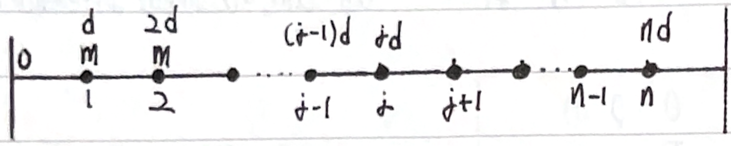
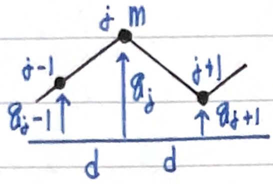
The Loaded String

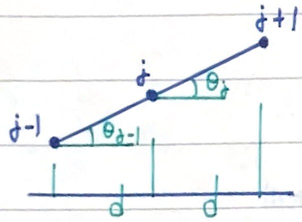


$L=(n+1)d$



if q_{j-1} q_j q_{j+1} are small, tension τ approximately constant

$$F_j=-\frac{\tau}{d}(q_j-q_{j-1})-\frac{\tau}{d}(q_j-q_{j+1})$$



$$F_j\approx-\tau\sin\theta_{j-1}+\tau\sin\theta_j$$
$$\approx-\tau\tan\theta_{j-1}+\tau\tan\theta_j$$

$$m\frac{d^2q_j}{dt^2}=\frac{\tau}{d}(q_{j-1}-2q_j+q_{j+1})\qquad \text{let } \frac{\tau}{d}=k$$

$$U=\frac{1}{2}\frac{\tau}{d}\sum_{j=1}^{n+1}(q_{j-1}-q_j)^2,\quad F_j=-\frac{\partial U}{\partial q_j}=-\frac{\tau}{2d}\frac{\partial}{\partial q_j}\left[(q_{j-1}-q_j)^2+(q_j-q_{j+1})^2\right]$$
$$=\frac{\tau}{d}(q_{j-1}-2q_j+q_{j+1})$$

$$E_k=\frac{1}{2}m\sum_{j=1}^{n}\left(\frac{dq_j}{dt}\right)^2\qquad \frac{dq_{n+1}}{dt}\equiv0\quad j=n+1$$

$$L=E_k-U=\frac{1}{2}\sum_{j=1}^{n+1}\left[m\left(\frac{dq_j}{dt}\right)^2-\frac{\tau}{d}(q_{j-1}-q_j)^2\right]$$
$$=\cdots+\frac{1}{2}m\left(\frac{dq_j}{dt}\right)^2-\frac{1}{2}\frac{\tau}{d}(q_{j-1}-q_j)^2-\frac{1}{2}\frac{\tau}{d}(q_j-q_{j+1})^2-\cdots$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right]-\frac{\partial L}{\partial q_j}=0,\qquad m\frac{d^2q_j}{dt^2}-\frac{\tau}{d}(q_{j-1}-2q_j+q_{j+1})=0$$

to solve the equation of motion, we substitute $q_j(t)=a_je^{i\omega t}$

$\Rightarrow -\frac{T}{d}a_{j-1} + \left(2\frac{T}{d} - m\omega^2\right)a_j - \frac{T}{d}a_{j+1} = 0$ $a_0 = a_{n+1} = 0$ linear difference equation

$$\begin{vmatrix} \lambda & -\frac{T}{d} & 0 & 0 & 0 & \cdots \\ -\frac{T}{d} & \lambda & -\frac{T}{d} & 0 & 0 & \cdots \\ 0 & -\frac{T}{d} & \lambda & -\frac{T}{d} & 0 & \cdots \\ 0 & 0 & -\frac{T}{d} & \lambda & -\frac{T}{d} & \cdots \\ 0 & 0 & 0 & \vdots & \vdots & \\ \vdots & \vdots & \vdots & & & \end{vmatrix} = 0 \qquad \lambda \equiv 2\frac{T}{d} - m\omega^2$$

for $n=1$, $\lambda = 0$ $\omega = \sqrt{\frac{2T}{md}} = \sqrt{\frac{2k}{m}}$

$n=2$ $\omega = \sqrt{\frac{2k \pm k}{m}}$

for large n, we try a solution $a_j = ae^{i(j\gamma - \delta)}$ phase a is real

$\Rightarrow -\frac{T}{d}e^{-i\gamma} + \left(2\frac{T}{d} - m\omega^2\right) - \frac{T}{d}e^{i\gamma} = 0$

$$\omega^2 = \frac{2T}{md} - \frac{T}{md}\left(e^{i\gamma} + e^{-i\gamma}\right) = \frac{2T}{md}(1 - \cos\gamma) = \frac{4T}{md}\sin^2\frac{\gamma}{2}$$

$$\omega_r = 2\sqrt{\frac{T}{md}}\sin\frac{\gamma_r}{2} \qquad r = 1\cdots n$$

$a_{jr} = \mathrm{Re}\left[a_r e^{i(j\gamma - \delta)}\right] = a_r \cos j(\gamma_r - \delta_r)$, $a_{jr} = 0$ for $j = 0 \Rightarrow a_{jr} = a_r\cos(j\gamma_r - \frac{\pi}{2})$
$= a_r \sin j\gamma_r$

boundary equation $a_{0r} = a_{(n+1)r} \equiv 0$

for $j = n+1$ $a_{(n+1)r} = 0 = a_r \sin(n+1)\gamma_r$, $(n+1)\gamma_r = s\pi$ $s = 1, 2 \cdots$

$$\gamma_r = \frac{s\pi}{n+1} = \frac{r\pi}{n+1}$$

$\Rightarrow a_{jr} = a_r \sin\left(j\frac{r\pi}{n+1}\right)$ $q_j = \sum_r B'_r a_{jr} e^{i\omega_r t} = \sum_r B'_r a_r \sin\left(\frac{r\pi}{n+1}j\right)e^{i\omega_r t}$
$= \sum_r B_r \sin\left(\frac{jr\pi}{n+1}\right) e^{i\omega_r t}$

$$\omega_r = 2\sqrt{\frac{T}{md}}\sin\frac{r\pi}{2(n+1)}$$