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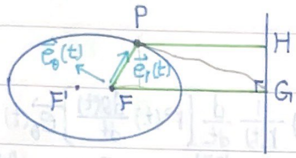
Gravitation

Kepler Problem



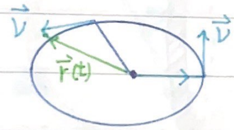
$$\frac{\overline{PF}}{\overline{PG}-\overline{PF}\cos\theta}=e, \text{ assume } \overline{PF}=r \quad \overline{FG}=s$$

$$r=se-er\cos\theta, \quad r(1+e\cos\theta)=se$$

$$r=\frac{se}{1+e\cos\theta}, \quad \frac{1}{r}=\frac{1+e\cos\theta}{se}=\frac{1}{se}+\frac{1}{s}\cos\theta=u$$

$$\vec{F}=m\vec{a}=m\left\{\left[\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right]\vec{e}_r(t)+\frac{1}{r(t)}\frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right]\vec{e}_\theta(t)\right\}$$

$\because$



$$\frac{\Delta area}{\Delta t}=\frac{1}{2}\vec{R}(t)\times\vec{V}(t), \quad \vec{R}(t)=r(t)\vec{e}_r(t), \quad \vec{V}(t)=\frac{dr(t)}{dt}\vec{e}_r(t)+r(t)\frac{d\theta(t)}{dt}\vec{e}_\theta(t)$$

$$=\frac{1}{2}r^2(t)\frac{d\theta(t)}{dt}\vec{e}_r(t)\times\vec{e}_\theta(t)$$

$$=\frac{1}{2}r^2(t)\frac{d\theta(t)}{dt}\vec{e}_z(t)=A\,\vec{e}_z(t) \text{ is a constant}$$

$$\therefore \frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right]=\frac{d}{dt}(2A)=0$$

$$\Rightarrow \vec{F}=m\left[\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt^2}\right)^2\right]\vec{e}_r(t), \text{ where } r(t)\left(\frac{d\theta(t)}{dt}\right)^2=\frac{4A^3}{r^3}$$

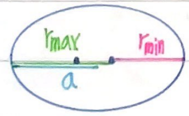
$$\begin{cases} r(t)=\dfrac{se}{1+e\cos\theta(t)} \\ u(t)=\dfrac{1}{se}+\dfrac{1}{s}\cos\theta(t) \end{cases}$$

$$\frac{dr(t)}{dt}=\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}=\underbrace{\frac{-1}{u^2}}_{r^2(t)}\left(-\frac{\sin\theta}{s}\right)\frac{2A}{r^2(t)}=\frac{\sin\theta(t)}{s}2A$$

$$\frac{d^2r(t)}{dt^2}=\frac{2A}{s}\cos\theta(t)\frac{d\theta(t)}{dt}=\frac{4A^2}{s}\frac{\cos\theta(t)}{r^2(t)}$$

$$\Rightarrow \vec{F}=m\left[\frac{4A^2}{s}\frac{\cos\theta}{r^2(t)}-\frac{4A^2}{r^3(t)}\right]=\frac{m4A^2}{r^3(t)\,s}\left[r(t)\cos\theta-s\right] \quad \text{since } \frac{r(t)\cos\theta-s}{r(t)}=-\frac{s-r(t)\cos\theta}{r(t)}=-\frac{1}{e}$$

$$=-\frac{4mA^2}{s\,e\,r^2(t)}\vec{e}_r(t) \leftarrow$$ universal gravitation

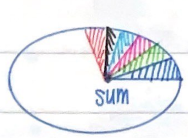


$$\theta=0, \quad r_{min}=\frac{se}{1+e}; \quad \theta=\pi, \quad r_{max}=\frac{se}{1-e}$$

$$2a=r_{min}+r_{max}=\frac{2se}{1-e^2}, \quad a=\frac{se}{1-e^2}, \quad a(1-e^2)=se$$

$$2c=r_{max}-r_{min}=\frac{2se^2}{1-e^2}, \quad c=ae$$

$$A=\frac{\pi ab}{T}, \quad 4A^2=\frac{4\pi^2a^2b^2}{T^2}=\frac{4\pi^2a^2a^2(1-e^2)}{T^2}=\frac{4\pi^2a^3se}{T^2}$$



$$\Rightarrow \vec{F}=-\frac{m4\pi^2a^3se}{T^2}\frac{1}{r^2(t)}\vec{e}_r(t)=-\frac{m4\pi^2a^3}{T^2}\frac{1}{r^2(t)}\vec{e}_r(t) \quad \text{and } \frac{4\pi^2a^3}{T^2}=GM$$

