NO.

DATE / /

Oscillations

Introduction

at the $x=0$ must be 0

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2F}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3F}{dx^3}\right)_0$$

$$\approx \left(\frac{dF}{dx}\right)_0 x$$

$$= -kx \quad \text{where } \underbrace{k}_{\oplus} \equiv \underbrace{-\left(\frac{dF}{dx}\right)_0}_{\ominus}$$

Simple Harmonic Oscillator

$$-kx = m\frac{d^2x}{dt^2}, \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x, \quad \text{we define } \omega_0^2 = \frac{k}{m}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$$\Rightarrow x(t) = A\sin(\omega_0 t - \delta)$$

or

$$x(t) = A\cos(\omega_0 t - \phi)$$

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega_0^2 A^2\cos^2(\omega_0 t - \delta) = \frac{1}{2}kA^2\cos^2(\omega_0 t - \delta)$$

$$dW = -F\,dx = kx\,dx \Rightarrow U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2(\omega_0 t - \delta)$$

$$E = E_k + U = \frac{1}{2}kA^2\left[\cos^2(\omega_0 t - \delta) + \sin^2(\omega_0 t - \delta)\right] = \frac{1}{2}kA^2$$

$\omega_0 T_0 = 2\pi$, $T_0 = 2\pi\sqrt{\frac{m}{k}}$ → the period of the SHM is independent of the amplitude (isochronous)

$\omega_0 = 2\pi f_0$ $\quad f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Harmonic Oscillations in Two Dimensions

$$\vec{F} = -k\vec{r}$$

$$\begin{cases} F_x = -kr\cos\theta = -kx \\ F_y = -kr\sin\theta = -ky \end{cases} \qquad \begin{cases} \dfrac{d^2x}{dt^2} + \omega_0^2 x = 0 \\ \dfrac{d^2y}{dt^2} + \omega_0^2 y = 0 \end{cases} \Rightarrow \quad \begin{aligned} x(t) &= A\cos(\omega_0 t - \alpha) \quad ① \\ y(t) &= B\cos(\omega_0 t - \beta) \end{aligned}$$

$$y(t) = B\cos(\omega_0 t - \alpha + \alpha - \beta)$$
$$= B\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - B\sin(\omega_0 t - \alpha)\sin(\alpha - \beta)$$

defining $\delta \equiv \alpha - \beta$ $\quad \cos(\omega_0 t - \alpha) = \frac{x}{A}$ from ①

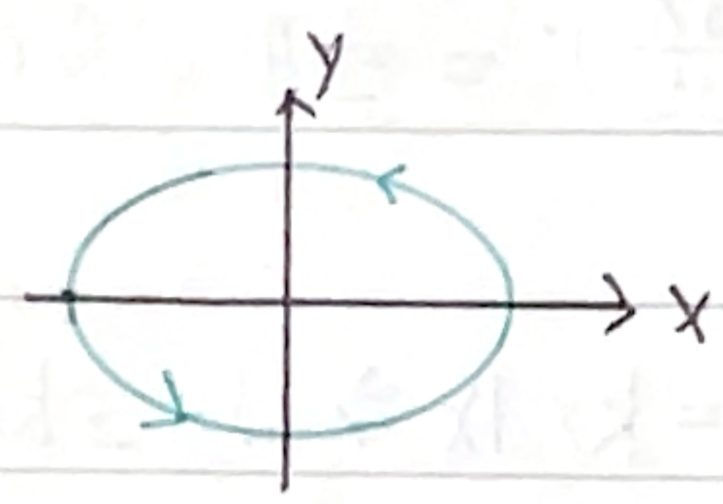
$\Rightarrow y = \frac{B}{A} x\cos\delta - B\sqrt{1 - \frac{x^2}{A^2}}\sin\delta$, $Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\sin\delta$

$\Rightarrow A^2y^2 - 2ABxy\cos\delta + B^2x^2\cos^2\delta = A^2B^2\sin^2\delta - B^2x^2\sin^2\delta$

$\Rightarrow B^2x^2 - 2ABxy\cos\delta + A^2y^2 = A^2B^2\sin^2\delta$

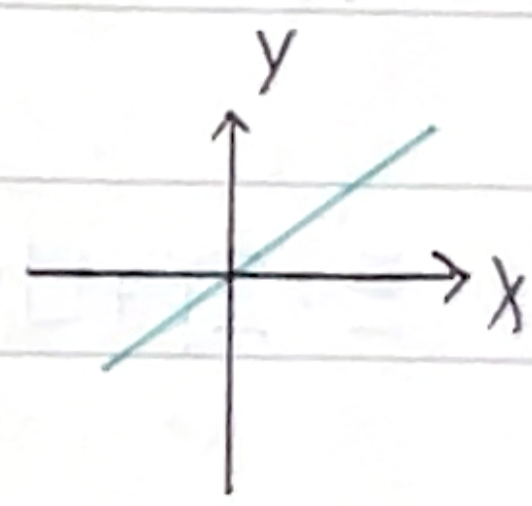
if δ is set equal to $\pm\frac{\pi}{2}$

$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ ellipse



if $A = B$ and if $\delta = \pm\frac{\pi}{2}$, $x^2 + y^2 = A^2$ circular

if $\delta = 0$, $B^2x^2 - 2ABxy + A^2y^2 = 0$, $(Bx - Ay)^2 = 0$, $y = \frac{B}{A}x$



if $\delta = \pm\pi$, $y = -\frac{B}{A}x$

in the general case $\quad x(t) = A\cos(\omega_x t - \alpha)$, $\quad y(t) = B\cos(\omega_y t - \beta)$ → Lissajous curve if ω_x and ω_y are commensurable ($\frac{\omega_x}{\omega_y}$ is rational fraction)

