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$$L_+L_-f = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right) e^{-i\phi}\left(\frac{\partial f}{\partial\theta}-i\cot\theta\frac{\partial f}{\partial\phi}\right)$$

$$= -\hbar^2 e^{i\phi}\left\{e^{-i\phi}\left[\frac{\partial^2 f}{\partial\theta^2}-i\left(-\csc^2\theta\frac{\partial f}{\partial\phi}+\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right]+ i\cot\theta\left[-ie^{-i\phi}\left(\frac{\partial f}{\partial\theta}-i\cot\theta\frac{\partial f}{\partial\phi}\right)+e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\partial\theta}-i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right]\right\}$$

$$= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2}+i\csc^2\theta\frac{\partial f}{\partial\phi}-i\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}+\cot\theta\frac{\partial f}{\partial\theta}-i\cot^2\theta\frac{\partial f}{\partial\phi}+i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}+\cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i(\csc^2\theta-\cot^2\theta)\frac{\partial}{\partial\phi}\right]f$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\frac{\partial}{\partial\phi}\right)f$$

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\cot^2\theta\frac{\partial^2}{\partial\phi^2}+i\frac{\partial}{\partial\phi}\right)-\hbar^2\frac{\partial^2}{\partial\phi^2}-\hbar\frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+(\cot^2\theta+1)\frac{\partial^2}{\partial\phi^2}+i\frac{\partial}{\partial\phi}-i\frac{\partial}{\partial\phi}\right) = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -l(l+1),\quad \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta\, Y$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y \Rightarrow \boxed{L^2 Y = \hbar^2 l(l+1)Y}$$

$$\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]+l(l+1)\sin^2\theta = m^2,\quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2,\quad \Phi(\phi) = e^{im\phi}$$

$$L_z\Phi = \frac{\hbar}{i}\frac{\partial\Phi}{\partial\phi} = \hbar m e^{im\phi} = \hbar m\Phi \Rightarrow \boxed{L_z Y = \hbar m Y}$$

$$L_z|l,l\rangle = \hbar l|l,l\rangle,\quad \frac{\hbar}{i}\frac{\partial}{\partial\phi}|l,l\rangle = \hbar l|l,l\rangle,\quad \frac{\partial|l,l\rangle}{\partial\phi} = il|l,l\rangle,\quad |l,l\rangle = f(\theta)e^{il\phi}$$

$$L_+|l,l\rangle = 0 = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)f(\theta)e^{il\phi} = 0,\quad \frac{df}{d\theta}e^{il\phi}+if\cot\theta\, il e^{il\phi} = 0$$

$$\frac{df}{d\theta} = l\cot\theta f,\quad \frac{df}{f} = l\cot\theta\, d\theta,\quad \int\frac{df}{f} = l\int\frac{\cos\theta}{\sin\theta}d\theta,\quad \ln f = l\ln(\sin\theta)+C$$

$$\Rightarrow f(\theta) = A\sin^l\theta$$

$$\Rightarrow Y_l^l(\theta,\phi) = |l,l\rangle = A(e^{i\phi}\sin\theta)^l$$

$$1 = A^2\int\sin^{2l}\theta\sin\theta\, d\theta\, d\phi = 2\pi A^2\int_0^{\pi}\sin^{2l+1}\theta\, d\theta = 4\pi A^2\frac{(2^l l!)^2}{(2l+1)!}$$

$$\Rightarrow A = \frac{(-1)^l}{2^{l+1}l!}\sqrt{\frac{(2l+1)!}{\pi}}$$

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Spin

$S^2 = S_x^2 + S_y^2 + S_z^2$

$S_\pm = S_x \pm i S_y$

$[S_x, S_y] = i\hbar S_z$ $\qquad S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$ $\quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ S can be integer or half-integers

$[S_y, S_z] = i\hbar S_z$ $\qquad S_z|s,m\rangle = m\hbar|s,m\rangle$ $\quad m_s = -s, -s+1, \ldots, s-1, s$

$[S_z, S_x] = i\hbar S_y$ $\qquad S_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s, m\pm1\rangle$

$[\hat{S}^2, \hat{S}_x] = [\hat{S}^2, \hat{S}_y] = [\hat{S}^2, \hat{S}_z] = 0$

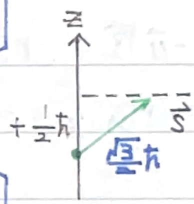
for electron $\uparrow : |\frac{1}{2}, \frac{1}{2}\rangle = \begin{bmatrix}1\\0\end{bmatrix} = \chi_+ \quad \alpha$

