Normal Coordinates

$$q_j(t)=\sum_r \alpha_r a_{jr} e^{i(\omega_r t-\delta_r)}, \quad q_j(t)=\sum_r B_r a_{jr} e^{i\omega_r t} \quad \text{let } \eta_r(t)=B_r e^{i\omega_r t}$$

$$q_j(t)=\sum_r a_{jr}\eta_r(t), \quad \frac{dq_j}{dt}=\sum_r a_{jr}\frac{d\eta_r}{dt}$$

$$T=\frac{1}{2}\sum_{jk} m_{jk}\frac{dq_j}{dt}\frac{dq_k}{dt}=\frac{1}{2}\sum_{jk} m_{jk}\sum_r a_{jr}\frac{d\eta_r}{dt}\sum_s a_{ks}\frac{d\eta_s}{dt}$$

$$=\frac{1}{2}\sum_{rs}\left(\sum_{jk} m_{jk}a_{jr}a_{ks}\right)\frac{d\eta_r}{dt}\frac{d\eta_s}{dt}$$

$$=\frac{1}{2}\sum_{rs}\delta_{rs}\frac{d\eta_r}{dt}\frac{d\eta_s}{dt}=\frac{1}{2}\sum_r\left(\frac{d\eta_r}{dt}\right)^2$$

$$U=\frac{1}{2}\sum_{jk}A_{jk}q_j q_k=\frac{1}{2}\sum_{rs}\left(\sum_{jk}A_{jk}a_{jr}a_{ks}\right)\eta_r\eta_s$$

$$=\frac{1}{2}\sum_{rs}\left(\omega_s^2\sum_{jk}m_{jk}a_{jr}a_{ks}\right)\eta_r\eta_s$$

$$=\frac{1}{2}\sum_{rs}\omega_s^2\delta_{rs}\eta_r\eta_s=\frac{1}{2}\sum_r\omega_r^2\eta_r^2$$

$$L=\frac{1}{2}\sum_r\left[\left(\frac{d\eta_r}{dt}\right)^2-\omega_r^2\eta_r^2\right]$$

$$\frac{\partial L}{\partial \eta_r}-\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_r}=0, \quad \frac{d^2\eta_r}{dt^2}+\omega_r^2\eta_r=0$$

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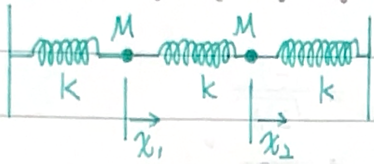
example



$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{2}kx_2^2$$
$$= \frac{1}{2}(k+k_{12})x_1^2 + \frac{1}{2}(k+k_{12})x_2^2 - k_{12}x_1x_2$$

$$A_{11} = \frac{\partial^2 U}{\partial x_1^2}\Big|_0 = k+k_{12} \quad A_{12} = \frac{\partial^2 U}{\partial x_1 \partial x_2}\Big|_0 = -k_{12} = A_{21} \quad A_{22} = \frac{\partial^2 U}{\partial x_2^2}\Big|_0 = k+k_{12}$$

$$\{A\} = \begin{bmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{bmatrix} \qquad \{m\} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

$$\begin{vmatrix} k+k_{12}-M\omega^2 & -k_{12} \\ -k_{12} & k+k_{12}-M\omega^2 \end{vmatrix} = 0$$

for $r=1$, $k=1$, we have

$$(A_{11} - \omega_1^2 m_{11})a_{11} + (A_{21} - \omega_1^2 m_{21})a_{21} = 0$$
$$(2k - \frac{3k}{M}\cdot M)a_{11} - ka_{21} = 0$$
$$a_{11} = -a_{21}$$

for $r=2$, $k=1$

$$(2k - \frac{k}{M}\cdot M)a_{12} - ka_{22} = 0$$
$$a_{12} = a_{22}$$

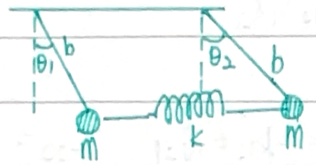
$$\begin{aligned} x_1 &= a_{11}\eta_1 + a_{12}\eta_2 \\ x_2 &= a_{21}\eta_1 + a_{22}\eta_2 \end{aligned} \rightarrow \begin{aligned} x_1 &= a_{11}\eta_1 + a_{22}\eta_2 \\ x_2 &= -a_{11}\eta_1 + a_{22}\eta_2 \end{aligned} \rightarrow \begin{aligned} \eta_2 &= \frac{1}{2a_{22}}(x_1+x_2) \\ \eta_1 &= \frac{1}{2a_{11}}(x_1-x_2) \end{aligned}$$