$$=-\frac{GMm}{r^2(t)}\vec{e}_r(t)$$

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Newton's Law of Universal Gravitation

problem : the motion orbit of the particle under the force is inversely proportional to the square of the distance from the source

$$\vec{F} = \frac{K}{r^2(t)}\vec{e}_r(t) = m\vec{a}(t) = m\left[\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right]\vec{e}_r(t) + m\frac{1}{r(t)}\frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right]\vec{e}_\theta(t)$$

since $\frac{1}{2}r^2(t)\frac{d\theta(t)}{dt} = A$, $\frac{d\theta(t)}{dt} = \frac{2A}{r^2(t)}$, therefore $\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2 = \frac{d^2r(t)}{dt^2} - \frac{4A^2}{r^3(t)}$

$$\Rightarrow \vec{F} = \frac{K}{r^2(t)}\vec{e}_r(t) = m\left[\frac{d^2r(t)}{dt^2} - \frac{4A^2}{r^3(t)}\right], \quad \frac{d^2r(t)}{dt^2} - \frac{4A^2}{r^3(t)} = \frac{K}{m}\frac{1}{r^2(t)}$$

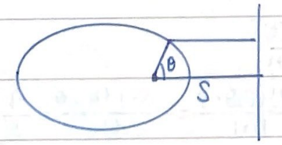
$$\frac{dr(t)}{dt} = \frac{dr}{du}\frac{du(\theta)}{d\theta}\frac{d\theta(t)}{dt} = -\frac{1}{u^2}\frac{du(\theta)}{d\theta}\frac{2A}{r^2(t)} = -2A\frac{du(\theta)}{d\theta}\Big|_{\theta=\theta(t)}$$

$$\frac{d^2r(t)}{dt^2} = \frac{d}{dt}\left[-2A\frac{du(\theta)}{d\theta}\right] = -2A\frac{d}{dt}\left[\frac{du(\theta)}{d\theta}\frac{d\theta(t)}{dt}\right] = -\frac{4A^2}{r^2(t)}\frac{d^2u(\theta)}{d\theta^2}\Big|_{\theta=\theta(t)}$$

$$\Rightarrow \vec{F} \Rightarrow -\frac{4A^2}{r^2(t)}\frac{d^2u(\theta)}{d\theta^2}\Big|_{\theta=\theta(t)} - \frac{4A^2}{r^3(t)} = \frac{K}{m}\frac{1}{r^2(t)}, \quad \frac{d^2u(\theta)}{d\theta^2} + \frac{1}{r(t)} = -\frac{K}{4A^2m}, \quad \frac{d^2u(\theta)}{d\theta^2} = -\left[\frac{K}{4A^2m} + \underbrace{\frac{1}{r(t)}}_{u(\theta)}\right]$$

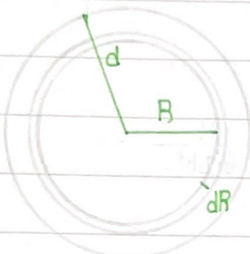
$$\Rightarrow w(\theta) = u(\theta) + \frac{K}{4A^2m} \qquad \frac{d^2w[\theta]}{d\theta^2} = -w[\theta] \Rightarrow w[\theta] = \frac{1}{C}\cos(\theta-\beta) = \frac{1}{r} + \frac{K}{4A^2m}$$

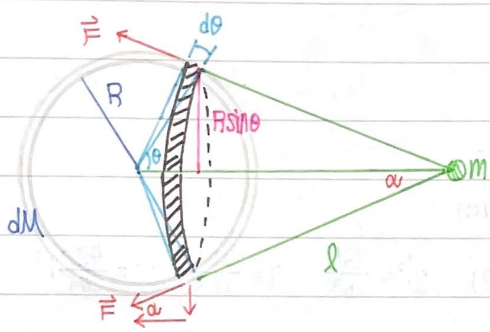
$C = s \quad \frac{K}{4A^2m} = \frac{1}{se}$



$$\frac{r}{s - r\cos\theta} = e, \quad r = se - er\cos\theta, \quad r(1+e\cos\theta) = se, \quad \frac{1}{r} = \frac{1}{se} + \frac{1}{s}\cos\theta$$

