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Waves

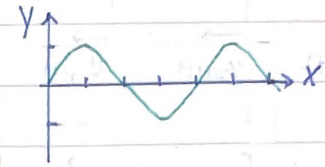
Mathematical Description of a Wave

$y(x,t) = y(x-vt, t=0)$

$y(x, t=0) = A\sin ax \qquad y(x=0, t=0) = A\sin a(0) = 0$

$y(\frac{\lambda}{2}, t=0) = A\sin(a\frac{\lambda}{2}) = 0$

$\Rightarrow a\frac{\lambda}{2} = \pi \; , \; a = \frac{2\pi}{\lambda}$



$y(x, t=0) = A\sin(\frac{2\pi}{\lambda}x)$

$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$ and $v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T}$

$= A\sin\left[2\pi(\frac{x}{\lambda} - \frac{t}{T})\right]$ angular wave number $k = \frac{2\pi}{\lambda}$

$= A\sin(kx - \omega t)$ in $+x$ direction $\qquad \omega = \frac{2\pi}{T} = 2\pi f$

$y(x,t) = A\sin(\underbrace{kx+\omega t}_{\text{phase}})$ in $-x$ direction

phase speed $\Rightarrow kx - \omega t = \text{constant} \qquad \frac{dx}{dt} = \frac{\omega}{k}$

transverse velocity of any particle in a transverse wave

$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t)$

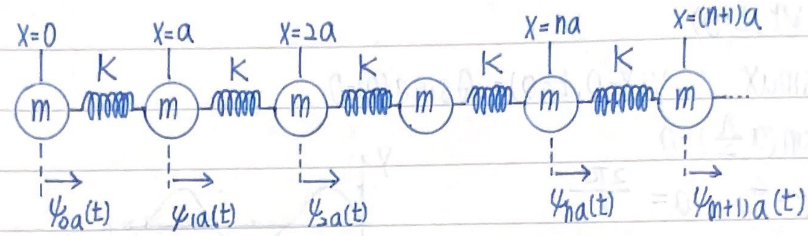
$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx-\omega t) = -\omega^2 y(x,t)$

$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx-\omega t) = -k^2 y(x,t)$

$$\Rightarrow \frac{\frac{\partial^2 y(x,t)}{\partial t^2}}{\frac{\partial^2 y(x,t)}{\partial x^2}} = \frac{\omega^2}{k^2} = v^2 \; , \quad \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$: wave equation

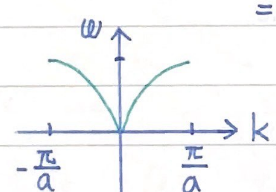
Speed of Sound Waves

Newton model

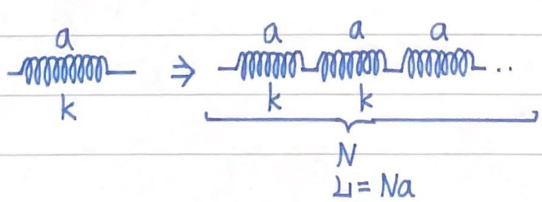


$$m\,\frac{d^2\psi_{na}(t)}{dt^2} = K\left[\psi_{n+1}(t)-\psi_n(t)\right] - K\left[\psi_n(t)-\psi_{n-1}(t)\right]$$
$$= K\left[\psi_{n+1}(t)+\psi_{n-1}(t)-2\psi_n(t)\right] \qquad n=0,1,2\ldots\infty$$
$$\psi_n(t) = A\cos(\omega t - kna) \qquad k\text{: wave number}$$

$$m\left[-\omega^2 A\cos(\omega t-kna)\right] = KA\left[\cos(\omega t-kna-ka)+\cos(\omega t-kna+ka)-2\cos(\omega t-kna)\right]$$
$$= KA\left[2\cos(\omega t-kna)\cos(ka)-2\cos(\omega t-kna)\right]$$
$$= KA\,2\cos(\omega t-kna)\left[\cos(ka)-1\right]$$
$$\Rightarrow m\omega^2 = 2K(1-\cos ka), \quad \omega^2 = \frac{K}{m}2(1-\cos ka)$$
$$= \omega_0^2\, 4\sin^2\frac{ka}{2} \quad\Rightarrow\quad \omega = \omega_0 2\sin\frac{ka}{2}$$



