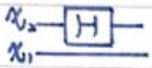
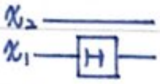
Controlled Operations

two qubit system

$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $|01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ $|10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $|11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$



$I\otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}$



$H\otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}$

CNOT

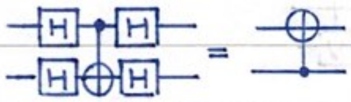
$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

$|c\rangle|t\rangle \to |c\rangle|t\oplus c\rangle$

in	out
00	00
01	01
10	11
11	10

since $HZH = X$
$CH \times CZ \times CH = CNOT$

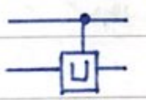
$CNOT = |0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X$



$(\frac{1}{\sqrt{2}})^2 \begin{bmatrix}H&H\\H&-H\end{bmatrix}\begin{bmatrix}I&0\\0&X\end{bmatrix}\begin{bmatrix}H&H\\H&-H\end{bmatrix}$

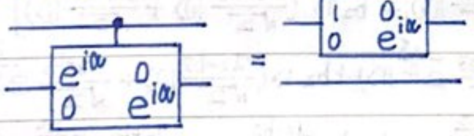
$= \frac{1}{2}\begin{bmatrix}H^2+HXH & H^2-HXH\\H^2-HXH & H^2+HXH\end{bmatrix} = \frac{1}{2}\begin{bmatrix}I+Z & I-Z\\I-Z & I+Z\end{bmatrix} = CNOT_{2,1}$

Controlled-U



$|c\rangle|t\rangle \to |c\rangle U^c|t\rangle$

Phase shift



$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to e^{i\alpha}|10\rangle$
$|11\rangle \to e^{i\alpha}|11\rangle$

an arbitrary single qubit unitary operator can be written as

$U = e^{i\alpha}R_{\hat{n}}(\theta) = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$

$= e^{i\alpha}\left[\cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(n_xX+n_yY+n_zZ)\right] = e^{i\alpha}\begin{bmatrix}\cos\frac{\theta}{2}-i\sin(\frac{\theta}{2})(n_x+n_z) & -n_y\sin\frac{\theta}{2}\\ n_y\sin\frac{\theta}{2} & \cos\frac{\theta}{2}-i\sin(\frac{\theta}{2})(n_x-n_z)\end{bmatrix}$

$= e^{i\alpha}\begin{bmatrix}[1-i\tan(\frac{\theta}{2})(n_x+n_z)]\cos\frac{\theta}{2} & -n_y\sin\frac{\theta}{2}\\ n_y\sin\frac{\theta}{2} & [1-i\tan(\frac{\theta}{2})](n_x-n_z)\end{bmatrix}$

$\frac{\gamma}{2}$

$\cos(\frac{\beta}{2}-\frac{\delta}{2})$

$\frac{\sin^2(\frac{\beta}{2}+\frac{\delta}{2})}{\tan^2\frac{\gamma}{2}}$

$= \begin{bmatrix}e^{i(\alpha-\frac{\beta}{2}-\frac{\delta}{2})}\cos\frac{\gamma}{2} & -e^{i(\alpha-\frac{\beta}{2}+\frac{\delta}{2})}\sin\frac{\gamma}{2}\\ e^{i(\alpha+\frac{\beta}{2}-\frac{\delta}{2})}\sin\frac{\gamma}{2} & e^{i(\alpha+\frac{\beta}{2}+\frac{\delta}{2})}\cos\frac{\gamma}{2}\end{bmatrix}$

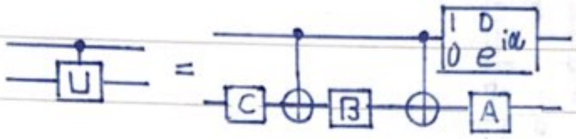
suppose U is a unitary gate, then there exist unitary operator A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$

set $A \equiv R_z(\beta) R_y(\frac{\gamma}{2})$ $\quad B \equiv R_y(-\frac{\gamma}{2}) R_z(-\frac{\delta+\beta}{2})$ $\quad C \equiv R_z(\frac{\delta-\beta}{2})$