$T_x = \frac{2\pi}{\omega_x}$ $\quad T_y = \frac{2\pi}{\omega_y}$, if $\frac{\omega_x}{\omega_y} = \frac{p}{q}$, $\frac{T_x}{T_y} = \frac{q}{p}$. $pT_x = qT_y$

motion is periodic

if $\frac{\omega_x}{\omega_y} = c$ which is irrational, $cT_x = T_y$, if T_x is $\mathbb{N}$ then T_y is irrational

NO.

DATE / /

Damped Oscillations

$$m\frac{d^2x}{dt}+b\frac{dx}{dt}+kx=0\ ,\qquad \frac{d^2x}{dt}+\frac{b}{m}\frac{dx}{dt}+\frac{k}{m}x=0\qquad \text{let } \beta=\frac{b}{2m}\quad \omega_0^2=\frac{k}{m}$$

$\uparrow$ damping parameter

$$\Rightarrow\ \frac{d^2x}{dt}+2\beta\frac{dx}{dt}+\omega_0^2x=0$$

let $x=e^{\lambda y}$, then $m\lambda^2e^{\lambda y}+2\beta\lambda e^{\lambda y}+\omega_0^2e^{\lambda y}=0$

$e^{\lambda y}(\lambda^2+2\beta\lambda+\omega_0^2)=0\quad \because e^{\lambda y}\neq 0\quad \therefore \lambda^2+2\beta\lambda+\omega_0^2=0$

thus $\lambda_1=\dfrac{-2\beta+\sqrt{4\beta^2-4\omega_0^2}}{2}=-\beta+\sqrt{\beta^2-\omega_0^2}$

$\lambda_2=-\beta-\sqrt{\beta^2-\omega_0^2}$

the general solution $x(t)=C_1e^{\lambda_1t}+C_2e^{\lambda_2t}$

$$=e^{-\beta t}\left(A_1e^{\sqrt{\beta^2-\omega_0^2}\,t}+A_2e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

underdamped motion

$\beta^2-\omega_0^2<0\ ,\ \beta^2<\omega_0^2\qquad$ we define $\omega_1^2=\omega_0^2-\beta^2$ where $\omega_1^2>0$

$\Rightarrow x(t)=e^{-\beta t}\left(A_1e^{i\omega_1t}+A_2e^{-i\omega_1t}\right)=A_1\cos(\omega_1t)+A_2\cos(\omega_1t)+i\sin(\omega_1t)A_1-iA_2\sin(\omega_1t)$

$=Ae^{-\beta t}\cos(\omega_1t-\delta)$

$=(A_1+A_2)\cos(\omega_1t)+(A_1-A_2)i\sin(\omega_1t)$

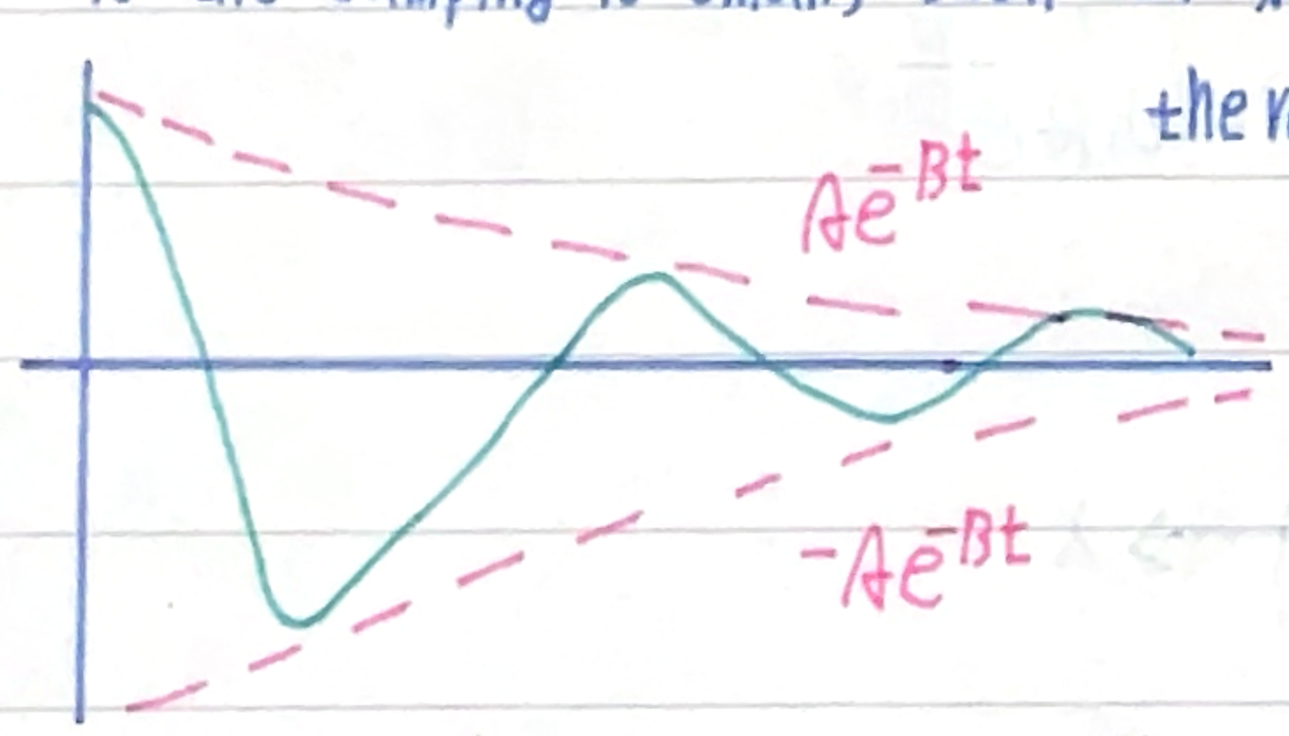
$=2\dfrac{A_1+A_2}{2}\cos(\omega_1t)-\dfrac{A_1-A_2}{2i}2\sin(\omega_1t)$