$e<1 \qquad e=1 \qquad e>1$



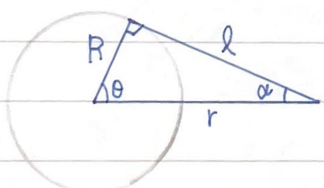


$$\vec{F} = \frac{dM \dfrac{(2\pi R\sin\theta)(Rd\theta)}{4\pi R^2} m G\cos\alpha}{l^2}$$

$$= -GdM\int \frac{2\pi R\sin\theta\, R}{4\pi R^2} m \frac{1}{l^2}\cos\alpha\, d\theta$$

$$= -\frac{G(dM)m}{2}\int \frac{\cos\alpha}{l^2}\sin\theta d\theta$$

since



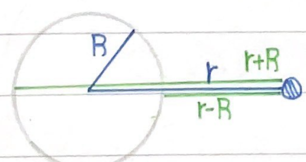
$$l^2 = R^2 + r^2 - 2Rr\cos\theta$$

$$2l(\theta)\frac{dl(\theta)}{d\theta} = 2Rr\sin\theta, \quad \sin\theta = \frac{l(\theta)}{Rr}\frac{dl(\theta)}{d\theta}$$

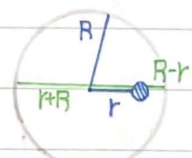
$$R^2 = l^2 + r^2 - 2rl\cos\alpha, \quad \cos\alpha = \frac{l^2+r^2-R^2}{2rl}$$

$$\therefore \vec{F} = -\frac{G(dM)m}{2}\int \frac{l^2(\theta)+r^2-R^2}{2rl(\theta)}\frac{l(\theta)}{Rr}\frac{dl(\theta)}{d\theta}\frac{1}{l^2(\theta)}d\theta = -\frac{GdMm}{4}\frac{1}{Rr^2}\int 1+\frac{r^2-R^2}{l^2}dl = -\frac{G(dM)m}{4Rr^2}\left(l-\frac{r^2-R^2}{l}\right)$$

$r>R$



$r<R$



$R+r-(r-R)+(R-r)-\dfrac{(r-R)(R+r)}{R-r}$

$r+R-\dfrac{r^2-R^2}{r+R}-(r-R)+\dfrac{r^2-R^2}{r-R}$

$$-\frac{G(dM)m}{4Rr^2}\left[l-\frac{r^2-R^2}{l}\right]_{r-R}^{r+R} \qquad -\frac{G(dM)m}{4Rr^2}\left[l-\frac{r^2-R^2}{l}\right]_{R-r}^{R+r}$$

$= r+R-(r-R)-(r-R)+r+R$

$= \cancel{r}+R\cancel{-r}+R\cancel{-r}+R\cancel{+r}+R$

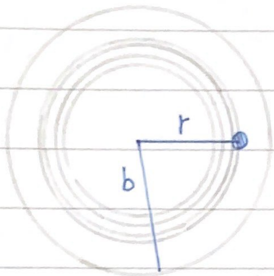
$=4R$

$R+\cancel{r}-r+R+R\cancel{-r}+R+\cancel{r}$

$$= -\frac{G(dM)m}{4Rr^2}4R \qquad = -\frac{G(dM)m}{4Rr^2}(2r-2r)$$

$$= -\frac{G(dM)m}{r^2} \qquad =0$$

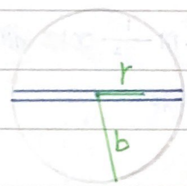
SEASON



$0<r<b$

$$-\frac{Gm\int dM}{r^2} = -\frac{Gm\frac{4}{3}\pi r^3\rho}{r^2} \qquad \rho = \frac{M}{\frac{4}{3}\pi b^3}$$

$$= \frac{-GMmr}{b^3}$$



$$m\frac{d^2r(t)}{dt^2} = -\frac{GMmr}{b^3}$$

$$\Rightarrow \frac{d^2r(t)}{dt^2} = -\frac{GM}{b^3}r(t)$$

$$\Rightarrow r(t) = b\sin(\omega t+\beta) \quad \omega^2 = \frac{GM}{b^3} \quad T = \frac{2\pi}{\omega} \quad T^2 = \frac{4\pi^2}{GM}b^3$$

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Laplace-Lenz-Ruge Vector

$m\vec{v}(t)\times 2A\vec{e}_z$

$$\frac{d}{dt}\left[m\vec{v}(t)\times 2A\vec{e}_z\right] = \frac{d\,m\vec{v}(t)}{dt}\times 2A\vec{e}_z + m\vec{v}(t)\times \overbrace{\frac{d\,2A\vec{e}_z}{dt}}^{0} = \vec{F}\times 2A\vec{e}_z$$

$$= \frac{K}{r^2(t)}\vec{e}_r(t)\times r^2(t)\frac{d\theta(t)}{dt}\vec{e}_z$$

$$= K\frac{d\theta(t)}{dt}\vec{e}_r(t)\times\vec{e}_z \quad = -K\frac{d\theta(t)}{dt}\vec{e}_\theta(t) = -K\frac{d}{dt}\vec{e}_r(t)$$