$XBX = X R_y(-\frac{\gamma}{2}) X X R_z(-\frac{\delta+\beta}{2}) X = R_y(\frac{\gamma}{2}) R_z(\frac{\delta+\beta}{2})$

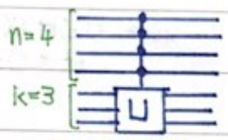
$AXBXC = R_z(\beta) R_y(\gamma) R_z(\delta)$

$\Rightarrow U = e^{i\alpha} AXBXC$ and $ABC = I$



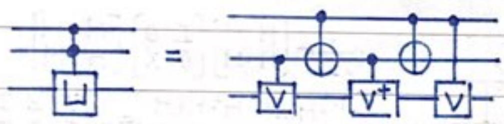
$C^n(U)|x_1 x_2 \cdots x_n\rangle|\psi\rangle = |x_1 x_2 \cdots x_n\rangle U^{x_1 x_2 \cdots x_n}|\psi\rangle$

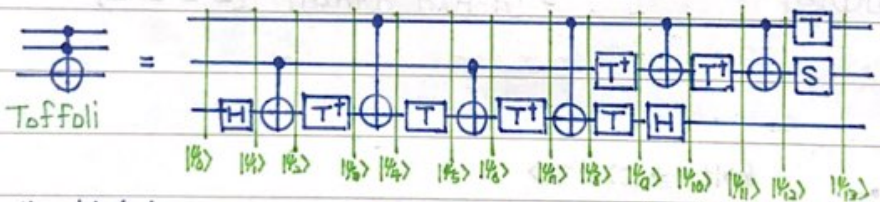
if $n=4$ and $k=3$



if $V^2 = U$, define $V \equiv \frac{(1-i)(I+iX)}{2}$

$C^2(U) =$





$|\psi_0\rangle = |\phi_1 \phi_2 \phi_3\rangle = (a_1|0\rangle + b_1|1\rangle)(a_2|0\rangle + b_2|1\rangle)(a_3|0\rangle + b_3|1\rangle)$

$|\psi_1\rangle = (a_1|0\rangle + b_1|1\rangle)(a_2|0\rangle + b_2|1\rangle)(\frac{a_3+b_3}{\sqrt{2}}|0\rangle - \frac{a_3-b_3}{\sqrt{2}}|1\rangle)$

$|\psi_2\rangle = (a_1|0\rangle + b_1|1\rangle)\left[a_2|0\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle) + b_2|1\rangle(\frac{a_3-b_3}{\sqrt{2}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}|1\rangle)\right]$

$|\psi_3\rangle = (a_1|0\rangle + b_1|1\rangle)\left[a_2|0\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle) + b_2|1\rangle(\frac{a_3-b_3}{\sqrt{2}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle)\right]$

$|\psi_4\rangle = a_1|0\rangle\left[a_2|0\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle) + b_2|1\rangle(\frac{a_3-b_3}{\sqrt{2}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle)\right] +$

$\quad b_1|1\rangle\, a_2|0\rangle(\frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}|1\rangle) + b_2|1\rangle(\frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle)\Big]$

$|\psi_5\rangle = a_1|0\rangle\left[a_2|0\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle) + b_2|1\rangle(\frac{a_3-b_3}{\sqrt{2}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}|1\rangle)\right] +$

$\quad b_1|1\rangle\left[a_2|0\rangle(\frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}} + \frac{a_3+b_3}{\sqrt{2}}e^{\frac{i\pi}{4}}|1\rangle) + b_2|1\rangle(\frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|0\rangle - \frac{a_3-b_3}{\sqrt{2}}e^{\frac{i\pi}{4}}|1\rangle)\right]$

