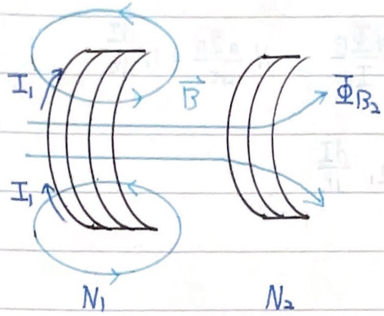
Inductance

Mutual Inductance



$\Phi_{B2} = \int \vec{B}_1 \cdot d\vec{A}_2$ and $\Phi_{B2} \propto I_1$, $N_2 \Phi_{B2} = M_{21} I_1$, M: proportionality constant

$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt} = M_{21} \frac{dI_1}{dt}$, $M_{21} = \frac{N_2 \Phi_{B2}}{I_1}$

$\Phi_{B1} = \int \vec{B}_2 \cdot d\vec{A}_1$ $\quad N_1 \Phi_{B_1} = M_{12} I_2$

$\mathcal{E}_1 = -N_1 \frac{d\Phi_{B1}}{dt} = M_{12} \frac{dI_2}{dt}$, $M_{12} = \frac{N_1 \Phi_{B1}}{I_2}$

if the coil are in a vaccum, the flux Φ_B through each turn of coil is directly proportional to the current I

in this discussion we will assume that any magnetic material present has constant K_m so that $\Phi \propto I$ and M_{21} depends on geometry only

M_{21} is always equal to M_{12}

$$[M] = [N] \frac{[\Phi_B]}{[I]} = \frac{Wb}{A} = \frac{V \cdot s}{A} = \Omega \cdot s = \frac{J}{A^2} = H$$

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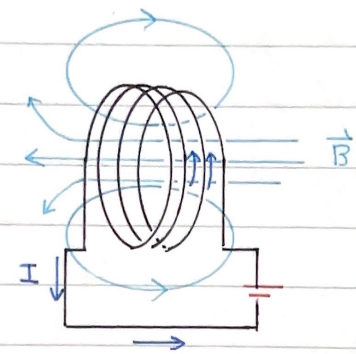
Self - Inductance and Inductors

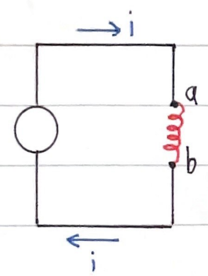
self - inductance



$$L = \frac{N\Phi_B}{I} \qquad N\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

$$\mathcal{E} = -L\frac{dI}{dt}$$

inductor

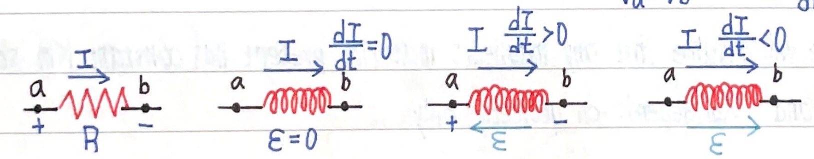


conservative field $\vec{E}_c$ non conservative field $\vec{E}_n$

$\vec{E}_{total} = \vec{E}_c + \vec{E}_n = \vec{0}$

$$\oint \vec{E}_n \cdot d\vec{l} = -L\frac{di}{dt} = \int_a^b \vec{E}_n \cdot d\vec{l} \quad , \text{ so that } \int_a^b \vec{E}_c \cdot d\vec{l} = L\frac{di}{dt}$$

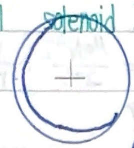
$$V_a - V_b = L\frac{di}{dt} = V_{ab}$$



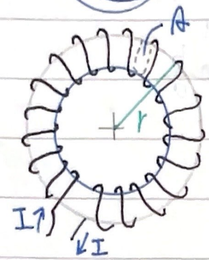
$V_{ab} = IR > 0$ $\quad V_{ab} = L\frac{dI}{dt} = 0$ $\quad V_{ab} = L\frac{dI}{dt} > 0$ $\quad V_{ab} = L\frac{dI}{dt} < 0$

in the following examples, we will assume that the circuit encloses only vacuum

an added complication is that with ferromagnetic materials the magnetization is in general not a linear function of magnetizing current , in our discussion we will ignore this complication and assume always that the inductance is constant toroidal solenoid I



