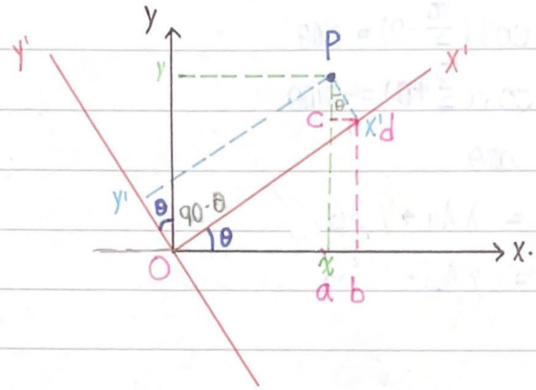
Matrices, Vectors, and Vector Calculus

Coordinate Transformations

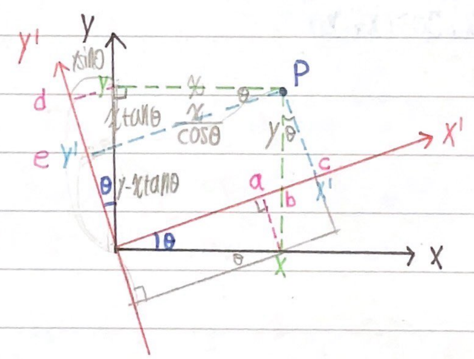


$$\vec{u}_1 = x\vec{e}_1 + y\vec{e}_2 = (|\overline{OB}| - |\overline{AB}|)\vec{e}_1 + (|\overline{BD}| + |\overline{CP}|)\vec{e}_2$$

$$= (x'\cos\theta - y'\sin\theta)\vec{e}_1 + (x'\sin\theta + y'\cos\theta)\vec{e}_2$$

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$



$$x' = \overline{Oa} + \overline{ab} + \overline{bc} = x\cos\theta + y\sin\theta = x\cos\theta + y\cos\left(\frac{\pi}{2} - \theta\right)$$

$$y' = \overline{Od} - \overline{de} = y\cos\theta - x\sin\theta = y\cos\theta + x\cos\left(\frac{\pi}{2} + \theta\right)$$

the angle between X-axis and the X-axis is denoted by (X, X)

and we define $\lambda \equiv \cos(X, X)$

↓

direction cosine

$$\lambda_{11} = \cos(x_1', x_1) = \cos(x, x) = \cos\theta$$
$$\lambda_{12} = \cos(x_1', x_2) = \cos(x, y) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\lambda_{21} = \cos(x_2', x_1) = \cos(y, x) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$
$$\lambda_{22} = \cos(x_2', x_2) = \cos(y, y) = \cos\theta$$
$$\Rightarrow \quad x = x\cos(x_1', x_1) + y\cos(x_1', x_2) = x\lambda_{11} + y\lambda_{12}$$
$$y = x\cos(x_2', x_1) + y\cos(x_2', x_2) = x\lambda_{21} + y\lambda_{22}$$

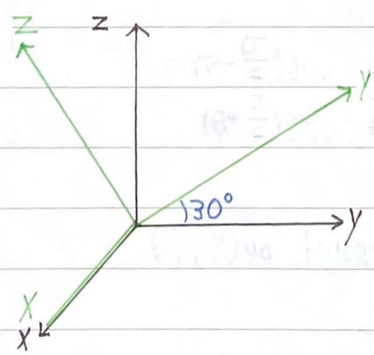
for three dimensions we have

$$x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3$$
$$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \quad \Rightarrow \quad x_i' = \sum_{j=1}^{3} \lambda_{ij}x_j, \quad i = 1,2,3$$
$$x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3$$

the inverse transformation

$$x_1 = x_1'\cos(x_1', x_1) + x_2'\cos(x_2', x_1) + x_3'\cos(x_3', x_1)$$
$$= \lambda_{11}x_1' + \lambda_{21}x_2' + \lambda_{31}x_3'$$
$$= \sum_{j=1}^{3} \lambda_{ji}x_j', \quad i = 1,2,3$$