$|\psi_6\rangle = a_1|0\rangle\left[a_2|0\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle) + b_2|1\rangle(\frac{a_3+b_3}{\sqrt{2}}|0\rangle - \frac{a_3-b_3}{\sqrt{2}}|1\rangle)\right] +$

$\quad b_1|1\rangle\left[a_2|0\rangle(\frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}|1\rangle) + b_2|1\rangle(\frac{a_3-b_3}{\sqrt{2}}e^{\frac{i\pi}{4}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle)\right]$

$$|\psi_7\rangle = a_1|0\rangle\Big[a_2|0\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle\Big) + b_2|1\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle\Big)\Big] +$$
$$b_1|1\rangle\Big[a_2|0\rangle\Big(\frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|0\rangle + \frac{a_3+b_3}{\sqrt{2}}|1\rangle\Big) + b_2|1\rangle \frac{a_3-b_3}{\sqrt{2}}e^{\frac{i\pi}{4}}0\rangle + \frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{2}}|1\rangle)\Big]$$

$$|\psi_8\rangle = a_1|0\rangle\Big[a_2|0\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle\Big) + b_2|1\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle\Big)\Big] +$$
$$b_1|1\rangle\ a_2|0\rangle\ \frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{-\frac{i\pi}{4}}|1\rangle) + b_2|1\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{\frac{i\pi}{4}}|1\rangle\Big)\Big]$$

$$|\psi_9\rangle = a_1|0\rangle\Big[a_2|0\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle\Big) + b_2 e^{-\frac{i\pi}{4}}|1\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle\Big)\Big] +$$
$$b_1|1\rangle\Big[a_2|0\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}|1\rangle\Big) + b_2 e^{-\frac{i\pi}{4}}|1\rangle\Big(\frac{a_3+b_3}{\sqrt{2}}e^{-\frac{i\pi}{2}}|0\rangle + \frac{a_3-b_3}{\sqrt{2}}e^{\frac{i\pi}{2}}|0\rangle\Big)\Big]$$

$$|\psi_{10}\rangle = a_1|0\rangle\big[a_2|0\rangle(a_3|0\rangle + b_3|1\rangle) + b_2 e^{-\frac{i\pi}{4}}|1\rangle(a_3|0\rangle + b_3|1\rangle)\big] + b_1|1\rangle\big[b_2 e^{-\frac{3}{4}i\pi}|0\rangle(b_3|0\rangle + a_3|1\rangle)$$
$$+ a_2 \quad |1\rangle(a_3|0\rangle + b_3|1\rangle)\big]$$

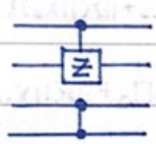
$$|\psi_{11}\rangle = a_1|0\rangle\big[a_2|0\rangle(a_3|0\rangle + b_3|1\rangle) + b_2 e^{-\frac{i\pi}{2}}|1\rangle(a_3|0\rangle + b_3|1\rangle)\big] + b_1 e^{-\frac{3i\pi}{4}}|1\rangle\big[b_2|0\rangle(b_3|0\rangle +$$
$$a_3|1\rangle) + a_2 e^{\frac{i\pi}{2}}|1\rangle(a_3|0\rangle + b_3|1\rangle)\big]$$

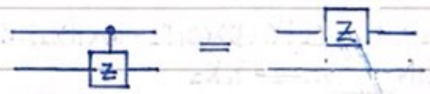
$$|\psi_{12}\rangle = a_1|0\rangle\big[a_2|0\rangle(a_3|0\rangle + b_3|1\rangle) + b_2 e^{-\frac{i\pi}{2}}|1\rangle(a_3|0\rangle + b_3|1\rangle)\big] + b_1 e^{-\frac{3i\pi}{4}}|1\rangle\big[a_2 e^{\frac{i\pi}{2}}|0\rangle(a_3|0\rangle + b_3|1\rangle)$$
$$+ b_2|1\rangle(b_3|0\rangle + a_3|1\rangle)\big]$$

$$|\psi_{13}\rangle = a_1|0\rangle\big[a_2|0\rangle(a_3|0\rangle + b_3|1\rangle) + b_2|1\rangle(a_3|0\rangle + b_3|1\rangle)\big] + b_1|1\rangle\big[a_2|0\rangle(a_3|0\rangle + b_3|1\rangle) + b_2|1\rangle(b_3|0\rangle + a_3|1\rangle)$$

