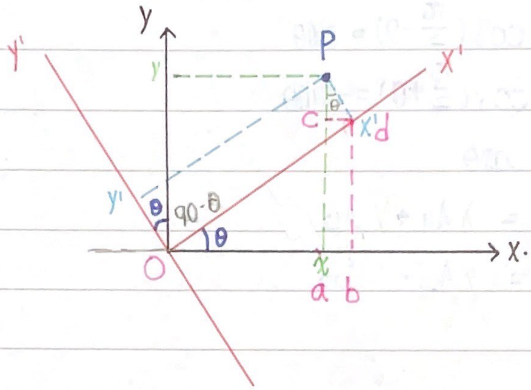
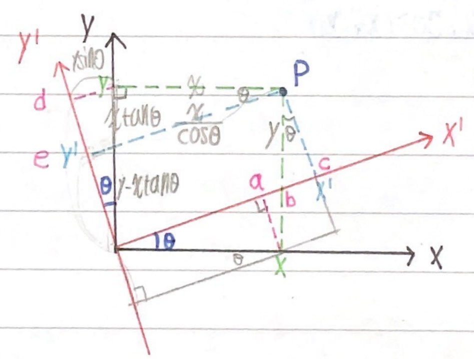
Matrices, Vectors, and Vector Calculus

Coordinate Transformations



$$\vec{u}_1 = x\vec{e}_1 + y\vec{e}_2 = (|\overline{OB}| - |\overline{AB}|)\vec{e}_1 + (|\overline{BD}| + |\overline{CP}|)\vec{e}_2$$
$$= (x'\cos\theta - y'\sin\theta)\vec{e}_1 + (x'\sin\theta + y'\cos\theta)\vec{e}_2$$

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$



$$x' = \overline{Oa} + \overline{ab} + \overline{bc} = x\cos\theta + y\sin\theta = x\cos\theta + y\cos(\tfrac{\pi}{2} - \theta)$$
$$y' = \overline{Od} - \overline{de} = y\cos\theta - x\sin\theta = y\cos\theta + x\cos(\tfrac{\pi}{2} + \theta)$$

the angle between x-axis and the x-axis is denoted by (x, x)

and we define $\lambda \equiv \cos(x, x)$

λ ↓ direction cosine

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$\lambda_{11} = \cos(x_1', x_1) = \cos(x, x) = \cos\theta$

$\lambda_{12} = \cos(x_1', x_2) = \cos(x, y) = \cos(\frac{\pi}{2} - \theta) = \sin\theta$

$\lambda_{21} = \cos(x_2', x_1) = \cos(y, x) = \cos(\frac{\pi}{2} + \theta) = -\sin\theta$

$\lambda_{22} = \cos(x_2', x_2) = \cos(y, y) = \cos\theta$

$\Rightarrow \; x = x\cos(x_1', x_1) + y\cos(x_1', x_2) = x\lambda_{11} + y\lambda_{12}$

$\quad\; y = x\cos(x_2', x_1) + y\cos(x_2', x_2) = x\lambda_{21} + y\lambda_{22}$

for three dimensions we have

$x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3$

$x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \quad \Rightarrow \quad x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j \; , \; i=1,2,3$

