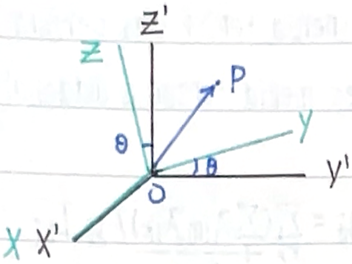
Further Properties of the Inertia Tensor



$$\overrightarrow{OP} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{cases} x = x' \\ y = y'\cos\theta + z'\sin\theta \\ z = -y'\sin\theta + z'\cos\theta \end{cases} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{R_1 \text{ or } \lambda_1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\lambda] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad x_i = \sum_j \lambda_{ji} x'_j \qquad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [\lambda^t] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad x'_i = \sum_j \lambda_{ij} x_j$$

$$L_k = \sum_l I_{kl}\,\omega_l \qquad L'_i = \sum_j I'_{ij}\,\omega'_j \qquad L_k = \sum_m \lambda_{mk} L'_m \qquad \omega_l = \sum_j \lambda_{jl}\,\omega'_j$$

$$\Rightarrow \sum_m \lambda_{mk} L'_m = \sum_l I_{lk} \sum_j \lambda_{jl}\,\omega'_j \xrightarrow{\times \lambda_{ik}} \sum_m \underbrace{\left(\sum_k \lambda_{ik}\lambda_{mk}\right)}_{\delta_{im}} L'_m = \sum_j \underbrace{\left(\sum_{kl} \lambda_{ik}\lambda_{jl} I_{kl}\right)}_{I'_{ij}} \omega'_j$$

$$\Rightarrow L'_i = I'_{ij}\,\omega'_j$$

$\Rightarrow I' = \lambda I \lambda^t = \lambda I \lambda^{-1}$ similarity transformation

$$\overrightarrow{L'} = [I']\,\overrightarrow{\omega'} \qquad \overrightarrow{L} = [I]\,\overrightarrow{\omega} \qquad \overrightarrow{x} = \lambda\,\overrightarrow{x'} \qquad \overrightarrow{x'} = R^t\,\overrightarrow{x}$$

$$R^t \overrightarrow{L} = [I'] R^t \overrightarrow{\omega}, \quad RR^t \overrightarrow{L} = R[I']R^t \overrightarrow{\omega}, \quad \overrightarrow{L} = R[I']R^t \overrightarrow{\omega} = [I]\,\overrightarrow{\omega}$$

$$\Rightarrow [I] = R[I']R^t$$

what condition must be satisfied if we take an arbitrary inertia tensor and perform a coordinate rotation in such a way that the transformed inertia tensor is diagonal

$$I'_{ij} = I_i \delta_{ij} \qquad I_i \delta_{ij} = \sum_{kl} \lambda_{ik} \lambda_{jl} I_{kl} \xrightarrow{\times \lambda_{im}} \sum_i I_i \lambda_{im} \delta_{ij} = \sum_{kl} \underbrace{\left(\sum_i \lambda_{im} \lambda_{ik}\right)}_{\delta_{mk}} \lambda_{jl} I_{kl}$$

$$I_j \lambda_{jm} = \sum_l \lambda_{jl} I_{ml} = \sum_l I_j \lambda_{jl} \delta_{ml} \ , \ \sum_l \left(I_{ml} - I_j \delta_{ml} \right) \lambda_{jl} = 0 \ , \ \left| I_{ml} - I \delta_{ml} \right| = 0$$

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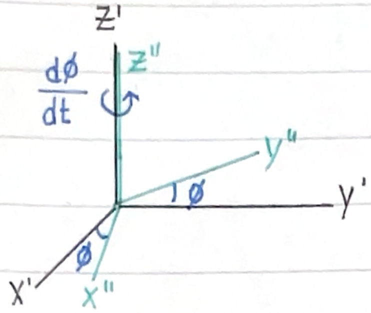
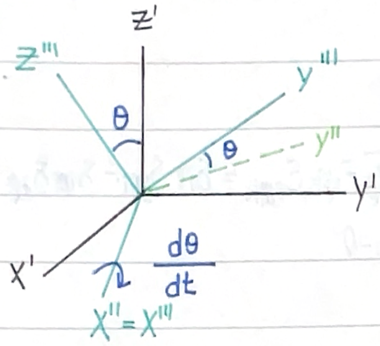
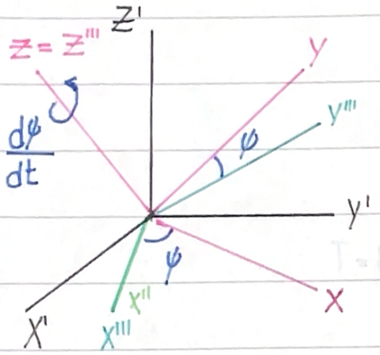
Eulerian Angles



