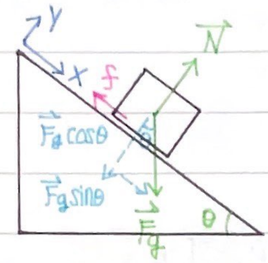
Netonian Mechanics - Single Particle

The Equation of Motion for a Particle



$\ddot{x}$

$\frac{d}{dt}\dot{x}^2 =$

$2\dot{x}\frac{dx}{dt}$

$$\vec{F}_{net} = \vec{F_g} + \vec{N} = m\frac{d^2\vec{r}}{dt^2}$$

y-direction $\quad -F_g\cos\theta + N = 0$

x-direction $\quad F_g\sin\theta = m\frac{d^2x}{dt^2}$ $\Rightarrow \frac{d^2x}{dt^2} = \frac{F_g}{m}\sin\theta = g\sin\theta$

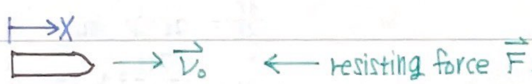
$\downarrow \times 2\frac{dx}{dt}$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2\frac{dx}{dt}g\sin\theta$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}^2\right) = 2g\sin\theta\frac{dx}{dt}$$

$$\Rightarrow \int_0^{v_0^2}\frac{d}{dt}\frac{dx^2}{dt} = 2g\sin\theta\int_0^{x_0}\frac{dx}{dt}$$

$$\Rightarrow v_0^2 = 2g\sin\theta\, x_0$$



$m\vec{a} = m\frac{d\vec{v}}{dt} = -km\vec{v}$, $\int_{v_0}^{v}\frac{d\vec{v}}{\vec{v}} = -k\int_0^t dt$, $\ln v = -kt + C$, $C = \ln v_0$

$\Rightarrow v = v_0 e^{-kt}$ $\quad \ln v - \ln v_0 = -k(t-0)$ $\quad v = v_0 e^{-kt}$

$\ln v - \ln v_0 = -kt$

$v = \frac{dx}{dt} = v_0 e^{-kt}$, $x = v_0\int_0^t e^{-kt}dt = -\frac{v_0}{k}e^{-kt} + C$ $\quad C = \frac{v_0}{k}$

$\Rightarrow x = \frac{v_0}{k}(1 - e^{-kt})$ $\quad \frac{d}{dt}e^{-kt} = -k e^{-kt}$

$\frac{dv}{dx} = \frac{dv}{dt}\frac{dt}{dx} = \frac{dv}{dt}\cdot\frac{1}{v}$, $v\frac{dv}{dx} = \frac{dv}{dt} = -kv$, $\frac{dv}{dx} = -k$, $v = v_0 - kx$

$x = v_0\left(-\frac{1}{k}e^{-kt} + \frac{1}{k}e^{0}\right)$

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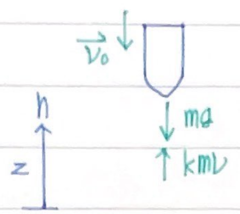
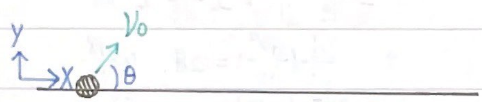


$$F = m\frac{dv}{dt} = -mg - kmv \quad , \quad \frac{dv}{kv+g} = -dt \quad , \quad \int \frac{dv}{kv+g} = \int -dt \quad , \quad \frac{\ln(kv+g)}{k} = -t - C$$

$$\Rightarrow kv + g = e^{-kt+kc} \quad , \quad v = \frac{dz}{dt} = -\frac{g}{k} + \frac{kv_0+g}{k} e^{-kt}$$

$$z = h - \frac{gt}{k} + \frac{kv_0+g}{k^2}(1 - e^{-kt})$$



x-direction $0 = m\frac{d^2x}{dt^2}$ y-direction $-mg = m\frac{d^2y}{dt^2}$

$\frac{dx}{dt} = v_0\cos\theta$ $\frac{dy}{dt} = -gt + v_0\sin\theta$

$x = v_0 t\cos\theta$ $y = -\frac{gt^2}{2} + v_0 t\sin\theta$

$$v = \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2} = \sqrt{v_0^2 + g^2t^2 - 2v_0 g t\sin\theta}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{v_0^2 t^2 + \frac{1}{4}g^2t^2 - v_0 g t^3 \sin\theta}$$

when $y=0$ $\quad y = t\left(-\frac{gt}{2} + v_0\sin\theta\right) = 0$

occurs for $\underset{0}{\overset{}{t}}$ and $t = T$ $\quad -\frac{gT}{2} + v_0\sin\theta = 0$

$$T = \frac{2v_0\sin\theta}{g}$$

$$x_{max} = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin 2\theta$$

$$y_{max}\left(t = \frac{T}{2}\right) = -\frac{1}{8}gT^2 + \frac{1}{2}v_0 T\sin\theta$$

