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Dynamics of a System of Particles

Linear Momentum of the System

the center of mass of a system move as if it were a single particle of mass equal to the total mass of the system, acted on by the total external force, and independent of the nature of the internal forces

force $f_\alpha = \sum_\beta f_{\alpha\beta}$ the force on the α th particle due to β th particle $\qquad F_\alpha = \underbrace{F_\alpha^e}_{\text{external force}} + f_\alpha$

$$\frac{dP_\alpha}{dt} = m_\alpha \frac{d^2 r_\alpha}{dt^2} = F_\alpha^e + f_\alpha \qquad \frac{d^2}{dt^2}\sum_\alpha m_\alpha r_\alpha = \sum_\alpha F_\alpha^e + \sum_{\substack{\alpha\ \beta \\ \alpha\neq\beta}}\sum f_{\alpha\beta}$$

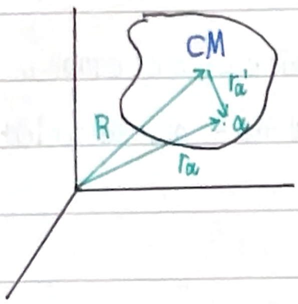
$$\begin{cases} \sum_\alpha F_\alpha^e \equiv F \\ \sum_{\substack{\alpha\ \beta \\ \alpha\neq\beta}}\sum f_{\alpha\beta} \equiv \sum_{\substack{\alpha \\ \beta\neq\alpha}} f_{\alpha\beta} = \sum_{\alpha<\beta} f_{\alpha\beta} + f_{\beta\alpha} = \sum_{\substack{\alpha \\ \beta\neq\alpha}} f_{\alpha\beta} + \left(-\sum_{\substack{\alpha \\ \beta\neq\alpha}} f_{\alpha\beta}\right) = 0 \end{cases} \Rightarrow M\frac{d^2R}{dt^2} = F$$

the linear momentum of the system is the same as if a single particle of mass M were located at the position of the center of mass and moving in the manner the center of mass moves

$$P = \sum_\alpha m_\alpha \frac{dr_\alpha}{dt} = \frac{d}{dt}\sum_\alpha m_\alpha r_\alpha = \frac{d}{dt}(MR) = M\frac{dR}{dt} \qquad \frac{dP}{dt} = M\frac{d^2R}{dt^2} = F$$

the total momentum for a system free of external force is constant and equal to the linear momentum of the center of mass

Angular Momentum of the System



$$\vec{r_\alpha} = \vec{R} + \vec{r'_\alpha} \qquad \vec{L_\alpha} = \vec{r_\alpha} \times \vec{p_\alpha}$$

$$\vec{L} = \sum_\alpha \vec{L_\alpha} = \sum_\alpha (\vec{r_\alpha} \times \vec{p_\alpha}) = \sum_\alpha \left(\vec{r_\alpha} \times m_\alpha \frac{d\vec{r_\alpha}}{dt}\right)$$

$$= \sum_\alpha (\vec{r'_\alpha} + \vec{R}) \times m_\alpha \left(\frac{d\vec{r'_\alpha}}{dt} + \frac{d\vec{R}}{dt}\right)$$

$$= \sum_\alpha m_\alpha \left[\left(\vec{r'_\alpha} \times \frac{d\vec{r'_\alpha}}{dt}\right) + \underbrace{\left(\vec{r'_\alpha} \times \frac{d\vec{R}}{dt}\right)}_{\sum_\alpha m_\alpha \vec{r'_\alpha} \times \frac{d\vec{R}}{dt}} + \underbrace{\left(\vec{R} \times \frac{d\vec{r'_\alpha}}{dt}\right)}_{\vec{R} \times \sum_\alpha m_\alpha \vec{r'_\alpha} = 0} + R \times \frac{d\vec{R}}{dt}\right]$$

$$= \sum_\alpha m_\alpha (\vec{r_\alpha} - \vec{R}) \times \cdots$$

$$= \left[\sum_\alpha m_\alpha \vec{r_\alpha} - \vec{R} \sum_\alpha m_\alpha\right] \times \cdots$$

$$= M\vec{R} - M\vec{R} = 0$$

the total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass

$$\vec{L} = M\vec{R} \times \frac{d\vec{R}}{dt} + \sum_\alpha \vec{r'_\alpha} \times \vec{p'_\alpha} = \vec{R} \times \vec{P} + \sum_\alpha \vec{r'_\alpha} \times \vec{p'_\alpha}$$

if the net resultant external torques about a given axis vanish, then the total angular momentum of the system about that axis remains constant in time