$\downarrow : |\frac{1}{2}, -\frac{1}{2}\rangle = \begin{bmatrix}0\\1\end{bmatrix} = \chi_- \quad \beta$

$\chi = a\chi_+ + b\chi_- \qquad |a|^2 + |b|^2 = 1$

$S^2 = \begin{bmatrix}c & d\\ e & f\end{bmatrix} \qquad S^2\chi_+ = \frac{1}{2}(\frac{1}{2}+1)\hbar^2\chi_+ \qquad \Rightarrow S^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$



$S_z = \begin{bmatrix}c & d\\ e & f\end{bmatrix} \qquad S_z\chi_+ = \frac{1}{2}\hbar\chi_+ \qquad \Rightarrow S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$S_z\chi_- = -\frac{1}{2}\hbar\chi_-$

$S_+ = \begin{bmatrix}c & d\\ e & f\end{bmatrix} \qquad S_+\chi_+ = 0 \qquad \Rightarrow S_+ = \hbar\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

$S_+\chi_- = \hbar\chi_+ \qquad \langle\alpha|S_+|\beta\rangle = \langle\beta|S_-|\alpha\rangle = \hbar$

$S_-\chi_+ = \hbar\chi_+ \qquad \Rightarrow S_- = \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$

$S_-\chi_- = 0$

$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$ Pauli matricex

$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \qquad \hat{S}_+\hat{S}_- = \hat{S}^2 - \hat{S}_z^2 + \hbar\hat{S}_z$

$\hat{S}_- = \hat{S}_x - i\hat{S}_y \qquad \hat{S}_-\hat{S}_+ = \hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z$

$S_x\chi = \lambda\chi$

$\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \lambda\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, $\lambda^2-(\frac{\hbar}{2})^2=0$, $\lambda=\pm\frac{\hbar}{2}$

$\chi_+ = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\lambda = \frac{\hbar}{2}$

$\chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, $\lambda = -\frac{\hbar}{2}$

$\Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = C_1\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + C_2\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1, -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = 0 + C_2\frac{1}{\sqrt{2}}\begin{bmatrix}1, -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$C_2 = \frac{1}{\sqrt{2}}(a-b) \quad C_1 = \frac{1}{\sqrt{2}}(a+b)$

Electron in a magnetic field

$\vec{\mu} = \gamma\vec{S} \qquad \gamma \simeq \frac{q}{m}$

$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot B_0\hat{z}$

$= -\gamma B_0 S_z = -\gamma B_0\frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

$\chi_+ = \begin{bmatrix}1\\0\end{bmatrix} \quad E_+ = -\frac{\hbar\gamma B_0}{2}$

$\chi_- = \begin{bmatrix}0\\1\end{bmatrix} \quad E_- = \frac{\hbar\gamma B_0}{2}$

$i\hbar\frac{\partial\chi}{\partial t} = H\chi$, $\chi(t) = a\chi_+ e^{-\frac{iE_+t}{\hbar}} + b\chi_- e^{-\frac{iE_-t}{\hbar}} = \begin{bmatrix}a e^{i\frac{\gamma B_0 t}{2}}\\ b e^{-i\frac{\gamma B_0 t}{2}}\end{bmatrix} = \begin{bmatrix}\cos(\frac{\alpha}{2}) e^{i\frac{\gamma B_0 t}{2}}\\ \sin(\frac{\alpha}{2}) e^{-i\frac{\gamma B_0 t}{2}}\end{bmatrix}$

$\langle S_x\rangle = \begin{bmatrix}\cos\frac{\alpha}{2} e^{-i\frac{\gamma B_0 t}{2}}, & \sin\frac{\alpha}{2} e^{i\frac{\gamma B_0 t}{2}}\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\cos\frac{\alpha}{2} e^{i\frac{\gamma B_0 t}{2}}\\ \sin\frac{\alpha}{2} e^{-i\frac{\gamma B_0 t}{2}}\end{bmatrix}$

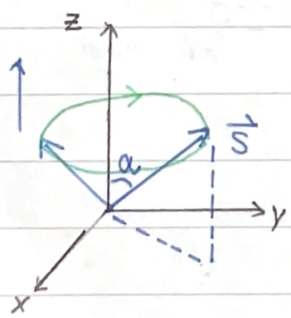
$= \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$

