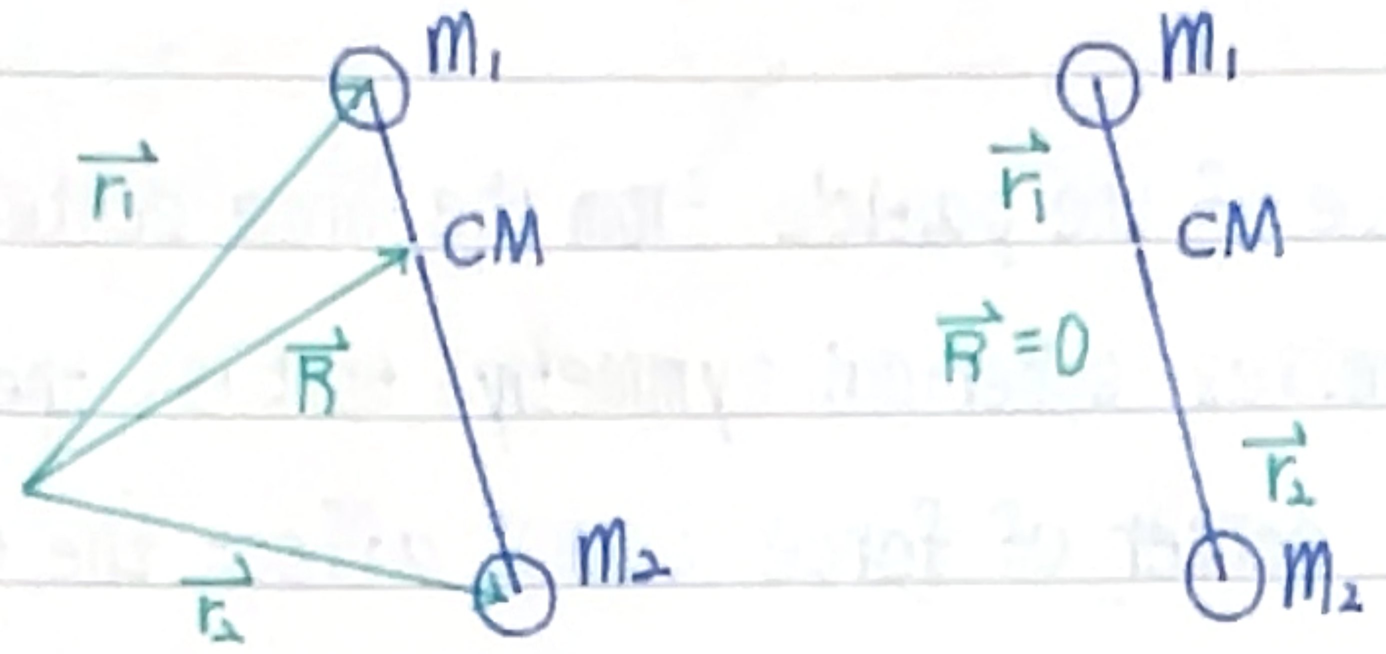
Central-Force Motion

Reduced Mass



$$\vec{r} = |r_1 - r_2|$$

$$L = \frac{1}{2} m_1 \left| \frac{d\vec{r_1}}{dt} \right|^2 + \frac{1}{2} m_2 \left| \frac{d\vec{r_2}}{dt} \right|^2 - U(\vec{r})$$

$$m_1 \vec{r_1} + m_2 \vec{r_2} = 0 \quad — (1) \qquad \vec{r} = \vec{r_1} - \vec{r_2} \quad — (2)$$

$(2) \times m_2 + (1)$, $\quad \vec{r_1} = \frac{m_2}{m_1+m_2} \vec{r}$

$(1) \times m_1 + (2)$, $\quad \vec{r_2} = -\frac{m_1}{m_1+m_2} \vec{r}$

$$\Rightarrow E_k = \frac{1}{2} m_1 \left(\frac{d\vec{r_1}}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\vec{r_2}}{dt}\right)^2$$

$$= \frac{1}{2} m_1 \left(\frac{d\vec{r_1}}{dt}\right)^2 + \frac{1}{2} m_2 \left(-\frac{m_1}{m_2} \frac{d\vec{r_1}}{dt}\right)^2$$

$$= \frac{1}{2} m_1 \left(\frac{d\vec{r_1}}{dt}\right)^2 \left(1 + \frac{m_1}{m_2}\right)$$

$$= \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2} \frac{d\vec{r}}{dt}\right)^2 \left(1 + \frac{m_1}{m_2}\right)$$

