decodable code. Hence, we can construct a square matrix $B \in \mathbb{F}^{L \times L}$ by putting side by side (L/N) copies of $C \times A$.
It remains to show that B is rigid. By theorem 4.3.6, we have $C \times A$ is $(r, (\delta sL)/(\log M)^{1+\alpha})$-rigid. Hence,

$$\mathcal{R}_B(r) \geq \frac{L}{N} \cdot \frac{\delta sL}{(\log M)^{1+\alpha}}$$

and by averaging over L rows, B must be $(r, (\delta sL)/[N \cdot (\log M)^{1+\alpha}])$-row rigid. Setting $\delta = 1$ or some suitable constant, we get the desired result. □

Remark 4.3.12 (Link to Friedman's result). Notice that in theorem 4.1.1, the first $N/2$ rows of A is $(r, (N/r)\log(N/r))$-row rigid. By corollary 4.3.10, we can construct a matrix $B \in \mathbb{F}^{N \times N}$ with

$$\mathcal{R}_B(r) = O\Big(\frac{N^2}{(\log N)r} \cdot \log(\frac{N}{r})\Big) \tag{4.3.2.3}$$

4.4 Linear Data Structure ❖

Theorem 4.4.1 ([DGW19])**.** *A data structure lower bound of $t \geq \log^c n$ in the group (linear) model for computing a linear map $M \in \mathbb{F}^{m \times n}$, even against data structures with arbitrarily small linear space $s = (1+\epsilon)n$, yields an $(\epsilon n', d)$-row-rigid matrix $M' \in \mathbb{F}^{m \times n'}$ with $\epsilon n' \geq d \geq \Omega(\log^{c-1} n)$. Moreover, if M is explicit, then $M' \in \mathbf{P}^{\mathbf{NP}}$.*

Chapter 5

Rigid Sets

So far we have seen the trade-off between the values of dimensions and distance that can be obtained by explicit sets of size n. The study of rigid sets aim to investigate the trade-off between the values of *size* and *distance,* when the value of *dimension* is fixed.

Definition 5.0.1. For $x \in \mathbb{F}_2^N, U \subseteq \mathbb{F}_2^N$, we define the **Hamming distance from** x **to** U by

$$\text{dist}(x, U) = \min_{u \in U} |x + u| \tag{5.0.0.1}$$

where $|v|$ denotes the Hamming weight of v.

Definition 5.0.2 (Rigid Sets)**.** A set $S \subseteq \mathbb{F}_2^N$ is called (N, k, d)**-rigid** if for every linear subspace $U \subseteq \mathbb{F}_2^N, \dim(U) = k$, we have

$$\max_{s \in S} \text{dist}(s, U) \geq d \tag{5.0.0.2}$$

Let $A \in \mathbb{F}_2^{M \times N}, M = |S|$ be the matrix whose rows are the elements of S. Notice that A is (k, d)-rigid if and only if S is (N, k, d)-rigid.

Theorem 5.0.3 ([SY11])**.** *Let q be a prime power. For every $0 \leq d \leq O(N)$, there exists an explicit set $(N, N/2, d)$-rigid set $S \subseteq \mathbb{F}_q^N$ of size $2^{O(d)N/d}$.*

Notation 5.0.4. Let $I \subseteq [N]$ be a set of coordinates. For a vector $x \in \mathbb{F}^N$, we write $x|_I$ to denote the vector x restricted to the coordinates in I. Similarly, for a linear subspace $U \subseteq \mathbb{F}^N$, we write $U|_I$ to denote the linear subspace U restricted to the coordinates in I.

Lemma 5.0.5. *Let q be a prime power and $U \subseteq \mathbb{F}_q^N$ be a linear subspace with $\dim(U) = k$, then*

$$\mathbb{P}_{x \in \{0,1\}^N}[x \in U] \leq \frac{1}{2^{N-k}} \tag{5.0.0.3}$$

Proof. Let $I \subseteq [N]$ be the set of coordinates such that $U|_I = \mathbb{F}_q^N$ and $J = [N] \setminus I$. Hence, we note that a vector $x = x|_I + x|_J \in U$ is uniquely determined by $x|_I$ because $x|_J$ is the zero vector. Hence, for a random vector $x \in \{0,1\}^N$, there is at most 2^{k-N} chance that x is in U. □

Lemma 5.0.6. *Let q be a prime power. For every $\epsilon > 0$, there exists a $\delta > 0$ such that for all linear subspaces $U \subseteq \mathbb{F}_q^N, \dim(U) \leq (1-\epsilon)N$, there exists a point $x \in \{0,1\}^n$ such that*

$$\text{dist}(x, U) \geq \delta N \tag{5.0.0.4}$$

Proof. Let U be given. We note that for a random vector $x \in \{0,1\}^N$,

$$\mathbb{P}[\text{dist}(x, U) \leq \delta N] = \mathbb{P}[\exists I \subseteq [N], |I| = (1-\delta)N \text{ such that } x|_I \in L|_I] \tag{5.0.0.5}$$

For a fixed set I with $|I| = (1-\delta)N$, we have by lemma 5.0.5, we have

$$\mathbb{P}[x|_I \in U|_I] = \frac{1}{2^{(1-\delta)N-(1-\epsilon)N}} = \frac{1}{2^{(\epsilon-\delta)N}} \tag{5.0.0.6}$$

Hence, by union bound on all possible set I of size $(1-\delta)N$, the probability

$$\mathbb{P}[\text{dist}(x,U) \leq \delta N] \leq \binom{N}{\delta N} \frac{1}{2^{(\epsilon-\delta)N}} \tag{5.0.0.7}$$

is negligible when δ is sufficiently smaller than ϵ. □

proof of theorem 5.0.3. As a consequence of lemma 5.0.6, let δ be the constant that for all linear subspace $U \subseteq \mathbb{F}_q^N, \dim(U) = N/2$, there exists a point p in $\mathbb{F}_q^N$ that is more than δN-far from U.
To obtain S, we first split the coordinates into cN/d disjoint sets $Z_1, Z_2, ..., Z_{\delta N/d}$, each of size d/δ. For each set Z_i, we let W_i be the set of all binary vectors $x_i \in \{0,1\}^N$ with support on this set Z_i. That is, each x_i has some value 1 on some coordinates in set Z_i and has 0 on every other coordinates. Let $S = \cup_i W_i$ consist of all these vectors. Hence,

$$|S| = 2^{O(d)} N/d \tag{5.0.0.8}$$

and every vector in $\mathbb{F}_q^N$ is the sum of at most $\delta N/d$ vectors in S.
Let a linear subspace $U \subseteq \mathbb{F}_q^N, \dim(U) = N/2$ be given. Suppose every vector in S is at most d-far from U. That is, any vector in S is the sum of one vector in U and at most d unit vectors. Because every vector v in $\mathbb{F}_q^N$ is the sum of at most $\delta N/d$ vectors in S, v must also be the sum of a vector in U and at most $d \times (\delta N/d) = \delta N$ unit vectors. Hence, no point p will be more than δN-far from U, which gives us a contradiction. □

Corollary 5.0.7 ([APY09])**.** *For every $0 \leq d \leq O(N)$, there exists an explicit $(N, N/2, d)$-rigid set $S \subseteq \mathbb{F}_2^N$ of size $2^{O(d)} N/d$.*

5.1 Strong Rigid Sets ❖

A new approach for constructing rigid sets by applying *U-polynomials* was introduced by Alon and Cohen [AC15].

Definition 5.1.1. For a subspace $U \subseteq \mathbb{F}_2^N$, the **U-polynomial** $p_{U,\rho} :\subseteq \mathbb{F}_2^N \to \mathbb{R}$ is defined as

$$p_{U,\rho}(x) = \frac{1}{W_\rho(U)} \cdot \sum_{u \in U} \rho^{|u|} \cdot (-1)^{\langle u,x \rangle} \tag{5.1.0.1}$$

where $W_\rho(U) = \sum_{u \in U} \rho^{|u|}$ is the **weight enumerator** of U with parameter $\rho \in (0,1)$.

Theorem 5.1.2 ([AC15])**.** *Let the parameter $\rho \in (0,1)$ and the linear subspace $U \subseteq \mathbb{F}_2^N$ be given. Then, for every $x \in \mathbb{F}_2^N$,*

$$\text{dist}(x,U) = \Omega(\log \frac{1}{p_{U^\perp,\rho}(x)}) \tag{5.1.0.2}$$

5.1.1 Fourier Analysis ❄

Before we prove theorem 5.1.2, we first show the following.

