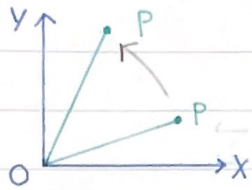
Rotation of Rigid Bodies and Dynamics of Rotational Motion

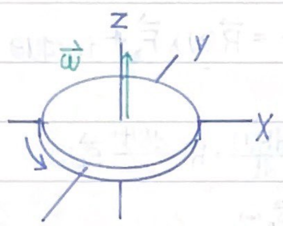
Kinematics

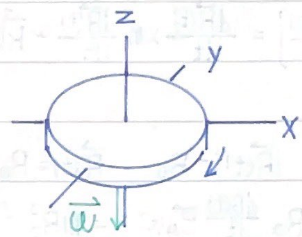
angular and acceleration



average angular velocity $\omega_{av\text{-}z} = \frac{\theta_2-\theta_1}{t_2-t_1} = \frac{\Delta\theta}{\Delta t}$

instantaneous angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$





average angular acceleration $\alpha_{av\text{-}z} = \frac{\omega_{2z}-\omega_{1z}}{t_2-t_1} = \frac{\Delta\omega_z}{\Delta t}$

instaneous angular acceleration $\alpha_z = \lim_{\Delta t\to 0}\frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt}$

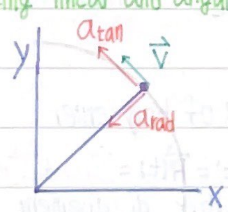
$= \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$

rotation with constant angular acceleration

$\omega_z = \omega_{0z} + \alpha_z t$ $\quad\quad$ $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_z t^2$

$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega_{0z})t$ $\quad\quad$ $\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta-\theta_0)$

relating linear and angular kinematics



$S = r\theta$, $\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$, $v = r\omega$

$a_{tan} = \frac{d\vec{v}}{dt} = r\frac{d\omega}{dt} = r\alpha$ speed not velocity

$a_{rad} = \frac{v^2}{r} = \omega^2 r$

$\alpha = \sqrt{\alpha_{tan}^2 + \alpha_{rad}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r^2\sqrt{\alpha^2+\omega^4}$

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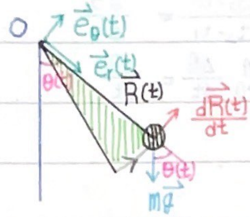
Angular Momentum

case I one particle



area $= \frac{1}{2}\vec{R}(t)\times\frac{d\vec{R}(t)}{dt}$

$\times 2m$ $\quad m\vec{R}(t)\times\frac{d\vec{R}(t)}{dt} = \vec{R}(t)\times m\vec{v}(t) = \vec{L}$

$$\frac{d}{dt}\left[\vec{R}(t)\times m\frac{d\vec{R}(t)}{dt}\right] = \frac{d\vec{R}(t)}{dt}\times m\frac{d\vec{R}(t)}{dt} + \vec{R}(t)\times\frac{d\,m\vec{v}(t)}{dt} = \vec{R}(t)\times\vec{F} \leftarrow \text{torque}$$

$\vec{F} = \vec{F_T} + m\vec{g}$ $\quad |\vec{R}(t)| = R_0$ $\quad \vec{R}(t) = R_0\vec{e_r}(t)$ $\quad \frac{d\vec{R}(t)}{dt} = R_0\frac{d\theta(t)}{dt}\vec{e_\theta}(t)$

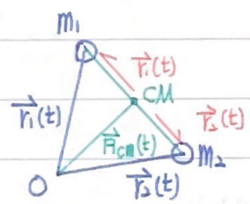
$$\vec{L} = mR_0\vec{e_r}(t)\times R_0\frac{d\theta(t)}{dt}\vec{e_\theta}(t) = mR_0^2\frac{d\theta(t)}{dt}\vec{e_r}(t)\times\vec{e_\theta}(t)$$

$$= mR_0^2\frac{d\theta(t)}{dt}\vec{e_z}$$

$$\frac{d\vec{L}}{dt} = mR_0^2\frac{d^2\theta(t)}{dt^2}\vec{e_z} = -mgR_0\sin\theta\,\vec{e_z}$$

$$\vec{R}(t)\times\vec{F} = \vec{R}(t)\times(\vec{F_T} + m\vec{g}) = \vec{R}(t)\times m\vec{g}$$

case II two particles



$$\vec{R}_{cm}(t) = \frac{m_1\vec{r_1}(t) + m_2\vec{r_2}(t)}{m_1+m_2}$$

$$\vec{r_1}'(t) = \vec{r_1}(t) - \vec{R}_{cm}(t) = \frac{m_2(\vec{r_1}-\vec{r_2})}{m_1+m_2} = \frac{m_2}{m_1+m_2}\vec{R}(t)$$

$$\vec{r_2}'(t) = \vec{r_2}(t) - \vec{R}_{cm}(t) = \frac{m_1(\vec{r_2}-\vec{r_1})}{m_1+m_2} = -\frac{m_1}{m_1+m_2}\vec{R}(t)$$

$$\Rightarrow m_1\vec{r_1}'(t) + m_2\vec{r_2}'(t) = \vec{0}$$

if there is no external force $\frac{d\vec{R}_{cm}(t)}{dt}$ = constane vector → interia of barycenter

momentum of barycenter $m_1\frac{d\vec{r_1}'(t)}{dt} + m_2\frac{d\vec{r_2}'(t)}{dt} = \vec{0}$ $\quad \vec{r_1}(t) - \vec{r_2}(t) = \vec{R}(t) = \vec{r_1}'(t) + \vec{r_2}'(t)$

relative discapement

$\vec{L_1} = m_1\vec{r_1}'(t)\frac{d\vec{r_1}'(t)}{dt}$ $\quad \vec{L_2} = m_2\vec{r_2}'(t)\frac{d\vec{r_2}'(t)}{dt}$