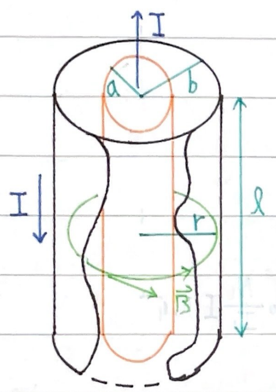
assume B is uniform across a cross section



$$B = \mu_0 \frac{NI}{2\pi r} \qquad \Phi_B = BA = \frac{\mu_0 N I A}{2\pi r}$$

$$L = N\frac{\Phi_B}{I} = \frac{\mu_0 N^2 A}{2\pi r}$$

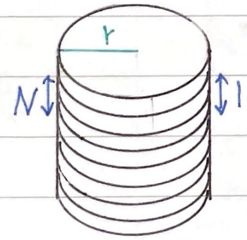
coaxial cable



$$B = \frac{\mu_0 I}{2\pi r} \qquad d\Phi_B = B\,dA = B\,l\,dr$$

$$\Phi_B = \int_a^b \frac{\mu_0 I}{2\pi r} l\,dr = \frac{\mu_0 I l}{2\pi r}\int_a^b \frac{dr}{r} = \frac{\mu_0 I l}{2\pi}\ln\frac{b}{a}$$

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

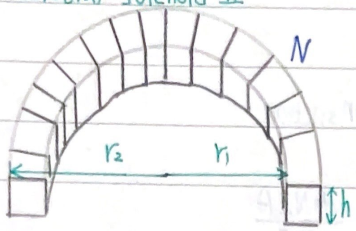


$$B = \mu_0 N I$$

$$\Phi_B = \mu_0 N I \cdot N \cdot \pi r^2 = \mu_0 N^2 \pi r^2 I$$

$$L = \frac{\Phi_B}{I} = \mu_0 N^2 \pi r^2$$

torodial solenoid II



$$B = \frac{\mu_0 N I}{2\pi r}$$

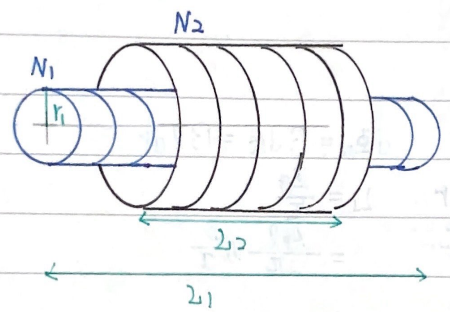
$$r_1 < r < r_2$$

$$\Phi_B = N\int_S \vec{B}\cdot d\vec{A} = \frac{\mu_0 N^2 I}{2\pi}\int \frac{1}{r}\,dA$$

$$= \frac{\mu_0 N^2 I h}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r}$$

$$= \frac{\mu_0 N^2 I h}{2\pi}\ln\frac{r_2}{r_1}$$

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{r_2}{r_1}$$



$$B = \mu_0 N I = \mu_0 \frac{N_1}{L_1} I \qquad \Phi_B = \frac{N_2\int \vec{B}\cdot d\vec{A}}{L_2} = \frac{N_2}{L_2}\mu_0\frac{N_1}{L_1} I \pi r_1^2$$

$$M = \frac{\mu_0 N_1 N_2 \pi r_1^2}{L_1 L_2}$$

Magnetic-Field Energy

$P = V_{ab} I = LI \frac{dI}{dt}$, $dU = P dt = LI dI$

$U = \int P dt = \int_0^I LI dI = L \int_0^I I dI$
$= \frac{1}{2} LI^2$

toroidal solenoid I

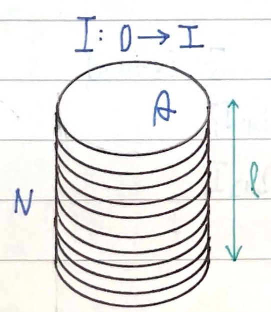
assume the cross-sectional area A is small enough that we can pretend that the magnetic field is uniform over the area