$=2\left[\cos\delta\cos(\omega_1t)+\sin\delta\sin(\omega_1t)\right]$

we cannot define a frequency when damping is present, because the motion is not periodic

we define $\omega_1=\dfrac{2\pi}{2T_1}$ where T_1 is the time between adjacent zero-axis crossings

if the damping is small, then $\omega_1=\sqrt{\omega_0^2-\beta^2}\approx\omega_0$



the ratio of the amplitude at two successive maxima

$$\frac{Ae^{-\beta T}}{Ae^{-\beta(T+t_1)}}=e^{\beta t_1}\ \text{decrement}$$

where first occurs at T and $\dfrac{2\pi}{\omega_1}$.

$\dfrac{dv}{dt}e^{\pi t}+(2\beta-2\beta)e^{\pi t}v=0\qquad \displaystyle\int\frac{de^{\pi t}}{e^{\pi t}}$

$\dfrac{dv}{dt}e^{\pi t}=(2\pi-2\pi)e^{-\pi t}v\qquad \ln v=-2\ln e^{-\pi t}-2\beta t$

$\dfrac{1}{v}dv=\dfrac{2\beta-2\beta}{e^{\pi t}}e^{\pi t}dt\qquad v=\dfrac{1}{e^{-2\pi t}}e^{-2\pi t}=1$

$\displaystyle\int\frac{1}{v}dv=\int\frac{-2\pi e^{-\pi t}}{e^{\pi t}}dt+\int 2\beta\,dt$

the total energy of a damped oscillator is

$$E(t) = \frac{1}{2} m \left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2} k x(t)^2$$

where $x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$

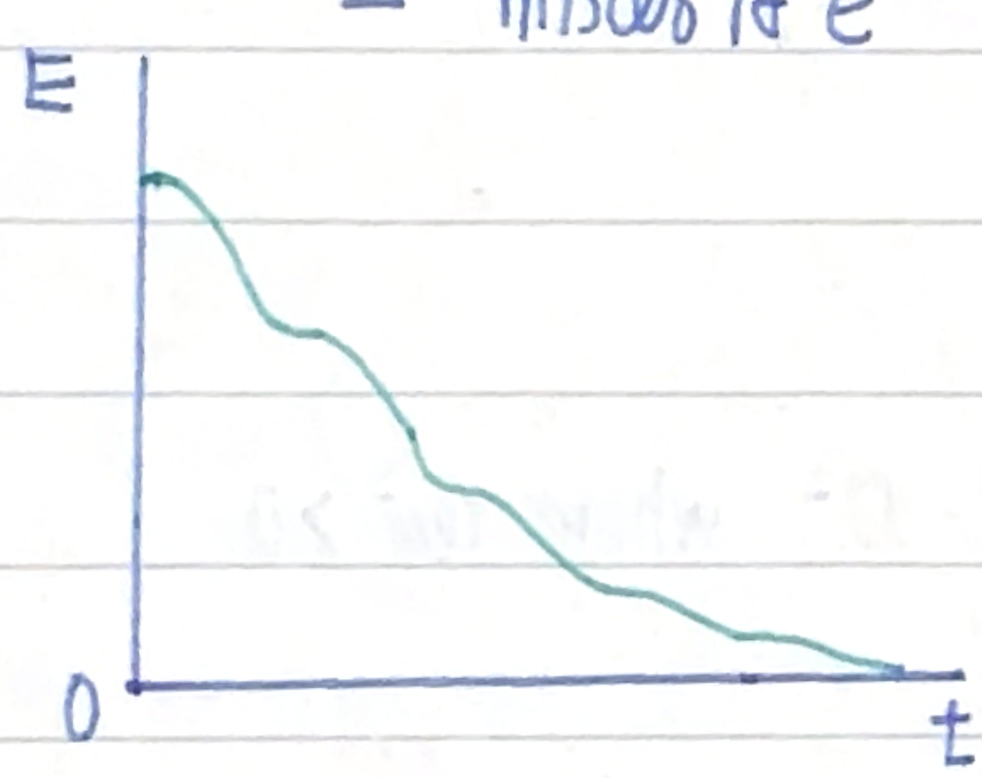
$$\frac{dx(t)}{dt} = A e^{-\beta t}\left[-\beta \cos(\omega_1 t - \delta) - \omega_1 \sin(\omega_1 t - \delta)\right]$$