$$\Rightarrow \frac{d}{dt}\frac{m\vec{v}(t)\times 2A\vec{e}_z}{-K} = \frac{d}{dt}\vec{e}_r(t) \quad , \quad \frac{d}{dt}\frac{m\vec{v}(t)\times 2A\vec{e}_z}{-K} - \frac{d}{dt}\vec{e}_r(t) = 0$$

$$\therefore \overrightarrow{LLR} = \frac{m\vec{v}(t)\times 2A\vec{e}_z}{-K} - \vec{e}_r(t)$$ along the major axes

$$\vec{R}(t)\cdot\overrightarrow{LLR} = |\vec{R}(t)||\overrightarrow{LLR}|\cos\theta = \vec{R}(t)\cdot\left(\frac{m\vec{v}(t)\times 2A\vec{e}_z}{-K} - \vec{e}_r(t)\right)$$ since $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

$$= \frac{m(\vec{R}(t)\times\vec{v}(t))\cdot 2A\vec{e}_z}{-K} - r(t)$$

conic section

$$= -\frac{m4A^2}{K} - r(t) \qquad \Leftrightarrow r(t)(1+|\overrightarrow{LLR}|\cos\theta) = -\frac{4mA^2}{K} = se$$

$$\overrightarrow{LLR}\cdot\overrightarrow{LLR} = \left[-\frac{m\vec{v}(t)\times 2A\vec{e}_z}{K} - \vec{e}_r(t)\right]^2 = 1 + \frac{m^2\vec{v}(t)\cdot\vec{v}(t)\,4A^2}{K^2} + 2\frac{\vec{e}_r(t)\cdot(m\vec{v}(t)\times 2A\vec{e}_z)}{K\ \ r(t)}$$

$$= 1 + m^2\frac{|\vec{v}(t)|^2 4A^2}{K^2} + 2\frac{m4A^2}{K}\left(m|\vec{v}(t)|^2 + 2\frac{K}{r(t)}\right)$$ assume $E = \frac{1}{2}m|\vec{v}(t)|^2 + \frac{K}{r(t)}$

$$= 1 + \frac{4A^2m}{K^2}2E$$

$$\Rightarrow |\overrightarrow{LLR}| = 1 + \frac{4A^2m}{K^2}2E = 1 + \frac{4A^2m}{K}\frac{2E}{K} = 1 - se\frac{2E}{K} = e^2 \quad , 1-e^2 = se\frac{2E}{K} \quad , \frac{K}{2E} = \frac{se}{1-e^2} = a$$

$$E = -\frac{K}{2a} \quad , K = -GMm$$

when $\theta = 0$ or π, $\vec{v}(t) = r(t)\frac{d\theta(t)}{dt}\vec{e}_\theta(t)$

$$\overrightarrow{LLR} = \left[-\frac{mr^3(t)\left[\frac{d\theta(t)}{dt}\right]^2}{K} - 1\right]\vec{e}_r(t) = ?$$

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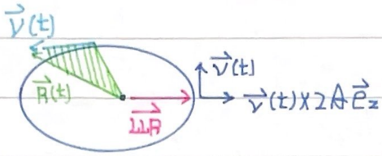
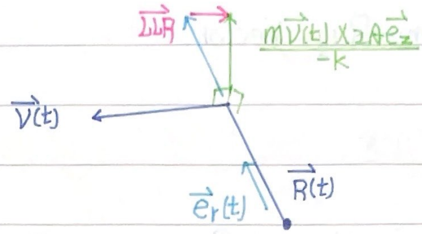
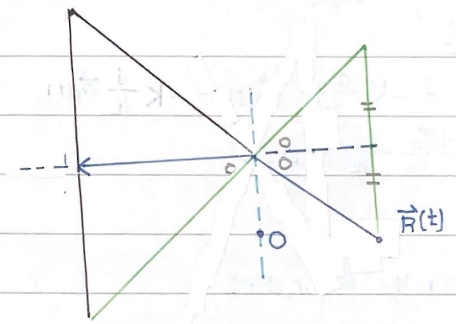
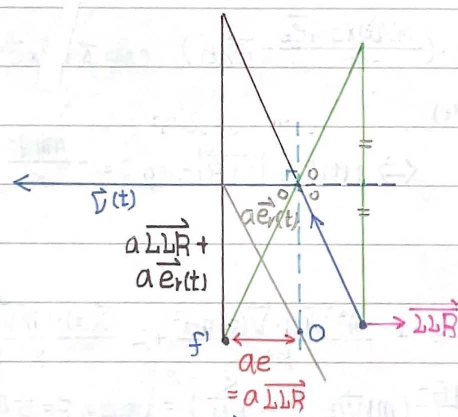








$$a\overrightarrow{LLR} = a\frac{m\vec{v}(t)\times 2A\vec{e}_z}{-k} - a\vec{e}_r(t)$$