Lormor frequency

$\langle S_y\rangle = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$

$\langle S_z\rangle = \frac{\hbar}{2}\cos\alpha$

$\vec{B} = B_0\hat{z}$



$\omega = \gamma B_0$

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Addition of Angular Momenta

suppose that we have two particles $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$ $\Rightarrow$ $|s_1 s_2 m_1 m_2\rangle$

$$S^{(1)2} |s_1 s_2 m_1 m_2\rangle = s_1(s_1+1)\hbar^2 |s_1 s_2 m_1 m_2\rangle$$
$$S^{(2)2} |s_1 s_2 m_1 m_2\rangle = s_2(s_2+1)\hbar^2 |s_1 s_2 m_1 m_2\rangle$$
$$S_z^{(1)} |s_1 s_2 m_1 m_2\rangle = m_1\hbar |s_1 s_2 m_1 m_2\rangle$$
$$S_z^{(2)} |s_1 s_2 m_1 m_2\rangle = m_2\hbar |s_1 s_2 m_1 m_2\rangle$$

total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$J_z |s_1 s_2 m_1 m_2\rangle = J_{z1} |s_1 s_2 m_1 m_2\rangle + J_{2z} |s_1 s_2 m_1 m_2\rangle$$
$$= \hbar(m_1+m_2) |s_1 s_2 m_1 m_2\rangle = \hbar m |s_1 s_2 m_1 m_2\rangle$$

$\Rightarrow$ $m = m_1 + m_2$

consider two spin $\frac{1}{2}$ particles ($e^- + p$) in the ground state of H

$$|\uparrow\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle \qquad m=1$$
$$|\uparrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \qquad m=0$$
$$|\downarrow\uparrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\rangle \qquad m=0$$
$$|\downarrow\downarrow\rangle = |\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \qquad m=-1$$

this doesn't look right: m is supposed to be $-j$ to $+j$, but there is an "extra" state $m=0$

we apply $J_- = J_-^{(1)} + J_-^{(2)}$

$$J_-|\uparrow\uparrow\rangle = J_-^{(1)} |\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle J_-^{(2)} |\uparrow\rangle = \hbar|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle\hbar|\downarrow\rangle = \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$
$$J_-|10\rangle = \frac{1}{\sqrt{2}}(J_-|\uparrow\rangle|\downarrow\rangle + J_-|\downarrow\uparrow\rangle + |\uparrow\rangle J_-|\downarrow\rangle + |\downarrow\rangle J_-|\uparrow\rangle)$$

$$\left.\begin{aligned} |11\rangle &= |\uparrow\uparrow\rangle \\ |10\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,-1\rangle &= |\downarrow\downarrow\rangle \end{aligned}\right\} \; j=1 \text{ (triplet)}$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad j=0 \text{ (singlet)}$$

$$J^2 = (J_1+J_2)\cdot(J_1+J_2) = J_1^2 + J_2^2 + 2J_1J_2$$

$$J_1J_2|\uparrow\downarrow\rangle = J_{1x}|\uparrow\rangle J_{2x}|\downarrow\rangle + J_{1y}|\uparrow\rangle J_{2y}|\downarrow\rangle + J_{1z}|\uparrow\rangle J_{2z}|\downarrow\rangle$$
$$= \frac{\hbar}{2}|\downarrow\rangle\frac{\hbar}{2}|\uparrow\rangle + \frac{\hbar}{2}i|\downarrow\rangle(-\frac{\hbar}{2}i)|\uparrow\rangle + \frac{\hbar}{2}|\uparrow\rangle(-\frac{\hbar}{2})|\downarrow\rangle$$
$$= \frac{\hbar^2}{4}\,2(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

$$J_1J_2|\downarrow\uparrow\rangle = \frac{\hbar^2}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$j_1 j_2 |10\rangle = \frac{1}{4}\hbar^2 |10\rangle$ $\qquad$ $j^2|10\rangle = 2\hbar^2|10\rangle$

$j_1 j_2 |00\rangle = -\frac{3}{4}\hbar^2 |00\rangle$ $\qquad$ $j^2|00\rangle = 0$

⇒ combining spin $\frac{1}{2}$ with spin $\frac{1}{2}$ to get spin 1 and spin 0

. if we combine spin j_1 with spin j_2,

the total spins $j = (j_1+j_2), (j_1+j_2-1), (j_1+j_2-2), \cdots |j_1-j_2|$ Clebsch-Gordan series

the combined state $|jm\rangle = \sum_{m=m_1+m_2} \underline{C^{j_1 j_2 j}_{m_1 m_2 m}} |j_1 j_2 m_1 m_2\rangle$