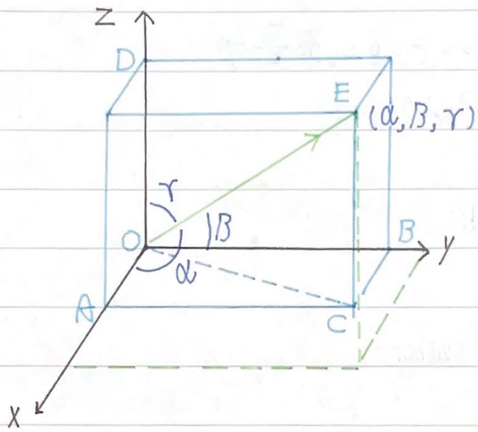
$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$



$$\lambda_{11} = \cos(x', x) = \cos(0°) = 1$$
$$\lambda_{12} = \cos(x', y) = \cos(90°) = 0$$
$$\lambda_{13} = \cos(x', z) = \cos(90°) = 0$$
$$\lambda_{21} = \cos(y', x) = \cos(90°) = 0$$
$$\lambda_{22} = \cos(y', y) = \cos(30°) \approx 0.866$$
$$\lambda_{23} = \cos(y', z) = \cos(60°) = 0.5$$
$$\lambda_{31} = \cos(z', x) = \cos(90°) = 0$$
$$\lambda_{32} = \cos(z', y) = \cos(120°) = -0.5$$
$$\lambda_{33} = \cos(z', z) = \cos(30°) \approx 0.866$$

$$\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & 0.5 \\ 0 & -0.5 & 0.866 \end{bmatrix}$$

Properties of Rotation Matrices

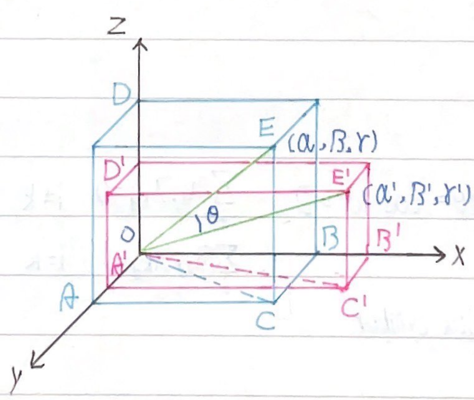


$\overline{OE}\cos\alpha = \overline{OA}$ $\qquad$ $\overline{OE}^2\left[\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right] = \overline{OA}^2+\overline{OB}^2+\overline{OD}^2$

$\overline{OE}\cos\beta = \overline{OB}$ $\qquad$ but $\overline{OA}^2+\overline{OB}^2 = \overline{OC}^2$ and $\overline{OC}^2+\overline{OD}^2 = \overline{OE}^2$

$\overline{OE}\cos\gamma = \overline{OD}$ $\qquad$ thus $\overline{OA}^2+\overline{OB}^2+\overline{OD}^2 = \overline{OE}^2$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$



$\overline{OE}^2+\overline{OE'}^2-2\,\overline{OE}\,\overline{OE'}\cos\theta = \overline{EE'}^2$

$\overline{EE'}^2 = (\overline{OB'}-\overline{OB})^2+(\overline{OA'}-\overline{OA})^2+(\overline{OD'}-\overline{OD})^2$

$= (\overline{OE'}\cos\beta'-\overline{OE}\cos\beta)^2+(\overline{OE'}\cos\alpha'-\overline{OE}\cos\alpha)^2+(\overline{OE'}\cos\gamma'-\overline{OE}\cos\gamma)^2$

$= \overline{OE'}^2(\cos^2\alpha'+\cos^2\beta'+\cos^2\gamma')+\overline{OE}^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$

$\quad -2\,\overline{OE'}\,\overline{OE}\,(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')$

$= \overline{OE'}^2+\overline{OE}^2-2\,\overline{OE'}\,\overline{OE}\,(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')$

$\Rightarrow \cos\theta = \cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'$

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because the x'-axis and the y'-axis is $\frac{\pi}{2}$ angle between

