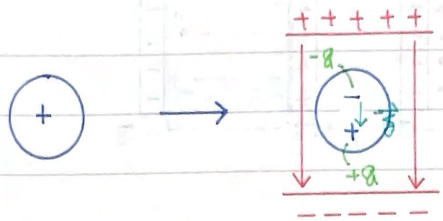
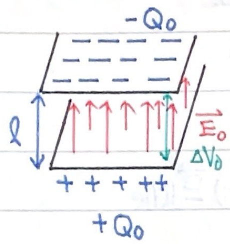
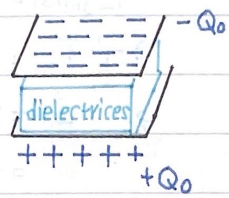
Capacitance and Dielectrics

Dielectrices



nucleus is polarized and have electric dipole moment





$$\Delta V_0 = \frac{1}{C_0} Q_0$$

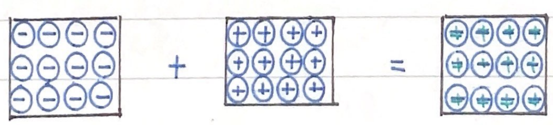
$$\Delta V = \frac{1}{K}\Delta V_0 = \frac{1}{K}\frac{Q_0}{C_0} = \frac{1}{C_{diele}} Q_0$$

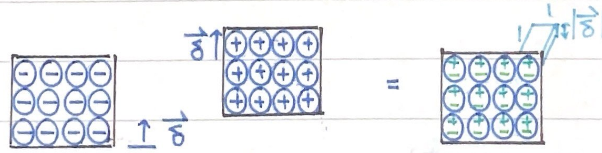
K: permittivity

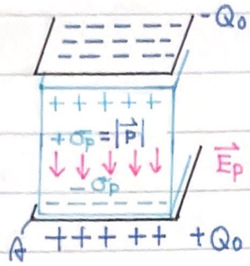
let $C_{diele} = K C_0$

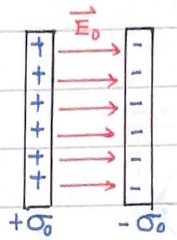
$$|\vec{E}_0| l = \Delta V_0 \qquad |\vec{E}_{diele}| l = \Delta V_{diele}$$

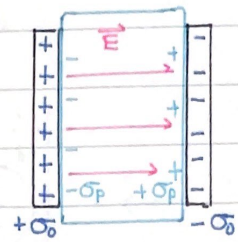
$$\vec{E} = \frac{1}{K}\vec{E}_0$$











$$E_0 = \frac{1}{\epsilon_0}\frac{Q_0}{A} = \frac{\sigma_0}{\epsilon_0} \qquad |\vec{E}_p| = \frac{\sigma_p}{\epsilon_0}$$

$$\sigma_0 = \frac{Q_0}{A} \qquad |\vec{E}_{didle}| = \vec{E}_0 + \vec{E}_p = \frac{\sigma_0 - \sigma_p}{\epsilon_0}$$

$$\Rightarrow \frac{1}{K}\vec{E}_0 = \vec{E}_0 + \vec{E}_p,$$

$$\Rightarrow \vec{E}_p = \left(\frac{1}{K} - 1\right)\vec{E}_0$$

$$= -\frac{K-1}{K}\vec{E}_0 = -(k-1)\vec{E}_{didle}$$

$$\Rightarrow \sigma_p = (k-1)\epsilon_0 |\vec{E}_{didle}| = \sigma_0\left(1 - \frac{1}{K}\right)$$