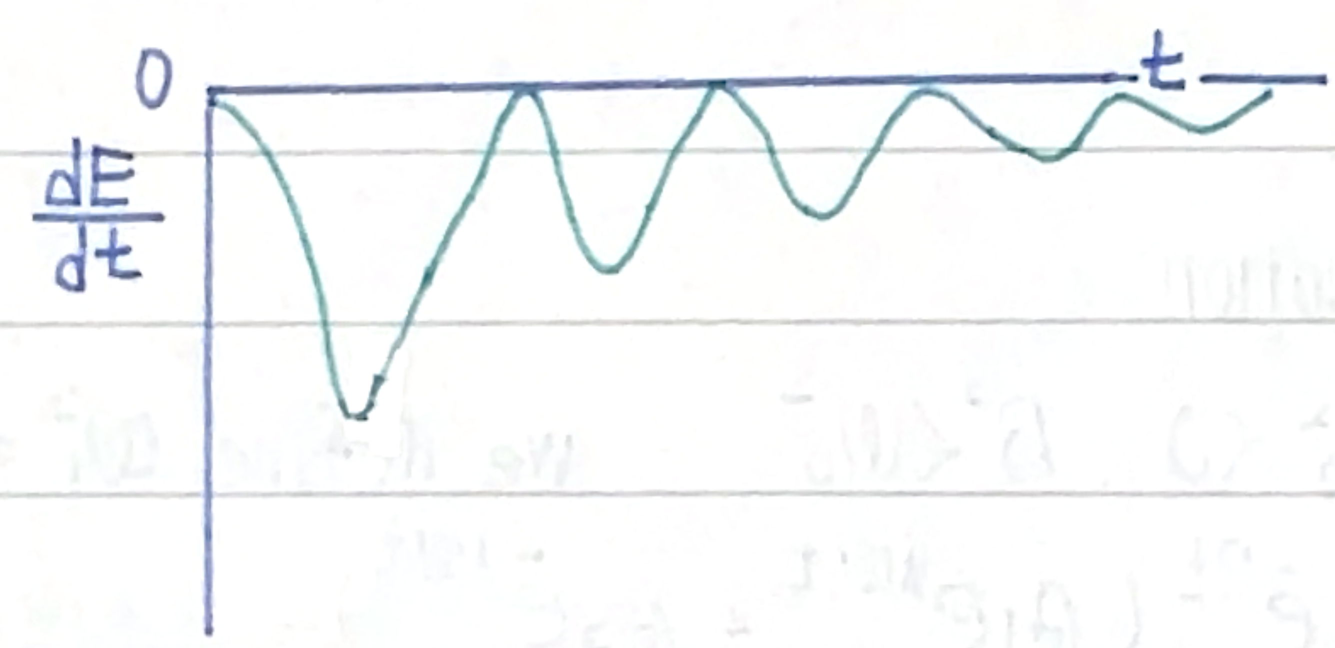
$$\Rightarrow E(t) = \frac{A^2}{2} e^{-2\beta t}\left[(m\beta^2 + k)\cos(\omega_1 t - \delta) + m\omega_1^2 \sin^2(\omega_1 t - \delta) + 2m\beta\omega_1 \sin(\omega_1 t - \delta)\cdot \cos(\omega_1 t - \delta)\right]$$

$$= \frac{mA^2}{2} e^{-2\beta t}\left[\beta^2 \cos 2(\omega_1 t - \delta) + \beta\sqrt{\omega_0^2 - \beta^2}\, \sin 2(\omega_1 t - \delta) + \omega_0^2\right]$$

$$\frac{dE}{dt} = \frac{mA^2}{2} e^{-2\beta t}\left[(2\beta\omega_0^2 - 4\beta^3)\cos 2(\omega_1 t - \delta) - 4\beta^2\sqrt{\omega_0^2 - \beta^2}\, \sin 2(\omega_1 t - \delta) - 2\beta\omega^2\right]$$