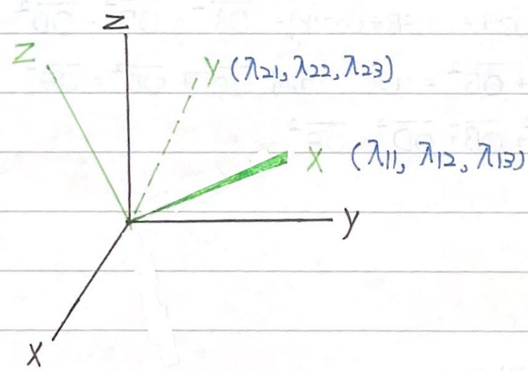
we have $\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\theta = \cos\frac{\pi}{2} = 0$

or $\sum_j \lambda_{ij}\lambda_{2j} = 0$ $\quad \sum_j \lambda_{ij}\lambda_{kj} = 0,\ i \neq k$

$\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1$ $\quad \sum_j \lambda_{ij}\lambda_{kj} = 1\ \ i=k$

$\Rightarrow \sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}$ Kronecker delta symbor

$$\delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$



$\Rightarrow \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\theta = \cos 90° = 0$, $\sum_j \lambda_{ij}\lambda_{kj} = 0 \quad i \neq k$

$\Rightarrow \lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1$, $\sum_j \lambda_{ij}\lambda_{kj} = 1 \quad i = k$

$\Rightarrow \sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}$ Kronecker delta symbor

$$\delta_{ik} \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$

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Matrix Operations

$$X_i' = \sum_j \lambda_{ij} X_j \Rightarrow X' = \lambda X$$

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

also can be written as $X_i = \sum_j \lambda_{ij}^t X_j'$

consider the orthogonal rotation matrix λ for the case of two dimensions

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

$$\lambda \lambda^t = \begin{bmatrix} \lambda_{11}^2 + \lambda_{12}^2 & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} & \lambda_{21}^2 + \lambda_{22}^2 \end{bmatrix}$$ and $\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2 + \lambda_{22}^2 = 1$

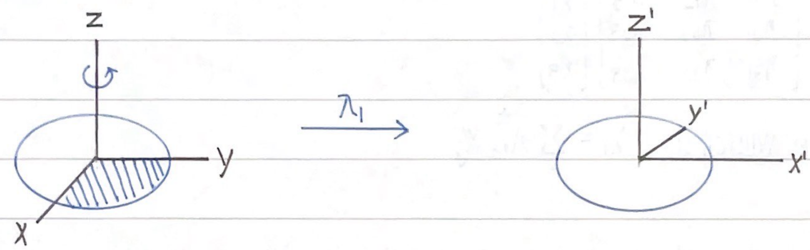
$\lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$ in orthogonal matrices

$$\Rightarrow \lambda^t = \lambda^{-1}$$

Geometrical Significance of Transformation Matrices

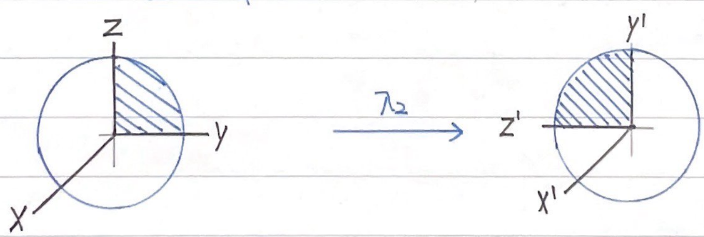
90° rotation about z axis



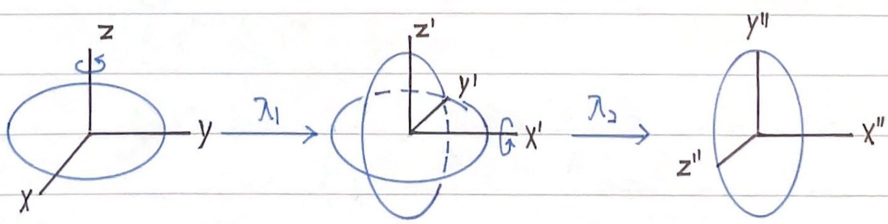
$\cos(x'_1, y) = \lambda_{12} = 1$ $\quad \cos(y', x) = \lambda_{21} = -1$ $\quad \cos(z', z) = \lambda_{33} = 1$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