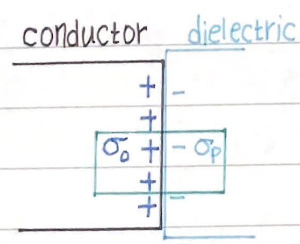
$$= 1 \times 1 \times |\vec{\delta}| \times \rho = 1 \times 1 \times |\vec{\delta}| \times Nq$$

assume there have N atoms in unit volume, each nucleus $+q$

$$= N\vec{p} = \vec{P}$$ polarization vector (dipole moment in unit volume)

$$\Rightarrow |\vec{P}| = \sigma_p = (k-1)\epsilon_0 |\vec{E}_{diele}| = \vec{P} \cdot \vec{n}$$

gauss's law in dielectrices



$$\vec{E} \cdot \vec{A} = \frac{(\sigma_0 - \sigma_p)A}{\epsilon_0} \quad \text{and} \quad \sigma_p = \sigma_0\left(1 - \frac{1}{K}\right)$$

$$\vec{E} \cdot \vec{A} = \frac{\sigma_0 A}{K\epsilon_0}, \quad EKA = \frac{\sigma_0 A}{\epsilon_0}$$

$$\oint K\vec{E} \cdot d\vec{A} = \frac{Q_{encl\text{-}free}}{\epsilon_0}$$

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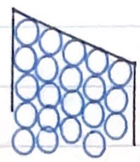

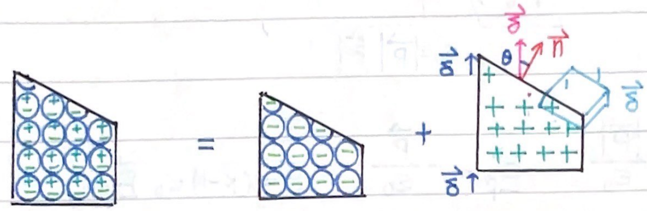

$$1 \times 1 \times |\vec{\delta}| \cdot \vec{n} = 1 \times 1 \times |\vec{\delta}| \cos\theta$$

$$\vec{P} \cdot \vec{n} = |N_q \vec{\delta}||\vec{n}| \cos\theta = 1 \times 1 \times |\vec{\delta}| \cos\theta \times N_q$$

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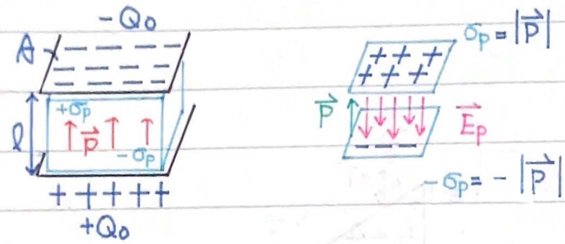
Calculating Dielectrices

case I



(a) $|\vec{E}_p| = \dfrac{\sigma_p}{\epsilon_0} = \dfrac{|\vec{P}|}{\epsilon_0}$ $\quad \vec{E}_p = -\dfrac{\vec{P}}{\epsilon_0}$ $\quad \vec{P} = (K-1)\,\epsilon_0\,\vec{E}_{diele}$

$$\vec{E}_{diele} = \vec{E}_0 + \frac{-\vec{P}}{\epsilon_0} = \vec{E}_0 + (K-1)\,\vec{E}_{diele}$$

$$1 + (K-1)\,\vec{E}_{diele} = \vec{E}_0 \ , \quad \vec{E}_{diele} = \frac{1}{K}\vec{E}_0$$

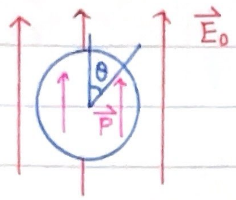
(b) $\sigma_p = \vec{P} \cdot \vec{n} = -(-\sigma_p)$

$\sigma_{lower\ plate} = \sigma_0 + (-\sigma_p) = \sigma_0 - \vec{P}$

$\sigma_{high\ plate} =$

$$\vec{E} = \frac{\sigma - P}{\epsilon_0} = \vec{E}_0 + \vec{E}_p$$

case II



(a) $\sigma_p = |\vec{P}|\cos\theta$

$$\vec{E}_p = \frac{\rho}{3\epsilon_0}\vec{d} = -\frac{\rho}{3\epsilon_0}\vec{\delta} = -\frac{Nq}{3\epsilon_0}\vec{\delta} = -\frac{\vec{P}}{3\epsilon_0}$$

