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Continuous System : Waves

Continuous String as a Limiting Case of the Loaded String

wish to allow the number of masses to become infinite $n \to \infty$

let $m \to 0$ $d \to 0$ $\quad \frac{m}{d} = \text{constant} \equiv \rho$

$n \to \infty$ $\quad d \to 0$ $\quad (n+1)d = L$

$$q_j(t) = \sum_r \underbrace{\eta_r(t)}_{B_r e^{i\omega_r t}} \sin\Big(\underbrace{\frac{j r \pi}{n+1}}_{= r\pi \frac{jd}{(n+1)d} = r\pi \frac{x}{L}}\Big) \Rightarrow q(x,t) = \sum_r \eta_r(t) \sin\Big(\frac{r\pi x}{L}\Big) = \sum_r B_r e^{i\omega_r t} \sin\Big(\frac{r\pi x}{L}\Big)$$

$$q(x,0) = \sum_r \mu_r \sin\Big(\frac{r\pi x}{L}\Big) \qquad \frac{dq(x,0)}{dt} = -\sum_r \omega_r \nu_r \sin\frac{r\pi x}{L}$$

$\times \sin\frac{s\pi x}{L}$ $\downarrow$ integrate

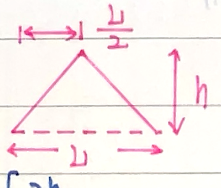
$$\int_0^L \sin\Big(\frac{r\pi x}{L}\Big) \sin\Big(\frac{s\pi x}{L}\Big)\, dx = \frac{L}{2}\delta_{rs}$$

$$\mu_r = \frac{2}{L}\int_0^L q(x,0) \sin\Big(\frac{r\pi x}{L}\Big) dx \qquad \nu_r = -\frac{2}{\omega_r L}\int_0^L \frac{dq(x,0)}{dt} \sin\Big(\frac{r\pi x}{L}\Big) dx$$

$$\omega_r = 2\sqrt{\frac{\tau}{md}} \sin\Big[\frac{r\pi}{2(n+1)}\Big] = \frac{2}{d}\sqrt{\frac{\tau}{\rho}} \sin\Big(\frac{rnd}{2L}\Big)$$

when $d \to 0$ $\quad \omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}}$

question



$$q(x,0) = \begin{cases} \frac{2h}{L}x & 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & \frac{L}{2} \le x \le L \end{cases} \qquad \frac{dq(x,0)}{dt} = 0$$

becasue relesed from rest, all the ν_r vanish

$$\mu_r = \frac{4h}{L^2}\int_0^{\frac{L}{2}} x \sin\Big(\frac{r\pi x}{L}\Big) dx + \frac{4h}{L^2}\int_{\frac{L}{2}}^L (L-x) \sin\Big(\frac{r\pi x}{L}\Big) dx$$

$$= \frac{8h}{r^2\pi^2} \sin\frac{r\pi}{2} \qquad \begin{cases} 0 & r \text{ even} \\ \frac{8h}{r^2\pi^2}(-1)^{\frac{1}{2}(r-1)} & r \text{ odd} \end{cases}$$

$$q(x,t) = \frac{8h}{\pi^2} \quad \sin\Big(\frac{\pi x}{L}\Big)$$

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Energy of a Vibrating String

$$E_k = \frac{1}{2}\rho\int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx = \frac{1}{2}\rho\int_0^L \left[\sum_r \frac{d\eta_r}{dt}\sin\left(\frac{r\pi x}{L}\right)\right]^2 dx$$

$$= \frac{1}{2}\rho\sum_{r,s}\frac{d\eta_r}{dt}\frac{d\eta_s}{dt}\underbrace{\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{rs}}$$

$$= \frac{\rho L}{4}\sum_{rs}\frac{d\eta_r}{dt}\frac{d\eta_s}{dt}\delta_{rs} = \frac{\rho L}{4}\sum_r\left(\frac{d\eta_r}{dt}\right)^2$$