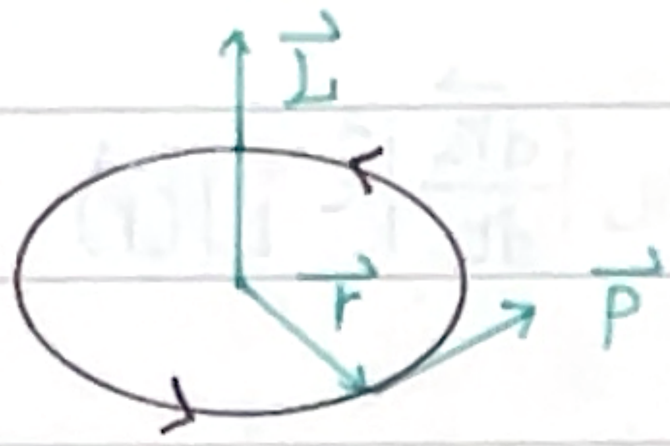
$$= \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \left(\frac{d\vec{r}}{dt}\right)^2 = \frac{1}{2} \mu \left(\frac{d\vec{r}}{dt}\right)^2, \quad \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\Rightarrow L = \frac{1}{2} \mu \left| \frac{d\vec{r}}{dt} \right|^2 - U(\vec{r})$$

Conservation Theorems — First Integrals of the Motion

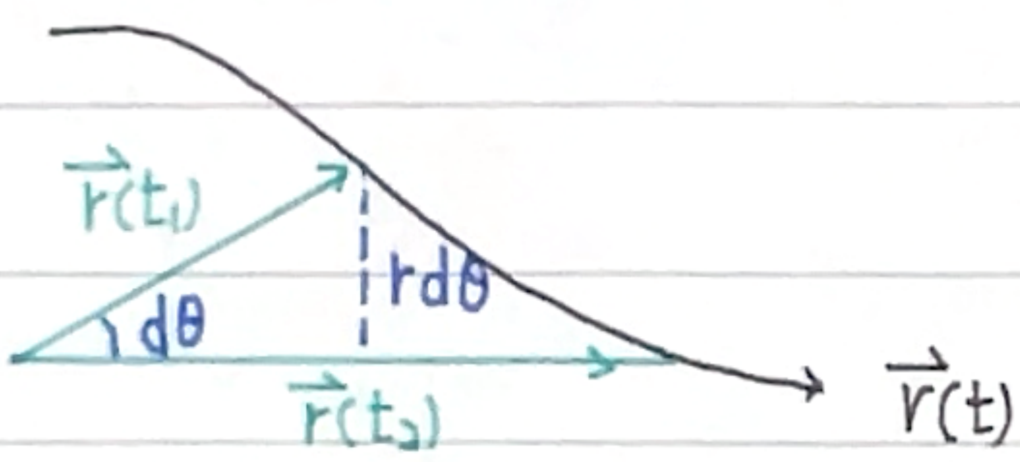
potential energy depends only on the distance of the particle from the force center and not on the orientation, the system possesses spherical symmetry; that is, the system's rotation about any fixed axis through the center of force cannot affect the equations of motion : $\vec{L} = \vec{r} \times \vec{p}$ = constant

$\Rightarrow \vec{r}$ and $\vec{p}$ always



$$\Rightarrow L = \frac{1}{2}\mu\left[\left(\frac{d\vec{r}}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - U(\vec{r})$$

$$\frac{dP_\theta}{dt} = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$ because L is cyclic in θ, $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \frac{d\theta}{dt} = \text{constant} = \ell$



$dA = \frac{1}{2} r^2 d\theta$, areal velocity $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{\ell}{2\mu}$ = constant → Kepler's second law

$\Rightarrow E_k + U = E_t$ = constant , $E_t = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + U(\vec{r})$

$$= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

Equations of Motion

$$\frac{d\vec{r}}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}} \quad , \quad d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

$$dt = \pm\frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

and $$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\frac{\ell}{\mu r^2}}{\pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

$$t - t_0 = \int_{r_0}^{r} \frac{dr}{\pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

$$\int d\theta = \pm\int \frac{\frac{\ell}{\mu r^2}dr}{\pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{2\mu^2 r^2}}} = \theta(r)$$

$$\Rightarrow \theta = \int_{r_0}^{r} \frac{dr}{\sqrt{r^2 \frac{2\mu E}{\ell^2} - \frac{2\mu U}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$ let $u = \frac{1}{r}$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu U}{\ell^2} - u^2}}$$