$$\vec{L_1}+\vec{L_2} = \frac{m_1m_2}{m_1+m_2}(\vec{r_1}(t)-\vec{r_2}(t))\times\frac{m_2}{m_1+m_2}\frac{d}{dt}\left[\vec{r_1}(t)-\vec{r_2}(t)\right] + \frac{m_2m_1}{m_1+m_2}(\vec{r_1}(t)-\vec{r_2}(t))\times\frac{m_1}{m_1+m_2}\frac{d}{dt}\left[\vec{r_2}(t)-\vec{r_1}(t)\right]$$

$$= \frac{m_1m_2}{m_1+m_2}(\vec{r_1}(t)-\vec{r_2}(t))\times\left[\frac{m_2}{m_1+m_2}\frac{d}{dt}(\vec{r_1}(t)-\vec{r_2}(t)) + \frac{m_1}{m_1+m_2}\frac{d}{dt}(\vec{r_1}(t)-\vec{r_2}(t))\right]$$

$$= \frac{m_1m_2}{m_1+m_2}(\vec{r_1}(t)-\vec{r_2}(t))\times\frac{d}{dt}(\vec{r_1}(t)-\vec{r_2}(t))\underbrace{\left[\frac{m_2}{m_1+m_2}+\frac{m_1}{m_1+m_2}\right]}_{1}$$

$\frac{1}{\mu}$, μ reduced mass $= \boxed{\frac{m_1m_2}{m_1+m_2}}\left[\vec{R}(t)\times\frac{d\vec{R}(t)}{dt}\right]$ $\Rightarrow \vec{R}(t)\perp\vec{L}$ $\quad \frac{d\vec{R}(t)}{dt}\perp\vec{L}$

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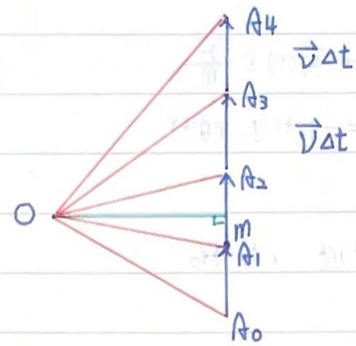
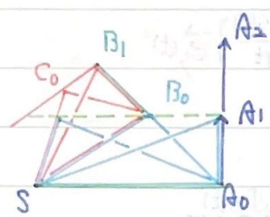
Area Law and Angular Momentum

from Kepler second law $r^2(t)\dfrac{d\theta(t)}{dt} = 2A$

the motion of the particle which do not under force



$\Delta OA_0A_1 = \Delta OA_1A_2 = \Delta OA_2A_3 = \cdots$



$\Delta SA_2A_1 = SA_0A_1 = \Delta SA_0B_0 \quad \because \overline{A_1B_0} \,//\, \overline{SA_0}$

$= \Delta SB_0B_1 \quad \because$ inertia

$= \Delta SB_0C_0 \quad \because \overline{C_0B_1} \,//\, \overline{SB_0}$

area law : force is the direction of the heart ,

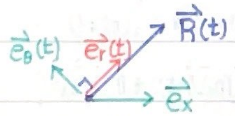
$$\frac{1}{2}\vec{R}(t)\times\vec{V}(t) = \frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}\left[\vec{R}(t)\times\vec{V}(t)\right] = \frac{d\vec{R}(t)}{dt}\times\vec{V}(t) + \vec{R}(t)\times\frac{d\vec{V}(t)}{dt} = \vec{R}(t)\times\frac{d\vec{V}(t)}{dt} = \vec{R}(t)\times\vec{a}(t) = \vec{0}$$

$$\Updownarrow$$

$$\vec{a}(t) \,//\, \vec{R}(t) \,//\, \vec{F}$$

$2A\vec{e}_z = \vec{R}(t)\times\vec{V}(t) \Rightarrow \vec{R}(t)\perp 2A\vec{e}_z$, $\vec{V}(t)\perp 2A\vec{e}_z \Rightarrow$ the trajectory of motion is a plane



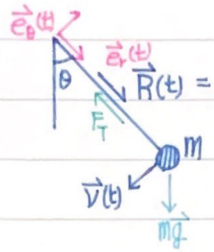
$\vec{R}(t) = r(t)\vec{e}_r(t)$

$\vec{V}(t) = \dfrac{dr(t)}{dt}\vec{e}_r(t) + r(t)\dfrac{d\theta(t)}{dt}\vec{e}_\theta(t) + 0\vec{e}_z$

$\vec{R}(t)\times\vec{V}(t) = r^2(t)\dfrac{d\theta(t)}{dt}\vec{e}_r(t)\times\vec{e}_\theta(t) = r^2(t)\dfrac{d\theta(t)}{dt}\vec{e}_z$

$\dfrac{d}{dt}\left[\vec{R}(t)\times\vec{V}(t)\right] = \dfrac{d}{dt}\left[r^2(t)\dfrac{d\theta(t)}{dt}\right]\vec{e}_z = 0$

the system do not obey area law



$\vec{R}(t) = r(t)\vec{e}_r(t) = R_0\,\vec{e}_r(t)$

$$\frac{d}{dt}\left[\vec{R}(t)\times\vec{V}(t)\right] = \frac{d\vec{R}(t)}{dt}\times\vec{V}(t) + \vec{R}(t)\times\frac{d\vec{V}(t)}{dt} = \vec{R}(t)\times\frac{\vec{F}}{m}$$

$$= \vec{R}(t)\times\frac{\vec{F}_T + m\vec{g}}{m} = \vec{R}(t)\times\vec{g} = -r(t)\,g\sin\theta(t)$$

$$\frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right] = -r(t)\,g\sin\theta(t) \qquad r(t) \text{ is a constant} \quad r(t)=R_0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{R_0}\sin\theta(t)$$

if θ is very small, $\frac{d^2\theta}{dt^2} = -\frac{g}{R_0}\theta(t)$, $\theta(t) = A\sin(\omega t+\phi)$