$$\lambda_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X'' = \lambda_\phi X'$$



$$\lambda_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad X''' = \lambda_\theta X''$$



$$\lambda_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X = \lambda_\psi X'''$$

$$X = \lambda_\psi X''' = \lambda_\psi \lambda_\theta X'' = \lambda_\psi \lambda_\theta \lambda_\phi X' = \lambda X', \quad \lambda = \lambda_\psi \lambda_\theta \lambda_\phi$$

$$\lambda = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\phi & \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

$$\omega_\phi = \frac{d\phi}{dt} = R_\psi R_\theta \begin{bmatrix} 0 \\ 0 \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \dot\phi \sin\theta\sin\psi \\ \dot\phi \sin\theta\cos\psi \\ \dot\phi \cos\theta \end{bmatrix}$$

$$\omega_\theta = \frac{d\theta}{dt} = R_\psi \begin{bmatrix} \dot\theta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot\theta\cos\psi \\ -\dot\theta\sin\psi \\ 0 \end{bmatrix}$$

$$\omega_\psi = \frac{d\psi}{dt} = \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix}$$

$$\vec{\omega} = \frac{d\vec\phi}{dt} + \frac{d\vec\theta}{dt} + \frac{d\vec\psi}{dt} = \begin{bmatrix} \frac{d\phi}{dt}\sin\theta\sin\psi + \frac{d\theta}{dt}\cos\psi \\ \frac{d\phi}{dt}\sin\theta\cos\psi - \frac{d\theta}{dt}\sin\psi \\ \frac{d\phi}{dt}\cos\theta + \frac{d\psi}{dt} \end{bmatrix}$$

Euler's Equations for a Rigid Body

(I) $\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L}$

$\tau_k = \frac{dL_k}{dt} + (\vec{\omega}\times\vec{L})_k \qquad L_k = I_k\omega_k$

$= I_k\frac{d\omega_k}{dt} + \sum_{lm}\epsilon_{klm}\omega_l L_m$

$\tau_k - I_k\frac{d\omega_k}{dt} - \sum_{lm}\epsilon_{klm}\omega_l\omega_m I_m = 0 \qquad \times\sum_k\epsilon_{ijk}$

$\sum_k\epsilon_{ijk}\left(\tau_k - I_k\frac{d\omega_k}{dt}\right) - \sum_{klm}\epsilon_{ijk}\epsilon_{klm}\omega_l\omega_m I_m = 0$

$\sum_k\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$\sum_k\epsilon_{ijk}\left(\tau_k - I_k\frac{d\omega_k}{dt}\right) - \sum_{klm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\omega_l\omega_m I_m = 0$

$\sum_k\epsilon_{ijk}\left(\tau_k - I_k\frac{d\omega_k}{dt}\right) - (\omega_i\omega_j I_j - \omega_j\omega_i I_i) = 0$

$(I_i - I_j)\omega_i\omega_j - \sum_k\left(I_k\frac{d\omega_k}{dt} - \tau_k\right)\epsilon_{ijk} = 0$

(II)

consider the force-free motion of a rigid body, $L = T - U = T$

$T = \frac{1}{2}\sum_i I_i\omega_i^2$

$\frac{\partial T}{\partial\psi} - \frac{d}{dt}\frac{\partial T}{\partial\dot\psi} = 0$, $\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\psi} - \frac{d}{dt}\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\dot\psi} = 0$

$\frac{\partial\omega_1}{\partial\psi} = \frac{d\phi}{dt}\sin\theta\cos\psi - \frac{d\theta}{dt}\sin\psi = \omega_2$ $\quad\frac{\partial\omega_2}{\partial\psi} = -\frac{d\phi}{dt}\sin\theta\sin\psi - \frac{d\theta}{dt}\cos\psi = -\omega_1$ $\quad\frac{\partial\omega_3}{\partial\psi} = 0$