$x_1 = x_2$ $\eta_1 = 0$ in phase $\omega_2 = \sqrt{\frac{k}{M}}$ symmetrical mode

$x_1 = -x_2$ $\eta_2 = 0$ out of phase $\omega_1 = \sqrt{\frac{3k}{M}}$ antisymmetrical mode

$$\vec{a}_1 = a_{11}(\vec{e}_1 - \vec{e}_2) \qquad \vec{a}_2 = a_{22}(\vec{e}_1 + \vec{e}_2) \rightarrow a_{11} = -a_{21} = \frac{1}{\sqrt{2M}} \qquad a_{12} = a_{22} = \frac{1}{\sqrt{2M}}$$

example



$$T = \frac{1}{2} m\left(b\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2} m\left(b\frac{d\theta_2}{dt}\right)^2$$

$$U = mgb(1-\cos\theta_1) + mgb(1-\cos\theta_2) + \frac{1}{2}k(b\sin\theta_1 - b\sin\theta_2)^2 \qquad \sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$= \frac{mgb}{2}(\theta_1^2 + \theta_2^2) + \frac{kb^2}{2}(\theta_1 - \theta_2)^2$$

$$\{m\} = \begin{Bmatrix} mb^2 & 0 \\ 0 & mb^2 \end{Bmatrix} \qquad \{A\} = \begin{Bmatrix} mgb + kb^2 & -kb^2 \\ -kb^2 & mgb + kb^2 \end{Bmatrix}$$

$$\begin{vmatrix} mgb + kb^2 - \omega^2 mb^2 & -kb^2 \\ -kb^2 & mgb + kb^2 - \omega^2 mb^2 \end{vmatrix} = 0$$

$$b^2(mg + kb - \omega^2 mb)^2 - (kb^2)^2 = 0$$

$$(mg + kb - \omega^2 mb)^2 = (kb)^2$$

$$mg + kb - \omega^2 mb = \pm kb$$

taking the plus sign $\omega = \omega_1$, $\omega_1^2 = \frac{g}{b}$

taking the minus sign $\omega = \omega_2$ $\omega_2^2 = \frac{g}{b} + \frac{2k}{m}$

putting into $\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0$

for $k=1$

$r=1$ $\left(mgb + kb^2 - \frac{g}{b} mb^2\right) a_{11} - kb^2 a_{21} = 0$ $\quad a_{11} = a_{21}$

$r=2$ $\left(mgb + kb^2 - \frac{g}{b} mb^2 - \frac{2k}{m} mb^2\right) a_{12} - kb^2 a_{22} = 0$ $\quad a_{12} = -a_{22}$

$$\begin{matrix} \theta_1 = a_{11}\eta_1 + a_{12}\eta_2 \\ \theta_2 = a_{21}\eta_1 + a_{22}\eta_2 \end{matrix} \rightarrow \begin{matrix} \theta_1 = a_{11}\eta_1 - a_{22}\eta_2 \\ \theta_2 = a_{21}\eta_1 + a_{22}\eta_2 \end{matrix} \rightarrow \begin{matrix} \eta_1 = \frac{1}{2a_{11}}(\theta_1 + \theta_2) \\ \eta_2 = \frac{1}{2a_{22}}(\theta_2 - \theta_1) \end{matrix}$$

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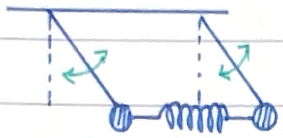

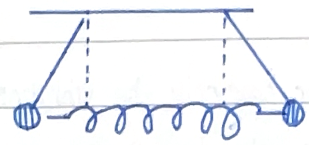

spring not compressed or extended

symmetric

normal mode 1

$\theta_1=\theta_2 \quad \eta_2=0$

$$\omega_1=\omega_0=\sqrt{\frac{g}{b}}$$

antisymmetric

normal mode 2

$\theta_1=-\theta_2 \quad \eta_1=0$