Theorem 5.1.3 ([AC15])**.** *Let $U \subseteq \mathbb{F}_2^N$ be a linear subspace. Then, for any parameter $\rho \in (0,1)$ and any point $x \in \mathbb{F}_2^N$,*

$$\text{dist}(x,U) \geq \left(\log \frac{1+\rho}{1-\rho}\right)^{-1} \cdot \log \frac{1}{p_{U^\perp,\rho}(x)} \tag{5.1.1.1}$$

We will need a few definitions and tools from Fourier analysis and error correcting codes first.

Definition 5.1.4. We define the **inner product** $\langle \cdot, \cdot \rangle$ on pairs of function $f, g : \mathbb{F}_2^N \to \mathbb{R}$ by

$$\langle f, g \rangle = 2^{-N} \sum_{x \in \mathbb{F}_2^N} f(x)g(x) \tag{5.1.1.2}$$

Definition 5.1.5 (Fourier Expansion)**.** Every function $f : \mathbb{F}_2^N \to \mathbb{R}$ can be uniquely expressed as a multilinear polynomial,

$$f(x) = \sum_{\alpha \in \mathbb{F}_2^N} \hat{f}(\alpha)\chi_\alpha(x) \tag{5.1.1.3}$$

where $\chi_\alpha(x) = (-1)^{\langle \alpha, x \rangle}$. This expression is called the **Fourier expansion** of f, and the real number $\hat{f}(\alpha) = \langle f, \chi_\alpha \rangle$ is called the **Fourier coefficient** of f on S. Collectively, the coefficients are called the **Fourier spectrum** of f.

Definition 5.1.6 (Noise Operator)**.** For $0 \geq \rho \geq 1$ and $f : \mathbb{F}_2^N \to \mathbb{R}$, we define the **noise operator** with parameter $\rho, T_\rho(f) : \mathbb{F}_2^N \to \mathbb{R}$ on the function f by

$$T_\rho(f)(x) = \sum_{y \in \mathbb{F}_2^N} \Big(\frac{1-\rho}{2}\Big)^{|y|} \cdot \Big(\frac{1+\rho}{2}\Big)^{N-|y|} \cdot f(x+y) \tag{5.1.1.4}$$

Proposition 5.1.7. $\widehat{T_\rho(f)}(\alpha) = \rho^{|\alpha|}\hat{f}(\alpha)$.

Definition 5.1.8. Let the parameter $\rho \in (0,1)$ and the linear subspace $U \subseteq \mathbb{F}_2^N$ be given. The function $\text{energy}_{U,\rho} : \mathbb{F}_2^N \to \mathbb{R}$ is defined as

$$\text{energy}_{U,\rho}(x) = \frac{1}{W_\rho(U)} \cdot \sum_{u \in U} \rho^{|u+x|} \tag{5.1.1.5}$$

Notice that $\text{energy}_{U,\rho}(x) \in (0,1]$ and $\text{energy}_{U,\rho}(x) = 1$ if and only if $x \in U$.

Theorem 5.1.9 (MacWilliam's Theorem [MS77])**.** *Let $U \subseteq \mathbb{F}_2^N$ be a linear subspace with $\dim U = k$. Then, for any parameter $\rho \in (0,1)$,*

$$W_\rho(U^\perp) = \frac{(1+\rho)^N}{2^k} \cdot W_{\frac{1-\rho}{1+\rho}}(U) \tag{5.1.1.6}$$

proof of theorem 5.1.3. We use $\mathbf{1}_U : \mathbb{F}_2^N \to \{0,1\}$ to denote the **indicator function** for U, i.e., $\mathbf{1}_U = 1$ if and only if $x \in U$. Then,

$$\begin{aligned}
T_\rho(\mathbf{1}_U)(x) &= \sum_{y \in \mathbb{F}_2^N} \Big(\frac{1-\rho}{2}\Big)^{|y|} \cdot \Big(\frac{1+\rho}{2}\Big)^{N-|y|} \cdot \mathbf{1}_U(x+y) \\
&= \Big(\frac{1+\rho}{2}\Big)^n \cdot \sum_{y \in \mathbb{F}_2^N} \Big(\frac{1-\rho}{1+\rho}\Big)^{|y|} \cdot \mathbf{1}_U(x+y) \\
&= \Big(\frac{1+\rho}{2}\Big)^n \cdot \sum_{u \in U} \Big(\frac{1-\rho}{1+\rho}\Big)^{|x+u|} \\
&= \Big(\frac{1+\rho}{2}\Big)^n \cdot W_{\frac{1-\rho}{1+\rho}}(U) \cdot \text{energy}_{U,\frac{1-\rho}{1+\rho}}(x)
\end{aligned}$$

Note that because U is a subspace, we have for $\alpha \notin U^\perp$, $\chi_\alpha(x) = 1$ for exactly half of the time and $\chi_\alpha(x) = -1$ for exactly the other half,

$$\langle \mathbf{1}_U, \chi_\alpha \rangle = \frac{1}{2^N}\Big(\sum_{x \in U} f(x)\chi_\alpha(x) + \sum_{x \notin U} f(x)\chi_\alpha(x)\Big) = \frac{1}{2^N}\sum_{x \in U} f(x)\chi_\alpha(x) = 0 \tag{5.1.1.7}$$

As for $\alpha \in U^\perp$,

$$\langle \mathbf{1}_U, \chi_\alpha \rangle = \frac{1}{2^N}\Big(\sum_{x\in U} f(x)\chi_\alpha(x) + \sum_{x\notin U} f(x)\chi_\alpha(x)\Big) = \frac{1}{2^N}\sum_{x\in U}\chi_\alpha(x) = \frac{1}{2^N}\cdot 2^k = 2^{k-N} \tag{5.1.1.8}$$

Therefore,

$$\widehat{\mathbf{1}_U}(\alpha) = \begin{cases} 2^{k-N}, & \alpha \in U^\perp \\ 0, & \text{otherwise} \end{cases} \tag{5.1.1.9}$$

By proposition 5.1.7,

$$\begin{aligned} T_\rho(\mathbf{1}_U)(x) &= \sum_{\alpha\in\mathbb{F}_2^N} \widehat{T_\rho(\mathbf{1}_U)}(\alpha)\cdot\chi_\alpha \\ &= \sum_{\alpha\in\mathbb{F}_2^N} \rho^{|\alpha|}\widehat{\mathbf{1}_U}(\alpha)\cdot\chi_\alpha \\ &= \sum_{\alpha\in U^\perp} \rho^{|\alpha|}\widehat{\mathbf{1}_U}(\alpha)\cdot\chi_\alpha \\ &= 2^{k-N}\sum_{\alpha\in U^\perp}\rho^{|\alpha|}\cdot\chi_\alpha \end{aligned} \tag{5.1.1.10}$$

By definition 5.1.1, we have

$$\begin{aligned} T_\rho(\mathbf{1}_U)(x) &= 2^{k-N}\sum_{\alpha\in U^\perp}\rho^{|\alpha|}\cdot\chi_\alpha \\ &= 2^{k-N}\cdot W_\rho(U^\perp)\cdot p_{U^\perp,\rho}(x) \end{aligned} \tag{5.1.1.11}$$

By MacWilliam's Theorem, 5.1.9, we have

$$\begin{aligned} T_\rho(\mathbf{1}_U)(x) &= 2^{k-N}\cdot W_\rho(U^\perp)\cdot p_{U^\perp,\rho}(x) \\ &= 2^{k-N}\cdot\frac{(1+\rho)^N}{2^k}\cdot W_{\frac{1-\rho}{1+\rho}}(U)\cdot p_{U^\perp,\rho}(x) \\ &= \Big(\frac{1+\rho}{2}\Big)^N\cdot W_{\frac{1-\rho}{1+\rho}}(U)\cdot p_{U^\perp,\rho}(x) \end{aligned} \tag{5.1.1.12}$$

Combining equation 5.1.1 and 5.1.1.12, we have

$$\text{energy}_{U,\frac{1-\rho}{1+\rho}}(x) = p_{U^\perp,\rho}(x) \tag{5.1.1.13}$$

Let $d = \text{dist}(x, U)$. Then there exists $w \in U$ such that $|x + w| = d$. By definition 5.1.8, we have

$$W_{\frac{1-\rho}{1+\rho}}(U)\cdot\text{energy}_{U,\frac{1-\rho}{1+\rho}}(x) = \sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u+x|} \tag{5.1.1.14}$$

Because U is a subspace,

$$\sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u+x|} = \sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u+x+w|} \tag{5.1.1.15}$$

