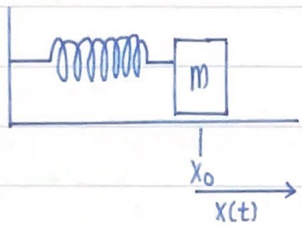
Periodic Motion

Simple Harmonic Motion

when the restoring force is directly proportional to the displacement from equilibrium, as given by $F_x = -kx$, the oscillation is called S.H.M



$$m\frac{d^2x(t)}{dt^2} = -kx(t)$$

$$\frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t) \quad \text{let } \omega^2 = \frac{k}{m}$$

$$= -\omega^2 x(t)$$

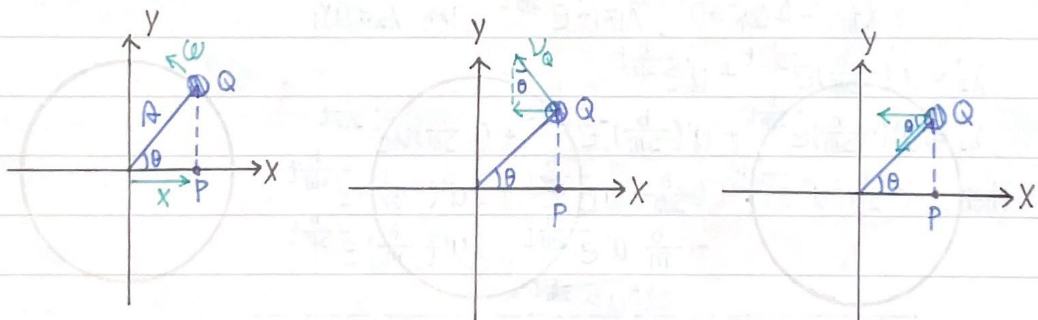
let $x = e^{\lambda t}$, then $\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0$, $e^{\lambda t}(\lambda^2 + \omega^2) = 0$, $\because e^{\lambda t} \neq 0 \quad \therefore \lambda^2 + \omega^2 = 0$

$\lambda^2 = -\omega^2$, $\lambda = \pm\omega i$

$x_1(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$x_2(t) = e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$

$$\frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin(\omega t) \qquad \frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos(\omega t)$$



$x = A\cos\theta \quad a_Q = \omega^2 A$

$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta = -\omega^2 x$

$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$x = A\cos(\omega t + \phi)$, $\overset{\text{phase}}{\theta} = \omega t + \overset{\text{phase constant}}{\phi}$ $\quad x_0 = A\cos\phi$

$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$

$v_{0x} = -\omega A\sin\phi$

$\frac{v_{0x}}{x_0} = -\omega\tan\phi$, $\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) \qquad A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}}$

Damped Oscillations

the decrease in amplitude caused by dissipative forces is called damping

$$m\frac{d^2x(t)}{dt^2} = -kx(t) - b\frac{dx(t)}{dt}, \quad \frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

let $x(t) = Ae^{\lambda t}$

$$A\lambda^2 e^{\lambda t} + \frac{b}{m}A\lambda e^{\lambda t} + \omega_0^2 A e^{\lambda t} = 0$$

$$Ae^{\lambda t}(\lambda^2 + \frac{b}{m}\lambda + \omega_0^2) = 0 \quad \text{since } Ae^{\lambda t} \neq 0$$

$$\lambda^2 + \frac{b}{m}\lambda + \omega_0^2 = 0$$

$$\lambda = \frac{-\frac{b}{m} \pm \sqrt{(\frac{b}{m})^2 - 4\omega_0^2}}{2}$$

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

overdamped

$$(\frac{b}{m})^2 - 4\omega_0^2 > 0, \quad \lambda_1 \text{ and } \lambda_2 < 0$$

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

$$u'(-\frac{b}{2m})e^{-\frac{b}{2m}t} + (\frac{b}{2m})^2 u e^{-\frac{b}{2m}t} + u'(-\frac{b}{2m})e^{-\frac{b}{2m}t} + u'' e^{-\frac{b}{2m}t}$$

critical damped

$$(\frac{b}{m})^2 - 4\omega_0^2 = 0, \quad x_1 = Ae^{-\frac{b}{2m}t} \quad \text{let } x_2 = ux_1$$