$x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3$

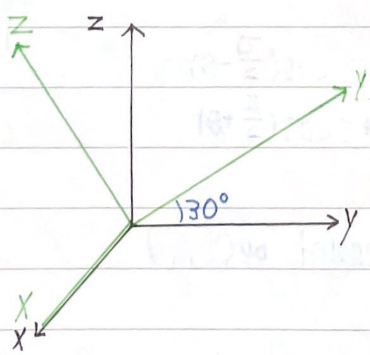
the inverse transformation

$x_1 = x_1'\cos(x_1', x_1) + x_2'\cos(x_2', x_1) + x_3'\cos(x_3', x_1)$

$\quad = \lambda_{11}x_1' + \lambda_{21}x_2' + \lambda_{31}x_3'$

$\quad = \sum_{j=1}^{3} x_{ji} x_j' \quad , i=1,2,3$

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$



$\lambda_{11} = \cos(x', x) = \cos(0°) = 1$

$\lambda_{12} = \cos(x', y) = \cos(90°) = 0$

$\lambda_{13} = \cos(x', z) = \cos(90°) = 0$

$\lambda_{21} = \cos(y', x) = \cos(90°) = 0$

$\lambda_{22} = \cos(y', y) = \cos(30°) \approx 0.866$

$\lambda_{23} = \cos(y', z) = \cos(60°) = 0.5$

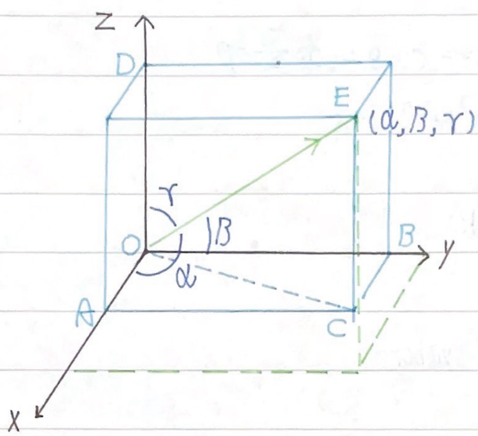
$\lambda_{31} = \cos(z', x) = \cos(90°) = 0$

$\lambda_{32} = \cos(z', y) = \cos(120°) = -0.5$

$\lambda_{33} = \cos(z', z) = \cos(30°) \approx 0.866$

$$\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & 0.5 \\ 0 & -0.5 & 0.866 \end{bmatrix}$$

Properties of Rotation Matrices

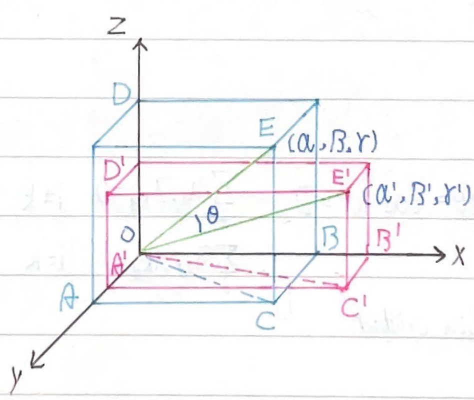


$\overline{OE}\cos\alpha = \overline{OA}$ $\qquad$ $\overline{OE}^2\left[\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right] = \overline{OA}^2 + \overline{OB}^2 + \overline{OD}^2$

$\overline{OE}\cos\beta = \overline{OB}$ $\qquad$ but $\overline{OA}^2 + \overline{OB}^2 = \overline{OC}^2$ and $\overline{OC}^2 + \overline{OD}^2 = \overline{OE}^2$

$\overline{OE}\cos\gamma = \overline{OD}$ $\qquad$ thus $\overline{OA}^2 + \overline{OB}^2 + \overline{OD}^2 = \overline{OE}^2$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



$\overline{OE}^2 + \overline{OE}'^2 - 2\,\overline{OE}\,\overline{OE}'\cos\theta = \overline{EE}'^2$

$\overline{EE}'^2 = (\overline{OB}' - \overline{OB})^2 + (\overline{OA}' - \overline{OA})^2 + (\overline{OD}' - \overline{OD})^2$

$= (\overline{OE}'\cos\beta' - \overline{OE}\cos\beta)^2 + (\overline{OE}'\cos\alpha' - \overline{OE}\cos\alpha)^2 + (\overline{OE}'\cos\gamma' - \overline{OE}\cos\gamma)^2$

$= \overline{OE}'^2(\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma') + \overline{OE}^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

$\quad - 2\,\overline{OE}'\,\overline{OE}\,(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$

$= \overline{OE}'^2 + \overline{OE}^2 - 2\,\overline{OE}'\,\overline{OE}\,(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$

$\Rightarrow \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$

because the x'-axis and the y'-axis is $\frac{\pi}{2}$ angle between

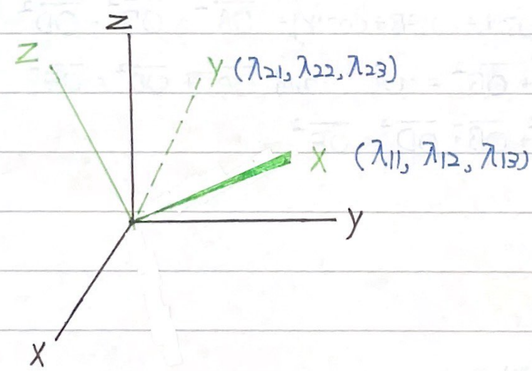
we have $\lambda_{11}\lambda_{21}+\lambda_{12}\lambda_{22}+\lambda_{13}\lambda_{23}=\cos\theta=\cos\frac{\pi}{2}=0$

or $\sum_j \lambda_{1j}\lambda_{2j}=0$ $\quad \sum_j \lambda_{ij}\lambda_{kj}=0,\ i\neq k$