$$\frac{dL_\alpha}{dt} = r_\alpha \times \frac{dP_\alpha}{dt} = r_\alpha \times \left(F_\alpha^e + \sum_\beta f_{\alpha\beta}\right)$$

$$\frac{d\vec{L}}{dt} = \sum_\alpha \frac{d\vec{L}_\alpha}{dt} = \sum_\alpha (r_\alpha \times F_\alpha^e) + \underline{\sum_{\substack{\alpha \\ \beta\neq\alpha}} (r_\alpha \times f_{\alpha\beta})}$$

same direction

$$\sum_{\alpha<\beta} (r_\alpha \times f_{\alpha\beta}) + (r_\beta \times f_{\beta\alpha}) = \sum_{\alpha<\beta} (r_\alpha - r_\beta) \times f_{\alpha\beta} = \sum_{\alpha<\beta} \overset{\frown}{r_{\alpha\beta}} \times f_{\alpha\beta} = 0$$

$$= \sum_\alpha \Gamma_\alpha^e = \underbrace{\Gamma^e}_{\text{external torques}}$$

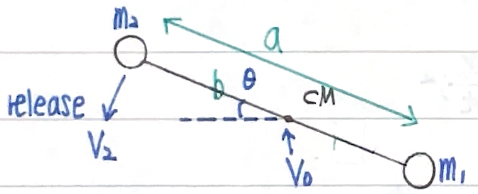
the total internal torque must vanish if the internal forces are central — that is, if $f_{\alpha\beta} = -f_{\beta\alpha}$.

and the angular momentum of an isolated system cannot be altered without the application of external forces

torque on the αth particle due to all the internal forces $\sum_\beta r_\alpha \times f_{\alpha\beta}$

$$\sum_{\substack{\alpha \\ \beta\neq\alpha}} (r_\alpha \times f_{\alpha\beta}) = \sum_{\alpha<\beta} (r_{\alpha\beta} \times f_{\alpha\beta}) = 0$$

example



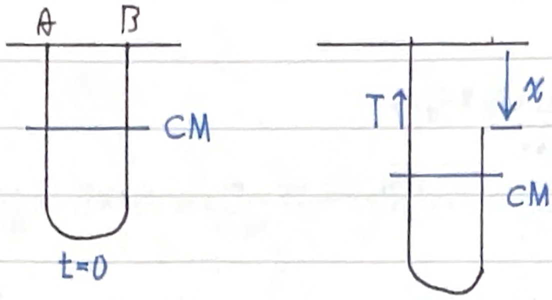
$m_2 > m_1$

$$b = \frac{m_1}{m_1+m_2} a \qquad V_2 = b\frac{d\theta}{dt} \qquad V_1 = (a-b)\frac{d\theta}{dt}$$

$$\text{centrifugal force} = \frac{m_2 \left(b\frac{d\theta}{dt}\right)^2}{b} = \text{tension} = m_2 b \left(\frac{d\theta}{dt}\right)^2 = m_2 \frac{m_1 a}{m_1+m_2} \left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{m_1 m_2 a}{m_1+m_2} \left(\frac{d\theta}{dt}\right)^2$$

example



density ρ free fall
length b
mass M

$$\frac{dP}{dt} = Mg - T \qquad P = \rho\frac{b-x}{2}\frac{dx}{dt}, \quad \frac{dP}{dt} = \frac{\rho}{2}\left[-\left(\frac{dx}{dt}\right)^2 + \frac{d^2x}{dt^2}(b-x)\right]$$

for free fall $x = \frac{1}{2}gt^2$

$$= \frac{\rho}{2}(gb - 3gx)$$

$$\frac{dx}{dt} = gt = \sqrt{2gx} \qquad = Mg - T$$

$$\frac{d^2x}{dt^2} = g \qquad \Rightarrow T = \frac{M}{2}g\left(\frac{b}{3}x + 1\right)$$

energy:

$$U(t=0) = U_0 = -\rho\frac{gb^2}{4}$$

$$U = -\frac{1}{4}\rho g(b^2 + 2bx - x^2)$$

$$E_k = \frac{\rho}{4}(b-x)\left(\frac{dx}{dt}\right)^2$$

and $E_k + U = U_0$

$$\frac{\rho}{4}(b-x)\left(\frac{dx}{dt}\right)^2 - \frac{1}{4}\rho g(b^2 + 2bx - x^2) = -\frac{1}{4}\rho g b^2$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{g(2bx - x^2)}{b-x} \qquad \frac{d^2x}{dt^2} = g + \frac{g(2bx - x^2)}{2(b-x)^2}$$

$$\Rightarrow E_k = \frac{Mg}{4b}\frac{1}{b-x}(2b^2 + 2bx - 3x^2)$$