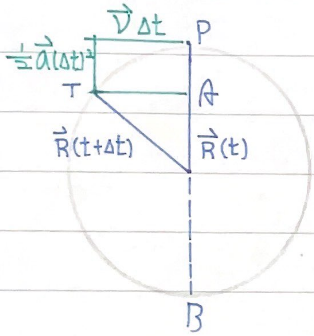
$$a\overrightarrow{LLR} + a\vec{e}_r(t) = a\frac{m\vec{v}(t)\times 2A\vec{e}_z}{-k}$$

$$\vec{F} = \frac{K}{r^2(t)}\vec{e}_r(t) = m\vec{a}(t) = m\left(\frac{d^2r(t)}{dt} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{e}_r(t) + m\frac{1}{r(t)}\frac{d}{dt}\left(r^2(t)\frac{d\theta(t)}{dt}\right)$$

$$\Rightarrow \frac{r^2(t)}{K} m\left[\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right]\vec{e}_r(t) = \vec{e}_r(t)$$

$$\Rightarrow \frac{mr^3(t)\left(\frac{d\theta(t)}{dt}\right)^2}{-K} - 1 = \frac{mr^2(t)}{-K}\frac{d^2r(t)}{dt^2} \qquad K = -GMm$$

$$>0 \quad \oplus \quad >0$$



$$\vec{R}(t+\Delta t) = \vec{R}(t) + \underbrace{\frac{d\vec{R}(t)}{dt}\Delta t}_{\vec{v}(t)\Delta t} + \underbrace{\frac{d^2\vec{R}(t)}{dt^2}\frac{\Delta t^2}{2!}}_{\vec{a}\frac{(\Delta t)^2}{2}}$$

$$(\overline{AT})^2 = \overline{PA}\,\overline{AB}$$

$$\vec{PA} = \frac{|\vec{v}|^2\Delta t^2}{\vec{AB}}, \quad \Delta t \to 0, \quad \vec{PA} \to 0, \quad \vec{AB} \to 2r \quad ? \quad |\vec{a}| = \frac{|\vec{v}|^2}{r}$$

$$|\vec{a}| = \frac{|\vec{v}|^2}{r}, \quad |\vec{v}| = \frac{2\pi r}{T}, \quad |\vec{v}| = \frac{4\pi^2r^2}{T^2} \text{ and } T^2 = kr^3$$

$$\Downarrow \qquad\qquad = \frac{4\pi^2r^2}{kr^3}$$

$$\frac{4\pi^2}{kr^2}$$

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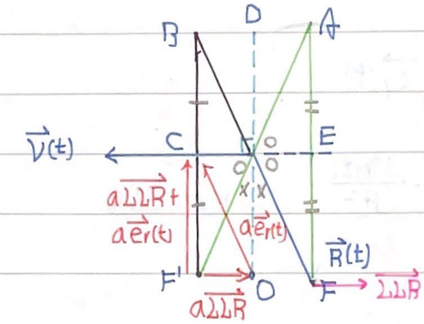


$$a\cdot\overrightarrow{LLR} = a\cdot\frac{m\vec{V}(t)\times 2A\vec{e_z}}{-k} - a\vec{e_r}(t) = \overrightarrow{F'O} = ae$$

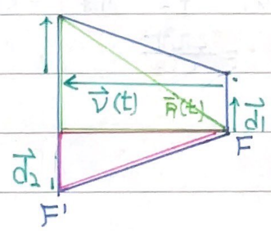
$$a\cdot\overrightarrow{LLR} + a\vec{e_r}(t) = a\,\frac{m\vec{V}(t)\times 2A\vec{e_z}}{-k} = \overrightarrow{F'C}$$

$2\overline{FE} = \overline{AF}$ AFF'B is isosceles trapezium

$\overline{AF} \neq \overline{BF'}$

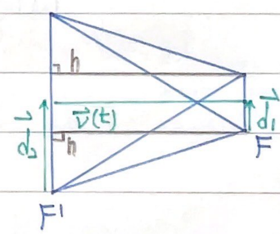
$\overline{AF'} = \overline{BF} = 2a$

$\overline{BF} \parallel \overline{CO}$



$\vec{R}(t)\cdot\vec{V}(t) = 2A$

$|\vec{V}(t)||\vec{d_1}| = 2A$



$$\left.\begin{array}{l} \triangle\ (|\vec{d_1}|+|\vec{d_2}|)^2 + h^2 = (2a)^2 \\ \triangle\ (|\vec{d_2}|-|\vec{d_1}|)^2 + h^2 = (2ae)^2 \end{array}\right] \quad 4|\vec{d_1}||\vec{d_2}| = 4a^2(1-e^2) = 4b^2$$