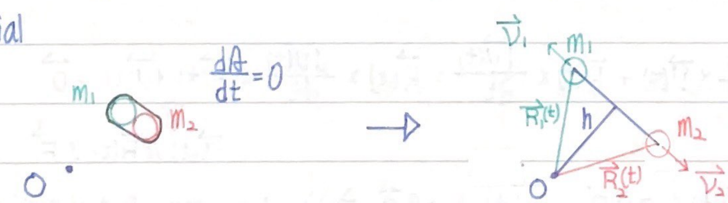
$$\vec{F} = m\vec{a} = m\left[\underbrace{\frac{d^2r(t)}{dt^2}}_{0} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right]\vec{e}_r(t) + m\frac{1}{r(t)}\frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right]\vec{e}_\theta(t)$$

$$\vec{F}_T + m\vec{g} = -mR\left(\frac{d\theta(t)}{dt}\right)^2\vec{e}_r(t) + mR\frac{d^2\theta(t)}{dt^2}\vec{e}_\theta(t)$$

$$\Rightarrow F_T - mg\cos\theta(t) = -mR\left(\frac{d\theta(t)}{dt}\right)^2 \qquad -mg\sin\theta(t) = mR\frac{d^2\theta(t)}{dt^2}$$

extending the area law to more than two particles system

initial



$m_1\vec{V}_1(t) + m_2\vec{V}_2(t) = 0$

$\vec{R}_1(t)\times\vec{V}_1(t) = \vec{h}\times\vec{V}_1(t)$

$\vec{R}_2(t)\times\vec{V}_2(t) = \vec{h}\times\vec{V}_2(t)$

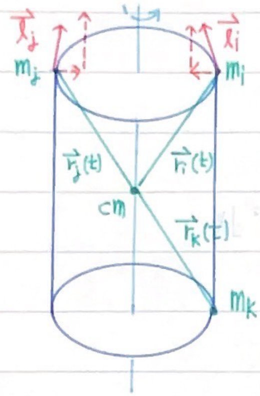
$\vec{R}_1(t)\times\vec{V}_1(t) + \vec{R}_2(t)\times\vec{V}_2(t) = \vec{h}\times\vec{V}_1(t) + \vec{h}\times\vec{V}_2(t) \neq 0$

$\times m$

$m_1\vec{R}_1(t)\times\vec{V}_1(t) + m_2\vec{R}_2(t)\times\vec{V}_2(t) = \vec{h}\times(m_1\vec{V}_1(t) + m_2\vec{V}_2(t))$

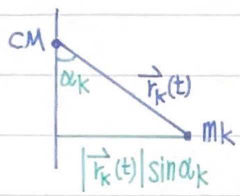
$= \vec{h}\times 0 = 0$

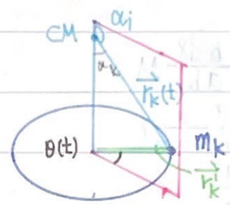
$\Rightarrow$ angular momentum conservation,
以質量調權之面積守恆



$|\vec{r}_i(t)-\vec{r}_j(t)|$ do not change with time

$$\vec{\omega} = \frac{d\theta(t)}{dt}\vec{e}_z$$





rotational velocity $|\vec{r}_k(t)|\sin\alpha_k \frac{d\theta}{dt} = \frac{d\vec{r}_k(t)}{dt}$

$$|\vec{\omega}||\vec{r}_k(t)|\sin\alpha_k = |\vec{\omega}\times\vec{r}_k(t)| = \left|\frac{d\vec{r}_k(t)}{dt}\right|$$

$$\frac{d\vec{r}_k(t)}{dt} = \vec{\omega}\times\vec{r}_k(t)$$

$$\sum_{i=1}^{N} m_i\vec{r}_i(t)\times\frac{d\vec{r}_i(t)}{dt} = \vec{L}_{cm}$$

$$= \sum \underbrace{m_i\vec{r}_i(t)}_{B}\times\underbrace{(\vec{\omega}\times\vec{r}_i(t))}_{C}$$

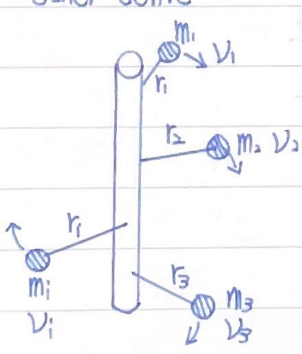
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

$$\vec{L}_{cm}\cdot\frac{\vec{\omega}}{|\vec{\omega}|}\,A = \frac{1}{|\vec{\omega}|}\sum m_i(\vec{\omega}\times\vec{r}_i(t))\cdot(\vec{\omega}\times\vec{r}_i(t)) = \frac{1}{|\vec{\omega}|}\sum m_i|\vec{\omega}|^2|\vec{r}_i(t)|^2\sin^2\alpha_i$$

$$= |\vec{\omega}|\underbrace{\left(\sum m_i|\vec{r}_i(t)|^2\sin^2\alpha_i\right)}_{\text{moment of inertia, } I}$$

$$= \omega\ I$$

other derive



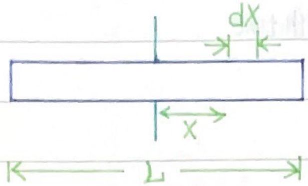
$$K = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$$

$$= \sum\frac{1}{2}m_ir_i^2\omega^2$$

$$= \frac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

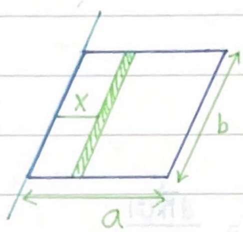
$$= \frac{1}{2}I\omega^2$$

Calculating Moment of Inertia



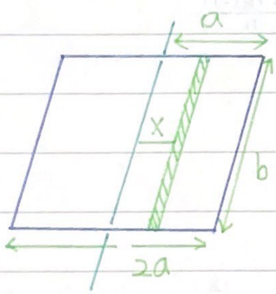
$$dm = \rho\, dx = \frac{M}{L}\, dx$$

$$I = \int r^2 dm$$
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \overset{\rho}{\frac{M}{L}}\, dx = \frac{M}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\, dx$$
$$= \frac{M}{L}\,\frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12}ML^2$$