$$\vec{E}_{diele} = \vec{E}_0 + \frac{\vec{P}}{3\epsilon_0} = \vec{E}_0 - \frac{K-1}{3\epsilon_0}\epsilon_0\vec{E}_{diele}$$

$$\left(1+\frac{K-1}{3}\right)\vec{E}_{diele} = \vec{E}_0$$

$$\vec{E}_{diele} = \frac{3}{K+2}\vec{E}_0$$

$$\vec{P} = 3\frac{K-1}{K+2}\epsilon_0\vec{E}_0, \quad \sigma_p = \vec{P}\cdot\vec{n} = 3\frac{K-1}{K+2}\epsilon_0|\vec{E}_0|\cos\theta$$

$$\vec{E}_p = -\frac{K-1}{K+2}\vec{E}_0$$

conductor is $K\to\infty$ of dielectric $\vec{E}_{diele} = 0$ $\vec{E}_p = -\vec{E}_0$ $\sigma_p = 3\epsilon_0|\vec{E}_0|\cos\theta$

(b)

$$\vec{P}_{total} = \vec{P}\times\frac{4}{3}\pi R^3 = \frac{4\pi R^3 P}{3}$$

$r > R$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}_t\cdot\vec{n}}{r^2} = \frac{R^3}{3\epsilon_0}\frac{P\cos\theta}{r^2}$$

$$\vec{E} = -\nabla V(\vec{r}) = \frac{P_t}{4\pi\epsilon_0}\frac{1}{r^3}(2\cos\theta+\sin\theta)$$

$$= \frac{P}{3\epsilon_0}\frac{R^3}{r^3}(2\cos\theta+\sin\theta)$$

$r \le R$

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r \quad r\le R$$

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \quad r>R$$

$$Q = \frac{4}{3}\pi R^3 nq$$

$$V(\vec{r}) = -\left(\int_\infty^R \vec{E}_2\cdot d\vec{r} + \int_R^r \vec{E}_1\cdot d\vec{r}\right)$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_\infty^R \frac{dr}{r^2} - \frac{Q}{4\pi\epsilon_0 R^3}\int_R^r r\,dr$$

$$= \frac{Q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right) = \frac{nqR^2}{6\epsilon_0}\left(3-\frac{r^2}{R^2}\right)$$

for $\oplus$ $r^2 = r_+^2 = r^2+\left(\frac{d}{2}\right)^2 - rd\cos\theta$

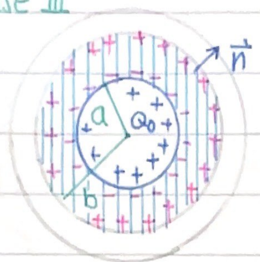
for $\ominus$ $r^2 = r_-^2 = r^2+\left(\frac{d}{2}\right)^2 + rd\cos\theta$

$$V(\vec{r}) = V_+ + V_- = \frac{nqR^2}{6\epsilon_0}\left[3-\frac{r^2+\left(\frac{d}{2}\right)^2-rd\cos\theta}{R^2}\right] + \frac{-nqR^2}{6\epsilon_0}\left[3-\frac{r^2+\left(\frac{d}{2}\right)^2-rd\cos\theta}{R^2}\right]$$

$$= \frac{nqdr\cos\theta}{3\epsilon_0} = \frac{Pr\cos\theta}{3\epsilon_0} = \frac{\vec{P}\cdot\vec{r}}{3\epsilon_0} = \frac{P}{3\epsilon_0}y$$

$$\vec{E} = -\vec{\nabla}\phi = -\left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\frac{Py}{3\epsilon_0} = -\frac{P}{3\epsilon_0}\vec{e}_y$$

case III



$\vec{P}$ is uniform here

Q_0 generate σ_{pa} at $a \le r \le b$, σ_{pa} generate $\vec{E}_p$

use Gauss's surface at $r = a + 0_+$

$\int \vec{E}_{diele} \cdot d\vec{A} = \frac{1}{\epsilon_0}(Q_0 + 4\pi a^2 \sigma_{pa})$ and $\vec{P} \cdot \vec{n} = \vec{P} \cdot (-\vec{e}_r) = (1-K)\epsilon_0 \vec{E}_{diele}$