$\lambda_{11}^2+\lambda_{12}^2+\lambda_{13}^2=1$ $\quad \sum_j \lambda_{ij}\lambda_{kj}=1 \quad i=k$

$\Rightarrow \sum_j \lambda_{ij}\lambda_{kj}=\delta_{ik}$ Kronecker delta symbor

$$\delta_{ik}=\begin{cases}0 & \text{if } i\neq k\\ 1 & \text{if } i=k\end{cases}$$



$\Rightarrow \lambda_{11}\lambda_{21}+\lambda_{12}\lambda_{22}+\lambda_{13}\lambda_{23}=\cos\theta=\cos 90°=0$, $\sum_j \lambda_{ij}\lambda_{kj}=0 \quad i\neq k$

$\Rightarrow \lambda_{11}^2+\lambda_{12}^2+\lambda_{13}^2=1$, $\sum_j \lambda_{ij}\lambda_{kj}=1 \quad i=k$

$\Rightarrow \sum_j \lambda_{ij}\lambda_{kj}=\delta_{ik}$ Kronecker delta symbor

$$\delta_{ik}\begin{cases}0 & \text{if } i\neq k\\ 1 & \text{if } i=k\end{cases}$$

Matrix Operations

$$x_i' = \sum_j \lambda_{ij} x_j \Rightarrow X' = \lambda X$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

also can be written as $x_i = \sum_j \lambda_{ij}^t x_j'$

consider the orthogonal rotation matrix λ for the case of two dimensions

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

$$\lambda \lambda^t = \begin{bmatrix} \lambda_{11}^2 + \lambda_{12}^2 & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} & \lambda_{21}^2 + \lambda_{22}^2 \end{bmatrix}$$ and $\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2 + \lambda_{22}^2 = 1$

$\lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$ in orthogonal matrices

$\Rightarrow \lambda^t = \lambda^{-1}$

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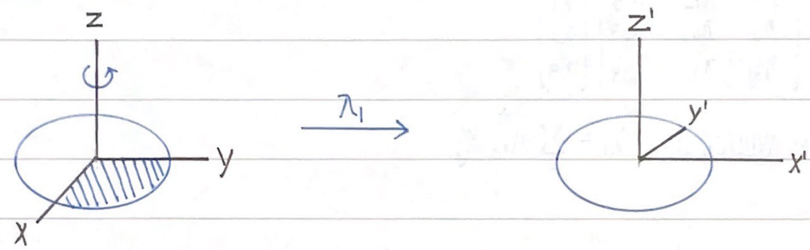
Geometrical Significance of Transformation Matrices

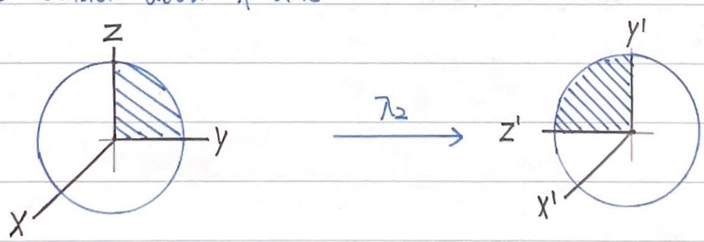
90° rotation about z axis



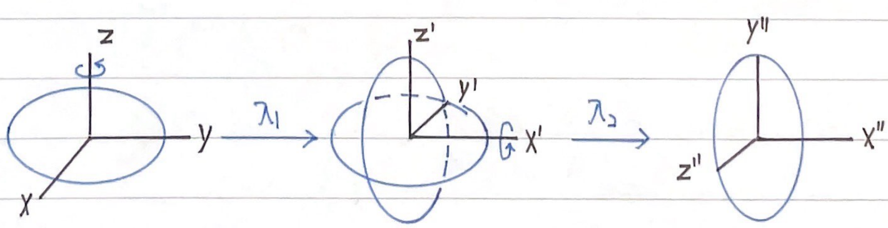
$\cos(x_1', y) = \lambda_{12} = 1 \qquad \cos(y', x) = \lambda_{21} = -1 \qquad \cos(z', z) = \lambda_{33} = 1$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