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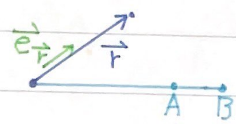
$$\frac{\vec{d_1}}{|\vec{d_1}|}\,\frac{a\cdot m\vec{V}(t)\times 2A\vec{e_z}}{-k} = \frac{a\,m(\vec{d_1}\times\vec{V}(t))\cdot 2A\vec{e_z}}{-k\ |\vec{d_1}|} = \frac{a m 2A\vec{e_z}\cdot 2A\vec{e_z}}{-k\ |\vec{d_1}|}$$

$$= \frac{aSe}{|\vec{d_1}|} = \frac{a\cdot a(1-e^2)}{|\vec{d_1}|} = \frac{b^2}{|\vec{d_1}|} = |\vec{d_2}|$$

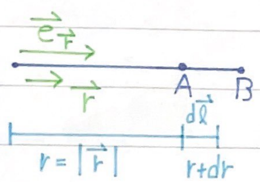
The Potential Energy of the Force is Inversely Proportional to the Square of the Distance From the Source

$$\vec{F} = \frac{K}{|\vec{r}|^2}$$



if O, A, B collinearity

$$\vec{F} = \frac{K}{|\vec{r}|^2}\vec{e}_{\vec{r}}, \quad d\vec{l} = dr\,\vec{e}_{\vec{r}}, \quad \vec{F}\cdot d\vec{r} = \frac{K}{r^2}dr$$

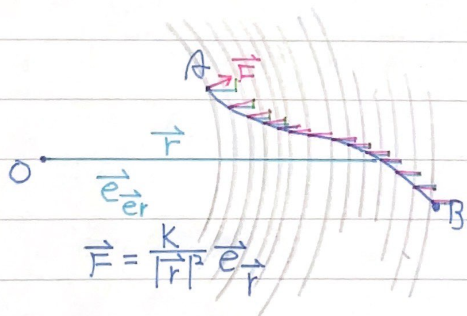


$$U(A) - U(B) = \int_A^B \vec{F}\cdot d\vec{l} = \int_{r_A}^{r_B} \frac{K}{r^2}dr = -\frac{K}{r}\Big|_{r=r_A}^{r=r_B} = \frac{K}{r_A} - \frac{K}{r_B}$$

assume $r_B \to \infty$, $U(B) = 0$

$$\Rightarrow U(A) = \frac{K}{r_A}$$

if O, A, B not collinearity



$$d\vec{l} = dr\,\vec{e}_r + d\vec{l}_\perp$$

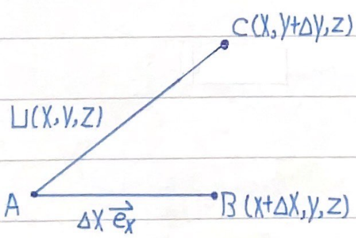
$$\vec{F}\cdot d\vec{l} = \frac{K}{r^2}dr$$

$$\vec{F} = \frac{K}{|\vec{r}|^2}\vec{e}_{\vec{r}}$$

$$\int_A^B \vec{F}\cdot d\vec{l} = \int_{r_A}^{r_B} \frac{K}{r^2}dr = \frac{K}{r_A} - \frac{K}{r_B} = U(A) - U(B)$$

assume $r_B \to \infty$ $\quad U(B) = 0$

$$\Rightarrow U(A) = \frac{K}{r_A}$$



$$U(x,y,z) - U(x+\Delta x,y,z) = \int_A^B \vec{F}\cdot d\vec{l} = F_x \Delta x$$

$$F_x = \lim_{\Delta x \to 0} -\frac{U(x+\Delta x,y,z) - U(x,y,z)}{\Delta x} = -\frac{dU}{dx}\Big|_{y,z\ \text{fixed}} = -\frac{\partial U(x,y,z)}{\partial x}$$

$$\Rightarrow F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z\right) = -\left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)U = -\nabla U$$

