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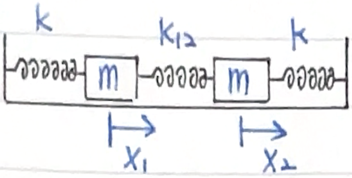
Coupled Oscillations

Two Coupled Harmonic Oscillators



$$m\frac{d^2x_1}{dt^2} = -kx_1 - k_{12}(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k_{12}(x_2 - x_1)$$

We expect the motion to be oscillatory, $x_1(t) = B_1 e^{i\omega t}$ $\quad x_2(t) = B_2 e^{i\omega t}$

$$\Rightarrow -m\omega^2 B_1 e^{i\omega t} + (k+k_{12})B_1 e^{i\omega t} - k_{12}B_2 e^{i\omega t} = 0$$

$$-m\omega^2 B_2 e^{i\omega t} + (k+k_{12})B_2 e^{i\omega t} - k_{12}B_1 e^{i\omega t} = 0$$

$$\Rightarrow \begin{matrix}(k+k_{12}-m\omega^2)B_1 - k_{12}B_2 = 0 \\ -k_{12}B_1 + (k+k_{12}-m\omega^2)B_2 = 0\end{matrix} \; ; \quad \begin{vmatrix} k+k_{12}-m\omega^2 & -k_{12} \\ -k_{12} & k+k_{12}-m\omega^2 \end{vmatrix} = 0$$

$$(k+k_{12}-m\omega^2)^2 - k_{12}^2 = 0 \;, \quad k+k_{12}-m\omega^2 = \pm k_{12} \quad \omega = \sqrt{\frac{k+k_{12}\pm k_{12}}{m}}$$

characteristic frequencies (or eigenfrequencies) $\omega_1 = \sqrt{\frac{k+2k_{12}}{m}}$ $\quad \omega_2 = \sqrt{\frac{k}{m}}$

$$\Rightarrow x_1(t) = C_1 e^{i\omega_1 t} + D_1 e^{-i\omega_1 t} + E_1 e^{i\omega_2 t} + F_1 e^{-i\omega_2 t}$$

$$x_2(t) = C_2 e^{i\omega_1 t} + D_2 e^{-i\omega_1 t} + E_2 e^{i\omega_2 t} + F_2 e^{-i\omega_2 t}$$

for $\omega = \omega_1$, $C_1 = -C_2$ $\; D_1 = -D_2$

for $\omega = \omega_2$, $E_1 = E_2$ $\; F_1 = F_2$

$$\Rightarrow x_1(t) = C_1 e^{i\omega_1 t} + D_1 e^{-i\omega_1 t} + E_1 e^{i\omega_2 t} + F_1 e^{-i\omega_2 t}$$

$$x_2(t) = -C_1 e^{i\omega_1 t} - D_1 e^{-i\omega_1 t} + E_1 e^{i\omega_2 t} + F_1 e^{-i\omega_2 t}$$

let $\eta_1 \equiv x_1 - x_2$ $\qquad x_1 = \frac{1}{2}(\eta_2 + \eta_1)$

$\quad\;\; \eta_2 \equiv x_1 + x_2$ $\qquad x_2 = \frac{1}{2}(\eta_2 - \eta_1)$

$$\Rightarrow m\left(\frac{d^2\eta_1}{dt^2}+\frac{d^2\eta_2}{dt^2}\right)+(k+2k_{12})\eta_1+k\eta_2=0 \quad \text{—— (1)}$$

$$m\left(\frac{d^2\eta_1}{dt^2}-\frac{d^2\eta_2}{dt^2}\right)+(k+2k_{12})\eta_1-k\eta_2=0 \quad \text{—— (2)}$$

$$(1)+(2) \quad m\frac{d^2\eta_1}{dt^2}+(k+2k_{12})\eta_1=0$$

$$(1)-(2) \quad m\frac{d^2\eta_2}{dt^2}+k\eta_2=0$$

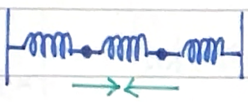
$$\Rightarrow \quad \eta_1(t)=G_1e^{i\omega_1 t}+H_1e^{-i\omega_1 t}$$

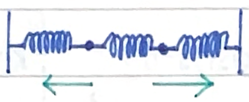
$$\eta_2(t)=G_2e^{i\omega_2 t}+H_2e^{-i\omega_2 t}$$

normal coordinates

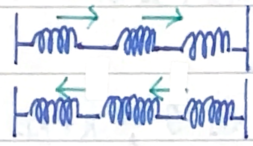
impose the special initial conditions

(1) $X_1(0)=-X_2(0)$, $\eta_2(0)=0$
$\frac{dX_1(0)}{dt}=-\frac{dX_2(0)}{dt}$, $\frac{d\eta_2(0)}{dt}=0$ $\Rightarrow$ $\eta_1(t)\neq 0$, $G_2=H_2=0$, $\eta_2(t)=0$ antisymmetrical





(2) $X_1(0)=X_2(0)$, $\eta_1(t)=0$
$\frac{dX_1(0)}{dt}=\frac{dX_2(0)}{dt}$, $\eta_2(t)\neq 0$



symmetrical

antisymmetrical mode has the higher f

if we hold m_2 fixed, $f=\sqrt{\frac{k+k_{12}}{M}}$

Weak Coupling

if $k_{12} \ll k$, $\frac{k_{12}}{2k} \equiv \varepsilon \ll 1$

$$\omega_1 = \sqrt{\frac{k+2k_{12}}{m}} = \sqrt{\frac{k}{m}}\sqrt{1+\frac{2k_{12}}{k}} = \sqrt{\frac{k}{m}}\sqrt{1+4\varepsilon} \cong \sqrt{\frac{k}{m}}(1+2\varepsilon)$$

$$\omega_0 = \sqrt{\frac{k+k_{12}}{m}} \cong \sqrt{\frac{k}{m}}(1+\varepsilon), \quad \sqrt{\frac{k}{m}} \cong \omega_0(1-\varepsilon)$$

$$\omega_1 \cong \sqrt{\frac{k}{m}}(1+2\varepsilon) \cong \omega_0(1-\varepsilon)(1+2\varepsilon) \cong \omega_0(1+\varepsilon)$$

$$\omega_2 \cong \sqrt{\frac{k}{m}} \cong \omega_0(1-\varepsilon)$$

initial conditions $X_1(0)=B$ $X_2(0)=0$ $\frac{dX_1(0)}{dt}=0$ $\frac{dX_2(0)}{dt}=0$

$C_1=D_1=E_1=F_1=\frac{1}{4}B$

$$X_1(t) = \frac{B}{4}\left[(e^{i\omega_1 t}+e^{-i\omega_1 t})+(e^{i\omega_2 t}+e^{-i\omega_2 t})\right] = \frac{B}{2}(\cos\omega_1 t+\cos\omega_2 t)$$

$$= B\left[\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\right]$$

$$= \underline{B\cos(\varepsilon\omega_0 t)}\cos(\omega_0 t) \qquad \frac{\omega_1+\omega_2}{2}=\omega_0 \quad \frac{\omega_1-\omega_2}{2}=\varepsilon\omega_0$$

$$X_2(t) = \underline{B\sin\varepsilon\omega_0 t}\,\sin\omega_0 t \qquad \Rightarrow \text{beats}$$

vary slowly with time

when $t=\frac{\pi}{2\varepsilon\omega_0}$ then $B\cos\varepsilon\omega_0 t=0$