Clebsch-Gordan coefficients

$$|j_1 j_2 m_1 m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m} |jm\rangle$$

Ex: $j_1 = \frac{1}{2}$ $\qquad j_2 = \frac{3}{2}$

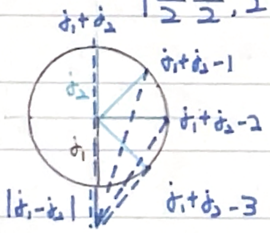
$j = 2, 1$ $\qquad (2j_1+1)\times(2j_2+1) = 2\times 4 = 8$

uncoupled: $|\frac{1}{2}\frac{3}{2}, \frac{1}{2}\frac{3}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, \frac{1}{2}\frac{1}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, \frac{1}{2}-\frac{1}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, \frac{1}{2}-\frac{3}{2}\rangle$

$|\frac{1}{2}\frac{3}{2}, -\frac{1}{2}\frac{3}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, -\frac{1}{2}\frac{1}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, -\frac{1}{2}-\frac{1}{2}\rangle$ $|\frac{1}{2}\frac{3}{2}, -\frac{1}{2}-\frac{3}{2}\rangle$

coupled: $|\frac{1}{2}\frac{3}{2}, 2\,2\rangle$ $|\frac{1}{2}\frac{3}{2}, 2\,1\rangle$ $|\frac{1}{2}\frac{3}{2}, 2\,0\rangle$ $|\frac{1}{2}\frac{3}{2}, 2\,-1\rangle$

$|\frac{1}{2}\frac{3}{2}, 2\,-2\rangle$ $|\frac{1}{2}\frac{3}{2}, 1\,1\rangle$ $|\frac{1}{2}\frac{3}{2}, 1\,0\rangle$ $|\frac{1}{2}\frac{3}{2}, 1\,-1\rangle$

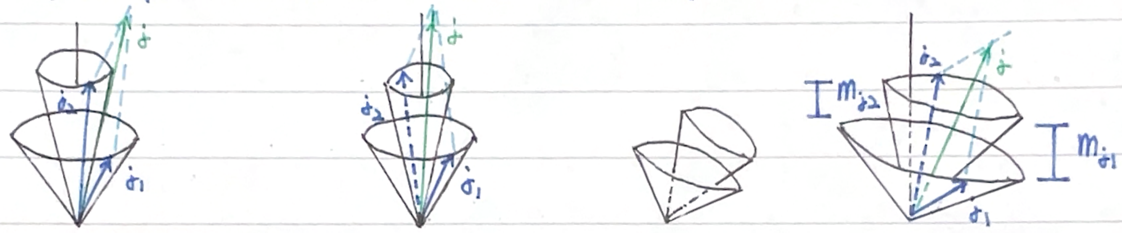


The Vector Model of Coupled Angular Momenta

length of the vector representing the total angular momentum is $\sqrt{j(j+1)}$, with j is one of the values permitted by the Clebsch-Gordan series

the projection of the total angular momentum on the z-axis is m_j

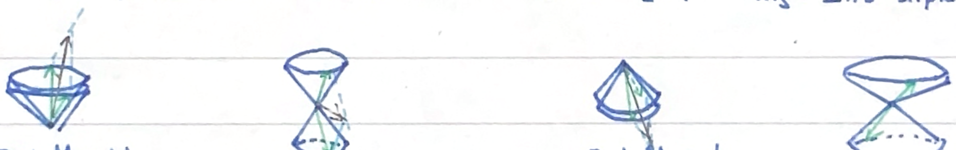
in the uncoupled picture (in which j is not specified), the individual components m_{j_1} and m_{j_2} may be specified, and their sum is eaual to m_j



two particles with spin $s=\frac{1}{2}$ (such as $2e^-$), $m_s=\pm\frac{1}{2}$, in the uncoupled picture, the e^- may be

$\alpha_1\alpha_2$ $\alpha_1\beta_2$ $\beta_1\alpha_2$ $\beta_1\beta_2$

in the coupled picture, the total spin $S = 1$ or 0, if $S=0$ $M_S=0$ singlet; $S=1$ $M_S=\pm1, 0$ triplet



$S=1$ $M_S=+1$ $S=1$ $M_S=0$ $S=1$ $M_S=-1$ $S=0$ $M_S=0$ 1L