$$x_2' = u(-\frac{b}{2m})e^{-\frac{b}{2m}t} + u'e^{-\frac{b}{2m}t}$$

$$x_2'' = 2u'(-\frac{b}{2m})e^{-\frac{b}{2m}t} + u(\frac{b}{2m})^2 e^{-\frac{b}{2m}t} + (-\frac{b}{2m})u''e^{-\frac{b}{2m}t}$$

then
$$(-\frac{b}{2m})u''e^{-\frac{b}{2m}t} + (\frac{b}{2m})^2 u e^{-\frac{b}{2m}t} + 2u'(-\frac{b}{2m})e^{-\frac{b}{2m}t}$$
$$-\frac{b^2}{2m^2} u e^{-\frac{b}{2m}t} \qquad u'(\frac{b}{m})e^{-\frac{b}{2m}t}$$
$$\omega_0^2 u e^{-\frac{b}{2m}t}$$

$$-\frac{b}{2m}u''e^{-\frac{b}{2m}t} = 0 \quad \text{let } v = u', \quad \frac{dv}{dt}e^{-\frac{b}{2m}t} = 0 \quad \dots \quad u = t$$

$$x_2 = te^{-\frac{b}{2m}t}, \quad x(t) = Ae^{-\frac{b}{2m}t} + Bte^{-\frac{b}{2m}t}$$

underdamped

$$(\frac{b}{m})^2 - 4\omega_0^2 < 0, \quad 4\omega_0^2 - (\frac{b}{m})^2 > 0, \quad \text{let } 4\omega^2 = 4\omega_0^2 - (\frac{b}{m})^2$$

$$\lambda_1 = -\frac{b}{2m} + i\omega \qquad x_1(t) = e^{\lambda_1 t} = e^{-\frac{b}{2m}t}e^{i\omega t}$$

$$\lambda_2 = -\frac{b}{2m} - i\omega \qquad x_2(t) = e^{\lambda_2 t} = e^{-\frac{b}{2m}t}e^{-i\omega t}$$

Forced Oscillations I

if we apply a periodically varying driving force with angular frequency ω to a damped harmonic oscillator : forced oscillation or driven oscillation

$$m\frac{d^2X(t)}{dt^2} = -kX(t) - b\frac{dX(t)}{dt} + F_0\cos\omega t$$

$$\frac{d^2X(t)}{dt^2} + \frac{b}{m}\frac{dX(t)}{dt} + \omega_0^2 X(t) = \frac{F}{m}\cos\omega t$$

let $X(t) = A\cos(\omega t+\alpha)$

then $-\omega^2 A\cos(\omega t+\alpha) - \frac{b}{m}A\omega\sin(\omega t+\alpha) + \omega_0^2 A\cos(\omega t+\alpha) = \frac{F}{m}\cos(\omega t)$

$$(\omega_0^2-\omega^2)A\cos(\omega t+\alpha) - \frac{b}{m}A\omega\sin(\omega t+\alpha) = \frac{F}{m}\cos(\omega t)$$

$$A\left[\underbrace{\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2}}}_{\cos B}\cos(\omega t+\alpha) + \underbrace{\frac{-\frac{b}{m}\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2}}}_{\sin B}\sin(\omega t+\alpha)\right]\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2}$$

$$= \frac{F_0}{m}\cos(\omega t)$$

$$\Rightarrow \cos(\omega t+\alpha-B)\,A\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2} = \frac{F_0}{m}\cos(\omega t)$$

$$\Rightarrow \alpha = B = \tan^{-1}\frac{-\frac{b}{m}\omega}{\omega_0^2-\omega^2} \qquad A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2}}$$

Forced Oscillations II

$$m\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+kx(t)=F_0\cos(\omega t) \quad \text{let } x=x_R(t)+i\,x_{im}(t)=x_{com}(t)$$

then $$m\frac{d^2x_R(t)}{dt^2}+b\frac{dx_R(t)}{dt}+kx_R(t)=F_0\cos(\omega t) \quad \text{—— (1)}$$

$$m\frac{d^2x_{im}(t)}{dt^2}+b\frac{dx_{im}(t)}{dt}+kx_{im}(t)=F_0\sin(\omega t) \quad \text{—— (2)}$$

$(1)+i(2)$: $$m\frac{d^2x_{com}(t)}{dt^2}+b\frac{dx_{com}(t)}{dt}+kx_{com}(t)=F_0e^{i\omega t} \quad \text{—— (3)}$$