Using triangle inequality, we have $|u + x + w| \le |u| + |x + w|$. Thus,

$$\begin{aligned} \sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u+x+w|} &\ge \sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u|+|x+w|} \\ &= \Big(\frac{1-\rho}{1+\rho}\Big)^d\sum_{u\in U}\Big(\frac{1-\rho}{1+\rho}\Big)^{|u|} \\ &= \Big(\frac{1-\rho}{1+\rho}\Big)^d\cdot W_{\frac{1-\rho}{1+\rho}}(U) \end{aligned} \tag{5.1.1.16}$$

In summary, we obtain

$$p_{U^\perp,\rho}(x) = \text{energy}_{U,\frac{1-\rho}{1+\rho}}(x) = \frac{1}{W_{\frac{1-\rho}{1+\rho}}(U)} \cdot \sum_{u\in U} \left(\frac{1-\rho}{1+\rho}\right)^{|u+x+w|} \geq \left(\frac{1-\rho}{1+\rho}\right)^d \tag{5.1.1.17}$$

which concludes the proof. □

5.1.2 Random Sets ❄

Proposition 5.1.10. *Let the parameter $\rho \in (0,1)$ and the linear subspace $U \subseteq \mathbb{F}_2^N, \dim U = N/2$ be given. Then*

$$W_\rho(U) \geq \left(\frac{1+\rho}{\sqrt{2}}\right)^N \tag{5.1.2.1}$$

Proof. Because U is a linear subspace with $\dim U = N/2$, there are exactly $2^{N/2}$ cosets $x+U$ of subspace U. For each coset, as $\rho < 1$,

$$\sum_{w\in x+U} \rho^{|w|} \leq \sum_{u\in U} \rho^{|u|} = W_\rho(U) \tag{5.1.2.2}$$

On the other hand, using binomial expansion, we have

$$(1+\rho)^N = \sum_{w\in\mathbb{F}_2^N} \rho^{|w|} = \sum_x \sum_{w\in x+U} \rho^{|w|} \leq \sum_x W_\rho(U) = 2^{N/2} W_\rho(U) \tag{5.1.2.3}$$

which concludes the proof. □

Notation 5.1.11. Let $\mathcal{P}_k$ denote the class of all U-polynomials $p_{U,\rho}$ with $\dim U = k$.

Proposition 5.1.12. *Let the parameter $\rho \in (\sqrt{2}-1, 1)$ and the linear subspace $U \subseteq \mathbb{F}_2^N, \dim U = N/2$ be given. Then, with high probability, for a random set $S \subset \mathbb{F}_2^N$ of size $O(N)$, the following holds: for every $p_{U,\rho} \in \mathcal{P}_{n/2}$,*

$$p_{U,\rho}(s) \leq 2^{\Omega(N)} \tag{5.1.2.4}$$

for at least half of the elements $s \in S$.

Proof. Let $p_{U,\rho} \in \mathcal{P}_{N/2}$ be given. Then,

$$\begin{aligned} \mathbb{E}_{x\sim\mathbb{F}_2^N}[p_{U,\rho}(x)] &= \mathbb{E}_{x\sim\mathbb{F}_2^N}\left[\frac{1}{W_\rho(U)} \cdot \sum_{u\in U} \rho^{|u|}(-1)^{\langle u,x\rangle}\right] \\ &= \frac{1}{W_\rho(U)} \cdot \sum_{u\in U} \left(\rho^{|u|} \cdot \mathbb{E}_{x\sim\mathbb{F}_2^N}[(-1)^{\langle u,x\rangle}]\right) \end{aligned} \tag{5.1.2.5}$$

Notice that $\mathbb{E}_{x\sim\mathbb{F}_2^N}[(-1)^{\langle u,x\rangle}] = 1$ if $u = 0$ and $\mathbb{E}_{x\sim\mathbb{F}_2^N}[(-1)^{\langle u,x\rangle}] = 0$ otherwise. By proposition 5.1.10,

$$\mathbb{E}_{x\sim\mathbb{F}_2^N}[p_{U,\rho}(x)] = \frac{1}{W_\rho(U)} \cdot \sum_{u\in U} \left(\rho^{|u|} \cdot \mathbb{E}_{x\sim\mathbb{F}_2^N}[(-1)^{\langle u,x\rangle}]\right) = \frac{1}{W_\rho(U)} \leq \left(\frac{\sqrt{2}}{1+\rho}\right)^N \tag{5.1.2.6}$$

Because $\rho > \sqrt{2}-1$, $\sqrt{2}/(1+\rho) < 1$. Hence, there exists a constant α such that

$$\mathbb{E}_{x\sim\mathbb{F}_2^N}[p_{U,\rho}(x)] \leq \left(\frac{\sqrt{2}}{1+\rho}\right)^N < 2^{\alpha N} \tag{5.1.2.7}$$

By Markov's inequality, we have

$$\mathbb{P}_{x\sim\mathbb{F}_2^N}[p_{U,\rho}(x) > 2^{\alpha N/2}] \leq 2^{-\alpha N/2} \tag{5.1.2.8}$$

For some $M \leq N$, let $x_1, ..., x_M$ be independent and uniformly random vectors in $\mathbb{F}_2^N$. Let $\mathcal{E}$ be the event that there exists some subset $S \subseteq [N], |S| = M/2$ such that for all $i \in S$, $p_{U,\rho}(x_i) > 2^{\alpha N/2}$. We have

$$\mathbb{P}_{x_1,\ldots,x_M\sim\mathbb{F}_2^N}[\mathcal{E}] \leq \binom{M}{M/2} \cdot \mathbb{P}_{x\sim\mathbb{F}_2^N}[p_{U,\rho}(x) > 2^{-\alpha N/2}]^{M/2} \leq \binom{M}{M/2} \cdot 2^{-\alpha NM/4} \leq 2^M \cdot 2^{-\alpha NM/4} \tag{5.1.2.9}$$

Set $M = O(N)$, we have that $\mathbb{P}_{x_1,\ldots,x_M\sim\mathbb{F}_2^N}[\mathcal{E}] = 2^{-\Omega(N^2)}$. □

5.1.3 Strong Rigid Sets ❄

Definition 5.1.13 (Strong Rigid Sets)**.** A set $S \subseteq \mathbb{F}_2^N$ is called **strong** (N,k,d)**-rigid** if for every linear subspace $U \subseteq \mathbb{F}_2^N, \dim(U) = k$, we have

$$\mathbb{E}_{s\sim S}[\text{dist}(s,U)] \geq d \tag{5.1.3.1}$$

Definition 5.1.14 (Small Biased Sets)**.** We say that a set $S \subseteq \mathbb{F}_2^N$ is ϵ-biased if for every nonzero $\alpha \in \mathbb{F}_2^N$,

$$\left|\mathbb{E}_{s\sim S}[(-1)^{\langle \alpha, s\rangle}]\right| \leq \epsilon \tag{5.1.3.2}$$

Theorem 5.1.15 ([AC15])**.** *For every $0 \leq d \leq cN$ for some suitable constant $0 < c < 1$. If $S \subseteq \mathbb{F}_2^N$ is an $\exp(-d)$-biased set, then S is $(N, N/2, d)$-strong rigid.*

1st proof. Let d be given. Let $S \subseteq \mathbb{F}_2^N$ be an ϵ-biased set with $\epsilon = \exp(-d)$ and $|S| = O(N/\epsilon^3)$. Let $U \subseteq \mathbb{F}_2^N$ be a linear subspace with $\dim U = N/2$. By proposition 5.1.10, for $\rho \in (\sqrt{2}-1, 1)$ and suitable constant d, we have

$$W_\rho(U) \geq \Big(\frac{1+\rho}{\sqrt{2}}\Big)^N \geq \exp(d) = \frac{1}{\epsilon} \tag{5.1.3.3}$$

Now,

$$\begin{aligned}
\mathbb{E}_{x\sim S}[p_{U,\rho}(x)] &= \mathbb{E}_{x\sim S}\left[\frac{1}{W_\rho(U)} \cdot \sum_{u\in U} \rho^{|u|}(-1)^{\langle u,x\rangle}\right] \\
&= \frac{1}{W_\rho(U)} \cdot \sum_{u\in U}\Big(\rho^{|u|} \cdot \mathbb{E}_{x\sim S}[(-1)^{\langle u,x\rangle}]\Big) \\
&= \frac{1}{W_\rho(U)} \cdot \Big(1 + \epsilon \sum_{u\in U, u\neq 0} \rho^{|u|}\Big) \\
&< \frac{1}{W_\rho(U)} \cdot \Big(1 + \epsilon \frac{1}{W_\rho(U)}\Big) \\
&= \frac{1}{W_\rho(U)} + \epsilon \\
&\leq 2\epsilon
\end{aligned} \tag{5.1.3.4}$$