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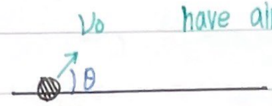
have air resistance



initial conditions

$$x(t=0) = 0 = y(t=0)$$

$$\frac{dx}{dt}(t=0) = v_0\cos\theta \equiv U$$

$$\frac{dy}{dt}(t=0) = v_0\sin\theta = V$$

$$m\frac{d^2x}{dt^2} = -km\frac{dx}{dt} \qquad m\frac{d^2y}{dt^2} = -km\frac{dy}{dt} - mg$$

$$x = \frac{U}{k}(1-e^{-kt}) \qquad y = -\frac{gt}{k} + \frac{kV+g}{k^2}(1-e^{-kt}) \qquad \frac{gt}{k} = \frac{kV+g}{k^2}(1-e^{-kt})$$

$$T = \frac{kV+g}{gk}(1-e^{-kT})$$

perturbation method

$$T = \frac{kV+g}{gk}\left(kT - \frac{1}{2}k^2T^2 + \frac{1}{6}k^3T^3 - \cdots\right) \qquad \frac{kV+g}{gk}kT - \frac{1}{2}\frac{kV+g}{gk}k^2T^2$$

$$\approx \frac{2\frac{V}{g}}{1+\frac{kV}{g}} + \frac{1}{3}kT^2$$

if we keep only terms in the expansion through k^3

$$\frac{1}{1+\frac{kV}{g}} = 1 - \frac{kV}{g} + \left(\frac{kV}{g}\right)^2 - \cdots$$ where we have kept only terms through k^2

$$\Rightarrow T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2}\right)k + O(k^2) \cong \frac{2V}{g}\left(1 - \frac{kV}{3g}\right)$$

$$T(k=0) = T_0 = \frac{2V}{g} = \frac{2v_0\sin\theta}{g}$$

$$x = \frac{U}{k}\left(kt - \frac{1}{2}k^2t^2 + \frac{1}{6}k^3t^3 - \cdots\right)$$

$$x(t=T) \cong U\left(T - \frac{1}{2}kT^2\right)$$

$$\cong \frac{2UV}{g}\left(1 - \frac{4kV}{3g}\right)$$

$$= R\left(1 - \frac{4kV}{3g}\right)$$ where $R = \frac{v_0^2}{g}\sin 2\theta$

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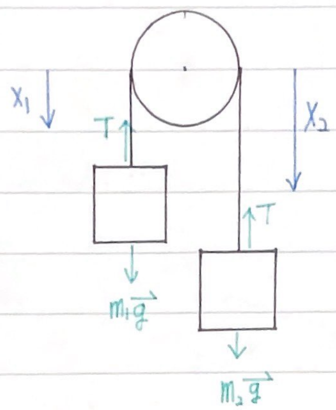

$$m_1\frac{d^2x_1}{dt^2} = m_1 g - T$$
$$m_2\frac{d^2x_2}{dt^2} = m_2 g - T$$
$$\frac{d^2x_2}{dt^2} = -\frac{d^2x_1}{dt^2}$$

$$m_1\frac{d^2x_1}{dt^2} = m_1 g - \left(m_2 g - m_2\frac{d^2x_2}{dt^2}\right)$$
$$= m_1 g - \left(m_2 g + m_2\frac{d^2x_1}{dt^2}\right)$$
$$\Rightarrow \frac{d^2x_1}{dt^2} = \frac{g(m_1-m_2)}{m_1+m_2} = -\frac{d^2x_2}{dt^2}$$

if $m_1 > m_2$ then $\frac{d^2x_1}{dt^2} > 0$

$$T = m_1 g - m_1\frac{d^2x_1}{dt^2} = m_1 g - m_1 g\frac{m_1-m_2}{m_1+m_2} = \frac{2m_1m_2 g}{m_1+m_2}$$

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Conservation Theorems

I. the total linear momentum $\vec{p}$ of a particle is conserved when the total force on it is zero

$$\frac{d\vec{P}}{dt}=0 \qquad \frac{d\vec{P}}{dt}\cdot\vec{S}=\underbrace{\vec{F}\cdot\vec{S}}_{W}=0 \Rightarrow \vec{P}\cdot\vec{S}=\text{constant}$$

II. the angular momentum of a particle subject to no torque is conserved

$$\vec{L}=\vec{r}\times\vec{p} \qquad \vec{\tau}=\vec{r}\times\vec{F}=\vec{r}\times\frac{d\vec{P}}{dt}=\vec{r}\times m\frac{d\vec{v}}{dt}$$

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}(\vec{r}\times\vec{p})=\frac{d\vec{r}}{dt}\times\vec{p}+\vec{r}\times\frac{d\vec{P}}{dt} \quad \text{but } \frac{d\vec{r}}{dt}\times\vec{P}=\frac{d\vec{r}}{dt}\times m\vec{v}=m(\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt})=0$$

$$=\vec{r}\times\frac{d\vec{P}}{dt}$$

$$=\vec{\tau}$$

III the total energy E of a particle in a conservative force field is a constant in time

$$W_{12}=\int_1^2\vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r}=m\frac{d\vec{V}}{dt}\cdot\frac{d\vec{r}}{dt}dt \qquad =\frac{m}{2}\frac{d}{dt}(v^2)dt$$