$$ka \geq 0 \quad \sin\frac{ka}{2} \approx \frac{ka}{2}, \quad \omega = 2\omega_0\frac{ka}{2} = \omega_0 a k$$
phase speed $v_{ph} = \dfrac{\omega}{k} = a\omega_0 = a\sqrt{\dfrac{K}{m}}$
$$= \sqrt{\frac{Ka}{\frac{m}{a}}} \quad \text{where } \frac{m}{a} = \rho_l \text{ linear density}$$



$$\frac{1}{K_L} = \frac{1}{K}+\frac{1}{K}+\frac{1}{K}+\cdots\frac{1}{K} = \frac{N}{K}$$
$$K_L = \frac{K}{N} = \frac{K}{\frac{L}{a}} \Rightarrow aK = LK_L$$

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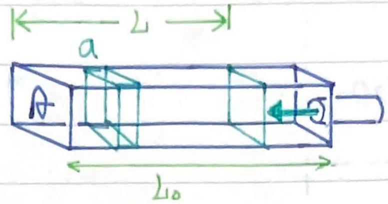


$F = k_L(L-L_0)$, $dF = k_L dL$ $\quad$ $F = PA$, $dF = A\,dP$ $\quad$ $\boxed{\begin{array}{l} V = AL \\ dV = A\,dL \end{array}}$

$= k_{L_0} dL$ $\quad$ $= A \frac{dP}{dV} dV$

$= A \frac{dP}{dV} A\,dL$

$\Rightarrow k_{L_0} = A^2 \frac{dP}{dV}\Big|_{V=V_0}$

$= -A^2 \frac{P_0}{V_0}$

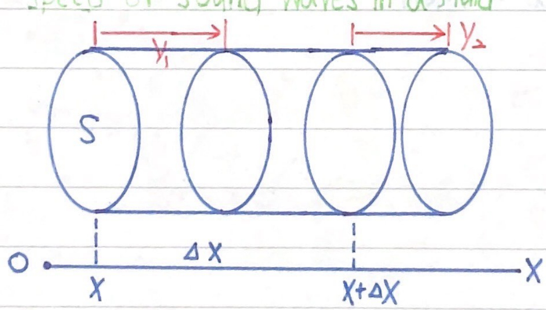
$\Rightarrow k_{L_0} L_0 L_0 = P_0 V_0$

$$\boxed{\begin{array}{l} PV = P_0V_0 \\ P = \frac{P_0V_0}{V} \\ \frac{dP}{dV}\Big|_{V=V_0} = \frac{-P_0V_0}{V^2} = \frac{-P_0}{V_0} \end{array}}$$

thus $\sqrt{\frac{ka}{\frac{m}{a}}} = \sqrt{\frac{k_{L_0} L_0}{\frac{m}{a}}} = \sqrt{\frac{k_{L_0} L_0^2}{\frac{m}{a} L_0}} = \sqrt{\frac{P_0V_0}{\rho_L L_0}} = \sqrt{\frac{P_0V_0}{\rho_V V_0}} = \sqrt{\frac{P_0}{\rho_V}} \approx 280\,m/s$

$$\boxed{\begin{array}{l} \rho_V \text{ volume density} \\ aA\rho_V = m,\ \frac{m}{a} = A\rho_V,\ \rho_L = A\rho_V \end{array}}$$

Speed of sound waves in a fluid



$V = S\Delta X$

$y_1 = y(X,t)$ $\quad$ $y_2 = y(X+\Delta X, t)$

if $y_2 > y_1$ volume ↑ pressure ↓

if $y_2 < y_1$ volume ↓ pressure ↑

if $y_2 = y_1$ no change

$\Delta V = S(y_2 - y_1) = S\left[y(X+\Delta X,t) - y(X,t)\right]$

$\frac{dV}{V} = \lim_{\Delta X \to 0} \frac{S\left[y(X+\Delta X,t) - y(X,t)\right]}{S\Delta X} = \frac{\partial y(X,t)}{\partial X}$