Because $\log(1/x)$ is a convex function, by Jensen's Inequality,

$$\mathbb{E}_{x\sim S}[\log \frac{1}{p_{U,\rho}(x)}] \geq \log \frac{1}{\mathbb{E}_{x\sim S}[p_{U,\rho}(x)]} > \log \frac{1}{2\epsilon} \tag{5.1.3.5}$$

Because U is a linear subspace with $\dim U = N/2$, its dual $U^\perp$ is also a linear subspace with $\dim U = N/2$. Hence, by theorem 5.1.2,

$$\mathbb{E}_{x\sim S}[\text{dist}(x,U)] = \mathbb{E}_{x\sim S}[\Omega\Big(\log \frac{1}{p_{U^\perp,\rho}(x)}\Big)] = \Omega\Big(\log \frac{1}{\epsilon}\Big) \tag{5.1.3.6}$$

In summary, S is a $(N, N/2, d)$-strong rigid set with $|S| = O(N/\epsilon^3) = N \cdot 2^{\Theta(d)}$. □

5.1.4 Bias Reduction ❄

Definition 5.1.16. (Statistical Distance) Let $x \sim X, y \sim Y$ be two random variables from two distributions X, Y on the same support S. We define the statistical distance between X and Y by

$$\text{SD}(X,Y) := \max_{A\subseteq S}\Big|\mathbb{P}[x \in A] - \mathbb{P}[y \in A]\Big| \tag{5.1.4.1}$$

Lemma 5.1.17 (The Parity Lemma [NN93])**.** *Let $S \subseteq \mathbb{F}_2^N$ is an ϵ-biased set. Let $T \subseteq [N]$ be an nonempty set of size k. Denote S_T the projection of S on the index set T. Then,*

$$\mathrm{SD}(S_T, \mathrm{U}_k) \leq \epsilon \cdot 2^{k/2} \tag{5.1.4.2}$$

This lemma says that the projection of a small biased set onto a small set of coordinates is close to the uniform distribution.

Lemma 5.1.18. *Let $S \subseteq \mathbb{F}_2^N$ is an ϵ-biased set. For every integer $c \geq 1$, let $c \cdot S$ denote the set $S + ... + S$ for c times. We have $c \cdot S$ is an ϵ^c-biased set.*

Proof. For every nonzero $\alpha \in \mathbb{F}_2^N$,

$$\begin{aligned} \left|\mathbb{E}_{s\sim c\cdot S}[(-1)^{\langle \alpha, s\rangle}]\right| &= \left|\mathbb{E}_{s_1,\ldots,s_c\sim S}[(-1)^{\langle \alpha, s_1+\ldots+s_c\rangle}]\right| \\ &= \left|\mathbb{E}_{s_1,\ldots,s_c\sim S}[\prod_{i=1}^{c}(-1)^{\langle \alpha, s_i\rangle}]\right| \\ &= \prod_{i=1}^{c}\left|\mathbb{E}_{s_i\sim S}[(-1)^{\langle \alpha, s_i\rangle}]\right| \leq \epsilon^c \end{aligned} \tag{5.1.4.3}$$

□

2nd proof of Theorem 5.1.15. Let $U \subseteq \mathbb{F}_2^N$ be a subspace with $\dim U = N/2$. Then,

$$\mathbb{P}_{x\in\mathbb{F}_2^N}[\mathrm{dist}(x, U) > N/10] > 0.6 \tag{5.1.4.4}$$

Let S be an 2^{-cd}-biased set where we choose constant c that satisfies

$$2^{-cN/20+N/2} < 0.1 \tag{5.1.4.5}$$

Let $S' = (N/20d) \cdot S$. Then, lemma 5.1.18, we have that S' is $2^{-cN/20}$-biased. By the Parity Lemma 5.1.17, we have

$$\mathrm{SD}(S', \mathrm{U}_N) \leq 2^{-cN/20}2^{N/2} < 0.1 \tag{5.1.4.6}$$

where U_N is the uniform distribution on $\mathbb{F}_2^N$. This gives us

$$\mathbb{P}_{x\sim S'}[\mathrm{dist}(x, U) > N/10] > 0.6 - 0.1 = 0.5 \tag{5.1.4.7}$$

By Markov's inequality, this implies

$$\mathbb{E}_{x\sim S'}[\mathrm{dist}(x, U)] > N/20 \tag{5.1.4.8}$$

Equivalently, let $k = (N/20d)$ because $S' = k \cdot S$,

$$N/20 < \mathbb{E}_{x\sim S'}[\mathrm{dist}(x, U)] = \mathbb{E}_{x_1,\ldots,x_k\sim S}[\mathrm{dist}(\sum_{i=1}^{k} x_i, U)] \tag{5.1.4.9}$$

Notice

$$\mathrm{dist}(\sum_{i=1}^{k} x_i, U) = |\sum_{i=1}^{k} x_i + u| \tag{5.1.4.10}$$

for some u. For all $i \in [k]$, let u_i be such that $|s_i + u_i| = \mathrm{dist}(s_i, U)$, then

$$\mathrm{dist}(\sum_{i=1}^{k} x_i, U) = |\sum_{i=1}^{k} x_i + u| \leq |\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} u_i| \leq \sum_{i=1}^{k} |x_i + u_i| = \sum_{i=1}^{k} \mathrm{dist}(s_i, U) \tag{5.1.4.11}$$

Then because each x_i are independent

$$\begin{aligned}\mathbb{E}_{x\sim S}[\text{dist}(x,U)] &= \frac{\sum_{i=1}^{k}\mathbb{E}_{x\sim S}[\text{dist}(x,U)]}{k}\\ &= \frac{\mathbb{E}_{x_1,\dots,x_k\sim S}[\sum_{i=1}^{k}\text{dist}(x_i,U)]}{k}\\ &\geq \frac{\mathbb{E}_{x_1,\dots,x_k\sim S}[\text{dist}(\sum_{i=1}^{k}x_i,U)]}{k}\\ &\geq \frac{\mathbb{E}_{x\sim S'}[\text{dist}(x,U)]}{k}\\ &> \frac{N/20}{k} = \frac{N/20}{N/20d} = d\end{aligned} \tag{5.1.4.12}$$

Hence, S is a $(N, N/2, d)$-strong rigid set. □

5.1.5 Expander Graphs ❄

Definition 5.1.19. If every vertex of a graph G has degree k, then G is said to be k-**regular**.

Let $G = (V.E)$ be an undirected D-regular on N vertices. Let A_G be the *normalised adjacency matrix* of G. That is, for any $u, v \in V$,

$$(A_G)_{u,v} = \frac{\text{number of edges between } u \text{ and } v}{D} \tag{5.1.5.1}$$

Definition 5.1.20. A D-regular graph G on N vertices is called a (N, D, λ)-**expander** if the absolute value of the second largest eigenvalue is λ.

Lemma 5.1.21 ([AC88])**.** *Let $G = (V, E)$ be an (N, D, λ)-expander. Then for any set $S \subseteq V$ with size $S = \alpha N$,*

$$\left|e(S) - \frac{1}{2}D\alpha^2 N\right| \leq \frac{1}{2}\lambda D\alpha(1-\alpha)N \tag{5.1.5.2}$$

Theorem 5.1.22 ([AR94])**.** *Let $S \subseteq \mathbb{F}_2^N$ be an ϵ-biased set. Let $G_S = (V, E)$ be a subgraph of the Boolean cube on $\mathbb{F}_2^N$ such that $V = \mathbb{F}_2^N$ and an edge connects a pair $u, v \in V$ if $u + v \in S$. Then G_S is a $(2^N, |S|, \epsilon)$-expander.*

Lemma 5.1.23. *Let $S \subseteq \mathbb{F}_2^N$ be an ϵ-biased set. Then for any subspace $U \subseteq \mathbb{F}_2^N$ with $\dim U = k$,*

$$\left|\frac{S\cap U}{S}\right| \leq \frac{1}{2^{N-k}} + \epsilon \tag{5.1.5.3}$$

Proof. Let $G_S = (V, E)$ be constructed as in theorem 5.1.22. Then G_S is a $(2^N, |S|, \epsilon)$-expander. Let $U \subset V = \mathbb{F}_2^N$ be a subspace with with $\dim U = k$. Then, for any vertex $u \in U$, we have the degree of u in the induced subgraph of G_S on U is

$$\left|\{s \in S | u + s \in U\}\right| = \left|\{s \in S | s \in U\}\right| = |U \cap S| \tag{5.1.5.4}$$

Hence,

$$e(U) = \frac{1}{2}\sum_{u\in u}\deg(u) = \frac{1}{2}\sum_{u\in u}|U\cap S| = \frac{1}{2}|U|\cdot|U\cap S| \tag{5.1.5.5}$$