$$\mathrm{Re}\left[\left(\frac{d\eta_r}{dt}\right)^2\right] = \mathrm{Re}\left\{\frac{d}{dt}\left[(\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\right]\right\}^2$$

$$= (-\omega_r\mu_r\sin\omega_r t - \omega_r\nu_r\cos\omega_r t)^2$$

$$\Rightarrow E_k = \frac{\rho L}{4}\sum_r \omega_r^2(\mu_r\sin\omega_r t + \nu_r\cos\omega_r t)^2$$

$$U = \frac{1}{2}\frac{\tau}{d}\sum_j (q_{j-1} - q_j)^2 = \frac{1}{2}\tau\sum_j\left(\frac{q_{j-1}-q_j}{d}\right)^2 d \quad \text{and } d\to 0$$

$$= \frac{1}{2}\tau\int_0^L \underbrace{\left(\frac{\partial q}{\partial x}\right)}_{=\sum_r \frac{r\pi}{L}\eta_r\cos\left(\frac{r\pi x}{L}\right)}^2 dx = \frac{1}{2}\tau\int_0^L\left[\sum_r\frac{r\pi}{L}\eta_r\cos\left(\frac{r\pi x}{L}\right)\right]^2 dx$$

$$= \frac{\tau}{2}\sum_{rs}\frac{r\pi}{L}\frac{s\pi}{L}\eta_r\eta_s\int_0^L\cos\left(\frac{r\pi x}{L}\right)\cos\left(\frac{s\pi x}{L}\right)dx$$

$$= \frac{\tau}{2}\sum_{rs}\frac{r\pi}{L}\frac{s\pi}{L}\eta_r\eta_s\frac{L}{2}\delta_{rs}$$

$$= \frac{\tau}{2}\sum_r\frac{r^2\pi^2}{L^2}\frac{L}{2}\eta_r^2 = \frac{\rho L}{4}\sum_r\omega_r^2\eta_r^2$$

$$= \frac{\rho L}{4}\sum_r\omega_r^2(\mu_r\cos\omega_r t - \nu_r\sin\omega_r t)^2$$

$$E = E_k + U = \frac{\rho L}{4}\sum_r\omega_r^2(\mu_r^2+\nu_r^2) = \frac{\rho L}{4}\sum_r\omega_r^2|B_r|^2$$

the average over one complete period of the fundamental vibration $r=1$

$$\langle E_k\rangle = \frac{\rho L}{4}\sum_r\omega_r^2\langle(\mu_r\underbrace{\sin\omega_r t}_{\frac{1}{2}} + \nu_r\underbrace{\cos\omega_r t}_{\frac{1}{2}})^2\rangle = \frac{\rho L}{8}\sum_r\omega_r^2(\mu_r^2+\nu_r^2)$$

$$\langle U\rangle = \frac{\rho L}{4}\sum_r\omega_r^2\langle(\mu_r\cos\omega_r t - \nu_r\sin\omega_r t)^2\rangle = \frac{\rho L}{8}\sum_r\omega_r^2(\mu_r^2+\nu_r^2)$$

$$\Rightarrow \langle T\rangle = \langle U\rangle$$

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General Solutions of the Wave Equation

$\Psi = \Psi(x,t)$ time-dependent wave equation

$\psi = \psi(x)$ time-independent wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 \Psi}{\partial t^2} = 0 , \quad \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = 0 \qquad \begin{aligned} \xi &\equiv x + vt \\ \eta &\equiv x - vt \end{aligned}$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial \Psi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial^2 \Psi}{\partial \xi^2} + 2\frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2}$$

and

$$\frac{1}{v}\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta} , \quad \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial \Psi}{\partial t}\right) = \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi}{\partial \eta}\right)$$

$$= \frac{\partial^2 \Psi}{\partial \xi^2} - 2\frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{\partial^2 \Psi}{\partial \eta^2}$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial \xi \partial \eta} = 0$$

$$\Psi = f(\xi) + g(\eta) = f(x+vt) + g(x-vt)$$

as time $t=0$ $\quad q(x,0) = f(x) + g(x)$