$$\simeq -m\beta\omega_0^2 A^2 e^{-2\beta t}$$





construct a general phase diagram

let $u = \omega_1 x$ $\quad w = \beta x + \frac{dx}{dt}$

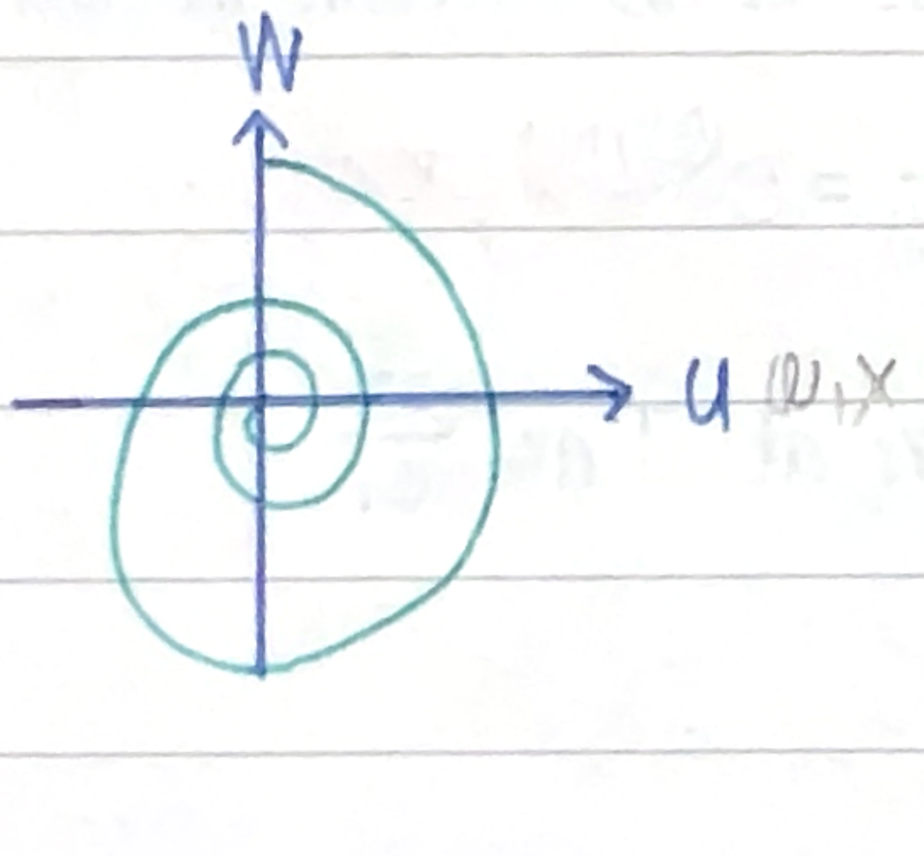
then $u = \omega_1 A e^{-\beta t} \cos(\omega_1 t - \beta)$

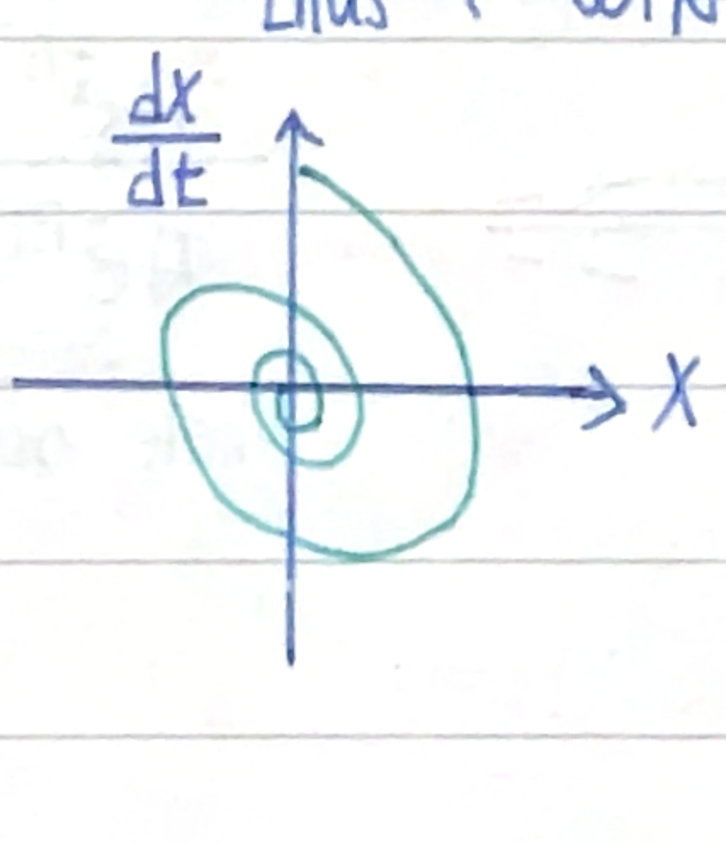
$w = -\omega_1 A e^{-\beta t} \sin(\omega_1 t - \delta)$

if we present u and w in polar coordinates

$\rho = \sqrt{u^2 + w^2} \quad \phi = \omega_1 t$

thus $\rho = \omega_1 A e^{-\frac{\beta}{\omega_1}\phi}$





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DATE / /

critically damped motion

$\beta^2 - \omega_0^2 = 0$, $\beta^2 = \omega_0^2$ $\Rightarrow$ $x_1 = e^{-\beta t}$, let $x_2 = ue^{-\beta t}$

