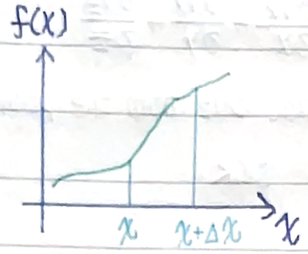
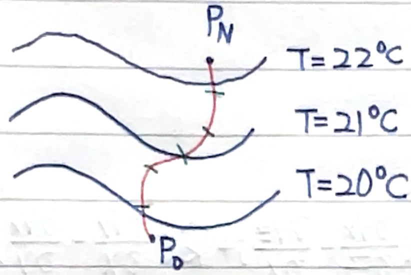
Vector Analysis

Differential Calculus



$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = f'(x)$$

$$df(x) = f'(x)dx$$



$$T(x_N, y_N, z_N) - T(x_0, y_0, z_0)$$

$$= \sum_{i=1}^{N} T(x_i, y_i, z_i) - T(x_{i-1}, y_{i-1}, z_{i-1}) \Rightarrow$$

$$\boxed{\begin{aligned} &= \frac{\partial T}{\partial x}\Big|_{(x_i - x_{i-1})} + \frac{\partial T}{\partial y}\Big|_{(y_i - y_{i-1})} + \frac{\partial T}{\partial z}\Big|_{(z_i - z_{i-1})} \\ &= \frac{\partial T}{\partial x}\vec{e}_x + \frac{\partial T}{\partial y}\vec{e}_y + \frac{\partial T}{\partial z}\vec{e}_z \equiv \nabla T \\ &(x_i - x_{i-1})\vec{e}_x + (y_i - y_{i-1})\vec{e}_y + (z_i - z_{i-1})\vec{e}_z = \Delta\vec{S}_i \end{aligned}}$$

$$\lim_{N\to\infty} = \lim_{\substack{N\to\infty \\ |\Delta \vec{S}_i| \to \infty}} \sum_{i=1}^{N} (\nabla T)_{P_i} \cdot \Delta \vec{S}_i$$

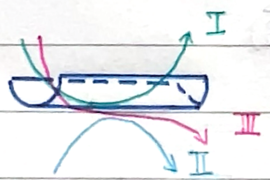
$$= \int_{P_0 P_N} (\nabla T)\cdot d\vec{S} = dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\cdot\underbrace{(dx\hat{x} + dy\hat{y} + dz\hat{z})}_{d\vec{l}}$$

$$= \nabla T \cdot d\vec{l}$$

Vector operator

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\nabla T = 0$ stationary point of the function $T(x,y,z)$: maximum, minimum, saddle

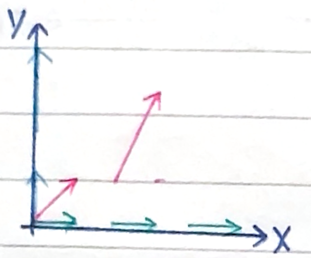


an ordinary vector
- by a scalar a: $a\vec{A}$
- by a vector $\vec{B}$: $\vec{A}\cdot\vec{B}$, $\vec{A}\times\vec{B}$

the operator
- on a scalar function T: ∇T gradient
- on a vector function $\vec{V}$: $\nabla\cdot\vec{V}$ divergence
- $\nabla\times\vec{V}$ curl

the divergence

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right) = \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$$

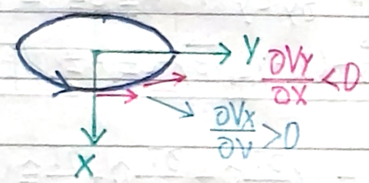


how much the vector $\vec{V}$ spreads out from the point

$\nabla\cdot\vec{E}>0$ source $\nabla\cdot\vec{E}<0$ sink

the curl

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}\left(\frac{\partial V_z}{\partial V_y}-\frac{\partial V_y}{\partial z}\right)+\hat{j}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\hat{k}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)$$



how much the vector $\vec{V}$ swirls around the point in question

the product rules

$\nabla[f(x)+g(x)] = \nabla f(x)+\nabla g(x)$ $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A}+\nabla\cdot\vec{B}$

