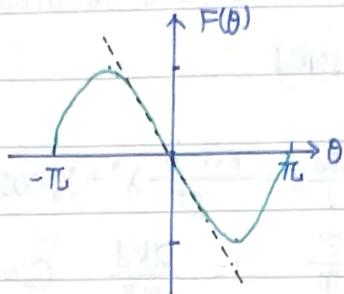
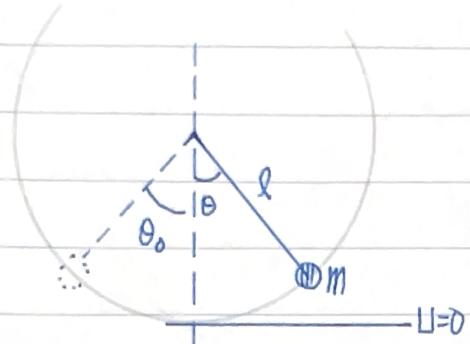


Plane Pendulum



$$F(\theta) = -mg \sin \theta$$

$$I \frac{d^2\theta}{dt^2} = \ell F \quad \text{where } I = ml^2$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \omega_0^2 \sin \theta = 0 \quad \text{where } \omega_0^2 = \frac{g}{l} \quad \text{if the amplitude of the motion is small}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \omega_0^2 \theta = 0 \quad T \cong 2\pi \sqrt{\frac{l}{g}}$$

$$\begin{cases} E_k = \frac{1}{2} I \omega^2 = \frac{1}{2} ml^2 \left(\frac{d\theta}{dt} \right)^2 \\ U = mgl(1 - \cos \theta) \end{cases} \quad \text{if we let } \theta = \theta_0 \text{ at the highest point}$$

$$\begin{aligned} E(\theta = \theta_0) &= 0 \\ U(\theta = \theta_0) &= E_k = mgl(1 - \cos \theta_0) \\ &= 2mgl \sin^2 \frac{\theta_0}{2} \end{aligned}$$

$$\Rightarrow \frac{1}{2} ml^2 \left(\frac{d\theta}{dt} \right)^2 = 2mgl \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right), \frac{d\theta}{dt} = 2\sqrt{\frac{g}{l}} \sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}$$

$$dt = \frac{1}{2} \sqrt{\frac{l}{g}} \sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}} d\theta \quad T = 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}$$

$$F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1+k^2 \sin^2 \theta}} \quad k^2 < 1 \quad \text{or if } z = \sin \theta \quad F(k, x) = \int_0^x \frac{dz}{\sqrt{1-z^2} \sqrt{1-k^2 z^2}}$$

$$z = \frac{\sin \frac{\theta_0}{2}}{\sin \frac{\theta}{2}} \quad k = \sin \frac{\theta_0}{2}, \quad dz = \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta_0}{2}} d\theta \quad d\theta = \frac{\sqrt{1-k^2 z^2}}{2k} dz$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}}$$

$$\text{for } |\theta_0| < \pi \quad \sin \frac{\theta_0}{2} = k \quad -1 < k < 1$$

$$(1-k^2 z^2)^{-\frac{1}{2}} = 1 + \frac{k^2 z^2}{2} + \frac{3k^4 z^4}{8} + \dots$$

$$\begin{aligned}
 T &= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dz}{\sqrt{1-z^2}} \left(1 + \frac{k^2 z^2}{2} + \frac{3k^4 z^4}{8} + \dots\right) \\
 &= 4\sqrt{\frac{l}{g}} \left(\frac{\pi}{2} + \frac{k^2}{2} \frac{1}{2} \frac{\pi}{2} + \frac{3k^4}{8} \frac{3}{8} \frac{\pi}{2} + \dots\right) \\
 &= 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots\right) \\
 &\cong 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right)
 \end{aligned}$$

if $|k| \leq 1$, $k = \sin \frac{\theta_0}{2}$
then $k \cong \frac{\theta_0}{2} - \frac{\theta_0^3}{48}$

$$\frac{d\theta}{dt} = 2\sqrt{\frac{g}{l}} \sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}} \quad \text{if } \theta \text{ and } \theta_0 \text{ are small, } \left(\sqrt{\frac{l}{g}} \frac{d\theta}{dt}\right)^2 + \theta^2 \cong \theta_0^2$$

if the total energy = E_0 , $E = 2mg l \sin^2 \frac{\theta_0}{2} \Rightarrow \theta_0 = \pm \pi$

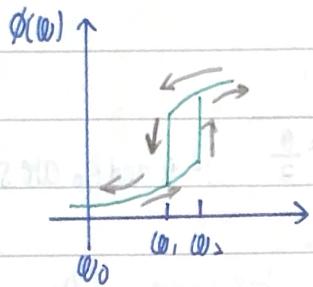
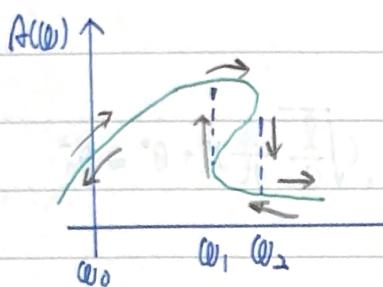
$$\frac{d\theta}{dt} = \pm 2\sqrt{\frac{g}{l}} \cos \frac{\theta}{2}$$

Jumps, Hysteresis, and Phase Lags

$$m \frac{d^2X}{dt^2} = -r \frac{dX}{dt} - kX + F_0 \cos \omega t$$

$$X(t) = A(\omega) \cos [\omega t - \phi(\omega)] \quad A(\omega) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (r\omega)^2}} \quad \tan[\phi(\omega)] = \frac{r\omega}{k-m\omega^2}$$

$$k(X) = (1+\beta X^2) k_0$$



We consider a simpler dependence of k