$\Rightarrow \frac{dx_2}{dt} = (-\beta)ue^{-\beta t} + u'e^{-\beta t}$

$\frac{d^2x_2}{dt^2} = \beta^2 ue^{-\beta t} + (-\beta)u'e^{-\beta t} + (-\beta)u'e^{-\beta t} + u''e^{-\beta t}$

$$\Rightarrow \quad \begin{array}{lll} u''e^{-\beta t} & -2\beta u'e^{-\beta t} & \beta^2 ue^{-\beta t} \\ & 2\beta u'e^{-\beta t} & -2\beta^2 ue^{-\beta t} \\ & & \omega_0^2 ue^{-\beta t} \end{array} = 0$$

$\frac{dv}{dt}e^{-\beta t} - \beta^2 ve^{-\beta t} + \omega_0^2 \int ve^{-\beta t} = 0$

$u''e^{-\beta t} - \beta^2 ue^{-\beta t} + \omega_0^2 ue^{-\beta t} = 0$ let $v = u'$

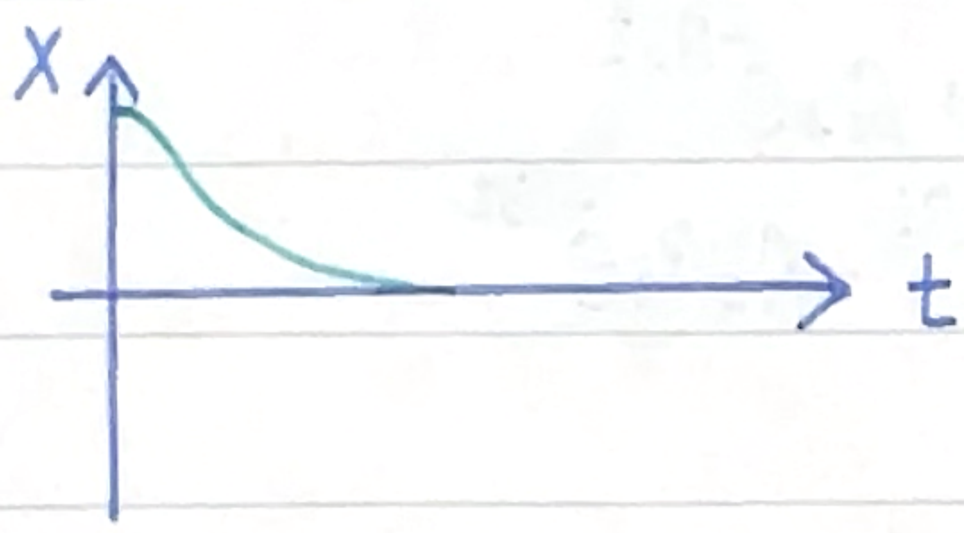
$\frac{dv}{dt}e^{-\beta t} = 0$, $\int \frac{dv}{dt}e^{-\beta t}dt = 0$, $\int e^{-\beta t}dv = 0$

$\downarrow$

$ve^{-\beta t} = 0$

$x_2 = te^{-\beta t}$

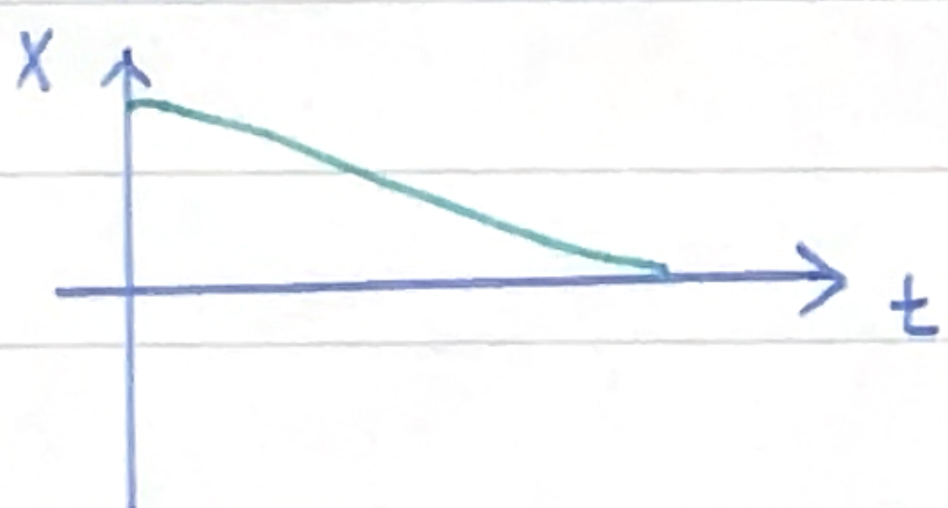
$\Rightarrow x(t) = (A + Bt)e^{-\beta t}$