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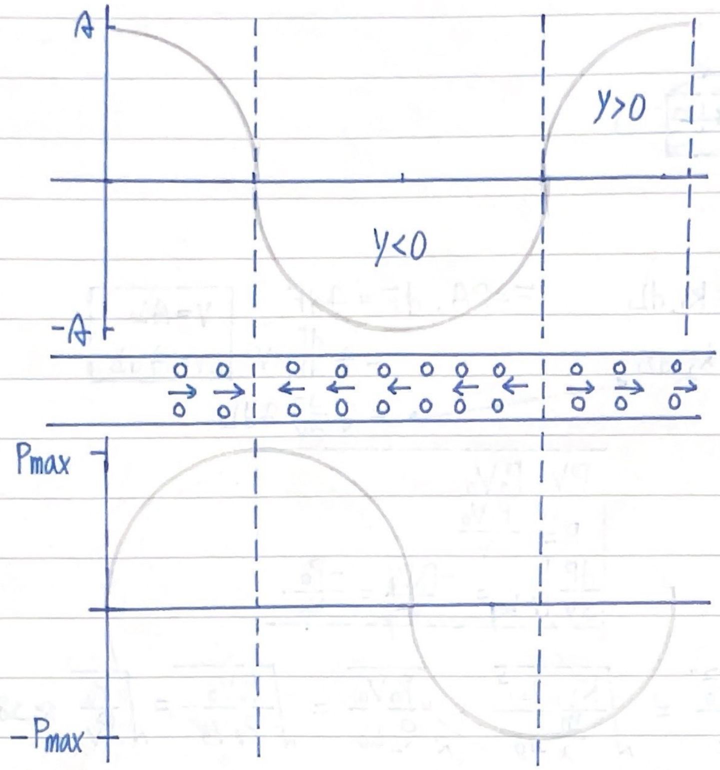

bulk modulus $B = \dfrac{-p(x,t)}{\frac{dV}{V}}$ $\quad p(x,t) = -B\dfrac{\partial y(x,t)}{\partial x}$

$$= BkA\sin(kx-\omega t)$$

$$P_{max} = BkA$$

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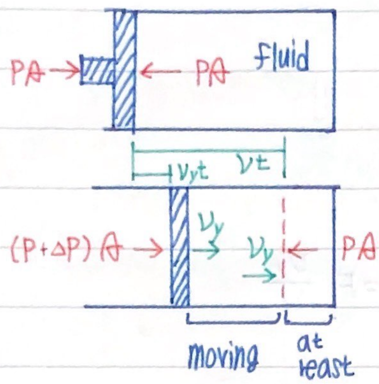

v wave speed

v_y speed of all portions of fluid

volume vtA

mass ρvtA

longitudinal momentum $(\rho vtA)v_y$

$$B = \frac{-\Delta P}{\frac{-Av_y t}{Avt}} \quad , \quad \Delta P = B\frac{v_y}{v}$$

longitudinal impulse $\Delta P A t = B\frac{v_y}{v}At$

$$\Rightarrow B\frac{v_y}{v}At = \rho vtAv_y \quad , \quad v = \sqrt{\frac{B}{\rho}}$$

speed of a longitudinal wave in a solid rod $= \sqrt{\frac{Y}{\rho}}$ Y : Young's modulus

speed of sound in a gas $v = \sqrt{\frac{\gamma RT}{M}}$ in an ideal gas M : molar mass

$B = \gamma P_0$ P_0 : equilibrium pressure of the gas

γ : ratio of heat capacities

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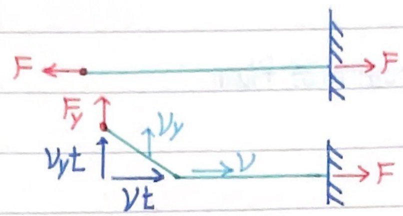
Speed of a Transverse Wave

First method



$F_y t = m v_y$ $\quad T=\sqrt{F^2+F_y^2}$ $\quad \frac{F_y}{F} = \frac{v_y t}{vt}$, $F_y = F\frac{v_y}{v}$