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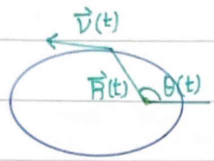

$$E = \frac{1}{2} m \vec{v}(t) \cdot \vec{v}(t) + \frac{K}{r(t)}$$

$$\text{and } \vec{v} = \frac{d\vec{R}(t)}{dt} = \frac{dr(t)}{dt}\vec{e}_r(t) + r(t)\frac{d\theta}{dt}\vec{e}_\theta(t)$$

$$\vec{v}\cdot\vec{v} = \left(\frac{dr(t)}{dt}\right)^2 + r^2(t)\left(\frac{d\theta(t)}{dt}\right)^2$$

$$E = \frac{m}{2}\left[\left(\frac{dr(t)}{dt}\right)^2 + r^2(t)\left(\frac{d\theta(t)}{dt}\right)^2\right] + \frac{K}{r(t)}$$

$$\text{when } r = \begin{cases} r_{max} \\ r_{min} \end{cases} \quad \frac{dr(t)}{dt} = 0 \qquad \text{and} \quad r^2(t)\frac{d\theta(t)}{dt} = 2A$$

$$\Rightarrow E = \frac{m}{2}\frac{4A^2}{r^2(t)} + \frac{K}{r(t)} \quad \text{when } r = r_{max} \text{ or } r_{min}$$

$$\Rightarrow E r^2(t) - K r(t) - 2mA^2 = 0$$

$$\Rightarrow r_\pm = \frac{K \pm \sqrt{K^2 + 4E2mA^2}}{2E} \qquad r_{max} + r_{min} = r_+ + r_- = \frac{K}{E}$$

$$\text{if } K<0, \quad E = \frac{K}{2a}, \quad K = -GMm$$

$$\text{and } r_{max} - r_{min} = 2ae = \frac{\sqrt{K^2+8EmA^2}}{E} = \frac{\sqrt{K^2+8EmA^2}}{\frac{K}{2a}} \Rightarrow e = \sqrt{1 + \frac{8EmA^2}{K^2}}$$

$$\Rightarrow e = \frac{r_{max} - r_{min}}{2a}$$

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$$\vec{F} = \frac{K}{r^2(t)}\,\vec{e}_r(t) = m\vec{a} = m\left\{\left[\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right]\vec{e}_r(t) + \frac{1}{r(t)}\frac{d}{dt}\left[r^2(t)\frac{d\theta(t)}{dt}\right]\vec{e}_\theta(t)\right\}$$

$$r^2(t)\frac{d\theta(t)}{dt} = 2A$$

$$\Rightarrow \frac{d^2r(t)}{dt^2} - \frac{4A^2}{r^3(t)} = \frac{K}{mr^2(t)}$$

$$E = \frac{m}{2}\left[\left(\frac{dr(t)}{dt^2}\right)^2 + r^2(t)\left(\frac{d\theta(t)}{dt}\right)^2\right] + \frac{K}{r(t)}$$

$$\Rightarrow \frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)} = \left(\frac{dr(t)}{dt^2}\right)^2$$

$$\Rightarrow \sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}} = \frac{dr(t)}{dt}$$

$$\Rightarrow 1 = \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}}\frac{dr(t)}{dt} = \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}}\frac{dr}{d\theta(t)}\frac{d\theta(t)}{dt} = \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}}\frac{2A}{r^2(t)}\frac{dr}{d\theta}$$

$$\int_{\theta(0)}^{\theta(t)} 1\,d\theta = \int \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}}\frac{2A}{r^2}\frac{dr(t)}{d\theta}d\theta = \int \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}}\frac{2A}{r^2}dr(t)$$

$$\boxed{u = \frac{1}{r(t)} \quad \frac{du}{dr} = -\frac{1}{r^2}} \quad = \int \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{K}{r(t)}\right) - \frac{4A^2}{r^2(t)}}} - 2A\,du = \int \frac{1}{\sqrt{\frac{2E}{m} - \frac{2K}{m}u - 4A^2u^2}} - 2A\,du$$

$$\boxed{\begin{aligned} W^2 &= \left(u + \frac{K}{4A^2m}\right)^2 \\ &= u^2 + \frac{K}{2A^2m}u + \frac{K^2}{16A^4m^2} \\ dW &= du \end{aligned}} \quad = \int \frac{-1}{\sqrt{\frac{2E}{4A^2m} - \frac{2K}{4A^2m}u - u^2}}\,du = \int \frac{1}{\sqrt{\frac{2E}{4A^2m} + \frac{K^2}{16A^2m^2} - W^2}}\,dW$$