$\frac{\partial\omega_1}{\partial\dot\psi} = \frac{\partial\omega_2}{\partial\dot\psi} = 0 \qquad \frac{\partial\omega_3}{\partial\dot\psi} = 1 \qquad \frac{\partial T}{\partial\omega_i} = I_i\omega_i$

$\Rightarrow I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{d}{dt}I_3\omega_3 = 0$, $\quad (I_1 - I_2)\omega_1\omega_2 - I_3\frac{d\omega_3}{dt} = 0$

$\Rightarrow (I_2 - I_3)\omega_2\omega_3 - I_1\frac{d\omega_1}{dt} = 0 \qquad (I_1 - I_2)\omega_1\omega_2 - I_3\frac{d\omega_3}{dt} = 0$

$(I_3 - I_1)\omega_3\omega_1 - I_2\frac{d\omega_2}{dt} = 0$ $\quad$ Euler's equation for force-free motion

Force-Free Motion of a Symmetric Top

a rigid body with $I_1 = I_2 \neq I_3$

$$(I_1 - I_3)\omega_2\omega_3 - I_1\frac{d\omega_1}{dt} = 0$$
$$(I_3 - I_1)\omega_3\omega_1 - I_1\frac{d\omega_2}{dt} = 0$$
$$I_3\frac{d\omega_3}{dt} = 0$$

$$\frac{d\omega_1}{dt} = -\frac{I_3 - I_1}{I_1}\omega_3\omega_2$$
$$\Rightarrow \frac{d\omega_2}{dt} = \frac{I_3 - I_1}{I_1}\omega_3\omega_1$$
$$\omega_3(t) = \text{constant}$$

let $\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3$,

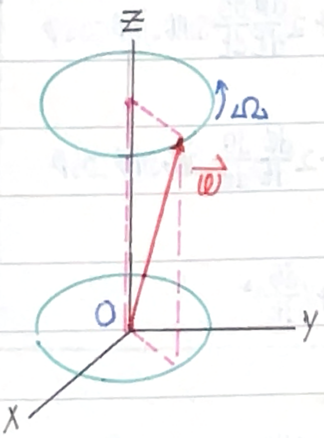
$$\frac{d\omega_1}{dt} + \Omega\omega_2 = 0 \quad \text{—(1)}$$
$$\frac{d\omega_2}{dt} - \Omega\omega_1 = 0 \quad \text{—(2)}$$

$(1) + (2)\times i$ $\quad \left(\frac{d\omega_1}{dt} + i\frac{d\omega_2}{dt}\right) - i\Omega(\omega_1 + i\omega_2) = 0$ $\quad$ define $\eta \equiv \omega_1 + i\omega_2$

$\Rightarrow \frac{d\eta}{dt} - i\Omega\eta = 0$, $\eta(t) = Ae^{i\Omega t}$, $\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$

$\Rightarrow \omega_1(t) = A\cos\Omega t$ $\quad |\vec{\omega}| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = \text{constant}$

$\quad \omega_2(t) = A\sin\Omega t$



$$\vec{L}\cdot(\vec{\omega}\times\vec{e_z}) = [L_1\ L_2\ L_3]\begin{bmatrix}\vec{e_x} & \vec{e_y} & \vec{e_z}\\ \omega_1 & \omega_2 & \omega_3\\ 0 & 0 & \vec{e_z}\end{bmatrix}$$
$$= [L_1\ L_2\ L_3]\begin{bmatrix}\omega_2\\ -\omega_1\\ 0\end{bmatrix} = L_1\omega_2 - L_2\omega_1$$
$$= I\omega_1\omega_2 - I_2\omega_1\omega_2 = 0$$

$\Rightarrow \vec{L}, \vec{\omega}, \vec{z}$-axis all lie in a plane

earth is slightly flattened near the poles , so $I_1 \cong I_3$ but $I_3 > I_1$

if Earth is considered to be a rigid body , then $\Omega \cong \frac{1}{300}\omega_3$ but the observed precession has an irregular period about 50 % greater than the predicted