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$$=m\frac{d\vec{V}}{dt}\cdot\vec{v}\,dt \qquad =d(\frac{1}{2}mv^2)$$

$$=\frac{m}{2}\frac{d}{dt}(\vec{V}\cdot\vec{V})dt$$

$$\Rightarrow W_{12}=(\frac{1}{2}mv^2)\Big|_1^2=\frac{1}{2}m(v_2^2-v_1^2)=E_{k2}-E_{k1}$$

$$\int_1^2\vec{F}\cdot d\vec{r}\equiv U_1-U_2 \qquad \text{and } \vec{F}=-\text{grad}\,U=-\nabla U$$

$$=-\int_1^2(\nabla U)\cdot dr$$

$$=-\int_1^2 dU$$

$$E=E_k+E_p\,,\quad \frac{dE}{dt}=\frac{dE_k}{dt}+\frac{dE_p}{dt} \qquad \begin{cases}\frac{dE_k}{dt}=\vec{F}\cdot\frac{d\vec{r}}{dt}\\ \frac{dE_p}{dt}=\sum_i\frac{\partial U}{\partial x_i}\frac{dx_i}{dt}+\frac{\partial U}{\partial t}=(\nabla U)\cdot\frac{d\vec{r}}{dt}+\frac{\partial U}{\partial t}\end{cases}$$

$$=\vec{F}\cdot\vec{r}+(\nabla U)\cdot\frac{d\vec{r}}{dt}+\frac{\partial U}{\partial t}$$

$$=\underbrace{(\vec{F}+\nabla U)}_{=0}\cdot\frac{d\vec{r}}{dt}+\frac{\partial U}{\partial t}$$

we do not consider velocity-dependent potentials here

0 if the total force is the conservative force

Energy

$$E = E_k + E_p = \frac{1}{2}mv^2 + U(x)$$

$$\Rightarrow v(t) = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E-U(x)]}$$

$$\Rightarrow dt = \pm\frac{dx}{\sqrt{\frac{2}{m}[E-U(x)]}}$$

integrating

$$\Rightarrow t - t_0 = \int_{x_0}^{x} \frac{\pm dx}{\sqrt{\frac{2}{m}[E-U(x)]}}$$

$U(x)$ express in a Taylor series

$$= U_0 + x\left(\frac{dU}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3U}{dx^3}\right)_0 + \cdots = \sum_{i=n+1}^{\infty} \frac{x^i}{i!}\left(\frac{d^iU}{dx^i}\right)_0$$

the quantity is to be evaluated at $x=0$

if $x=0$ is an equilibrium point, then $\left(\frac{dU}{dx}\right)_0 = 0$

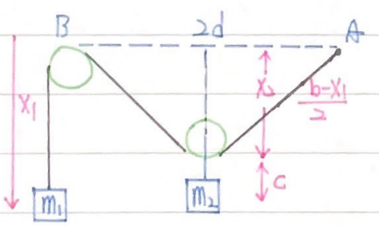
$$\Rightarrow U(x) = \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3U}{dx^3}\right)_0 + \cdots$$

near the $x=0$, the value of x is small, and each term is considerably small

$$\Rightarrow U(x) \simeq \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0$$

$\left(\frac{d^2U}{dx^2}\right)_0 > 0$ stable equilibrium $\left(\frac{d^2U}{dx^2}\right)_0 < 0$ unstable equilibrium

$$\left(\frac{d^2U}{dx^2}\right)_0 = 0 \Rightarrow \frac{dU}{dx} = -\frac{1}{n!}\left(\frac{d^{n+1}U}{dx^{n+1}}\right)_0 x^n = F(x)$$



let $U=0$ along the $\overline{AB}$

$$U = -m_1 g x_1 - m_2 g(x_2 + c) -$$

$$= -m_1 g x_1 - m_2 g\sqrt{\frac{(b-x_1)^2}{4} - d^2} - m_2 g c$$

$$\left(\frac{dU}{dx_1}\right)_0 = -m_1 g + \frac{m_2 g(b-x_0)}{4\sqrt{\frac{(b-x_0)^2}{4} - d^2}} = 0,\ 4m_1\sqrt{\frac{(b-x_0)^2}{4} - d^2} = m_2(b-x_0),\ (b-x_0)^2(4m_1^2 - m_2^2) = 16m_1^2 d^2$$

$$x_0 = b - \frac{4m_1 d}{\sqrt{4m_1^2 - m_2^2}}$$

$$\frac{d^2U}{dx_1^2} = \frac{-m_2 g}{4\sqrt{\frac{(b-x_1)^2}{4} - d^2}} + \frac{m_2 g(b-x_1)^2}{16\sqrt{\left[\frac{(b-x_1)^2}{4} - d^2\right]^3}} \qquad \left(\frac{d^2U}{dx_1^2}\right)_0 = \frac{g(4m_1^2 - m_2^2)^{\frac{3}{2}}}{4m_2^2 d}$$