$\nabla[kf(x)] = k\nabla f(x)$ $\nabla\cdot(k\vec{A}) = k(\nabla\cdot\vec{A}) = (\nabla k)\cdot\vec{A}+f(\nabla\cdot\vec{A})$

$\nabla[f(x)g(x)] = f(x)\nabla g(x)+g(x)\cdot\nabla f(x)$

$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})+(\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A}$

$\nabla\cdot(\vec{A}\times\vec{B}) = B\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$

$\nabla\times(\vec{A}+\vec{B}) = \nabla\times A+\vec{\nabla}\times\vec{B}$

$\nabla\times(u\vec{A}) = (\nabla u)\times\vec{A}+u(\nabla\times\vec{A})$

$\nabla\times(\vec{A}\times\vec{B}) = (\nabla\cdot\vec{B})\vec{A}-(\nabla\cdot\vec{A})\vec{B}+(\vec{B}\cdot\nabla)\vec{A}-(\vec{A}\cdot\nabla)\vec{B}$

$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f-f\nabla g}{g^2}$ $\nabla\times\left(\frac{\vec{A}}{g}\right) = \frac{g(\nabla\times\vec{A})+\vec{A}\times(\nabla g)}{g^2}$

$\nabla\cdot\left(\frac{\vec{A}}{g}\right) = \frac{g(\nabla\cdot\vec{A})-\vec{A}\cdot(\nabla g)}{g^2}$

prove $\nabla[f(x)g(x)]$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{z}\right)[f(x)g(x)]=\frac{\partial(fg)}{\partial x}\hat{i}+\frac{\partial(fg)}{\partial y}\hat{j}+\frac{\partial(fg)}{\partial z}\hat{k}$$

$$=f\frac{\partial g}{\partial x}\hat{i}+g\frac{\partial f}{\partial x}\hat{i}+f\frac{\partial g}{\partial y}\hat{j}+g\frac{\partial f}{\partial y}\hat{j}+f\frac{\partial g}{\partial z}\hat{k}+g\frac{\partial f}{\partial z}\hat{k}$$

$$=f\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)+g\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)$$

$$=f(x)\nabla g(x)+g(x)\nabla f(x)$$

prove $\nabla\cdot(\vec{A}\cdot\vec{B})$

$$\Rightarrow \nabla(A_xB_x+A_yB_y+A_zB_z)=\frac{\partial(A_xB_x+A_yB_y+A_zB_z)}{\partial x}\hat{i}+\frac{\partial(A_xB_x+A_yB_y+A_zB_z)}{\partial y}\hat{j}+\cdots$$

$$=\left(A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_y}{\partial x}+B_y\frac{\partial A_y}{\partial x}+A_z\frac{\partial B_z}{\partial x}+B_z\frac{\partial A_x}{\partial x}\right)\hat{i}+$$

$$A_x\frac{\partial B_x}{\partial y}+B_x\frac{\partial A_x}{\partial y}+A_y\frac{\partial B_y}{\partial y}+B_y\frac{\partial A_y}{\partial y}+A_z\frac{\partial B_z}{\partial y}+B_z\frac{\partial A_x}{\partial y})\hat{j}+$$

$$A_x\frac{\partial B_x}{\partial z}+B_x\frac{\partial A_x}{\partial z}+A_y\frac{\partial B_y}{\partial z}+B_y\frac{\partial A_y}{\partial z}+A_z\frac{\partial B_z}{\partial z}+B_z\frac{\partial A_x}{\partial z})\hat{k}$$