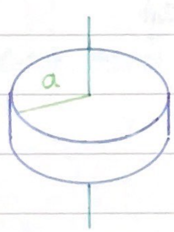


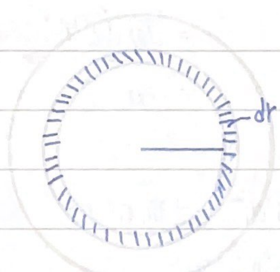
$$dm = \rho\, dx = \frac{M}{ab}\, b\, dx$$

$$I = \int x^2 \frac{b\, dx}{ab} M$$
$$= \frac{M}{a}\int_0^a x^2\, dx$$
$$= \frac{M}{a}\,\frac{a^3}{3}$$
$$= \frac{1}{3}Ma^2$$



$$I = \int_{-a}^{a} x^2 \frac{Mb}{2ab}\, dx$$
$$= \frac{M}{2a}\int_{-a}^{a} x^2\, dx$$
$$= \frac{M}{2a}\,\frac{2a^3}{3}$$
$$= \frac{1}{3}Ma^2$$





$$I = \int r^2 \frac{2\pi r\, dr}{\pi a^2} M$$
$$= \frac{2M}{a^2}\int_0^a r^3\, dr$$
$$= \frac{2M}{a^2}\,\frac{a^4}{4}$$
$$= \frac{1}{2}Ma^2$$

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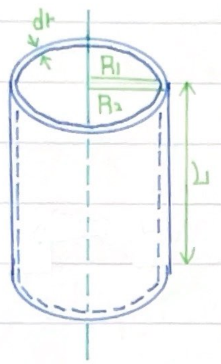


$$I=\int r^2 dm = \int_{R_1}^{R_2} r^2 \underbrace{\rho(2\pi r L dr)}_{M} = 2\pi\rho L\int_{R_1}^{R_2} r^3 dr = \frac{2\pi\rho L}{4}(R_2^4-R_1^4)$$

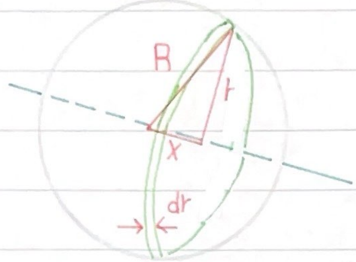
$$\boxed{\begin{array}{l} dm=\rho dV \\ \text{and } V=\pi r^2 L \\ \quad \Rightarrow dV = 2\pi r L dr \end{array}}$$

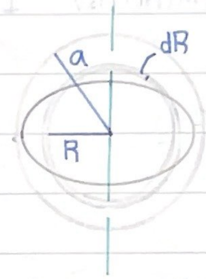
$$= \frac{\pi\rho L}{2}(R_2^2-R_1^2)(R_2^2+R_1^2)$$

since $V=\pi L(R_2^2-R_1^2)$

$M=\rho V$

$$=\frac{1}{2}M(R_1^2+R_2^2)$$





$r=\sqrt{R^2-x^2}$

$dV=\pi r^2 dx = \pi(R^2-x^2)dx$

$dm=\rho\pi(R^2-x^2)dx$

$dI=\frac{1}{2}r^2 dm = \frac{1}{2}(R^2-x^2)\pi\rho(R^2-x^2)dx$

$\quad = \frac{\pi\rho}{2}(R^2-x^2)^2 dx$

$I = 2\,\frac{\pi\rho}{2}\int_0^R (R^2-x^2)dx$

$\quad = \frac{8}{15}\pi\rho R^5 \qquad \rho=\frac{M}{V}=\frac{3M}{4\pi R^3}$

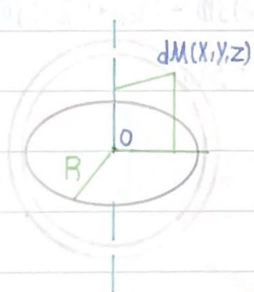
$\quad = \frac{2}{5}MR^2$

$\int \frac{2}{3}R^2 dM\ R \qquad$ and $\ dM=\frac{M}{\frac{4}{3}\pi a^3}4\pi R^2 dR$

$=\frac{2}{3}\frac{M}{\frac{4}{3}\pi a^3}4\pi\int_0^a R^4 dR$

$=\frac{2}{3}\frac{M}{\frac{4}{3}\pi a^3}\frac{a^5}{5}$

$=\frac{2}{5}Ma^2$



$x^2+y^2+z^2=R^2$

$I_z=\int(x^2+y^2)dM$

$I_y=\int(x^2+z^2)dM$

$I_x=\int(y^2+z^2)dM$

$I_x=I_y=I_z=\frac{I_x+I_y+I_z}{3}$

$\frac{1}{3}I = \frac{2}{3}\int(x^2+y^2+z^2)dM$

$\quad = \frac{2}{3}R^2\int dM$

$\quad = \frac{2}{3}MR^2$

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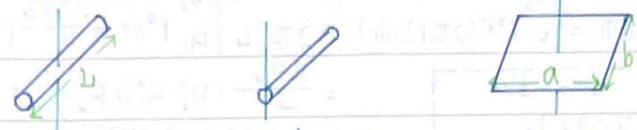
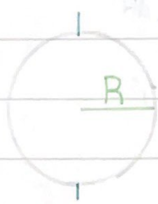
$\frac{1}{3}\times 2\times\frac{M}{2}\times\left(\frac{L}{2}\right)^2=\frac{?}{3}$