$L = \frac{\mu_0 N^2 A}{2\pi r}$ $\quad U = \frac{1}{2} LI^2 = \frac{\mu_0 N^2 A}{4\pi r} I^2$

energy per unit volume, magnetic energy density, $u = \frac{U}{2\pi r A} = \frac{\mu_0}{2} \frac{N^2 I^2}{4\pi^2 r^2}$ and $B = \frac{\mu_0 N I}{2\pi r}$

$= \frac{B^2}{2\mu_0}$

$= \frac{B^2}{2\mu}$ when the material inside the toroid is not vacuum



$B = \mu_0 n I \quad \Phi = n l B A = LI$

$P = \frac{dW}{dt} = -I\epsilon = I \frac{d\Phi_B}{dt} = I \frac{d}{dt} LI$, $dW = I L dt$

$W = \int dW = \int_0^I I L dI = \frac{1}{2} LI^2 = U$

$L = \frac{\Phi}{I} = \frac{nlBA}{I}$, $I = \frac{B}{\mu_0 N}$

$U = \frac{1}{2} \frac{nlBA}{I} I^2 = \frac{1}{2} nlBA \frac{B}{\mu_0 N} = \frac{1}{2\mu_0} B^2 A l$

$u = \frac{U}{Al} = \frac{B^2}{2\mu_0}$

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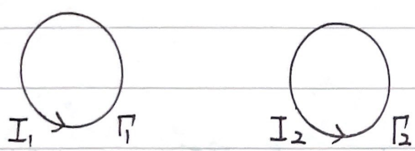
torodial solenoid II

$$B = \frac{\mu_0 N I}{2\pi r} \quad r_1 < r < r_2 \qquad u = \frac{\vec{B}\cdot\vec{B}}{2\mu_0} = \frac{1}{2\mu_0}\frac{\mu_0^2 N^2 I^2}{4\pi^2 r^2}$$

$$U = \int_{r_1}^{r_2} u \cdot 2\pi r h \, dr = 2\pi h \frac{\mu_0^2 N^2 I^2}{2\mu_0 4\pi^2}\int_{r_1}^{r_2}\frac{r\,dr}{r^2}$$

$$= \frac{\mu_0 h N^2 I^2}{4\pi}\ln\frac{r_2}{r_1} = \frac{1}{2}LI^2$$

$$\Rightarrow L = \frac{\mu_0 h N^2}{2\pi}\ln\frac{r_2}{r_1}$$



(i) Γ_1: $0 \to I_1$ $\Gamma_2 : I_2 = 0$

$$\frac{dW_{11}}{dt} = I_1 L_{11}\frac{dI_1}{dt}$$

$$W_{11} = \frac{1}{2}L_{11}I_1^2$$

(i) $\Gamma_1 : I_1 = 0$ Γ_2: $0 \to I_2$

$$W_{22} = \frac{1}{2}L_{22}I_2^2$$

(ii) $\Gamma_2 : 0 \to I_2$ $\Gamma_1 : I_1$ fixed

$$\frac{dW_{22}}{dt} = I_2 L_{22}\frac{dI_2}{dt}$$

$$\frac{dW_{2\to1}}{dt} = I_1 L_{2\to1}\frac{dI_2}{dt}$$

$$\int\left(\frac{dW_{22}}{dt} + \frac{dW_{21}}{dt}\right)dt = \frac{1}{2}L_{22}I_2^2 + I_1L_{21}I_2$$