▨ $+\left(-B_y\frac{\partial A_x}{\partial y}-B_z\frac{\partial A_x}{\partial z}\right)\hat{i}+\left(-B_z\frac{\partial A_y}{\partial z}-B_x\frac{\partial A_y}{\partial x}\right)\hat{j}+\left(-B_x\frac{\partial A_z}{\partial x}-B_y\frac{\partial A_z}{\partial y}\right)\hat{k}=\vec{B}\times(\nabla\times\vec{A})$

▨ $+\left(-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}\right)\hat{i}+\left(-A_z\frac{\partial B_y}{\partial z}-A_x\frac{\partial B_y}{\partial x}\right)\hat{j}+\left(-A_x\frac{\partial B_z}{\partial x}-A_y\frac{\partial B_z}{\partial y}\right)\hat{k}=\vec{A}\times(\nabla\times\vec{B})$

▨ $=(\vec{B}\cdot\nabla)\vec{A}$ ▨ $=(\vec{A}\cdot\nabla)\vec{B}$

prove $\nabla\cdot(\vec{A}\times\vec{B})$

$$\Rightarrow\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left[(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}\right]$$

$$=\frac{\partial(A_yB_z-A_zB_y)}{\partial x}+\frac{\partial(A_zB_x-A_xB_z)}{\partial y}+\frac{\partial(A_xB_y-A_yB_x)}{\partial z}$$

$$=A_y\frac{\partial B_z}{\partial x}+B_z\frac{\partial A_y}{\partial x}-A_z\frac{\partial B_y}{\partial x}-B_y\frac{\partial A_z}{\partial x}+A_z\frac{\partial B_x}{\partial y}+B_x\frac{\partial A_z}{\partial y}-A_x\frac{\partial B_z}{\partial y}-B_z\frac{\partial A_x}{\partial y}+$$

$$A_x\frac{\partial B_y}{\partial z}+B_y\frac{\partial A_x}{\partial z}-A_y\frac{\partial B_x}{\partial z}-B_x\frac{\partial A_y}{\partial z}$$

$$=B_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+B_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+B_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+ \quad \vec{B}\cdot(\nabla\times\vec{A})$$

$$-A_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)-A_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-A_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right) \quad \vec{A}\cdot(\nabla\times\vec{B})$$

prove $\nabla\cdot(f\vec{A})$

$$\Rightarrow\frac{\partial}{\partial x}(fA_x)+\frac{\partial}{\partial y}(fA_y)+\frac{\partial}{\partial z}(fA_z)=\left(\frac{\partial f}{\partial x}A_x+f\frac{\partial A_x}{\partial x}\right)+\left(\frac{\partial f}{\partial y}A_y+f\frac{\partial A_y}{\partial y}\right)+\left(\frac{\partial f}{\partial z}A_z+f\frac{\partial A_z}{\partial z}\right)$$

$$=(\nabla f)\cdot\vec{A}+f(\nabla\cdot\vec{A})$$

prove $\nabla\times(\vec{A}\times\vec{B})$

$$\Rightarrow\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times(\vec{A}\times\vec{B})=(\hat{i}+\hat{j}+\hat{k})\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial y}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial y}+\cdots\right.$$

$$=(\hat{i}+\hat{j}+\hat{k})\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{A}}{\partial z}\right)\times\vec{B}+(\hat{i}+\hat{j}+\hat{k})\times\vec{A}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{A}}{\partial z}\right) \quad \vec{a}\times(\vec{b}\times\vec{c})$$

$$=\left[(\hat{i}+\hat{j}+\hat{k})\cdot\vec{B}\right]\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{A}}{\partial z}\right)-(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{A}}{\partial z}\right)\cdot\vec{B}+\cdots \quad =(\vec{a}\cdot\vec{c})\cdot\vec{b}-(\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$=(\nabla\cdot\vec{B})\cdot\vec{A}-(\nabla\cdot\vec{A})\cdot\vec{B}+(\vec{B}\cdot\nabla)\cdot\vec{A}-(\nabla\cdot\vec{B})\cdot\vec{A}$$