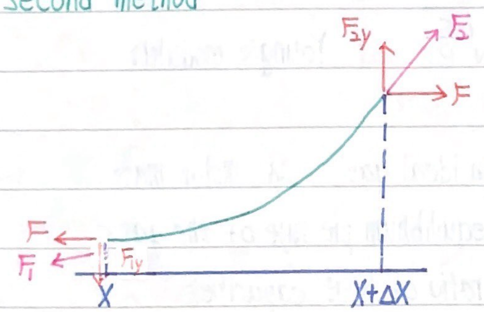
transverse impulse $F_y t = F\frac{v_y}{v}t$

transverse momentum $(\mu v t)v_y$ $\quad \mu$: mass per unit length

$\Rightarrow F\frac{v_y}{v}t = \mu v t\, v_y$

$\Rightarrow v = \sqrt{\frac{F}{\mu}}$

Second method



$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x$ $\quad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$ $\quad F_y = F_{1y}+F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$

$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu\Delta x \frac{\partial^2 y}{\partial t^2}$, $\quad \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$

take the limit as $\Delta x \to 0$ $\quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$

comparing the wave function $\quad v = \sqrt{\frac{F}{\mu}}$

$$= \sqrt{\frac{\text{restoring force returning the system to equilibrium}}{\text{inertia resisting the return to equilibrium}}}$$

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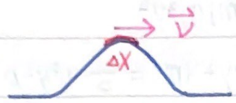
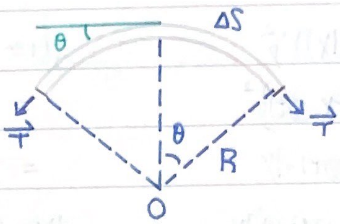
third method





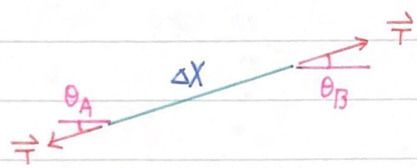
$$F_r = 2T\sin\theta \approx 2T\theta$$

$$m = \mu\Delta S = 2\mu R\theta$$

$$F_r = m\frac{v^2}{R} \Rightarrow 2T\theta = 2\mu R\theta\frac{v^2}{R}, \quad T = \mu v^2$$

$$v = \sqrt{\frac{T}{\mu}}$$

fourth method



$$\sum F_y = T\sin\theta_B - T\sin\theta_A = T(\sin\theta_B - \sin\theta_A) \approx T(\tan\theta_B - \tan\theta_A)$$
$$= T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$\sum F_y = ma_y = \mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) = T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A}{\Delta x}, \quad \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

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Energy in Wave Motion

$$dk = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}(\mu dx)v_y^2$$
$$= \frac{1}{2}\mu dx\left[-\omega A\cos(kx-\omega t)\right]^2$$
$$= \frac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)\,dx$$

when $t=0$, $dk = \frac{1}{2}\mu\omega^2A^2\cos^2(kx)dx$

$$dU = \frac{1}{2}(\Delta m)\omega^2y^2$$
$$= \frac{1}{2}\omega^2y^2dm = \frac{1}{2}\omega^2y^2\mu dx$$
$$= \frac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)\,dx$$
$$dU = \frac{1}{2}\mu\omega^2A^2\cos^2(kx)dx$$

$$k_\lambda = \int dk = \frac{1}{2}\mu\omega^2A^2\int_0^\lambda \cos^2(kx)dx$$
$$= \frac{1}{2}\mu\omega^2A^2\left[\frac{1}{2}x + \frac{1}{4k}\sin 2kx\right]\Big|_0^\lambda$$
$$= \frac{1}{4}\mu\omega^2A^2\lambda$$

$$U_\lambda = \int dU = \frac{1}{2}\mu\omega^2A^2\int_0^\lambda \cos^2(kx)dx$$
$$= \frac{1}{4}\mu\omega^2A^2\lambda$$

$$E_{total} = k_\lambda + U_\lambda = \frac{1}{2}\mu\omega^2A^2\lambda = E_\lambda$$

$$P = \frac{E_\lambda}{T} = \frac{1}{2}\mu\omega^2A^2\frac{\lambda}{T} = \frac{1}{2}\mu\omega^2A^2v$$

$$F_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}$$

$$\text{power} = \frac{W}{t} = \frac{\vec{F}\cdot\vec{d}}{t} = \vec{F}\cdot\vec{v} = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

let $y(x,t) = A\cos(kx-\omega t)$

$$\frac{\partial y(x,t)}{\partial x} = -kA\sin(kx-\omega t)$$
$$\frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t)$$