90° rotation about x-axis

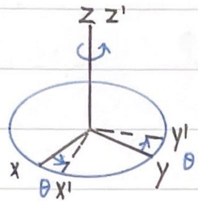


$$\lambda_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

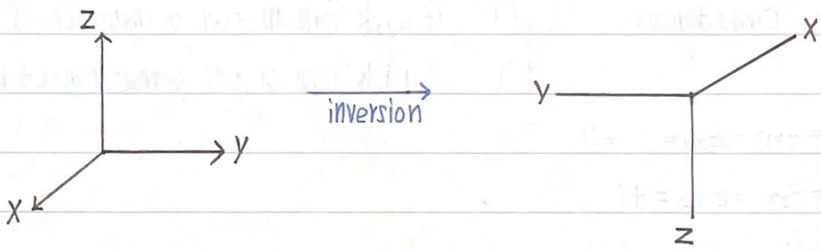


$x' = \lambda_1 x \,, \quad x'' = \lambda_2 x' \quad \Rightarrow \quad x'' = \lambda_2 \lambda_1 x \quad \Rightarrow \quad x'' = \lambda_3 x$

$\lambda_4 = \lambda_1 \lambda_2 \neq \lambda_3$



$$\lambda_5 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\lambda_6 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = |\lambda_5| = 1$ proper rotation

$|\lambda_6| = -1$ improper rotation

all orthogonal matrices must have a determinant equal to either $+1$ or -1

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Vector Product of Two Vectors

$$\vec{C} = \vec{A} \times \vec{B}$$

$$C_i \equiv \sum_{j,k} \varepsilon_{ijk} A_j B_k$$

permutation symbol ε_{ijk} (or Levi-Civita density)

$$\varepsilon_{ijk} \begin{cases} 0 & \text{if any index is equal to any other index} \\ +1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3 \\ 1 & \text{if } i,j,k \text{ form an odd permutation of } 1,2,3 \end{cases}$$

$$\varepsilon_{122} = \varepsilon_{313} = \varepsilon_{211} = \cdots = 0$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\Rightarrow C_1 = \sum_{jk} \varepsilon_{1jk} A_j B_k = \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$$

$$C_2 = A_3 B_1 - A_1 B_3$$

$$C_3 = A_1 B_2 - A_2 B_1$$

$$A^2 B^2 \sin^2\theta = A^2 B^2 - A^2 B^2 \cos^2\theta = \left(\sum_i A_i^2\right)\left(\sum_i B_i^2\right) - \left(\sum_i A_i B_i\right)^2$$

$$= (A_2 B_3 - A_3 B_2)^2 + (A_3 B_1 - A_1 B_3)^2 + (A_1 B_2 - A_2 B_1)^2$$

$$= C_1^2 + C_2^2 + C_3^2$$

$$\vec{A} \cdot (\vec{B} \times \vec{D}), \quad (\vec{B} \times \vec{D})_i = \sum_{jk} \varepsilon_{ijk} B_j D_k \Rightarrow \vec{A} \cdot (\vec{B} \times \vec{D}) = \sum_{ijk} \varepsilon_{ijk} A_i B_j D_k$$

$$\vec{D} \cdot (\vec{A} \times \vec{B}) = \sum_{ijk} \varepsilon_{ijk} D_i A_j B_k = \sum_{ijk} -\varepsilon_{jik} D_i A_j B_k = \sum_{ijk} \varepsilon_{jki} A_j B_k D_i$$

$$\Rightarrow \vec{A} \cdot \vec{B} \times \vec{D} = \vec{D} \cdot (\vec{A} \times \vec{B})$$

$$* \quad \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{A}\vec{B}\vec{C}$$

$$\vec{A} \times \vec{B} \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \vec{A} \cdot [\vec{B} \times (\vec{C} \times \vec{D})] = \vec{A} \cdot [(\vec{B} \cdot \vec{D})\vec{C} - (\vec{B} \cdot \vec{C})\vec{D}] = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [(\vec{A} \times \vec{B}) \cdot \vec{D}]\vec{C} - [(\vec{A} \times \vec{B}) \cdot \vec{C}]\vec{D}$$

$$= (ABD)C - (ABC)D - (ACD)B - (BCD)A$$