$4\pi a^2 |\vec{E}_{diele}| = \frac{1}{\epsilon_0}\left[Q_0 + 4\pi a^2 (1-K)\epsilon_0 |\vec{E}_{diele}|\right]$

$4\pi a^2 [(K-1)+1] |\vec{E}_{diele}| = \frac{Q_0}{\epsilon_0}$

$\Rightarrow$ when $r = a + 0_+$ $\quad \vec{E}_{diele} = \frac{1}{4\pi\epsilon_0} \frac{Q_0}{a^2} \frac{1}{K} \vec{e}_r$

$\sigma_{pa} = (1-K)\epsilon_0 |\vec{E}_{diele}|_{r=a+0_+} = \left(\frac{1}{K} - 1\right)\frac{Q_0}{4\pi a^2}$

$Q_{pa} = 4\pi a^2 \sigma_{pa} = \left(\frac{1}{K} - 1\right) Q_0$

$Q_0 + Q_{pa} = \frac{1}{K} Q_0$, $\quad \vec{E}_{diele} = \frac{1}{4\pi\epsilon_0} \frac{Q_0}{a^2} \frac{1}{K} \vec{e}_r$

use Gauss's surface at $r = b + 0_+$

$\sigma_{pb} = \vec{P}_{r=b+0_+} \cdot \vec{e}_r = (K-1)\epsilon_0 \vec{E}_{diele} \cdot \vec{e}_r$

$= (K-1) \frac{1}{4\pi b^2} \frac{Q_0}{K} = \left(1 - \frac{1}{K}\right) \frac{Q_0}{4\pi b^2}$

$4\pi b^2 \sigma_{pb} = \left(1 - \frac{1}{K}\right) Q_0 = Q_{pb} = -Q_{pa}$

when $r > b$ $\quad \int \vec{E}_{diele} \cdot d\vec{A} = \frac{1}{\epsilon_0}(Q_0 + 4\pi a^2 \sigma_{pa} + 4\pi b^2 \sigma_{pb}) = \frac{Q_0}{\epsilon_0}$

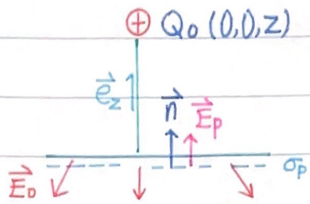
$\vec{E}_{diele} = \frac{1}{4\pi\epsilon_0} \frac{Q_0}{r^2} \vec{e}_r$

let $a \to 0$, then Q_0 is a point charge and Q_0

$Q_0 + Q_{pa} = \frac{1}{K} Q_0$

$r \le b$ $\quad \vec{E}_{diele} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{Q_0}{K} \vec{e}_r$

case IV



$$P_z = \vec{P}\cdot\vec{n} = \sigma_p = (k-1)\epsilon_0 \vec{E}_{diele}$$

$$= (k-1)\epsilon_0 \Big(\underbrace{-\frac{1}{4\pi\epsilon_0}\frac{Q_0}{z^2+r^2}\frac{z}{(z^2+r^2)^{\frac{1}{2}}}}_{E_z} - \frac{\sigma_p}{2\epsilon_0}\Big)$$

$$\left[1+\frac{(k-1)\epsilon_0}{2\epsilon_0}\right]P_z = \frac{k-1}{4\pi}\frac{-Q_0 z}{(z^2+r^2)^{\frac{3}{2}}}$$

$$P_z = \frac{k-1}{k+1}\frac{-Q_0}{2\pi}\frac{z}{(z^2+r^2)^{3/2}} = \sigma_p$$

$$\int \sigma_p 2\pi r\,dr = \frac{k-1}{k+1}(-Q_0)$$

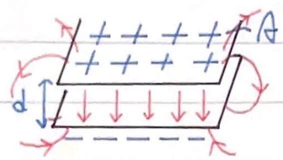
$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Q\frac{k-1}{k+1}(-Q_0)}{(2z)^2}\vec{e}_z$$

$$\vec{E} = \frac{1-k}{k+1}Q_0 + Q_0 = \frac{2}{k+1}Q_0$$

Capacitance

case I

$$C = \frac{Q}{V_{ab}}$$



$$\vec{E}_+ = \frac{\sigma}{2\epsilon_0} \qquad \vec{E}_- = \frac{-\sigma}{2\epsilon_0}$$