$x_{com}(t)=x_0e^{i\omega t}$ $\quad \frac{dx_{com}(t)}{dt}=i\omega x_0e^{i\omega t}$ $\quad$ into $\frac{(3)}{m}$

$$\Rightarrow (i\omega)^2x_0e^{i\omega t}+\frac{b}{m}i\omega x_0e^{i\omega t}+\omega_0^2x_0e^{i\omega t}=\frac{F_0}{m}e^{i\omega t}$$

$$x_0\left(\omega_0^2-\omega^2+\frac{b}{m}i\omega\right)=\frac{F_0}{m}$$

$$\Rightarrow x_0=\frac{F_0}{m}\frac{1}{\omega_0^2-\omega^2+\frac{b}{m}i\omega}=\frac{F_0}{m}\frac{(\omega_0^2-\omega^2)-\frac{b}{m}i\omega}{\left[(\omega_0^2-\omega^2)+\frac{b}{m}i\omega\right]\left[(\omega_0^2-\omega^2)-\frac{b}{m}i\omega\right]}$$

$x_0=Ae^{i\alpha}$ $$x_0(t)=\frac{F_0}{m}\frac{(\omega_0^2-\omega^2)-\frac{b}{m}i\omega}{(\omega_0^2-\omega^2)+(\frac{b}{m}\omega)^2}$$

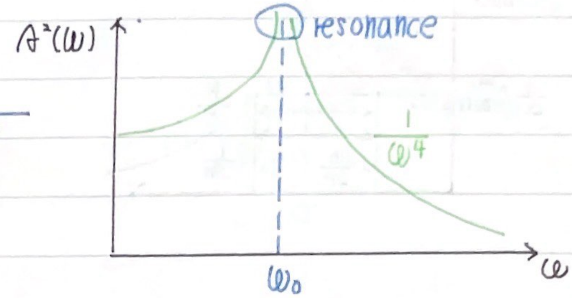
$$=\frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)+\frac{b^2\omega^2}{m^2}}}\frac{(\omega_0^2-\omega^2)-\frac{b}{m}i\omega}{\sqrt{(\omega_0^2-\omega^2)+(\frac{b\omega}{m})^2}}$$

$$=Ae^{i\alpha}$$

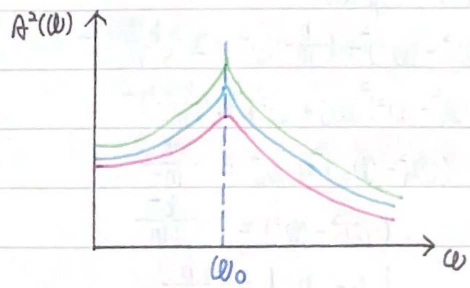
$$A=\frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(\frac{b\omega}{m})^2}} \qquad \cos\alpha=\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)+(\frac{b}{m}\omega)^2}}$$

when $b=0$

$$A^2 = \frac{F_0^2}{m^2} \frac{1}{(\omega_0^2-\omega^2)^2}$$



when $b \neq 0$



$$\text{quality factor } Q = \left.\frac{\text{energy stored}}{\text{energy dissipated per cycle}}\right|_{\omega=\omega_0} = \frac{\frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}kx^2(t)}{\underbrace{\left[\left(b\frac{dx(t)}{dt}\right)\cdot\frac{dx(t)}{dt}\right]}_{\text{power}}\frac{2\pi}{\omega}}$$

assume $x(t)=A\cos(\omega t+\alpha)$

$$= \left.\frac{\frac{1}{2}m\omega^2A^2\sin(\omega t+\alpha)+\frac{1}{2}m\omega_0^2A^2\cos^2(\omega t+\alpha)}{b\omega^2A^2\sin^2(\omega t+\alpha)\ \frac{2\pi}{\omega}}\right|_{\omega=\omega_0}$$

$$= \frac{\frac{1}{2}m\omega_0^2A^2}{\frac{2\pi}{\omega_0}b\omega_0^2A^2\sin^2(\omega t+\alpha)}$$