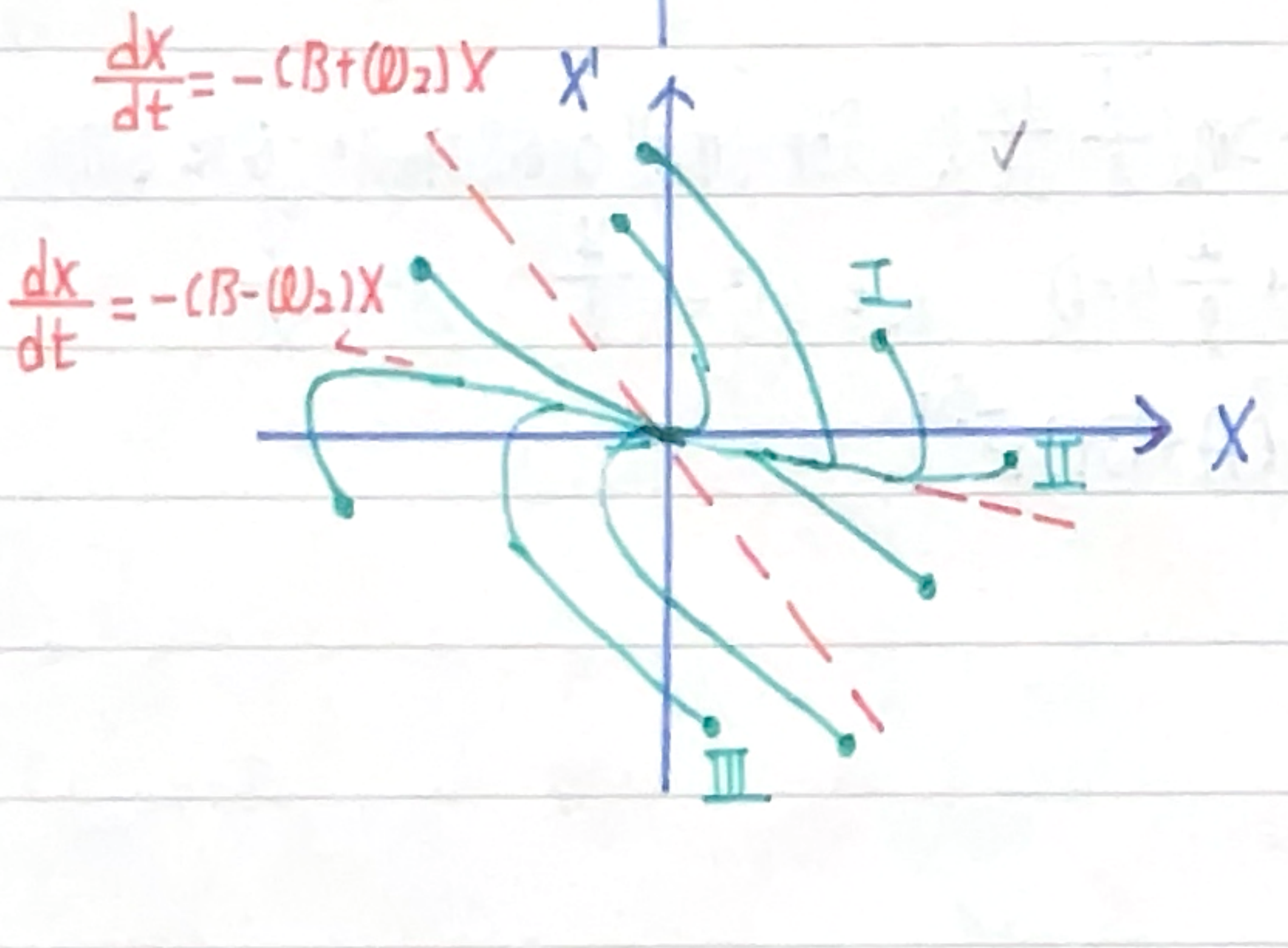


ovderdamped motion

$\beta^2 - \omega_0^2 > 0$, $\beta^2 > \omega_0^2$

$x(t) = e^{-\beta t}(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t})$ where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$