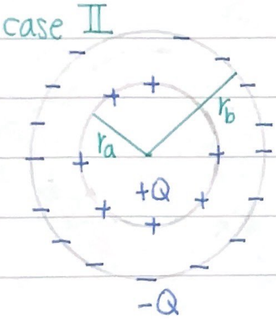
$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\sigma}{2\epsilon_0}\vec{e}_y + \frac{-\sigma}{2\epsilon_0}(-\vec{e}_y) = \frac{\sigma}{\epsilon_0}\vec{e}_y$$

$$V = -\int_d^0 \vec{E}\cdot d\vec{y} = -\frac{\sigma}{\epsilon_0} y\Big|_d^0 = \frac{\sigma}{\epsilon_0} d = \frac{1}{\epsilon_0}\frac{Qd}{A}$$

$$C = \frac{Q}{\frac{Qd}{\epsilon_0 A}} = \frac{\epsilon_0 A}{d}$$

case II



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\vec{e}_r$$

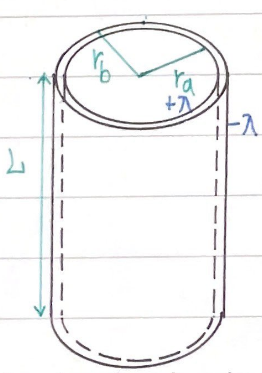
$$V = -\int_b^a \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\Big|_b^a = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

$$C = 4\pi\epsilon_0\frac{r_a r_b}{r_b - r_a} = \epsilon_0\frac{A_{gm}}{d}, \quad A_{gm} = 4\pi r_a r_b, \quad d = r_b - r_a$$

when $b \to \infty$ $\quad C = \lim_{b\to\infty} 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a} = 4\pi\epsilon_0\frac{r_a r_b}{r_b} = 4\pi\epsilon_0 r_a$

case III



$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

$$V = -\int_b^a \vec{E}\cdot d\vec{r} = -\int_b^a \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln r\Big|_b^a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

$$C = \frac{\lambda L}{V} = \frac{2\pi\epsilon_0 L}{\ln\frac{r_b}{r_a}} \qquad \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\frac{r_b}{r_a}}$$

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Energy Storage in Capacitors

$$P = \mathcal{E} i = i^2 R + \frac{i}{C} q$$

battery $\int_0^\infty P\,dt = \int_0^\infty \mathcal{E} i\,dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-\frac{t}{RC}}\,dt = \frac{\mathcal{E}^2}{R}\left[-RC e^{-\frac{t}{RC}}\right]_0^\infty = C\mathcal{E}^2$

resistor $\int_0^\infty P\,dt = \int_0^\infty i^2 R\,dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-\frac{2t}{RC}}\,dt = \frac{\mathcal{E}^2}{R}\left[-\frac{RC}{2} e^{-\frac{t}{RC}}\right]_0^\infty = \frac{1}{2}C\mathcal{E}^2$

the final charge on the capacitor is $Q = C\mathcal{E}$

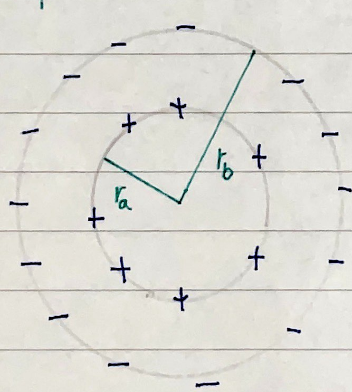
the energy stored is $U = \frac{Q^2}{2C} = \frac{1}{2}C\mathcal{E}^2$