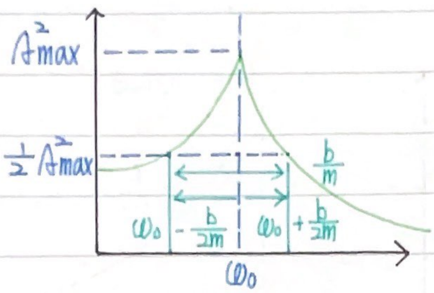
to get the average value $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t+\alpha)dt = \frac{1}{\frac{2\pi}{\omega}}\int_0^{\frac{2\pi}{\omega}}\frac{1-\cos2(\omega t+\alpha)}{2}dt = \frac{1}{2}$

$$\langle Q\rangle = \frac{\langle\frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2+\frac{1}{2}kx^2(t)\rangle}{\langle\left[b\frac{dx(t)}{dt}\cdot\frac{dx(t)}{dt}\right]\frac{2\pi}{\omega}\rangle} = \left.\frac{\frac{1}{2}\left(\frac{1}{2}m\omega^2A^2+\frac{1}{2}m\omega_0^2A^2\right)}{\frac{1}{2}b\omega^2A^2\ \frac{2\pi}{\omega}}\right|_{\omega=\omega_0}$$

$$= \frac{\frac{1}{2}m\omega_0^2}{\frac{1}{2}b\omega_0^2\frac{2\pi}{\omega_0}} = \frac{m}{b}\frac{\omega_0}{2\pi}$$

NO.

DATE / /



$$A^2 = \frac{F_0^2}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega)^2}$$

for $\frac{1}{2}A^2_{max}$

When $\omega \to \omega_0$ $\quad (\omega_0^2-\omega^2)^2+(\frac{b}{m}\omega_0)^2 = 2\,(\frac{b\omega_0}{m})^2$

$$(\omega_0-\omega)^2(\omega_0+\omega)^2 = (\frac{b\omega_0}{m})^2$$

$$(\omega_0-\omega)^2 4\omega_0^2 = \frac{b^2\omega_0^2}{m^2}$$

$$(\omega_0^2-\omega^2) = \frac{b^2}{4m}$$

$$|\omega-\omega_0| = \frac{b}{2m}$$

$$\langle Q\rangle = \frac{m}{b}\frac{\omega_0}{2\pi} = \frac{\omega_0}{2|\omega-\omega_0|}\frac{1}{2\pi}$$

$$= \frac{\omega_0}{\Delta\omega}\frac{1}{4\pi}$$

Energy in Oscillations

SHM

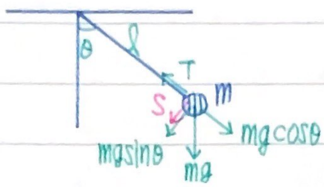
$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$
$$= \frac{1}{2}m\left[-\omega A\sin(\omega t+\phi)\right]^2 + \frac{1}{2}k\left[A\cos(\omega t+\phi)\right]^2$$
$$= \frac{1}{2}kA^2\sin^2(\omega t+\phi) + \frac{1}{2}kA^2\cos^2(\omega t+\phi)$$
$$= \frac{1}{2}kA^2$$

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2-x^2} \qquad v_{max} = \sqrt{\frac{k}{m}}A = \omega A$$

damped oscillations

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$
$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt} = mv_x a_x + kxv_x$$
$$= v_x(ma_x + kx) \qquad = v_x(-bv_x)$$
$$= -bv_x^2$$

The Simple Pendulum



$$F_\theta = -mg\sin\theta = m\frac{d^2S}{dt^2} \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

$$= -mg\theta \quad = -mg\frac{x}{l} \quad \text{if } \theta \text{ is small}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{mg}{L}}{m}} = \sqrt{\frac{g}{L}}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right)$$

The Physical Pendulum

in a same case

$$\tau_z = -mgl\sin\theta \quad \text{if } \theta \text{ is small} \quad \tau_z = -mg\theta$$

$$-mgd\theta = I\alpha_z = I\frac{d^2\theta}{dt^2}, \quad \frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{mgd}{I}}$$

Applications of SHM

angular SHM (Torsional Pendulum)

$$\tau_z = -k\theta \quad , k \text{ torsion constant}$$

$$\sum \tau_z = I\alpha_z = I\frac{d^2\theta}{dt^2}, \quad \frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta, \quad \omega = \sqrt{\frac{k}{I}}$$

$$\theta = \Theta\cos(\omega t + \phi)$$

vibrations of molecules

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^6\right] \qquad F_r = -\frac{dU}{dr} = U_0\left(\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^6}{r^7}\right) = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^7\right]$$

$$r = R_0 + x$$

$$F_r = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^7\right] = 12\frac{U_0}{R_0}\left[\underbrace{\frac{1}{(1+\frac{x}{R_0})^{13}}}_{\simeq\, 1+(-13)\frac{x}{R_0}} - \underbrace{\frac{1}{(1+\frac{x}{R_0})^7}}_{\simeq\, 1+(-7)\frac{x}{R_0}}\right] \approx -\frac{72U_0}{R_0^2}x$$