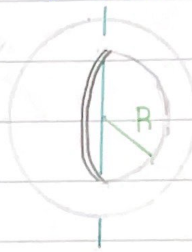


$I=\frac{1}{12}ML^2$ $\quad I=\frac{1}{3}ML^2$ $\quad I=\frac{1}{12}M(a^2+b^2)$

$I=\frac{1}{3}Ma^2$ $\quad I=\frac{1}{2}M(R_1^2+R_2^2)$ $\quad I=\frac{1}{2}MR^2$

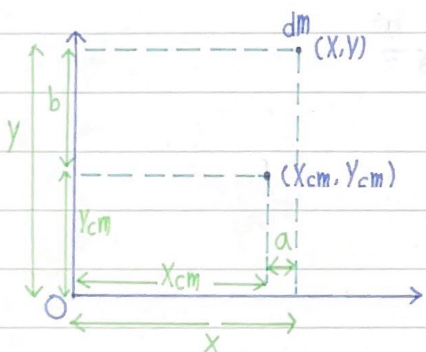
$I=MR^2$





$I=\frac{2}{5}MR^2$ $\quad I=\frac{2}{3}MR^2$

Parallel-Axis Theorem



$$I=\int r^2dm=\int(x^2+y^2)dm$$
$$=\int\left[(a+x_{cm})^2+(b+y_{cm})^2\right]dm$$
$$=\int(a^2+b^2)dm+2x_{cm}\int a\,dm+$$
$$2y_{cm}\int b\,dm+(x_{cm}^2+y_{cm}^2)\int dm$$

since $a=x-x_{cm}$

$$\sum a\,m_i=\sum m_i x-\sum m_i\frac{\sum m_i x_i}{\sum m_i}$$
$$=0$$

$$=\int(a^2+b^2)dm+(x_{cm}^2+y_{cm}^2)\int dm$$
$$=I_{cm}+Md^2$$

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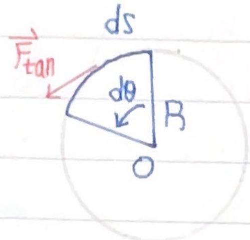
Energy, Work and Power in Rotational Motion



$$dW = \vec{F}\cdot ds = \overbrace{(F\sin\phi)R}^{\tau}\,d\theta = \tau\, d\theta$$

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

since $F_{tan} = m a_{tan} = m r \alpha_z$

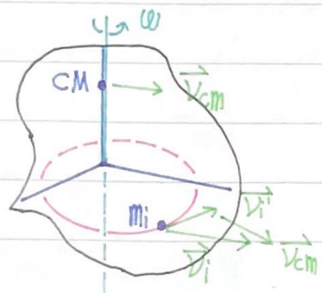
$$\hookrightarrow F_{tan}\, r = m r^2 \alpha_z$$

$$\Rightarrow \tau = I\alpha_z$$

thus $\tau\, d\theta = I\alpha_z\, d\theta = I\frac{d\omega_z}{dt}\, d\theta = I\frac{d\theta}{dt}\, d\omega_z = I\omega_z\, d\omega_z$

$$W = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega = P$$



$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$ → velocity of the particle relative to CM

$$K_i = \frac{1}{2}m_i(\vec{v}_i\cdot\vec{v}_i)$$

$$= \frac{1}{2}m_i(\vec{v}_{cm}+\vec{v}_i')\cdot(\vec{v}_{cm}+\vec{v}_i')$$

$$= \frac{1}{2}m_i(\vec{v}_{cm}\cdot\vec{v}_{cm} + 2\vec{v}_{cm}\cdot\vec{v}_i' + \vec{v}_i'\cdot\vec{v}_i')$$

$$= \frac{1}{2}m_i(v_{cm}^2 + 2\vec{v}_{cm}\cdot\vec{v}_i' + v_i'^2)$$

$$K = \sum K_i = \sum\left(\frac{1}{2}m_i v_{cm}^2\right) + \sum(m_i\vec{v}_{cm}\cdot\vec{v}_i') + \sum\frac{1}{2}m_i v_i'^2$$

$$= \frac{1}{2}\left(\sum m_i\right)v_{cm}^2 + \vec{v}_{cm}\cdot\underbrace{\sum m_i\vec{v}_i'}_{0} + \sum\left(\frac{1}{2}m_i v_i'^2\right)$$

$$= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

$L = I\omega$

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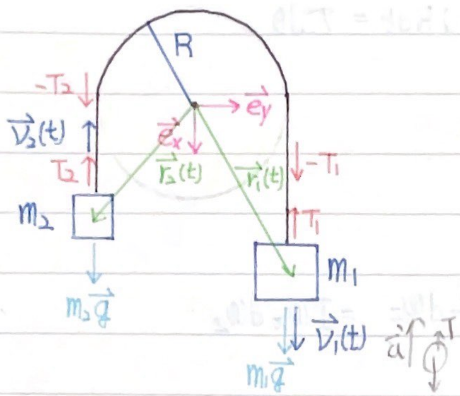
Applications of Angular Momentum