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DATE / /

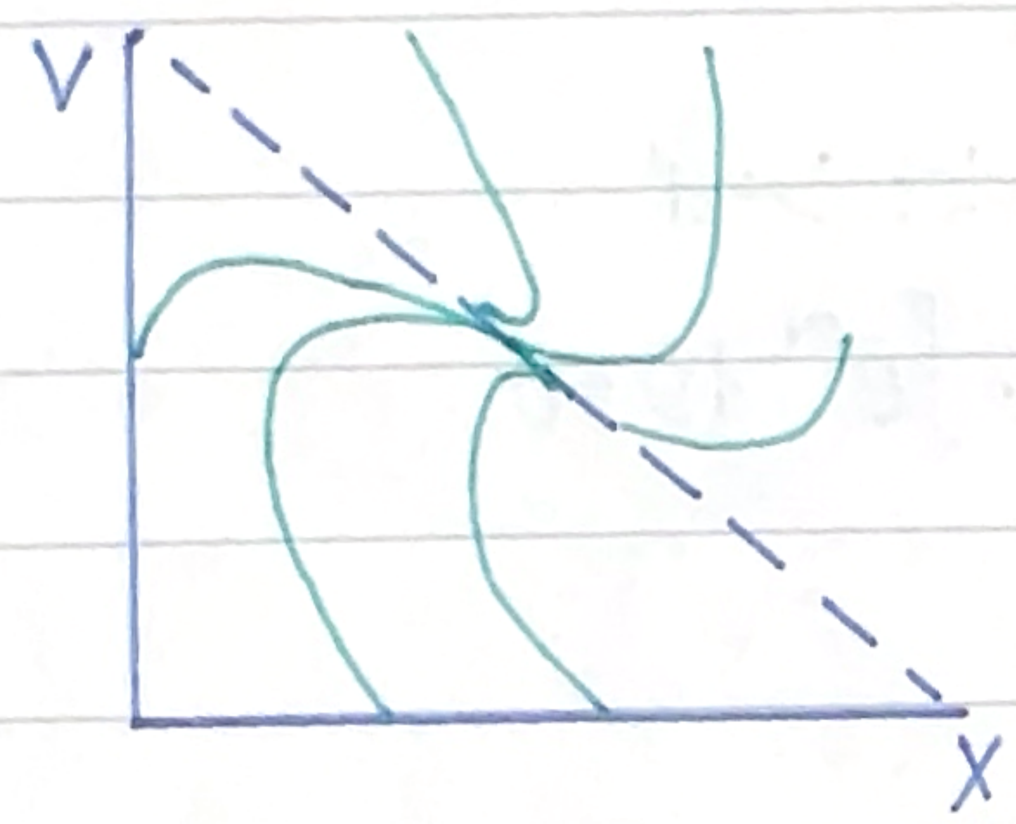
critically damped $X(t) = (A+Bt)e^{-\beta t}$

$V(t) = [B - \beta(A+Bt)]e^{-\beta t}$

$A = X_0 \quad B = V_0 + \beta X_0$

$\lim_{t\to\infty} V(t) = -\beta B t e^{-\beta t}$

$\lim_{t\to\infty} X(t) = B t e^{-\beta t}$



for overdamped $X(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$

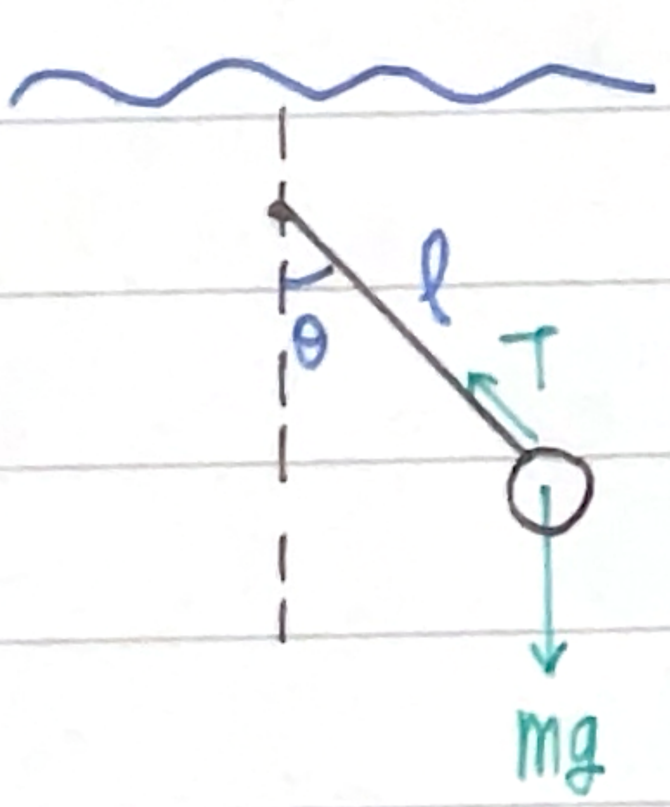
$V(t) = -A_1\beta_1 e^{-\beta t} \div A_2\beta_2 e^{-\beta_2 t}$

(a) at $t=0$, $X_0 = A_1 + A_2$

$V_0 = -A_1\beta_1 - A_2\beta_2$

(b) when $A_1 = 0$, we have $V_0 = -\beta_2 X_0$ and $V(t) = -\beta_2(t)$

for $A_1 \neq 0$, $V(t) \to -\beta_1 A_1 e^{-\beta t} = -\beta_1 X$ as $t\to\infty$ since $\beta_1 < \beta_2$



$F_{res} = 2m\sqrt{\frac{g}{l}}\, l \frac{d\theta}{dt}$

$F = ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - 2m\sqrt{\frac{g}{l}}\frac{d\theta}{dt}l$ for small oscillations $\theta \approx \sin\theta$

$\frac{d^2\theta}{dt^2} + 2\sqrt{\frac{g}{l}}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0$ let $\omega_0^2 = \frac{g}{l}$ $\beta^2 = \frac{g}{l}$

$\Rightarrow \omega_0^2 = \beta_0^2$, $\theta(t) = (A+Bt)e^{-\beta t}$