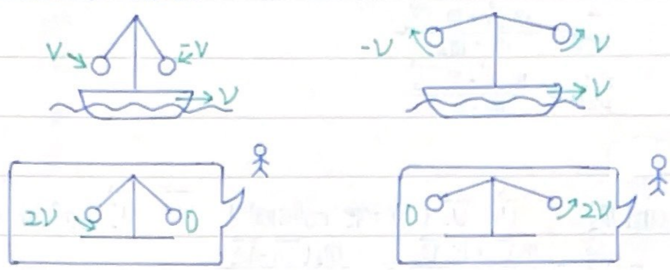
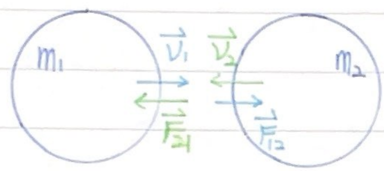
Momentum

Introduction

Descartes : momentum $m\vec{v}(t)$

Huygens :





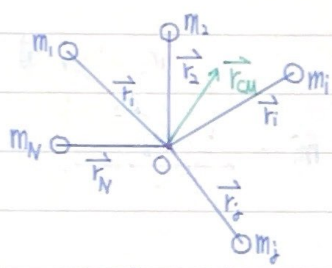
$$\frac{d m_1\vec{v}_1(t)}{dt} = \vec{F}_{21} \qquad \frac{d m_2\vec{v}_2(t)}{dt} = \vec{F}_{12}$$

$$\frac{d m_1\vec{v}_1(t)}{dt} + \frac{d m_2\vec{v}_2(t)}{dt} = \vec{F}_{21} + \vec{F}_{12} = 0$$

$$\frac{d}{dt}\left[m_1\vec{v}_1(t) + m_2\vec{v}_2(t)\right] = 0$$

$$m_1\vec{v}_1(t) + m_2\vec{v}_2(t) = \text{constant}$$

measure mass by mechanics (momentum): internial mass



$$\vec{r}_{cm}(t) = \frac{m_1\vec{r}_1(t) + m_2\vec{r}_2(t) + \cdots + m_N\vec{r}_N(t)}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i(t)}{\sum_{i=1}^{N} m_i}$$

$$\left(\sum_{i=1}^{N} m_i\right)\vec{r}_{cm}(t) = \sum_{i=1}^{N} m_i\vec{r}_i \quad , \quad \left(\sum_{i=1}^{N} m_i\right)\frac{d\vec{r}_{cm}(t)}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} = \sum_{i=1}^{N} m_i\vec{v}_i(t)$$

$$\frac{d}{dt}\left[\left(\sum_{i=1}^{N} m_i\right)\frac{d\vec{r}_{cm}(t)}{dt}\right] = \sum_{i=1}^{N}\frac{d m_i\vec{v}_i(t)}{dt} = \sum_{i=1}^{N}\vec{F}_i = \sum_{i=1}^{N}\Big(\sum_{\substack{j=1\\ i\neq j}}^{N}\vec{F}_{ij}\Big) + \sum_{i=1}^{N}\vec{F}_{i,ext} \qquad \boxed{\vec{F}_i = \sum_{\substack{j=1\\ j\neq i}}^{N}\vec{F}_{ij} + \vec{F}_{i,ext}}$$

since $\vec{F}_{ij} + \vec{F}_{ji} = 0$, then $= \sum_{i=1}^{N}\vec{F}_{i,ext} = \sum_{i=1}^{N} m_i \frac{d^2\vec{r}_{cm}(t)}{dt^2}$

if $\vec{F}_{i,ext} = m_i\vec{g}$, then $\frac{d^2\vec{r}_{cm}(t)}{dt^2} = \vec{g}$

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Collisions of Two Particles

momentum conservation $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{v}_{cm}' = \frac{m_1\vec{v}_1' + m_2\vec{v}_2'}{m_1+m_2}$$

the center velocity of m_1 and m_2 : $\vec{u}_1$ $\vec{u}_2$ (before collisions) $\vec{u}_1'$ $\vec{u}_2'$ (after collisions)

$$\vec{u}_1 = \vec{v}_1 - \vec{v}_{cm} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1+m_2}$$