$\Rightarrow$ $\text{power}(x,t) = Fk\omega A^2\sin^2(kx-\omega t) = \sqrt{\mu F}\,\omega^2A^2\sin^2(kx-\omega t)$

$\Rightarrow$ $P_{max} = \sqrt{\mu F}\,\omega^2A^2$ $\qquad$ $P_{average} = \frac{1}{2}\sqrt{\mu F}\,\omega^2A^2$

intensity: the time average rate at which energy is transported by the wave, per unit area

$$I_1 = \frac{P}{4\pi r_1^2}, \qquad \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

Sound Intensity

$$v_y(x,t) = \omega A \sin(kx-\omega t)$$

$$\rho(x,t)\, v_y(x,t) = [BkA\sin(kx-\omega t)][\omega A\sin(kx-\omega t)]$$

$$= B\omega k A^2 \sin^2(kx-\omega t)$$

$$I = \frac{1}{2} B\omega k A^2 = \frac{1}{2} A^2 B \omega \frac{\omega}{v} \qquad \boxed{\omega = vk}$$

$$= \frac{1}{2} A^2 \omega^2 \sqrt{\frac{B^2}{v^2}} = \frac{1}{2} A^2 \omega^2 \sqrt{\rho B} \qquad \boxed{v^2 = \frac{B}{\rho}}$$

$$= \frac{\omega B^2 k^2 A^2}{2Bk} = \frac{\omega P_{max}^2}{2Bk}$$

$$= \frac{v P_{max}^2}{2B} = \frac{P_{max}^2}{2\rho v} \qquad \boxed{\frac{v^2}{B} = \frac{1}{\rho}}$$

$$= \frac{P_{max}^2}{2\sqrt{\rho B}}$$

sound intensity level $\beta = (10\ \mathrm{dB}) \log \frac{I}{I_0}$ $\quad I_0 : 10^{-12}\ \mathrm{W/m^2}$

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The Rate of Energy Transfer by The Wave

$\psi_n(t) = A\cos(\omega t - kna)$

$$\frac{1}{a}\left\{\frac{1}{2}m\left(\frac{d\psi_n(t)}{dt}\right)^2 + \frac{K}{2}\left[\psi_{n+1}(t) - \psi_n(t)\right]^2\right\}$$
$$= \frac{1}{a}\left\{\frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - kna) + \frac{K}{2}A^2\left[\cos^2(\omega t - kna) + \cos^2(\omega t - kna - ka) - 2\cos(\omega t - kna)\cos(\omega t - kna - ka)\right]\right\}$$

$$\frac{1}{a}\left\langle\left\{\frac{1}{2}m\left(\frac{d\psi_n(t)}{dt}\right)^2 + \frac{K}{2}\left[\psi_{n+1}(t) - \psi_n(t)\right]^2\right\}\right\rangle$$
$$= \frac{1}{a}\quad \frac{1}{2}m\omega^2 A^2 \frac{1}{2} + \frac{K}{2}A^2\left(\frac{1}{2} + \frac{1}{2} - \cos ka\right)$$
$$= \frac{1}{a}KA^2 \sin^2\frac{ka}{2}$$

average of energy stored per unit length

$$\boxed{\omega^2 = 4\frac{K}{m}\sin^2\frac{ka}{2}}$$

$$\boxed{\omega = 2\omega_0 \sin\frac{ka}{2}}$$

$$\text{power} = FV = \frac{d\psi_n(t)}{dt}K\left[\psi_{n+1}(t) - \psi_n(t)\right] = -KA\omega\sin(\omega t - kna)K\left[A\cos(\omega t - kna - ka) - A\cos(\omega t - kna)\right]$$

$$\left\langle\frac{d\psi_n(t)}{dt}K\left[\psi_{n+1}(t) - \psi_n(t)\right]\right\rangle = \left\langle -KA^2\omega\sin(\omega t - kna)\sin(\omega t - kna)\sin ka\right\rangle$$

$$\boxed{\begin{array}{l}\dfrac{1}{\frac{2\pi}{\omega}}\displaystyle\int_0^{\frac{2\pi}{\omega}}\sin^2\omega t\,dt = \frac{1}{2} \\ \dfrac{1}{\frac{2\pi}{\omega}}\displaystyle\int_0^{\frac{2\pi}{\omega}}\sin\omega t\cos\omega t\,dt = 0\end{array}}$$