Case I



$\vec{v_2}(t) = -\vec{v_1}(t)$

Netonian mechanics

$$\frac{d}{dt} m_1 \vec{v_1}(t) = m_1\vec{g} - \vec{T_1} \quad \text{—— (1)}$$

$$\frac{d}{dt} m_2 \vec{v_2}(t) = \frac{-d m_2 \vec{v_1}(t)}{dt} = m_2\vec{g} - \vec{T_2} \quad \text{—— (2)}$$

$$\tau = \frac{d}{dt} I \vec{\omega}(t) = -\frac{d}{dt}\left(\frac{1}{2}MR^2\right)\vec{\omega}(t) = R\vec{T_1} - R\vec{T_2} \quad \text{—— (3)}$$

and $R\vec{\omega} = \vec{v_1}(t)$ rolling without slipping

(1)−(2) $\quad (m_1+m_2)\frac{d\vec{v_1}(t)}{dt} = (m_1-m_2)\vec{g} + (\vec{T_2}-\vec{T_1})$

$$= (m_1-m_2)\vec{g} - \frac{1}{2}\frac{1}{R}\frac{d}{dt}MR^2\vec{\omega}(t) \quad \text{by (3)}$$

$$= (m_1-m_2)\vec{g} - \frac{1}{2}M\frac{d\vec{v_1}(t)}{dt}$$

$$\left(m_1+m_2+\frac{M}{2}\right)\frac{d\vec{v_1}(t)}{dt} = (m_1-m_2)\vec{g}$$

$$\frac{d\vec{v_1}(t)}{dt} = \frac{(m_1-m_2)\vec{g}}{m_1+m_2+\frac{M}{2}}$$

change rate of angular momentum

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times m\vec{v}) = \vec{r}\times m\vec{a} = \vec{\tau}$$

$$\frac{d}{dt}\left[m_1\vec{R}\vec{v_1}(t) + m_2\vec{R}\vec{v_2}(t) + \frac{1}{2}MR^2\vec{\omega}(t)\right] = m_1\vec{g}\vec{R} - m_2\vec{g}\vec{R}$$

$$\left(m_1+m_2+\frac{1}{2}M\right)\frac{d\vec{v_1}(t)}{dt} = (m_1-m_2)\vec{g}$$

$$\frac{d\vec{v_1}(t)}{dt} = \frac{(m_1-m_2)\vec{g}}{m_1+m_2+\frac{1}{2}M}$$

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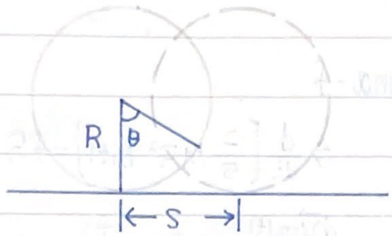
Case II



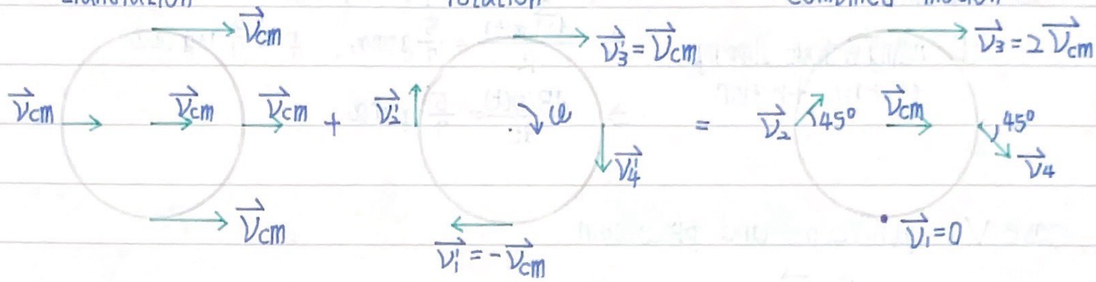
$\vec{v}_{cm} = R\vec{\omega}$ only if there is rolling without slipping

$S = R\theta$

$\vec{v}_{cm} = \frac{dS}{dt} = R\frac{d\theta}{dt} = R\omega$

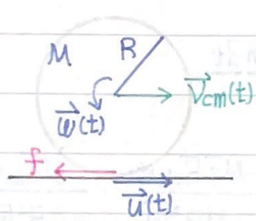
$\alpha_{cm} = \frac{d\vec{v}_{cm}}{dt} = R\frac{d\omega}{dt} = R\alpha$

translation rotation combined motion



$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MR^2\omega^2$$
$$= \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

case III



(i) $\frac{dM\vec{v}_{cm}(t)}{dt} = f$ ← kinetic friction force
direction opposite with $\vec{u}(t)$
$= -\mu_k M\vec{g}$

(ii) $\frac{dI\vec{\omega}(t)}{dt} = -\vec{R}f$ ← center of mass is not interia

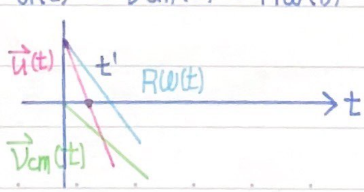
(iii) $\vec{u}(t) = \vec{v}_{cm}(t) + R\vec{\omega}(t)$

(i) ⇒ $\vec{v}_{cm}(t) - \vec{v}_{cm}(0) = \mu_k \vec{g} t$

(ii) ⇒ $\frac{dI\vec{\omega}(t)}{dt} = -\vec{R}\mu_k M\vec{g}$, $\frac{d}{dt}\left[\frac{2}{5}MR^2\omega(t)\right] = -\vec{R}\mu_k M\vec{g}$

$\vec{R}\omega(t) - \vec{R}\omega(0) = -\frac{5}{2}\mu_k \vec{g} t$

(iii) ⇒ $\vec{u}(t) = \vec{v}_{cm}(t) + \vec{R}\omega(0) - \frac{7}{2}\mu_k \vec{g} t$, $t' = \frac{2}{7}\frac{\vec{R}\vec{\omega}(0) + \vec{v}_{cm}(0)}{\mu_k g}$