$$dW = v\,dq = \frac{q\,dq}{C}$$

$$W = \int_0^W dw = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{2}\frac{Q^2}{C} = U = \frac{1}{2}CV^2 \overset{Q=CV}{=} \frac{1}{2}QV$$

u, energy density $\frac{\frac{1}{2}CV^2}{Ad} = \frac{1}{2}\epsilon_0 E^2$ $V = Ed$

example



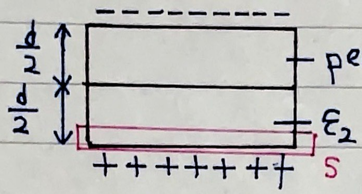
$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

$u = \frac{1}{2}\epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$ is not uniform

$$U = \int u\,dV = \int_{r_a}^{r_b} \frac{Q^2}{32\pi^2\epsilon_0 r^4} 4\pi r^2\,dr$$

$$= \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

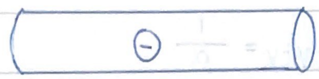
case I



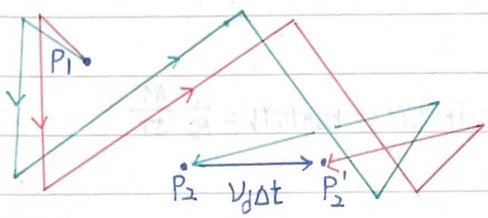
$D_1 = D_2$, $\varepsilon_1 E_1 = \varepsilon_2 E_2$

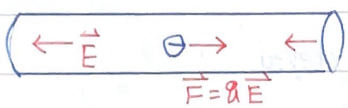
$\int_S D \cdot dA$, $D_2 A = Q_f$ $\quad E_2 = \frac{Q_f}{\varepsilon_2 A}$

Current

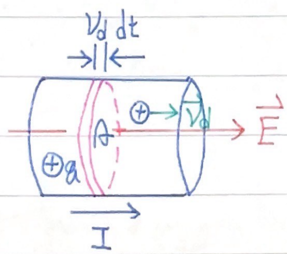


no $\vec{E}$





$I = \frac{dQ}{dt}$



suppose there are n moving charged particles per unit volume

$dQ = q(nAv_d\,dt) = nqv_dA\,dt$ $\quad I = \frac{dQ}{dt} = nqv_dA$

current density $J = \frac{I}{A} = nqv_d$ $\quad I = n|q|v_dA$

$\vec{J} = nq\vec{v_d}$

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Resistivity

resistivity $\rho = \frac{E}{J}$ conductivity $= \frac{1}{\rho}$

$\rho(T) = \rho_0 [1 + \alpha (T - T_0)]$

α: temperature coefficient of resistivity $= \frac{1}{\rho_0} \frac{\Delta \rho}{\Delta T}$

Resistance

$\vec{E} = \rho \vec{J}$ when Ohm's law is obeyed, $\vec{J} = \sigma \vec{E}$ (σ: conductivity)

$\because I = JA$, $V = EL$ $\Rightarrow \frac{V}{L} = \rho \frac{I}{A}$, $V = \rho \frac{L}{A} I$, $V = RI$

$R = \rho \frac{L}{A}$

$R(T) = R_0 [1 + \alpha (T - T_0)]$ 25.67

Energy and Power in Electric Circuits

$dQ = I dt$

$V_{ab} dQ = V_{ab} I dt$ $\Rightarrow P = V_{ab} I = I^2 R = \frac{V_{ab}^2}{R}$

Theory of Metallic Conduction

$\vec{J} = nq \vec{v_d}$ $\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$ $v_0 = 0$ for an average electron

$\vec{v} = \vec{v_0} + \vec{a} \tau$, τ: the average time between collisions

$\vec{v}_{av} = \vec{a} \tau = \frac{q\tau}{m} \vec{E}$

$\vec{v_d} = \frac{q\tau}{m} \vec{E}$, $\vec{J} = \frac{nq^2\tau}{m} \vec{E}$, $\rho = \frac{m}{ne^2\tau}$