(ii) $\Gamma_1 : 0 \to I_2$ $\Gamma_2 : I_2$ fixed

$$W_{11} = \frac{1}{2}L_{11}I_1^2$$

$$W_{12} = I_2 L_{12} I_1$$

$$W_{11} + W_{22} + W_{21} = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + I_1L_{21}I_2 = W_{22} + W_{11} + W_{12} = \frac{1}{2}L_{22}I_2^2 + \frac{1}{2}L_{11}I_1^2 + I_2L_{21}I_1$$

$$\Rightarrow W_{21} = W_{12} = I_1 I_2 L_{12} = I_1 I_2 L_{21}$$

$$\Rightarrow L_{12} = L_{21}$$

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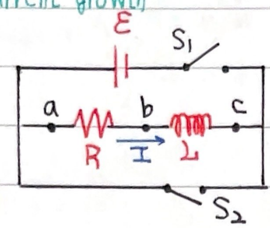
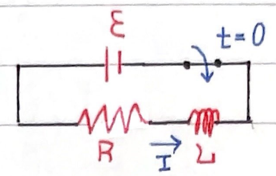
The R-L Circuit

Current growth





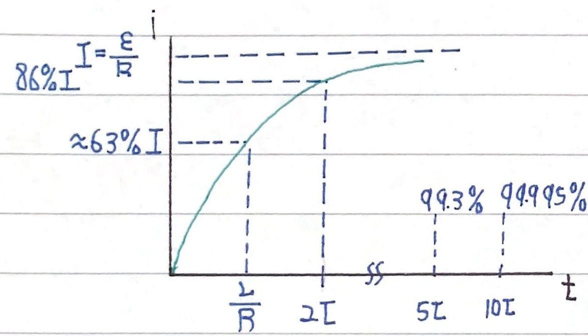
$V_{ab}=iR \qquad v_{bc}=L\frac{di}{dt}$

$\mathcal{E}-iR-L\frac{di}{dt}=0$, $\frac{di}{dt}=\frac{\mathcal{E}-iR}{L}=\frac{\mathcal{E}}{L}-\frac{R}{L}i$ $\left(\frac{di}{dt}\right)_{initial}=\frac{\mathcal{E}}{L}$, $L\uparrow \ \frac{di}{dt}\downarrow$

$\frac{di}{i-\frac{\mathcal{E}}{R}}=-\frac{R}{L}dt$, $\int_0^i \frac{di'}{i'-\frac{\mathcal{E}}{R}}=-\int_0^t \frac{R}{L}dt'$, $\ln\frac{i-\frac{\mathcal{E}}{R}}{-\frac{\mathcal{E}}{R}}=-\frac{R}{L}t$

$i=\frac{\mathcal{E}}{R}\left(1-e^{-\frac{R}{L}t}\right)$, $\frac{di}{dt}=\frac{\mathcal{E}}{L}e^{-\frac{R}{L}t}$ $\quad t=0 \quad i=0 \quad \frac{di}{dt}=\frac{\mathcal{E}}{L}$

$t\to\infty \quad i\to\frac{\mathcal{E}}{R} \quad \frac{di}{dt}\to 0$



when τ, time constant, $\Rightarrow t=\frac{L}{R}$, $i=(1-\frac{1}{e})$

$P=\mathcal{E}i=i^2R+Li\frac{di}{dt}$

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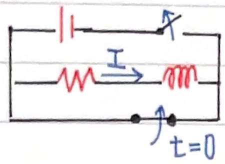
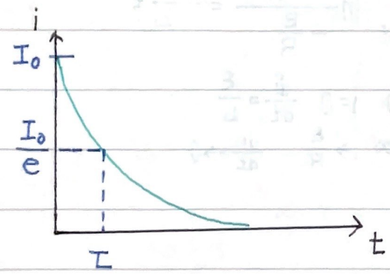
current decay



$$iR + L\frac{di}{dt} = 0 \qquad I(t=0) = \frac{\mathcal{E}}{R}$$

$$\frac{di}{dt} = -\frac{R}{L}i \ , \quad \frac{di}{i} = -\frac{R}{L}dt \ , \quad \int_{I_0}^{i}\frac{di'}{i'} = -\frac{R}{L}\int_0^t dt'$$