By lemma 5.1.21, we have

$$|U|\cdot|U\cap S| = 2e(U) \leq |S|\left(\frac{|U|}{2^N}\right)^2 2^N + \epsilon\cdot|S|\cdot|U| \tag{5.1.5.6}$$

Hence,

$$\frac{|U\cap S|}{|S|} \leq \frac{|U|}{2^N} + \epsilon \tag{5.1.5.7}$$

□

3rd proof of Theorem 5.1.15. Let $U \subset \mathbb{F}_2^N$ be a subspace with with $\dim U = N/2$. We partition the N unit vectors of $\mathbb{F}_2^N$ into $8d$ disjoint sets

$$B_1, ..., B_{8d}$$

each of size $N/8d$. For each subset $I \subset [8d]$ with $|I| = 2d$, we define

$$U_I = \text{span}(U \cup \bigcup_{i \in I} B_i) \tag{5.1.5.8}$$

Hence, we have that for each I, $\dim U_I \leq 3N/4$ and for every vector x, $\text{dist}(x, U_I) \leq 2d$ for some I. Let $S \subseteq \mathbb{F}_2^N$ be an ϵ-biased set. By lemma 5.1.23, for every I, we have

$$|S \cap U_I| \leq |S| \cdot (\frac{1}{2^{N-3N/4}} + \epsilon) = |S| \cdot (\frac{1}{2^{N/4}} + \epsilon) \tag{5.1.5.9}$$

Because there are at most

$$\binom{8d}{2d} < 120^d$$

such sets I, which covers all x such that $\text{dist}(x, U) \leq 2d$. Thus there are at most

$$120^d |S| \cdot (\frac{1}{2^{N/4}} + \epsilon)$$

vectors x in S such that $\text{dist}(x, U) \leq 2d$. Set $\epsilon = 1/(120^d \cdot 4)$, then

$$120^d |S| \cdot (\frac{1}{2^{N/4}} + \epsilon) \leq 120^d |S| \cdot (\frac{1}{2^{N/4}} + \frac{1}{120^d \cdot 4}) \leq |S| \cdot (\frac{2^{7d}}{2^{N/4}} + 1/4) \leq \frac{|S|}{2} \tag{5.1.5.10}$$

for $c \leq 1/28$ and $d \leq cN$. Hence, we have that at most half of the vectors in S are at most $2d$-far from U, which gives us

$$\mathbb{E}_{x \sim S}[\text{dist}(x, U)] \geq d \tag{5.1.5.11}$$

and S is a $(N, N/2, d)$-strong rigid set. □

5.1.6 Unbalanced Expanders ❄

Let $G = (L, R, E)$ be a bipartite graph with $|L| = m, |R| = n$ and left degree d. For a set $S \subseteq L$, we define

$$\Gamma(S) = \{r \in R | \exists s \in S \text{with} (s, r) \in E\} \tag{5.1.6.1}$$

and

$$\Gamma_1(S) = \{r \in R | \nexists s \in S \text{with} (s, r) \in E\} \tag{5.1.6.2}$$

Definition 5.1.24 (Bipartite Expander)**.** We call G a $(k_{\max}, 1-\epsilon)$**-bipartite expander** if for every $S \subseteq L, |S| \leq (k_{\max}$, it holds that $|\Gamma(S)| > (1-\epsilon)d|S|$.

Definition 5.1.25 (Unique Neighbour Expander)**.** We call G a $(k_{\max}, 1-\epsilon)$**-unique neighbour expander** if for every $S \subseteq L, |S| \leq (k_{\max}$, it holds that $|\Gamma_1(S)| > (1-\epsilon)d|S|$.

We call a bipartite expander a *unbalanced expander* if $m >> n$. Using probabilistic methods, we can show that for every $n, d, k_{\max}$ such that $k_{\max} = O(n/d)$ and for every constant $\epsilon > 0$, there exists a $(k_{\max}, 1-\epsilon)$-bipartite expander with

$$m = k_{\max} \cdot \Big(\frac{n}{d \cdot k_{\max}}\Big)^{\Omega(d)}$$

By the following proposition, this bipartite expander is also a $(k_{\max}, 1-2\epsilon)$-unique neighbour expander.

Proposition 5.1.26. *Every $(k_{\max}, 1-\epsilon)$-bipartite expander is a $(k_{\max}, 1-2\epsilon)$-unique neighbour expander.*

Theorem 5.1.27. *Let $G = (L, R, E)$ be a $(k_{\max}, 2/3)$-bipartite expander with $L = [m], R = [n]$ and left degree $4d$. For every $l \in L$, define a vector $c_l \in \mathbb{F}_2^N$ by, for $i \in [n]$,*

$$(c_l)_i = \begin{cases} 1, & \text{if } (l,i) \in E; \\ 0, & \text{otherwise} \end{cases}$$

If we have

$$\sum_{i=0}^{k_{\max}/2} \binom{m}{i} > 2^k,$$

then the set $C = \{c_l | l \in L\}$ is (n, k, d)-rigid.

Proof. Let $G = (L, R, E)$ be a $(k_{\max}, 2/3)$-bipartite expander with $L = [m], R = [n]$ and left degree $4d$. By proposition 5.1.26, we have that G is also a $(k_{\max}, 1/3)$-unique neighbour expander. Let $U \subseteq \mathbb{F}_2^N$ be a subspace with $\dim U = k$. Assume, for contradiction, that C is not (n, k, d)-rigid, i.e., for every $c \in C$, there exists a $u_c \in U$ such that $|c + u_c| \leq d$. Define

$$U' := \{u_c | c \in C\} \tag{5.1.6.3}$$

where if there is more than one u_c such that $|c + u_c| \leq d$, then we choose u_c arbitrarily.

Claim 5.1.28. $|U'| = |C| = m$.

Proof. Let $c, c' \in C, c \neq c'$ be given. It suffices to show that $u_c \neq u_{c'}$. Assume on the contrary that $u_c = u_{c'}$. Then

$$|c + c'| \leq |c + u_c| + |c' + u_{c'}| + |u_c + u_{c'}| \leq 2d$$

Because G is a $(k_{\max}, 1/3)$-unique neighbour expander, with $S := \{c, c'\}$,

$$|c + c'| = |\Gamma_1(S)| \geq \frac{1}{3} \cdot 4d \cdot 2 > 2d \tag{5.1.6.4}$$

which gives us a contradiction. ∎

We now define

$$U'' := \Big\{ \sum_{i=1}^{t} u_i \Big| t \in [k_{\max}/2], u_1, ..., u_t \in U' \Big\}. \tag{5.1.6.5}$$

Claim 5.1.29. $|U''| = \sum_{i=0}^{k_{\max}/2} \binom{m}{i}$.

Proof. Notice that it suffices to prove that for every $\emptyset \neq S \subseteq U'', |S| \leq k_{\max}$,

$$\sum_{u \in S} u \neq 0. \tag{5.1.6.6}$$

Suppose on the opposite $\sum_{u \in S} u = 0$. Then

$$|\sum_{u \in S} c_u| \leq \sum_{u \in S} |c_u + u| + \Big| \sum_{u \in S} u \Big| \leq d \cdot |S|$$

while as G is a $(k_{\max}, 1/3)$-unique neighbour expander,

$$|\sum_{u \in S} c_u| = |\Gamma_1(S)| \geq \frac{1}{3} \cdot 4d \cdot |S| > d \cdot |S|$$

∎

If

$$|U''| = \sum_{i=0}^{k_{\max}/2} \binom{m}{i} > 2^k$$

then $|U''| > |U'|$. On the other hand, because $U'' \subseteq U'$, we have $|U''| \leq |U'|$, which is a contradiction. Hence, it must be the case that C is (n, k, d)-rigid. □

5.2 Linear Data Structure and Rigidity ❖

5.2.1 Systematic Linear Data Structure Model ❄

The task of the **inner product problem** is to preprocess a vector $v \in \mathbb{F}_2^N$ to compute inner products. The queries are specified by a set $Q \subseteq \mathbb{F}_2^N$, called **query set**, and the data structure compute the inner product $\langle q, v\rangle$ of v and any *query* $q \in Q$. During preprocessing, a *systematic linear data structure model* stores v and k extra bits, which are the evaluations of k linear functions $\langle a_1, v\rangle, ..., \langle a_k, v\rangle$ where $a_1, ..., a_k \in \mathbb{F}_2^N$. To compute the answer on query q, the data structure outputs a linear combination of these k bits and any d entries from v.