90° rotation about x-axis

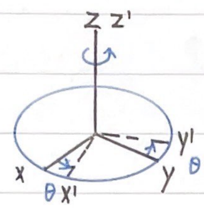


$$\lambda_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

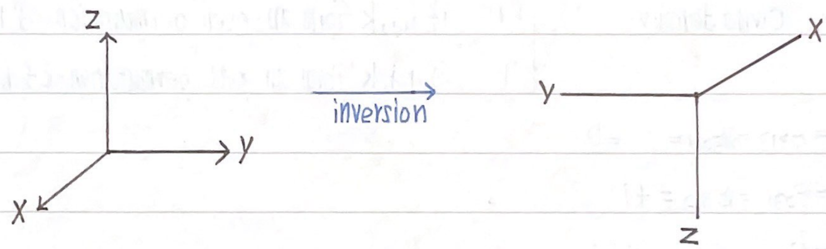


$x' = \lambda_1 x$, $x'' = \lambda_2 x' \Rightarrow x'' = \lambda_2 \lambda_1 x \Rightarrow x'' = \lambda_3 x$

$\lambda_4 = \lambda_1 \lambda_2 \neq \lambda_3$



$$\lambda_5 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\lambda_6 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = |\lambda_5| = 1$ proper rotation

$|\lambda_6| = -1$ improper rotation

all orthogonal matrices must have a determinant equal to either +1 or -1

Vector Product of Two Vectors

$$\vec{C} = \vec{A} \times \vec{B}$$

$$C_i \equiv \sum_{j,k} \varepsilon_{ijk} A_j B_k$$

permutation symbol (or Levi-Civita density) ε_{ijk} $\begin{cases} 0 & \text{if any index is equal to any other index} \\ +1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3 \\ 1 & \text{if } i,j,k \text{ form an odd permutation of } 1,2,3 \end{cases}$

$$\varepsilon_{122} = \varepsilon_{313} = \varepsilon_{211} = \cdots = 0$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\Rightarrow C_1 = \sum_{jk} \varepsilon_{1jk} A_j B_k = \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$$

$$C_2 \qquad = A_3 B_1 - A_1 B_3$$

$$C_3 \qquad = A_1 B_2 - A_2 B_1$$

$$A^2 B^2 \sin^2\theta = A^2 B^2 - A^2 B^2 \cos^2\theta = (\sum_i A_i^2)(\sum_i B_i^2) - (\sum_i A_i B_i)^2$$

$$= (A_2 B_3 - A_3 B_2)^2 + (A_3 B_1 - A_1 B_3)^2 + (A_1 B_2 - A_2 B_1)^2$$

$$= C_1^2 + C_2^2 + C_3^2$$

$$\vec{A} \cdot (\vec{B} \times \vec{D}), \quad (\vec{B} \times \vec{D})_i = \sum_{jk} \varepsilon_{ijk} B_j D_k \Rightarrow \vec{A} \cdot (\vec{B} \times \vec{D}) = \sum_{ijk} \varepsilon_{ijk} A_i B_j D_k$$

$$\vec{D} \cdot (\vec{A} \times \vec{B}) = \sum_{ijk} \varepsilon_{ijk} D_i A_j B_k = \sum_{ijk} -\varepsilon_{jik} D_i A_j B_k = \sum_{ijk} \varepsilon_{jki} A_j B_k D_i$$

$$\Rightarrow \vec{A} \cdot \vec{B} \times \vec{D} = \vec{D} \cdot (\vec{A} \times \vec{B})$$

$$* \quad \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{A}\vec{B}\vec{C}$$

$$\vec{A} \times \vec{B} \times \vec{C} = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \vec{A} \cdot [\vec{B} \times (\vec{C} \times \vec{D})] = \vec{A} \cdot [(\vec{B} \cdot \vec{D})\vec{C} - (\vec{B} \cdot \vec{C})\vec{D}] = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [(\vec{A} \times \vec{B}) \cdot \vec{D}]\vec{C} - [(\vec{A} \times \vec{B}) \cdot \vec{C}]\vec{D}$$

$$= (ABD)C - (ABC)D - (ACD)B - (BCD)A$$