$$\ln\frac{i}{I_0} = -\frac{R}{L}t \ , \quad i = I_0 e^{-\frac{R}{L}t}$$



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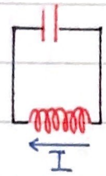
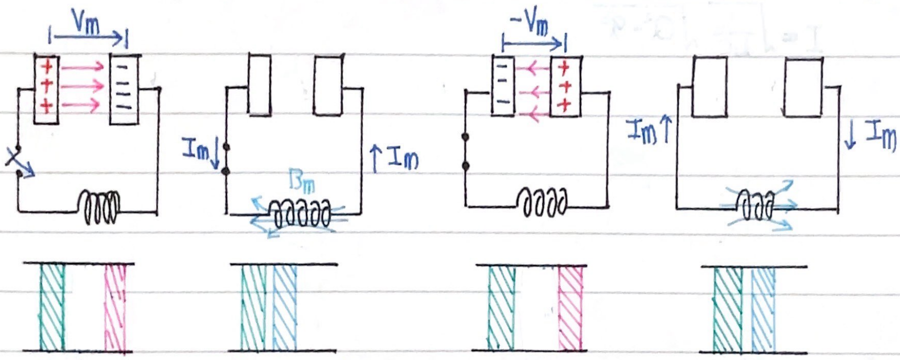
The L-C Circuit





$E = U_L + U_C$ $\quad$ $E = U_B + U_E$

$t = 0$, $t = T$ $\quad$ $t = \frac{1}{4}T$ $\quad$ $t = \frac{1}{2}T$ $\quad$ $t = \frac{3}{4}T$

$$-L\frac{di}{dt} - \frac{q}{C} = 0$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

$$U = U_E + U_B = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \frac{Q_{max}^2}{2C}$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{1}{2}Li^2\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\frac{d^2q}{dt^2} = -\frac{q}{LC} = -\omega^2 q \qquad \omega^2 = \frac{1}{LC}$$

let $q = e^{\lambda t}$, then $\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0$, $e^{\lambda t}(\lambda^2 + \omega^2) = 0$ $\because e^{\lambda t} \neq 0$, $\lambda^2 + \omega^2 = 0$

$\lambda^2 = -\omega^2$, $\lambda = \pm \omega i$

$q_1 = e^{i\omega t} = \cos\omega t + i\sin\omega t$, $\quad q = Q_{max}\cos(\omega t + \phi)$

$$i = \frac{dq}{dt} = -\omega Q_{max}\sin(\omega t + \phi) = -I_{max}\sin(\omega t + \phi)$$

$$U = U_B + U_E = \frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q_{max}^2}{2C}$$

$$= \frac{Q_{max}^2}{2C}\cos^2\omega t + \frac{1}{2}LI_{max}^2 \sin^2(\omega t)$$

$$\Rightarrow \frac{Q_{max}^2}{2C} = \frac{LI_{max}^2}{2}$$

$$I = \sqrt{\frac{1}{LC}}\sqrt{Q^2 - q^2}$$

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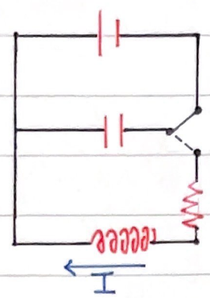
The L-R-C Series Circuit



$-iR - L\frac{di}{dt} - \frac{q}{C} = 0$, $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$

let $q = Ae^{\lambda t}$, $A\lambda^2 e^{\lambda t} + \frac{R}{L}A\lambda e^{\lambda t} + \omega_0^2 A e^{\lambda t} = 0$

$Ae^{\lambda t}(\lambda^2 + \frac{R}{L}\lambda + \omega_0^2) = 0$ and $\lambda^2 + \frac{R}{L}\lambda + \omega_0^2 = 0$

R small : underdamped

R large : overdamped

when $R^2 < \frac{4L}{C}$, $q = Ae^{-\frac{R}{2L}t}\cos(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \phi)$

$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$