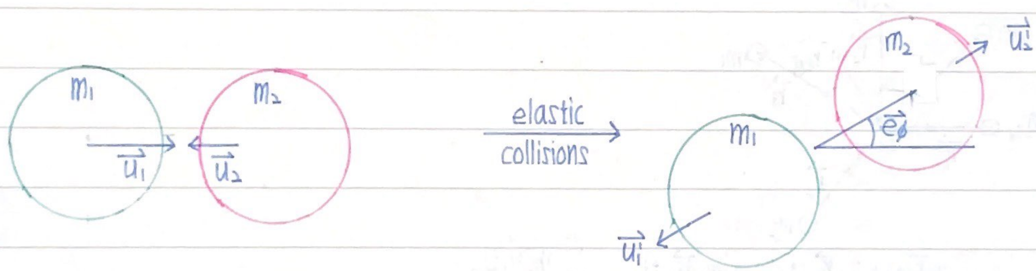
$$\vec{u}_2 = \vec{v}_2 - \vec{v}_{cm} = \vec{v}_2 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{m_1(\vec{v}_2 - \vec{v}_1)}{m_1+m_2}$$

$$\vec{u}_1' = \vec{v}_1' - \vec{v}_{cm}' = \frac{m_2(\vec{v}_1' - \vec{v}_2')}{m_1+m_2}$$

$$\vec{u}_2' = \vec{v}_2' - \vec{v}_{cm}' = \frac{m_1(\vec{v}_2' - \vec{v}_1')}{m_1+m_2}$$

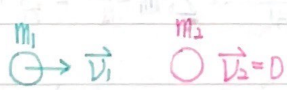
$$m_1\vec{u}_1 + m_2\vec{u}_2 = 0 \qquad m_1\vec{u}_1' + m_2\vec{u}_2' = 0$$

$$\frac{|\vec{v}_1' - \vec{v}_2'|}{|\vec{v}_1 - \vec{v}_2|} = e \qquad e=1 \text{ elastic collisions}$$



only direction change

$$|\vec{v}_1' - \vec{v}_2'| = |\vec{v}_1 - \vec{v}_2|\vec{e}_\phi$$



$$\vec{v}_{cm} = \frac{m_1\vec{v}_1}{m_1+m_2} = \vec{v}_{cm}'$$

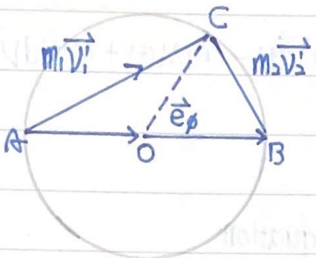
$$\vec{v}_1' - \vec{v}_2' = |\vec{v}_1|\vec{e}_\phi \qquad \vec{v}_1' = \vec{u}_1' + \vec{v}_{cm}' = \frac{m_2(\vec{v}_1' - \vec{v}_2')}{m_1+m_2} + \vec{v}_{cm} = \frac{m_2|\vec{v}_1|\vec{e}_\phi}{m_1+m_2} + \frac{m_1\vec{v}_1}{m_1+m_2}$$

$$\vec{v}_2' = \vec{u}_2' + \vec{v}_{cm}' = \frac{-m_1|\vec{v}_1|\vec{e}_\phi}{m_1+m_2} + \frac{m_1\vec{v}_1}{m_1+m_2}$$

$$m_1\vec{v_1'} = \frac{m_1m_2|\vec{v_1}|\vec{e_\phi}}{m_1+m_2} + \frac{m_1m_1\vec{v_1}}{m_1+m_2}$$

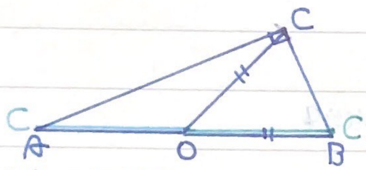
$$m_2\vec{v_2'} = \frac{-m_1m_2|\vec{v_1}|\vec{e_\phi}}{m_1+m_2} + \frac{m_2m_1\vec{v_1}}{m_1+m_2}$$

let $\frac{m_1m_1\vec{v_1}}{m_1+m_2}$ be $\vec{AO}$, $\frac{m_2m_1\vec{v_1}}{m_1+m_2}$ be $\vec{OB}$ and $\frac{m_1m_2|\vec{v_1}|\vec{e_\phi}}{m_1+m_2}$ be $\vec{OC}$



$$\frac{|\vec{AO}|}{|\vec{OB}|} = \frac{|\vec{AO}|}{|\vec{OC}|} = \frac{m_1}{m_2}$$