Definition 5.2.1 (Systematic Linear Model)**.** Let the systematic linear data structure model be defined above. For a set $Q \subseteq \mathbb{F}_2^N$, we define the **time** $T(Q,k)$ by

$$T(Q,k) := \max_{v\in\mathbb{F}_2^N} \Big(\min\{d| \text{ can compute } \langle q, v\rangle \forall q \in Q \text{ as a linear combination of } k \text{ extra bits and any } d \text{ bits of } v\}\Big) \tag{5.2.1.1}$$

where we are only allowed to output a linear function of k precomputed linear functions of v along with any d bits of v.

Theorem 5.2.2 ([RR20])**.** *A set $Q \subseteq \mathbb{F}_2^N$ is (N,k,d)-rigid if and only if $T(Q,k) \geq d$.*

Proof. ($\Leftarrow$) Suppose that Q is not (N,k,d)-rigid. That is, there is a k-dimensional subspace $U \subseteq \mathbb{F}_2^N$ such that $\text{dist}(q,U) < d$ for all $q \in Q$. Let $v \in \mathbb{F}_2^N$ be the input data and $\{b_1, ..., b_k\}$ be a basis of U. Let the data structure store $\langle b_1, v\rangle, ..., \langle b_k, v\rangle$. Then, there exists a $q_u \in U$ such that

$$\text{dist}(q, q_u) < d \tag{5.2.1.2}$$

Because q_u is a linear combination of $\{b_1, ..., b_k\}$, we have $T(Q,k) < d$.
($\Rightarrow$) Suppose that Q is (N,k,d)-rigid. Let $\{e_1, ..., e_N\}$ be the standard basis and $t = T(Q,k)$ be the query time. Let the evaluations of k linear functions $\langle a_1, v\rangle, ..., \langle a_k, v\rangle$, where $a_1, ..., a_k \in \mathbb{F}_2^N$, be given and let $U = \text{span}(a_1, ..., a_k)$. Because Q is (N,k,d)-rigid, there exists a $q^* \in Q$ such that

$$\text{dist}(q^*, U) \geq d \tag{5.2.1.3}$$

Let q^* be the query and assume that we can access bits $v_{i_1}, ..., v_{i_t}$ of v. Let

$$V = \text{span}(a_1, ..., a_k, e_{i_1}, ..., e_{i_t}) \tag{5.2.1.4}$$

Then

$$\text{dist}(q^*, U) \leq \text{dist}(q^*, V) + t \tag{5.2.1.5}$$

It remains to show that $\text{dist}(q^*, V) = 0$, which would imply $d \leq \text{dist}(q^*, U) \leq t = T(Q,k)$. Suppose that $\text{dist}(q^*, V) \geq 1$, there exists a vector $y \in \mathbb{F}_2^N$ such that $\langle y, q^*\rangle = 1$ and $\langle y, x\rangle = 0$ for all $x \in V$. This implies

$$\langle y + v, x\rangle = \langle y, x\rangle + \langle v, x\rangle = \langle v, x\rangle \tag{5.2.1.6}$$

However,

$$\langle q^*, y + v\rangle \neq \langle q^*, v\rangle \tag{5.2.1.7}$$

which implies that the output on query q^* will err either on v or $y + v$. □

5.2.2 Linear Data Structure Model ❄

A **linear data structure model**, on the other hand, stores s bits, which are the evaluations of s linear functions $\langle a_1, v\rangle, ..., \langle a_s, v\rangle$ where $a_1, ..., a_s \in \mathbb{F}_2^N$. To compute the answer on query q, the data structure outputs a linear combination of these s bits. Notice that the systematic linear model is different, as the query algorithm for the

systematic model is not charged for accessing the k precomputed bits. The **time** $LT(Q,s)$ for linear model is defined to be

$$LT(Q,s) := \max_{v\in\mathbb{F}_2^N} \Big(\min\{d| \text{ can compute } \langle q,v\rangle \forall q\in Q \text{ as a linear combination of any } d \text{ bits chosen from the } s \text{ stored bits}\}\Big) \tag{5.2.2.1}$$

The following proposition gives a simple comparison between the linear model and the systematic linear model. In the linear model, we can simply add the n bits of v by $\langle e_i, v\rangle, i\in[n]$. Taking into account the k precomputed bits that the systematic model can access without charge, we obtain the desired $LT(Q,n+k)\le d+k$.

Proposition 5.2.3. *If $T(Q,k)\le d$, then $LT(Q,N+k)\le d+k$.*

Lemma 5.2.4. *Let $S\subseteq\mathbb{F}_2^N$ be (N,k,d)-rigid of size m. Then there exists a set $S'\subseteq\mathbb{F}_2^{2k}$ of size at most $m\cdot\lceil N/2k\rceil$ that is $(2k,k,dk/N)$-rigid. If S is explicit, then S can be computed in poly(n) time.*

The following proof is similar to that of Theorem 5.0.3.

Proof. Let $r=N/2k$. Without loss of generality, assume that r is an integer. We first split the coordinates into r blocks $Z_1,Z_2,...,Z_r$. Let $S_i, i\in[r]$ be the set obtain from S by projecting each vector $v\in S$ to the ith block. Let $S'=\bigcup_i S_i$. We show that S' is $(2k,k,dk/N)$-rigid. Suppose not. Then there is a subspace $V\subseteq\mathbb{F}_2^{2k}$ such that $\text{dist}(v,V)<dk/N$ for all $v\in S'$. Because every vector $s\in S$ is the sum of at most r vectors in S', then we have that there is a subspace $U\subseteq\mathbb{F}_2^N$, where each $u\in U$ is a vector of r copies of a vector $v\in V$, such that for all $s\in S$, $\text{dist}(s,U)\le(dk/N)\cdot r=(dk/N)\cdot(N/2k)<d$, which is a contradiction. □

Theorem 5.2.5 ([RR20])**.** *Let $k=LT(Q,3N/2)$ and $Q\subseteq\mathbb{F}_2^N$ of size m be an explicit query set. Then there exists a $(k,k/2,k^2/(4N))$-rigid set $Q'\subset\mathbb{F}_2^N$ with size $m\cdot\lceil n/k\rceil$ if $k\ge 2\sqrt{n}$. Moreover, Q' is also explicit.*

Proof. Because $k=LT(Q,3N/2)$ and $k\le n$, we have that $LT(Q,N+k/2)\ge k$. By proposition 5.2.3, $T(Q,k/2)\ge k/2$. By Theorem 5.2.2, we have Q is $(N,k/2,k/2)$ rigid. By lemma lem: obtain rigid sets from rigid sets, we have that there is a set Q' of size at most $m\cdot\lceil N/k\rceil$ and is $(k,k/2,k^2/(4N))$-rigid. □

Chapter 6

Complexity Theory

6.1 FNP ❖

The complexity class FNP is the function-problem extension of the decision-problem class NP. Formally, a relation $R(x,y)$ is in FNP if there exists a non-deterministic polynomial-time Turing machine M such that for any input x, $M(x)$ outputs y such $R(x,y)=1$ or rejects if no such y exists.

Theorem 6.1.1 ([AC19])**.** *There is an absolute constant $\delta > 0$ such for all prime powers $q = p^r$ and all constants $\epsilon > 0$, there is a $\mathbf{P}^{\mathbf{NP}}$ machine M such that, for infinitely many N , on input 1^N, M outputs an $N \times N$ matrix $H_N \in \{0,1\}^{N\times N}$ such that $\mathcal{R}_{H_N}(2^{n^{1/4-\epsilon}}) \geq \delta N^2$ over $\mathbb{F}_q$.*

Theorem 6.1.2 ([BHPT20])**.** *There is a constant $0 < \delta < 1$ such that there is an FNP-machine that for infinitely many N, on input 1^N outputs an $N \times N$ matrix that is $(\delta \cdot N^2, 2^{\log N/\Omega(\log\log N)})$-rigid.*

Chapter 7

Non-Rigidity

7.1 Error Correcting Codes ❖

We first show a random generator matrix G is a good code with high probability. The observation is that for any nonzero vector $v \in \mathbb{F}_q^k$, the vector vG has entries that are distributed uniformly and independently in $\mathbb{F}_q^N$. Let $\text{Vol}_q(d-1,N)$ denote the volume of the Hamming ball of vectors of length N and Hamming weight at most d.

Proposition 7.1.1. *If $\text{Vol}_q(d-1,N) < q^{N-k}$ then there exists linear code of dimension k and distance at least d in $\mathbb{F}_q^N$.*

Proof. Let G be a generator matrix whose entries are chosen uniformly at random. Let $v \in \mathbb{F}_q^k$ be a nonzero vector. Then the probability that the vector vG has at most $d-1$ entries is thus $(\text{Vol}_q(d-1,N)/q^N)$. As there are $q^k - 1$ nonzero vectors, as long as

$$(q^k - 1) \cdot \frac{\text{Vol}_q(d-1,N)}{q^N} < 1$$

there must exists a linear code of dimension k and distance at least d in $\mathbb{F}_q^N$. □

Hence, if $N >> d$, then $\text{Vol}_q(d-1,N)/(q^{N-k}) << 1$, which means a random generator matrix G is a good code with high probability.