Controlled-Z

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$





$C\text{-}Z_{1,2}|\psi_1\rangle \otimes |\psi_2\rangle$
$= |10\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z\big[(a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle)\big]$
$= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle - b_1b_2|11\rangle$

$C\text{-}Z_{21}|\psi_1\rangle \otimes |\psi_2\rangle$
$= (I \otimes |0\rangle\langle 0| + Z\,|1\rangle\langle 1|)\big[(a_1 0\rangle + b_1 1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle)\big]$
$= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle - b_1b_2|11\rangle$

SWAP

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

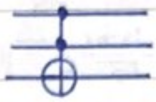
$|01\rangle \longrightarrow |10\rangle$
$|10\rangle \longrightarrow |01\rangle$

$SWAP_{ij} = CONT_{ij}\,CONT_{ji}\,CONT_{ij}$

Ternary Operator

Toffoli CCNOT
quantum AND

$$
\begin{array}{c|c}
 & 8 \\
8 & \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & & \vdots \\ \vdots & \vdots & 1 & \ddots & 1 \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix}
\end{array}
$$

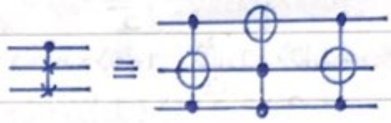


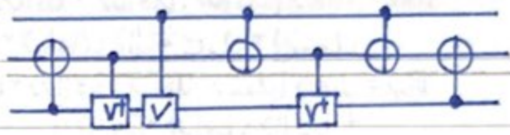
$(x, y, z) \mapsto (x, y, (z \oplus xy))$

Fredkin CSWAT

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & 0 & 1 & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & 0 & 1 & 0 & 1 & 0 \\
 & & & \vdots & 0 & 1 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$|0, AB\rangle \rightarrow |0, AB\rangle$





$CX_1C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) X_1 (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2)(|1\rangle\langle 0| I_2 + |0\rangle\langle 1| X_2)$
$= (|0\rangle\langle 1| + |1\rangle\langle 0|) X_2 = X_1 X_2$

$CY_1C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) Y_1 (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) i (|1\rangle\langle 0| I_2 - |0\rangle\langle 1| X_2)$
$= i(-|0\rangle\langle 1| + |1\rangle\langle 0|) X_2 = Y_1 X_2$

$CZ_1C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) Z_1 (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2)(|0\rangle\langle 0| I_2 - |1\rangle\langle 1| X_2)$
$= (|0\rangle\langle 0| - |1\rangle\langle 1|) I_2 = Z_1$

$CX_2C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) X_2 |0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2)(|0\rangle\langle 0| X_2 + |1\rangle\langle 1| I_2)$
$= I_1 X_2 = X_2$

$CY_2C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) Y_2 (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2)(|0\rangle\langle 0| Y_2 - i |1\rangle\langle 1| Z_2)$
$= (|0\rangle\langle 0| - |1\rangle\langle 1|) Y_2 = Z_1 Y_2$

$CZ_2C = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) Z_2 |0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2) = (|0\rangle\langle 0| I_2 + |1\rangle\langle 1| X_2)(|0\rangle\langle 0| Z_2 + i |1\rangle\langle 1| Y_2)$
$= (|0\rangle\langle 0| - |1\rangle\langle 1|) Z_2 = Z_1 Z_2$