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solving the problem formally, is not always a particable problem

let $U = ar^{n+1}$,

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu a}{\ell^2} u^{-n-1} - u^2}}$$

if $n=1$ $\quad \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu a}{\ell^2}\frac{1}{u^2} - u^2}}$ $\quad$ let $u^2 = x \quad du = \frac{dx}{2\sqrt{x}}$

$$= \theta_0 - \frac{1}{2}\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2\mu E}{\ell^2}x - \frac{2\mu a}{\ell^2} - x^2}}$$ harmonic oscillator

$n=-2$ inverse-square-law force

$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$, $\mu\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{\partial U}{\partial r} = F(r)$

let $u = \frac{1}{r}$

$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{\mu r^2}{\ell} = -\frac{\mu}{\ell}\frac{dr}{dt}$

$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\frac{dr}{dt}\right) = \frac{d}{dt}\left(-\frac{\mu}{\ell}\frac{dr}{dt}\right)\Big/\frac{d\theta}{dt} = \left(-\frac{\mu}{\ell}\frac{d^2r}{dt^2}\right)\Big/\frac{d\theta}{dt} = -\frac{\frac{\mu}{\ell}}{\frac{\ell}{\mu r^2}}\frac{d^2r}{dt^2} = -\frac{\mu^2}{\ell^2}r^2\frac{d^2r}{dt^2}$

$\Rightarrow \frac{d^2r}{dt^2} = -\frac{\ell^2}{\mu^2}\frac{1}{r^2}\frac{d^2u}{d\theta^2} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}$ $\quad r\left(\frac{d\theta}{dt}\right)^2 = r\left(\frac{\ell}{\mu r^2}\right)^2 = \frac{\ell^2}{\mu^2}\frac{1}{r^3} = \frac{\ell^2}{\mu^2}u^3$

$\Rightarrow F(r) = \mu\left(-\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}\right) - \mu\left(\frac{\ell^2}{\mu^2}u^3\right)$, $\quad \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(r)$

$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(r)$

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example : Find the law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{a\theta}$, and determine $r(t)$, $\theta(t)$ and total energy

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-a\theta}}{k}\right) = -\frac{-ae^{-a\theta}}{k} \qquad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{a^2 e^{-a\theta}}{k} = \frac{a^2}{r}$$

$$F(r) = -\frac{l^2}{\mu r^2}\left(\frac{a^2}{r} + \frac{1}{r}\right)$$

$$= -\frac{l^2}{\mu r^3}(a^2+1)$$

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2a\theta}}, \quad e^{2a\theta} d\theta = \frac{l}{\mu k^2} dt, \quad \frac{1}{2a} e^{2a\theta} = \frac{l}{\mu k^2} t + C'$$

$$e^{2a\theta} = \frac{2alt}{\mu k^2} + C, \quad \theta(t) = \frac{1}{2a} \ln\left(\frac{2al}{\mu k^2} t + C\right)$$

$$\Downarrow$$

$$\frac{r^2}{k^2} = \frac{2alt}{\mu k^2} + C, \quad r(t) = \sqrt{\frac{2al}{\mu} t + k^2 C}$$

$$U(r) = -\int F dr = \frac{l^2}{\mu}(a^2+1)\int r^{-3} dr = -\frac{l^2(a^2+1)}{2\mu}\frac{1}{r^2} \quad \text{let } U(\infty) = 0$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{l}{\mu r^2} \qquad \frac{dr}{dt} = \frac{dr}{d\theta}\frac{l}{\mu r^2} = ake^{a\theta}\frac{l}{\mu r^2} = \frac{al}{\mu r}$$

$$E = \frac{1}{2}\mu\left(\frac{al}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2(a^2+1)}{2\mu r^2}$$

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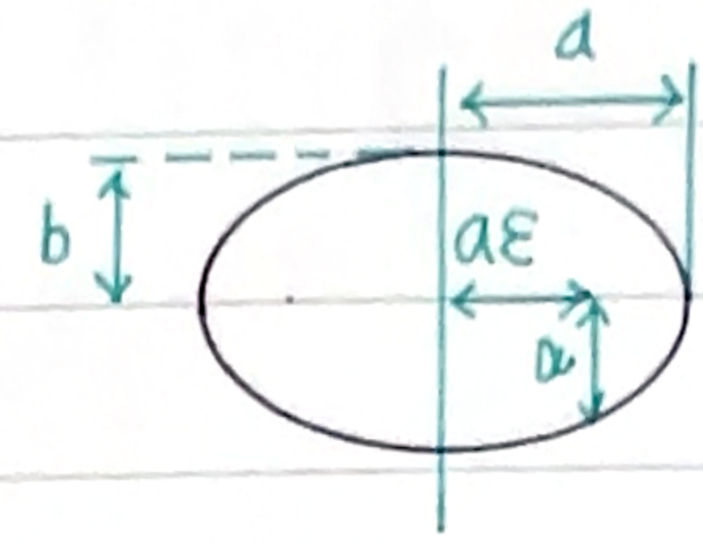
half major axes $a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}$

half minor axes $b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$



$r_{min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}$ $\quad r_{max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$