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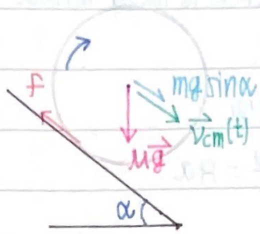
case IV



$$\frac{dM\vec{v}_{cm}(t)}{dt} = Mg\sin\alpha - f$$

$$\frac{d\,I\omega(t)}{dt} = fR \quad \Rightarrow \frac{d}{dt}\left[\frac{2}{5}MR^2\omega(t)\right] = fR$$

$$\frac{dR\omega(t)}{dt} = \frac{5}{2}\frac{f}{M} = \frac{d\vec{v}_{cm}(t)}{dt} = g\sin\alpha - \frac{f}{M}$$

$$\Rightarrow \left(1+\frac{5}{2}\right)\frac{f}{M} = g\sin\alpha, \quad \frac{f}{M} = \frac{2}{7}g\sin\alpha$$

$$\Rightarrow \frac{d\vec{v}_{cm}(t)}{dt} = \frac{5}{7}g\sin\alpha, \quad f = \frac{2}{7}Mg\sin\alpha$$

$$\Rightarrow \frac{dR\omega(t)}{dt} = \frac{5}{7}g\sin\alpha$$

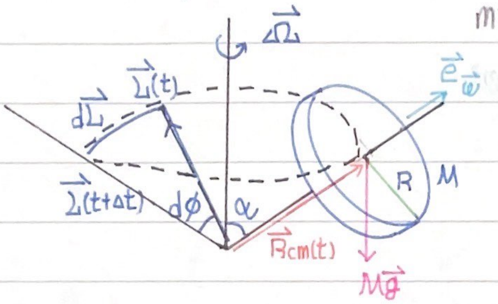
$\vec{u}(t) = \vec{v}_{cm}(t) - R\omega(t)$

$= 0$ rolling without slipping

← static friction

case V gyroscopes and precession

$m \times \vec{v}\,\vec{r}$



$\vec{\tau} = \vec{R}_{cm}(t) \times M\vec{g}$

$\frac{d\vec{L}}{dt} = \vec{R}_{cm}(t) \times M\vec{g}$

$\left|\frac{d\vec{L}}{dt}\right| = |\vec{L}(t)|\sin\alpha|\vec{\Omega}|$

$= |\vec{R}_{cm}(t)||M\vec{g}|\sin\alpha$

$\vec{L} = I\vec{\omega}$

$I = cMR^2$

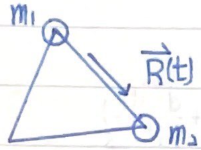
$$d\phi = \frac{d\vec{L}}{\vec{L}} = \frac{\vec{\tau}\,dt}{\vec{L}} = \frac{M\vec{g}\,\vec{R}_{cm}\,dt}{\vec{L}}$$

$$\vec{\Omega} = \frac{d\phi}{dt} = \frac{M\vec{g}\vec{R}_{cm}(t)}{I\vec{\omega}} = \frac{M\vec{g}\vec{R}_{cm}(t)}{cMR^2\vec{\omega}} = \frac{\vec{g}\vec{R}_{cm}(t)}{cR^2\vec{\omega}}$$

$$\frac{1}{2}m_1\left(\frac{d\vec{r}_1(t)}{dt}\right)^2+\frac{1}{2}m_2\left(\frac{d\vec{r}_2(t)}{dt}\right)^2=\frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\frac{d\vec{R}(t)}{dt}\right)^2+\frac{1}{2}\left(\frac{m_2m_1}{m_1+m_2}\frac{d\vec{R}(t)}{dt}\right)^2m_2$$

$$=\frac{1}{2}\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}\left(\frac{d\vec{R}(t)}{dt}\right)^2=\frac{1}{2}\mu\left(\frac{d\vec{R}(t)}{dt}\right)^2$$

case III



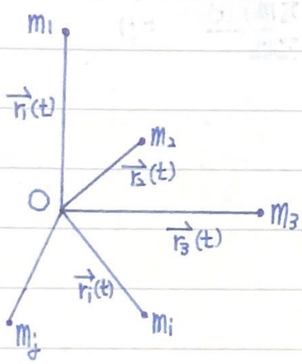
$$\frac{-Gm_1m_2}{|\vec{R}(t)|^2}\frac{\vec{R}(t)}{|\vec{R}(t)|}=m_2\frac{d^2\vec{r}_2(t)}{dt^2}\ \text{—(1)}=-\frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{R}(t)}{dt^2}$$

$$\frac{-Gm_1m_2}{|\vec{R}(t)|^2}\frac{\vec{R}(t)}{|\vec{R}(t)|}=m_1\frac{d^2\vec{r}_1(t)}{dt^2}\ \text{—(2)}=\frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{R}(t)}{dt^2}$$

$$\mu\frac{d^2\vec{R}(t)}{dt^2}=-\frac{Gm_1m_2}{|\vec{R}(t)|^2}\frac{\vec{R}(t)}{|\vec{R}(t)|}=\frac{K}{|\vec{R}(t)|^2}\vec{e}_r(t)$$ ask period?

$$\frac{d}{dt}(\vec{L}_1+\vec{L}_2)=\frac{d}{dt}\left[\mu\vec{R}(t)\times\frac{d\vec{R}(t)}{dt}\right]=\vec{R}(t)\times\mu\frac{d^2\vec{R}(t)}{dt^2}=\vec{R}(t)\times\vec{F}=0$$ angular momentum conservation

case IV multiparticle



$$\vec{l}_i(t)=\vec{r}_i(t)\times m_i\frac{d\vec{r}_i(t)}{dt}$$

$$\frac{d}{dt}\left(\sum\vec{l}_i(t)\right)=\sum_{i=1}^{N}\frac{d\vec{r}_i(t)}{dt}\times m_i\frac{d\vec{r}_i(t)}{dt}+\vec{r}_i(t)\times\frac{d\,m\vec{v}_i(t)}{dt}$$

$$=\sum_{i=1}^{N}\vec{r}_i(t)\times\vec{F}_i$$

$$=\sum_{i=1}^{N}\vec{r}_i(t)\times\sum_{\substack{j=1\\ j\neq i}}^{N}\vec{F}_{ij}+\sum\vec{r}_i(t)\times\vec{F}_{i,ext}$$