$$= -KA^2\omega\frac{1}{2}\sin ka$$
$$= -KA^2 2\omega_0 \sin\frac{ka}{2}\left(\sin\frac{ka}{2}\cos\frac{ka}{2}\right)$$

$$\frac{d\omega}{dk} = \omega_0\cos\frac{ka}{2}$$

$$\text{rate of energy transfer } V_E = \frac{\text{average of work}}{\text{average of energy stored per unit length}} = \frac{d\omega}{dk} = \text{group velocity} = a\omega_0\cos\frac{ka}{2}$$

phase speed $V_{ph} = \frac{\omega}{k}$

if $ka \ll 1$, $\omega = 2\omega_0 \sin\frac{ka}{2} \cong \omega_0 ak$

$\frac{\omega}{k} = \omega_0 a$ non-dispersion transmission medium

$V_E = V_G$ dispersion transmission medium

Group Velocity

$$A\cos(\omega t - kna) + A\cos\left[(\omega+\Delta\omega)t - (k+\Delta k)na\right]$$

$$= 2A\cos\alpha\cos\beta$$

let $\alpha+\beta = \omega t - kna$

$\alpha-\beta = (\omega+\Delta\omega)t - (k+\Delta k)na$

$$\alpha = \frac{2\omega+\Delta\omega}{2} - \frac{2k+\Delta k}{2}na \approx \omega t + kna$$

$$\beta = \frac{\Delta\omega t}{2} + \frac{\Delta k}{2}na$$

$$= 2A\cos(\omega t - kna)\cos\left(\frac{\Delta\omega t}{2} + \frac{\Delta k}{2}na\right)$$

$$V_G = \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Standing Waves and Normal Modes of a String

$y_1 = A\sin(kx - \omega t)$ $\quad y_2 = A\sin(kx + \omega t)$

$y = y_1 + y_2 = 2A\sin kx\cos\omega t$

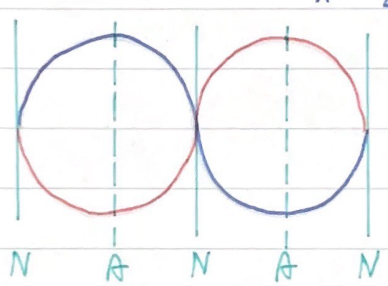
$= A_{sw}\sin kx\cos\omega t$

nodes $\sin kx = 0$, $kx = 0, \pi, 2\pi, 3\pi \cdots$

$k = \frac{2\pi}{\lambda}$, $x = 0, \frac{\lambda}{2}, \frac{3}{2}\lambda \cdots \frac{n}{2}\lambda$

antinodes $\sin kx = \pm 1$, $kx = \frac{\pi}{2}, \frac{3\pi}{2} \cdots$

$x = \frac{n\lambda}{4}$



N : nodes

A : antinodes

$L = n\frac{\lambda}{2}$, $n = 1, 2, 3 \cdots$ string fixed at both ends

$\lambda = \frac{2L}{n}$

$f_n = \frac{v}{2L}n = nf_1 = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$ $\quad$ fundamental frequency $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$

harmonics

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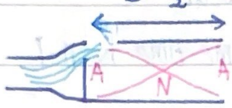
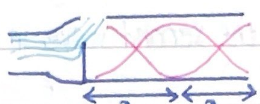
Standing Sound Waves and Normal Modes