We are now ready to show that some specific generating matrix of a good code can have low rigidity.

fix the general case

Theorem 7.1.2 (Dvir)**.** *For every sufficiently small constant $\varepsilon > 0$, all sufficiently large k, and every $d \in [k/4]$, there exist an $n = O(k)$ and a k-by-n matrix M such that every non-zero linear combination of the rows of M has Hamming weight $(?? \pm \varepsilon)n$ but the matrix M has rigidity at most $O(kn/d)$ with respect to rank $10d\log(k/d)$.*

Proof. Let $\varepsilon > 0$ be given. Let $k >> d$ be chosen later. Let $r = 10d\log(k/d)$, $m = k + r$, $c = O(\log(1/\varepsilon))$ and $n = \Omega(m/\varepsilon^2))$. Let H be a random m-by-n matrix with each entry set to 1 with probability $p = c/d$ independently.

Next, choose sufficiently large r such that for a random k-by-r matrix G' with entires uniformly chosen in $\mathbb{F}_q$ and any linear combination of its rows yields a vector of weight at least d with high probability. Let $G = [I_k|G']$, where I_k is the identity matrix. Then the code generated by G has distance at least d. Denoting the top k rows of H by S and the remaining rows by H' we get $GH = IS + G'H' = S + G'H'$.

We first show that with high probability, the matrix GH generates a good code in which all non-zero codewords have weight $(?? \pm \varepsilon)n$. Let x be a nonzero vector in $\mathbb{F}_q^k$. Observe that xG can be seen as a vector whose entries, except for the first k ones, are distributed uniformly and independently in $\mathbb{F}_q^m$. Let $v = xGH$. By definition of H, each entry in v is a sum of at least d independent random variables that are each nonzero with probability $p = c/d$. Hence, each entry in v is nonzero with probability $?? \pm \exp(-\Omega(c))$.

Now we show that GH does not have rigidity $2p \cdot kn = 2c \cdot kn/d$ with respect to rank $r = 10d\log(k/d)$. Observe that, with high probability, the matrix S has weight at most $2p \cdot kn = 2c \cdot kn/d$. On the other hand, G' is an k-by-r matrix, which implies that $G'H'$ has rank at most $r = 10d\log(k/d)$. Hence, $GH = S + G'H'$ does not have rigidity $2c \cdot kn/d$ with respect to rank $r = 10d\log(k/d)$. □

7.2 Polynomial Methods ❖

- Construct a low rank matrix M approximating H_n, using the polynomial methods.
- Observe that that the errors in M are highly concentrated on a relatively small number of rows and columns.
- The rows and columns can be "corrected" in a way that increases the rank of M by only a small amount.
- Each row of the matrix left over will have a small number of erroneous entries.

7.2.1 Hadamard Matrices ❄

Theorem 7.2.1 ([AW17])**.** *For every field $\mathbb{F}$, for every sufficiently small $\varepsilon > 0$, and for all n, we have*

$$\mathcal{R}_H(2^{n-f(\varepsilon)n}) \leq 2^{n(1+\varepsilon)} \tag{7.2.1.1}$$

Lemma 7.2.2 ([AW15])**.** *For any integers n, r, k with $n \geq r + k$ and any integers $c_1, \ldots, c_r$, there is a multivariate polynomial $p : \{0,1\}^n \to \mathbb{Z}$ of degree $r-1$ with integer coefficients such that $p(\vec{z}) = c_i$ for all $\vec{z} \in \{0,1\}^n$ with Hamming weight $|\vec{z}| = k + i$.*

Lemma 7.2.3. *For every field $\mathbb{F}$, for every $0 < \varepsilon < 1/2$, there is a multilinear polynomial $p(x_1, ..., x_n, y_1, ..., y_n)$ over $\mathbb{F}$ with at most $2^{n-\Omega(\varepsilon^2 n)}$ monomials, such that for all $\vec{x}, \vec{y} \in \{0,1\}^n$, with $\langle \vec{x}, \vec{y} \rangle \in [2\varepsilon n, (1/2+\varepsilon)n]$,*

$$p(\vec{x}, \vec{y}) = (-1)^{\langle \vec{x}, \vec{y} \rangle}$$

Lemma 7.2.4. *For every vector $x \in \{0,1\}^n$ with $|x| \in [(1/2-a)n, (1/2+a)n]$, and any parameters $a, b \in (0, 1/5)$, the probability that a uniformly random vector y from $\{0,1\}^n$ satisfies both*

- $|y| \in [(1/2-a)n, (1/2+a)n]$, *and*
- $\sum_{k=1}^{n} x_k y_k \leq bn$

is at most $(2an+1)(bn+1) \cdot 2^{(f(a,b)-1)n}$, where f is a function such that $f(a,b) \to 0$ as $a, b \to 0$.

Lemma 7.2.5. *Let M' be a matrix of rank r which is equal to M except in at most k columns and l rows. Then the rank of M is at most $r + k + l$.*

Corollary 7.2.6. *Let T be any $2^n \times 2^n$ matrix. Let $a \in (0, 1/2)$, and let M be a $2^n \times 2^n$ matrix of rank r, indexed by n-bit vectors. There is a $2^n \times 2^n$ matrix M' of rank at most $r + 4 \cdot n \cdot 2^{n-\Omega(a^2 n)}$ such that $M'(v_i, v_j) = T(v_i, v_j)$ on all $v_i, v_j \in \{0,1\}^n$ where at least one of the following holds:*

- $|v_i| \notin [(1/2-a)n, (1/2+a)n]$,
- $|v_j| \notin [(1/2-a)n, (1/2+a)n]$, *or,*
- $M(v_i, v_j) = T(v_i, v_j)$.

7.2.2 Croot-Lev-Pach Lemma ❄

Let

$$\mathcal{M}_d(q,n) = \{x_1^{a_1} \ldots x_n^{a_n} | 0 \leq a_i \leq q-1, \sum_{i=1}^{n} a_i \leq d\}$$

and $m_d(q,n) = |\mathcal{M}_d(q,n)|$.

Lemma 7.2.7 ([CLP17]). *Let $p : \mathbb{F}_q^n \to \mathbb{F}_q$ be a polynomial of degree at most d and set matrix $M \in \mathbb{F}_q^{q^n \times q^n}$ with $M_{x,y} = p(x+y)$ for $x, y \in \mathbb{F}_q^n$. Then*

$$\text{rank}(M) \leq 2 \cdot m_{\lfloor d/2 \rfloor}(q, n)$$

Corollary 7.2.8. *For any prime power q and any $\varepsilon > 0$, there exists $\delta > 0$ such that for sufficiently large n, we have*

$$m_{(1-\delta)(q-1)n} \geq q^n - q^{\varepsilon n}$$

Lemma 7.2.9. *Let $f : \mathbb{F}_q^n \to \mathbb{F}_q$. Then for all $d \leq n$, there exists a polynomial $p : \mathbb{F}_q^n \to \mathbb{F}_q$ of degree at most d satisfying*

$$|\{x \in \mathbb{F}_2^n | f(x) \neq p(x)\}| \leq q^n - m_d(q, n)$$

Theorem 7.2.10 ([DE19]). *Let $f : \mathbb{F}_q^n \to \mathbb{F}_q$ and set matrix $M \in \mathbb{F}_q^{q^n \times q^n}$ with $M_{x,y} = f(x+y)$ for $x, y \in \mathbb{F}_q^n$.. Then for all any $\varepsilon > 0$, there exists an $\varepsilon' > 0$ such that for sufficiently large n,*

$$\mathcal{R}_M^{\mathbb{F}_q}(N^{1-\varepsilon'}) \leq N^{1+\varepsilon}$$

where $N = q^n$.

7.3 Kronecker Products ❖

Definition 7.3.1 (Kronecker Products). If $A = (a_{ij})$ is an $m \times n$ matrix and B is a $p \times q$ matrix, then the **Kronecker product** $A \otimes B$ is the $pm \times qn$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Definition 7.3.2 (Row-column rigidity). For a matrix $A \in \mathbb{F}^{N \times N}$ and a target rank $0 \leq r \leq N$ let $R_{\mathbb{F}}^{rc}(A, r)$ be the minimal t for which there exists $Z \in \mathbb{F}^{N \times N}$ such that $\text{rank}(A - Z) \leq r$ and every row and column of Z has at most t non-zero entries.