$$=\sum_{i=1}^{N}\vec{r}_i(t)\times\vec{F}_{i,ext}$$

$\vec{F}_{ij}=-\vec{F}_{ji}\ //\ \vec{r}_i(t)-\vec{r}_j(t)$

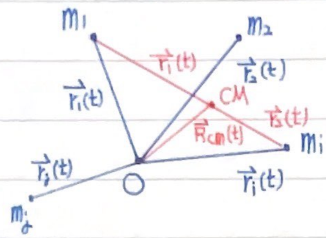
$\vec{r}_i(t)\times\vec{F}_{ji}+\vec{r}_j(t)\times\vec{F}_{ij}=[\vec{r}_i(t)-\vec{r}_j(t)]\times\vec{F}_{ij}=0$

if $\vec{F}_{i,ext}=m\vec{g}$, $\frac{d}{dt}\sum_{i=1}^{N}\vec{l}_i(t)=\sum_{i=1}^{N}\vec{r}_i(t)\times\vec{F}_{i,ext}=\sum_{i=1}^{N}\vec{r}_i(t)\times m_i\vec{g}$

$$=\left[\sum_{i=1}^{N}m_i\vec{r}_i(t)\right]\times\vec{g}=\left(\sum_{i=1}^{N}m_i\right)\vec{r}_{cm}(t)\times\vec{g}$$

$$=\vec{r}_{cm}(t)\times\left(\sum_{i=1}^{N}m_i\right)\vec{g}$$

Angular Momentum and Moment of Inertia



$$\vec{r}_i(t) = \vec{R}_{cm}(t) + \vec{r}_i'(t)$$

$$\sum_{i=1}^{N} \vec{l}_i = \sum \vec{r}_i(t) \times m_i \frac{d\vec{r}_i(t)}{dt} = \sum\left[\vec{R}_{cm}(t) + \vec{r}_i'(t)\right] \times m_i \left(\frac{d\vec{R}_{cm}(t)}{dt} + \frac{d\vec{r}_i'(t)}{dt}\right)$$

$$= \left(\sum_{i=1}^{N} m_i\right) \vec{R}_{cm}(t) \times \frac{d\vec{R}_{cm}}{dt} + \overset{①}{\sum \vec{R}_{cm}(t) \times m_i \frac{d\vec{r}_i'(t)}{dt}} + \overset{②}{\left(\sum_{i=1}^{N} m_i \vec{r}_i'(t)\right)} \frac{d\vec{R}_{cm}(t)}{dt}$$

$$+ \sum_{i=1}^{N} \vec{r}_i'(t) \times m_i \frac{d\vec{r}_i'(t)}{dt}$$

①

$$\because \left(\sum \vec{R}_{cm}(t)\right) \times m_i \frac{d\vec{r}_i'(t)}{dt} = \left(\sum \frac{\sum m_i \vec{r}_i(t)}{\sum m_i}\right) \times m_i \left(\frac{d\vec{r}_i(t)}{dt} - \frac{d}{dt}\frac{\sum m_i \vec{r}_i(t)}{\sum m_i}\right) = 0$$

$$\frac{1}{\sum m_i}\frac{d}{dt}\left(\sum m_i \vec{r}_i(t)\right)$$

$$\frac{d}{dt}\sum \vec{r}_i(t)$$

②

$$\because \sum_{i=1}^{N} m_i \vec{r}_i'(t) = \sum_{i=1}^{N} m_i\left(\vec{r}_i(t) - \vec{R}_{cm}(t)\right) = \sum m_i \vec{r}_i(t) - \sum m_i \frac{\sum m_i \vec{r}_i(t)}{\sum m_i} = 0$$

$$\Rightarrow \left(\sum_{i=1}^{N} m_i\right) \vec{R}_{cm}(t) \times \frac{d\vec{R}_{cm}}{dt} + \sum_{i=1}^{N} \vec{r}_i'(t) \times m_i \frac{d\vec{r}_i'(t)}{dt}$$

orbit angular momentum　　spin angular momentum

$$\frac{d}{dt}\left(\sum_{i=1}^{l} \vec{l}_i\right) = \sum \vec{r}_i(t) \times \vec{F}_{i,ext}$$

$$= \frac{d}{dt}\left[\left(\sum_{i=1}^{N} m_i\right) \vec{R}_{cm}(t) \times \frac{d\vec{R}_{cm}}{dt} + \sum_{i=1}^{N} m_i \vec{r}_i'(t) \times \frac{d\vec{r}_i'(t)}{dt}\right]$$

$$= \left(\sum_{i=1}^{N} m_i\right) \vec{R}_{cm}(t) \frac{d^2\vec{R}_{cm}}{dt^2} + \sum_{i=1}^{N} m_i \vec{r}_i'(t) \times \frac{d^2\vec{r}_i'(t)}{dt^2}$$

$$= \vec{R}_{cm}(t) \times \sum_{i=1}^{N} \vec{F}_{i,ext} + \frac{d}{dt}\vec{L}_{cm}$$

and $\sum \vec{r}_i(t) \times \vec{F}_{i,ext}$

$$= \sum_{i=1}^{N} \left(\vec{R}_{cm}(t) + \vec{r}_i'(t)\right) \times \vec{F}_{i,ext}$$

$$= \vec{R}_{cm}(t) \times \sum_{i=1}^{N} \vec{F}_{i,ext} + \sum_{i=1}^{N} \vec{r}_i'(t) \times \vec{F}_{i,ext}$$

$$\Rightarrow \frac{d}{dt}\vec{L}_{cm} = \sum_{i=1}^{N} \vec{r}_i'(t) \times \vec{F}_{i,ext}$$

$$\vec{L}_{cm} = \sum m_i \vec{r}_i'(t) \times \frac{d\vec{r}_i'(t)}{dt}$$