a pressure node is always a displacement antinode

a pressure antinode is always a displacement node

$L = \frac{\lambda}{2}$





$\frac{\lambda}{2}$ $\frac{\lambda}{2}$

$f_n = n\frac{v}{2L}$ $n = 1, 2, 3 \cdots$

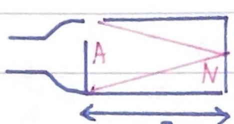
$= n f_1$

$f_1 = \frac{v}{2L}$

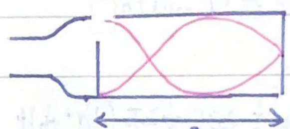
fundamental

$f_2 = 2\frac{v}{2L} = 2f_1$

second harmonic



$L = \frac{\lambda}{4}$



$L = 3\frac{\lambda}{4}$

$f_n = n\frac{v}{4L}$ $n = 1, 3, 5 \cdots$

$= n f_1$

fundamental $f_1 = \frac{v}{4L}$ third harmonic $f_3 = 3\frac{v}{4L} = 3f_1$

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Beats

what happens when two waves with equal amplitude but slightly different f

① suppose $f_a > f_b$, that is $T_a < T_b$

let n be the number of cycles of the first wave in time T_{beat}

then the number of cycles of the second wave in the same time is $n-1$

$\Rightarrow T_{beat} = nT_a = (n-1)T_b$

$T_{beat} = \frac{T_a T_b}{T_b - T_a} = \frac{1}{T_a} - \frac{1}{T_b}$

$\Rightarrow f_{beat} = f_a - f_b$

② $y_a(t) = A\sin\pi f_a t \quad y_b(t) = -A\sin 2\pi f_b t$

$y(t) = y_a(t) + y_b(t)$

$= \underline{2A\sin\frac{1}{2}2\pi(f_a-f_b)t}\ \underline{\cos\frac{1}{2}2\pi(f_a+f_b)t}$

$f_{beat} = 2(\frac{1}{2}(f_a - f_b))$

$y_1 = A\sin(\frac{\pi}{2} - \omega_1 t) = A\cos(2\pi f_1 t)$ set $\phi = \frac{\pi}{2}$ $x=0$

$y_2 = A\sin(\frac{\pi}{2} - \omega_2 t) = A\cos(2\pi f_2 t)$

$y = A(\cos 2\pi f_1 t + \cos 2\pi f_2 t)$

$= 2A\cos 2\pi(\frac{f_1 - f_2}{2})t\ \cos 2\pi\frac{f_1+f_2}{2}t$

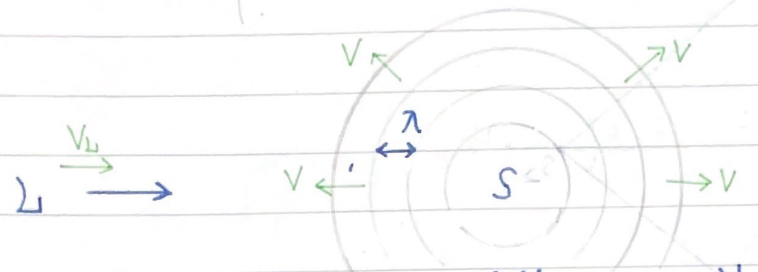
$\cos a + \cos b = 2\cos(\frac{a-b}{2})\cos(\frac{a+b}{2})$

$y_{envelope} = 2A\cos 2\pi(\frac{f_1-f_2}{2})t$

$\cos 2\pi\frac{f_1 - f_2}{2}t = \pm 1$ maximum

$f_{beat} = |f_1 - f_2|$

The Doppler Effect

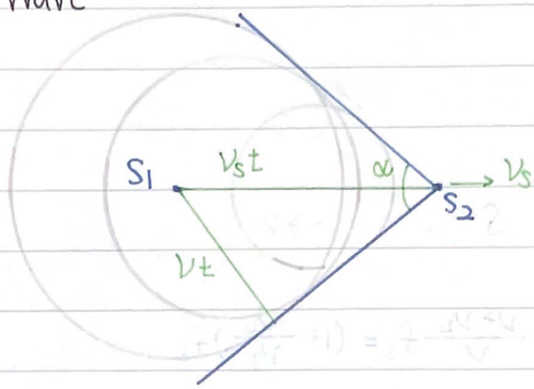


$\lambda = \frac{v}{f_S}$ $f_L = \frac{v + v_L}{\lambda} = \frac{v + v_L}{v} f_S = \left(1 + \frac{v_L}{v_S}\right) f_S$

$\lambda_{in} = \frac{v}{f_S} - \frac{v_S}{f_S}$ $\lambda_{behind} = \frac{v + v_S}{f_S}$

$f_L = \frac{v + v_L}{\lambda_{behind}} = \frac{v + v_L}{v + v_S} f_S$

Shock Wave



$$\sin\alpha = \frac{vt}{v_s t} = \frac{v}{v_s}$$