case I $m_1=m_2$

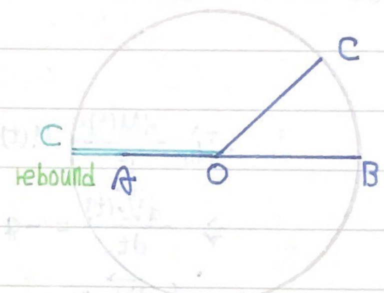


$m_2\vec{v_2'}=0$

A do not collide with B

all the momentum of A transport to B

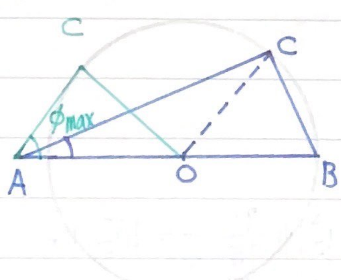
case II $m_2>m_1$



rebound

ex: α particle scattering

case III $m_1>m_2$



$$\sin(\phi_{max}) = \frac{m_2}{m_1}$$

$$\max(m\vec{v_2'}) = 2\frac{m_1m_2\vec{v_1}}{m_1+m_2} \Rightarrow \max(\vec{v_2'}) = \frac{m_1\vec{v_1}}{m_1+m_2}$$

ex: discovery neutron

$$\frac{\max(|\vec{v_H}|)}{\max(|\vec{v_N}|)} = \frac{3.3\times10^9}{4.7\times10^8} = \frac{\frac{1}{m_1+1}}{\frac{1}{m_1+14}}, \quad m_1=1.15$$

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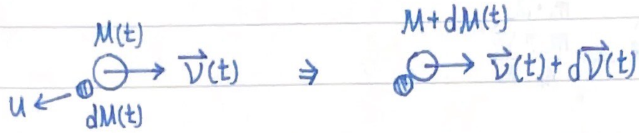
Rocket Propulsion : the masses of parts of the system change with time



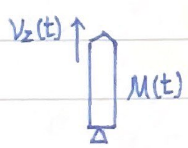
$$[M(t)+dM(t)][\vec{V}(t)+d\vec{V}(t)] - M(t)\vec{V}(t) - \vec{u}\,dM(t)$$

$$= M(t)\vec{V}(t) + dM(t)\vec{V}(t) + M(t)\,d\vec{V}(t) + \underbrace{dM(t)\,d\vec{V}(t)}_{\text{ignore}} - M(t)\vec{V}(t) - \vec{u}\,dM(t)$$

$$= -[\vec{u} - \vec{V}(t)]\,dM(t) + M(t)\,d\vec{V}(t)$$

$$-\underbrace{[\vec{u} - \vec{V}(t)]}_{\text{relative velocity}}\frac{dM(t)}{dt} + M(t)\frac{d\vec{V}(t)}{dt} = \sum \vec{F}_i \leftarrow \text{Tsiolkovsky equation}$$

$$\Rightarrow \vec{U}_{ul}\frac{dM(t)}{dt} + M(t)\frac{d\vec{V}(t)}{dt} = \sum \vec{F}_i$$



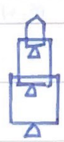
$$-\vec{U}_{z1}\frac{dM(t)}{dt} + M(t)\frac{d\vec{V}_z(t)}{dt} = -M(t)g$$

$$\Rightarrow \frac{d\vec{V}_z(t)}{dt} = -g + \vec{U}_{z1}\frac{1}{M(t)}\frac{dM(t)}{dt}$$

$$\Rightarrow \int \frac{d\vec{V}_z(t)}{dt}dt = \int -g\,dt + \vec{U}_{z1}\int_0^t \frac{dM(t)}{M(t)}\frac{1}{dt}dt$$

$$\Rightarrow \underbrace{V_z(t) - V_z(0)}_{\gtrsim 7.9\text{ km/s}} = -gt + \underbrace{\vec{U}_{z1}}_{2\sim3\text{ km/s}}\ln\frac{M(t)}{M(0)}$$

3-stage launch vehicle



shell	fuel	weight	accumulation
1	6	7	7
10	60	70	77
100	600	700	777

$$|\vec{U}_{z1}|\ln\frac{777}{777-600} = |\vec{U}_{z1}|\,1.48$$

$$|\vec{U}_{z1}|\ln\frac{77}{17} = |\vec{U}_{z1}|\,1.51$$

$$|\vec{U}_{z1}|\ln\frac{7}{1} = |\vec{U}_{z1}|\,1.95$$