Phase Space and Liouville's Theorem

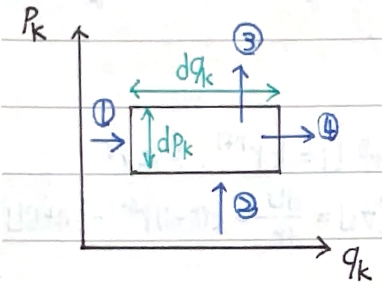
phase space refers to the plotting of both a particle's momentum and position on a two dimensional graph

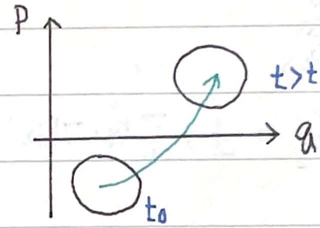
for a large collection of particles:

define a density in phase space ρ

the number N of systems whose representative points lie within a volume dv of phase space

$N = \rho\, dv$ where $dv = dq_1 dq_2 \cdots dq_s\, dp_1 \cdots dp_s$ — s is the number of degrees of freedom





$t > t_0$ the number of dots inside the surface is a constant in time

① $\rho \frac{dq_k}{dt} dp_k$ ② $\rho \frac{dp_k}{dt} dq_k$ $① + ② = \rho\left(\frac{dq_k}{dt} dp_k + \frac{dp_k}{dt} dq_k\right)$

$$③ + ④ = \left[\rho \frac{dq_k}{dt} + \frac{\partial}{\partial q_k}\left(\rho \frac{dq_k}{dt}\right) dq_k\right] dp_k + \left[\rho \frac{dp_k}{dt} + \frac{\partial}{\partial p_k}\left(\rho \frac{dp_k}{dt}\right) dp_k\right] dq_k$$

$$(③ + ④) - (① + ②) = \frac{\partial \rho}{\partial t} dq_k dp_k = -\left[\frac{\partial}{\partial q_k}\left(\rho \frac{dq_k}{dt}\right) + \frac{\partial}{\partial p_k}\left(\rho \frac{dp_k}{dt}\right)\right] dq_k dp_k$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_{k=1}^{s}\left(\frac{\partial \rho}{\partial q_k}\frac{dq_k}{dt} + \rho \frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \rho}{\partial p_k}\frac{dp_k}{dt} + \rho \frac{\partial \dot{p}_k}{\partial p_k}\right) = 0$$

$$\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_{k}\left(\frac{\partial \rho}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial \rho}{\partial p_k}\frac{dp_k}{dt}\right) = 0$$ total time derivative of ρ

$$\Rightarrow \frac{d\rho}{dt} = 0$$ Liouville's theorem

Virial Theorem

consider a collection of particles whose position vectors $\vec{r}_\alpha$ and momenta $\vec{P}_\alpha$ are both bounded (remain finite for all values of the time)

$$S \equiv \sum_\alpha \vec{P}_\alpha \cdot \vec{r}_\alpha \quad , \quad \frac{dS}{dt} = \sum_\alpha \left(\vec{P}_\alpha \cdot \frac{d\vec{r}_\alpha}{dt} + \frac{d\vec{P}_\alpha}{dt} \cdot \vec{r}_\alpha \right)$$

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dS}{dt}\,dt = \frac{S(\tau)-S(0)}{\tau}$$

if the system' motion is periodic, then $S(t)=S(0)$
even if the system not periodicity, because S is by hypothesis a bounded function, we can make $\langle \dot{S} \rangle$ as small as desired by allowing τ to become large

$$\Rightarrow \left\langle \sum_\alpha \vec{P}_\alpha \cdot \frac{d\vec{r}_\alpha}{dt} \right\rangle = -\left\langle \sum_\alpha \frac{d\vec{P}_\alpha}{dt} \cdot \vec{r}_\alpha \right\rangle$$

$$\Rightarrow \left\langle 2\sum_\alpha E_{k\alpha} \right\rangle = -\left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle$$

$$\Rightarrow \langle T \rangle = \underbrace{-\frac{1}{2}\left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle}_{\text{viral}} = \frac{1}{2}\left\langle \sum_\alpha \vec{r}_\alpha \cdot \nabla U_\alpha \right\rangle$$

if $U = kr^{n+1}$, $F \propto r^n$

$$\vec{r}\cdot\nabla U = \frac{dU}{dr} = k(n+1)r^{n+1} = (n+1)U$$

$$= \frac{n+1}{2}\langle U \rangle$$

example consider an ideal gas containing N atoms in a container of volume V pressure P and absolute temperature T_1

$$\langle T \rangle = \frac{3}{2}NkT_1 \qquad d\vec{F}_\alpha = -\vec{n}\,P\,dA$$

↳ unit vector $\vec{n}_{out}$

$$-\frac{1}{2}\left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle = \frac{P}{2}\int \vec{n}\cdot\vec{r}\,dA = \frac{P}{2}\int \nabla\cdot\vec{r}\,dV = 3\int dV = 3V\frac{P}{2}$$

$$\Rightarrow \frac{3}{2}NkT_1 = \frac{3}{2}PV \quad , \quad PV = NkT$$