Theorem 7.3.3 ([Alm21]). *For any field $\mathbb{F}$, positive integer $q > 1$, matrices $M_1, \ldots, M_n \in \mathbb{F}^{q \times q}$, and sufficiently small $\varepsilon > 0$, the Kronecker product $M := \bigotimes_{\ell=1}^n M_\ell \in \mathbb{F}^{N \times N}$ for $N = q^n$ has*

$$R_{\mathbb{F}}^{rc}(M, N^{1-O(2^{-q} q \log(q) \cdot \varepsilon^2 / \log^2(1/\varepsilon))}) \leq N^{\varepsilon},$$

where the O hides a universal constant. In particular, if $q \leq O(\log n)$, then M is not Valiant-rigid.

Chapter 8

Appendix

8.1 Picking a random generating matrix ❖

Let n, k, d be given and we choose a random generating matrix of a $[n, k, d]$-code in $\mathcal{F}_2$. We claim that there is a $k \times k$ random full rank submatrix.

To see this, we first write the standard form $[I|A]$ of a generating matrix of a $[n, k, d]$-code in $\mathcal{F}_2$, where I is the $k \times k$ identity matrix and A is a $k \times n-k$ matrix. Extending this to all possible codewords, we then obtain a $2^k \times n$ matrix as below.

Proposition 8.1.1. *C consists all possible linear combinations of the rows of I other than those in I.*

Proof. To see this, suppose there are two rows i and j such that $C_i = C_j$; because I defines a basis of the k-dimensional vector space, $C_i = C_j$ consists of a unique linear combination of the rows of I, meaning that $D_i = D_j$. Since no two codewords are the same in our extended matrix, we obtain a contradiction. □

$\begin{matrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{matrix}$	A
C	D

Now, we observe that picking a random generating matrix is equivalent to picking k independent random rows in our extended matrix. When restricting our attention to the first k columns, this gives us a random $k \times k$ full rank matrix.

8.2 How to get an almost fair coin? ❖

Take d independent tosses of a biased coin, which lands heads with probability $c/d, c \leq 0.5d$. Now, you count the number of heads you observe, and if this number is odd, you write down "head"; and you write down "tail" otherwise. Surprisingly, you can obtain an almost fair coin this way.
If you are a computer science student, you probably would phrase this problem differently. Observe a sequence of bits $\mathbf{x} = x_1x_2...x_d$. Each bit x_i is 1 with probability $c/d, c \leq 0.5d$ and is 0 otherwise. Then the parity of this sequence has only exponential bias. Concretely,

$$\mathbb{P}[\chi(x) = 1] = \frac{1}{2} \pm \exp(\Omega(c))$$

This claim turns out extremely easy to show with a basic introduction of *finite, irreducible, ergodic Markov chains.*

8.2.1 Crash Course Markov Chains ❄

Let S be a finite state space with positive integers and T be a subset of $[0, \infty)$. A **stochastic process** is a collection of random variables $\{X_n : n \in T\}$ which take values from S. Let $\{X_n, n = 0, 1, 2, ...\}$ be a stochastic process that takes on values from S. If $X_n = i$, we say that the process is in state i at time n. Suppose that

$$\mathbb{P}[X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0] = \mathbb{P}[X_{n+1} = j|X_n = i] = p_{i,j}$$

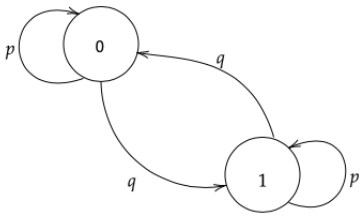
for all states $i_0, i_1, ..., i, j \in S$ and all $n \geq 0$. Such a stochastic process is called a **finite Markov chain**. The **transition matrix P**$= (p_{i,j})$ is a $|S| \times |S|$ matrix of transition probabilities

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

In our example, we have two states, 0 and 1. Let $p = 1 - c/d$ and $q = c/d$. If we are at state 0 during time n, i.e. $\chi(x_1x_2...x_n) = 0$, then we will remain in state 0 with probability p and transition to state 1 with probability 1. Thus, we obtain our transition matrix

$$\mathbf{Q} = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

We say that a finite Markov chain is **irreducible** if and only if its graph representation is a strongly connected graph. Apparently, Q is irreducible, whose graph representation is shown below.



Periodicity

The **period** of a state i is the largest common devisor of the set $\{n : p_{i,i}(n) > 0, n \geq 1\}$. We write $d(i) = \gcd\{n : p_{i,i}(n) > 0, n \geq 1\}$. We call state i **periodic** if $d(i) > 1$ and **aperiodic** if $d(i) = 1$. We notice that in **Q**, $p_{i,i} > 0$ for $i = 0, 1$ in our example. This means that in each step, we can get back to the last state with positive probability. In this case, our Markov chain is *aperiodic.*

Recurrence

Let $f_{i,i}(n) = \mathbb{P}[X_n = i, X_k \neq i \text{ for } 0 < k < n | X_0 = i]$ and let $f_{i,i}$ be the probability that given $X_0 = i$, $X_n = i$ for some $n > 0$. That is,

$$f_{i,i} = \sum_{n=1}^{\infty} f_{i,i}(n)$$

State i is said to be **recurrent** if $f_{i,i} = 1$; on the other hand, we say that state i is **transient** if $f_{i,i} < 1$. For a recurrent state, the **mean recurrence time** μ_i is define as

$$\mu_i = \sum_n n f_{i,i}(n)$$

A recurrent state i is called **positive recurrent** if $\mu_i < \infty$. Notice that a recurrent state i can have infinite mean recurrence time, in this case, we call such a state **null recurrent**. A state is said to be **ergodic** if it is positive recurrent and aperiodic. A Markov chain is ergodic if all its states are ergodic.

In our example, if we start from state 0, then in the next coin flip, we either get 0 with probability p, or get 1 with probability q. If we get a 1, we must wait until the next time to flip a 1 to get back to 0. That is,

$$\mu_0 = \sum_n n f_{0,0}(n) = p + q \sum_{k=0}^{\infty} (k+1) p^k q$$

Similarly, we obtain

$$\mu_1 = \sum_n n f_{1,1}(n) = p + q \sum_{k=0}^{\infty} (k+1) p^k q$$

Notice that $\sum_{k=0}^{\infty} k p^k q$ is the expectation of a geometric distribution with probability q and $\sum_{k=0}^{\infty} p^k q$ is the sum of the probability mass of the same geometric distribution. Thus, as $\sum_{k=0}^{\infty}(k+1)p^k q = \sum_{k=0}^{\infty} k p^k q + \sum_{k=0}^{\infty} p^k q = 1/q + 1$,

$$\mu_0 = \mu_1 = p + q \cdot (1/q + 1) = 2$$

Hence, both states of our Markov chain are positive recurrent. In fact, this Markov chain is *ergodic*.

Stationary distribution

A stationary distribution of a Markov chain is a probability distribution $\bar{\pi}$ such that $\bar{\pi}\mathbf{P} = \bar{\pi}$. In fact, any finite, irreducible, and ergodic Markov chain has a unique **stationary distribution** $\bar{\pi} = (\pi_0, \pi_1, ..., \pi_n)$, with $\pi_0 + \pi_1 + ... + \pi_n = 1$ (Theorem 7.7 in [MU17]). Hence, for our example,

$$\bar{\pi}\mathbf{Q} = (\pi_0, \pi_1) \begin{bmatrix} p & q \\ q & p \end{bmatrix} = \bar{\pi}, \pi_0 + \pi_1 = 1$$

gives us $\pi_0 = \pi_1 = 1/2$. This tells us that *if you flip a biased coin for an infinite number of times, you can get a fair coin.* In other words,

$$\mathbb{P}[\chi(x_1 x_2 ...) = 1] = 1/2$$

How fast does the probability converge to $1/2$ with respect to the number of coin tosses? In fact, this convergence is geometric.

Theorem 8.2.1 (Theorem 12.5 in [MU17])**.** *Let $\bar{\pi}_i^n$ represent the distribution of the state of the chain starting at state i after n steps. Let $\mathbf{P}$ be the transition matrix for a finite, irreducible, aperiodic Markov chain. Let m_j be the smallest entry in the jth column of the matrix, and let $m = \sum_j m_j$. Then for all i and n,*

$$||\bar{\pi}_i^n - \bar{\pi}|| \leq (1 - m)^n.$$

Now we are ready to conclude our motivating example. Since $c \leq 0.5d$, we have $q < p$. Thus, if we take d coin tosses,

$$||\bar{\pi}_0^n - \bar{\pi}|| \leq (1 - 2q)^d = (1 - 2 \cdot \frac{c}{d})^d \leq \exp(-2c)$$

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