$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{l}{2\mu}$, $dt = \frac{2\mu}{l}dA$

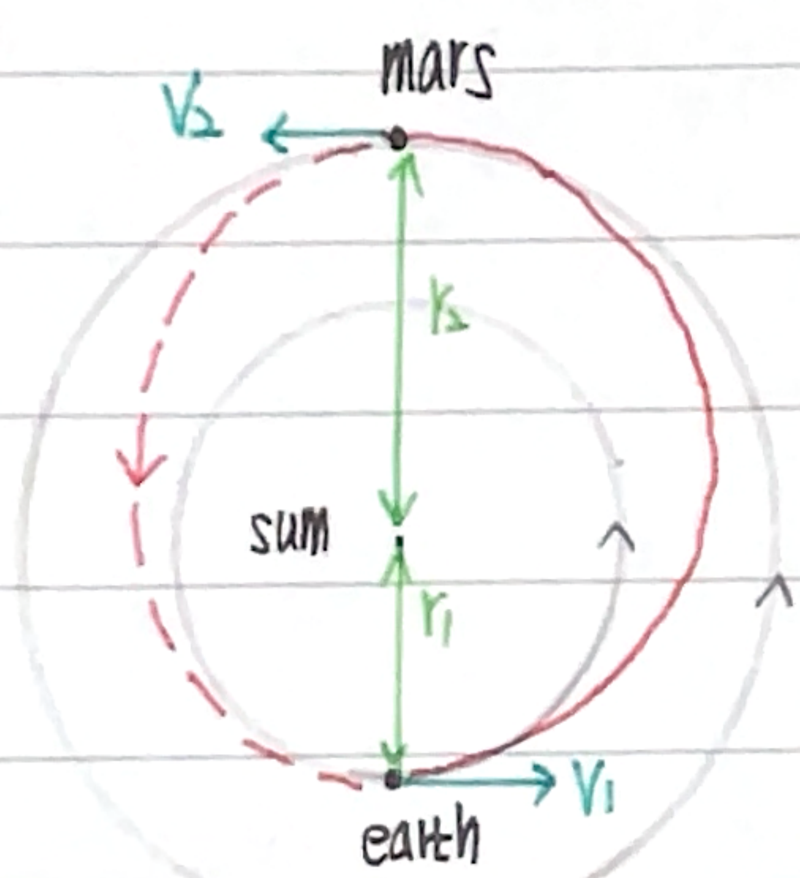
$\int_0^{\tau} dt = \frac{2\mu}{l}\int_0^{A} dA$, $\tau = \frac{2\mu}{l}A = \frac{2\mu}{l}\pi ab = \frac{2\mu}{l}\pi\frac{k}{2|E|}\frac{l}{\sqrt{2\mu|E|}}$

$= \pi k\sqrt{\frac{\mu}{2}}|E|^{-\frac{3}{2}} = 2\pi\sqrt{\frac{\mu}{k}}a^{\frac{3}{2}}$

$\tau^2 = \frac{4\pi^2\mu}{k}a^3$ Kepler's third law

for $F(r) = -\frac{Gm_1m_2}{r^2} = -\frac{k}{r^2}$ $\quad = \frac{4\pi^2a^3}{G(m_1+m_2)} \simeq \frac{4\pi^2a^3}{Gm_2}$ $\quad m_1 \ll m_2$

Orbital Dynamics



$E_t = -\frac{k}{2r_1} = \frac{1}{2}mV_1^2 - \frac{k}{r_1}$, $V_1 = \sqrt{\frac{k}{mr_1}}$

for ellipse

$2a = r_1 + r_2$, $E = -\frac{k}{r_1+r_2} = \frac{1}{2}mV_{t1}^2 - \frac{k}{r_1}$

$V_{t1} = \sqrt{\frac{2k}{mr_1}\frac{r_2}{r_1+r_2}}$

$\Delta V_1 = V_{t1} - V_1 = \sqrt{\frac{2k}{mr_1}\frac{r_2}{r_1+r_2}} - \sqrt{\frac{k}{mr_1}}$

similarly $V_2 = \sqrt{\frac{k}{mr_2}}$ $\quad V_{t2} = \sqrt{\frac{2k}{mr_2}\frac{r_1}{r_1+r_2}}$