$$\boxed{\begin{aligned} B^2 &= \frac{2E}{4A^2m} + \frac{K^2}{16A^2m^2} \\ B &= \frac{\sqrt{K^2 + 8EA^2m}}{4A^2m} \\ &= \frac{-K\sqrt{1 + \frac{8EA^2m}{K^2}}}{4A^2m} \end{aligned}} \quad \begin{aligned} &= \int \frac{-1}{\sqrt{B^2 - W^2}}\,dW = \int \frac{-1}{\sqrt{1 - \frac{W^2}{B^2}}}\frac{dW}{B} \\ &= \int \frac{-1}{\sqrt{1 - P^2}}\,dP \\ &= \cos^{-1}\frac{W}{B}\Big|_{W(0)}^{W(t)} = \cos^{-1}\frac{W(t)}{B} - \underbrace{\cos^{-1}\frac{W(0)}{B}}_{C} \end{aligned} \quad \boxed{P = \frac{W}{B}}$$

$$\Rightarrow \theta(t) - \theta(0) + C = \cos^{-1}\frac{W(t)}{B}, \quad \cos\overbrace{[\theta(t) - \theta(0) + C]}^{\phi(t)} = \frac{W(t)}{B} = \frac{\frac{1}{r(t)} + \frac{K}{4A^2m}}{-\frac{K}{4A^2m}\sqrt{1 + \frac{8EA^2m}{K^2}}} = \cos\phi(t)$$

$$\Rightarrow \frac{1}{r(t)} + \frac{K}{4A^2m} = -\frac{K}{4A^2m}\sqrt{1 + \frac{8EA^2m}{K^2}}\cos\phi(t), \quad \frac{1}{r(t)} = -\frac{K}{4A^2m}\left(\sqrt{1 + \frac{8EA^2m}{K^2}} + 1\right)\cos\phi(t)$$

$$\frac{1}{Se} = -\frac{K}{4A^2m} \qquad e^2 = 1 + \frac{8EA^2m}{K^2}$$

$$\Rightarrow r(t) = \frac{Se}{1 + e\cos\phi(t)}$$ conic section

Gravitational Potential Energy

$$W_{grav} = \int_{r_1}^{r_2} F\,dr$$

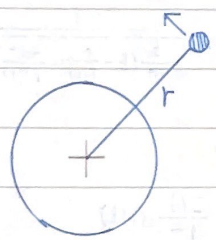
$$= \int_{r_1}^{r_2} -\frac{Gm_Em}{r^2}\,dr = -Gm_Em\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{Gm_Em}{r_2} - \frac{Gm_Em}{r_1}$$

$$= Gm_Em\,\frac{r_1-r_2}{r_1r_2}$$

if the body stays close to the earth

$$= Gm_Em\,\frac{r_1-r_2}{R_E^2} \quad \text{and} \quad g = \frac{Gm_E}{R_E^2}$$

$$= mg(r_1-r_2)$$



$$\frac{Gm_Em}{r^2} = m\frac{v^2}{r}, \quad v = \sqrt{\frac{Gm_E}{r}}$$

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{Gm_E}}$$

$E = K + U$

$$= \frac{1}{2}m\vec{v}^2 + \left(-\frac{Gm_Em}{\vec{r}}\right)$$

$$= \frac{1}{2}m\frac{Gm_E}{\vec{r}} - \frac{Gm_Em}{r}$$

$$= -\frac{Gm_Em}{2r} \quad \text{(circular orbits)}$$

$$= -\frac{Gm_Em}{2a}$$ where a is semimajor axis length elliptical orbitis

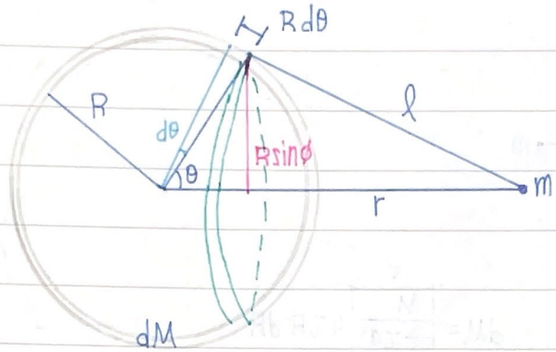
$$\frac{1}{2}m\vec{v}^2 - \frac{Gm_Em}{R_E} = -\frac{Gm_Em}{r_{max}}$$

$$\vec{v}^2 = 2Gm_Em\left(\frac{\ }{R_E} - \frac{1}{r_{max}}\right)$$

let $r_{max} \to \infty$

$\Rightarrow$ espace space $\vec{v} = \sqrt{\frac{2Gm_E}{R_E}}$

Spherical Mass Distributions



$$-\frac{Gm}{\ell}\int \frac{Rd\theta\, 2\pi R\sin\theta}{4\pi R^2}\, dM = -Gm(dM)\int \frac{\sin\theta\, d\theta}{2\ell}$$

$$= \frac{-GmdM}{2Rr}\int \frac{\ell(\theta)\frac{d\ell(\theta)}{d\theta}d\theta}{\ell(\theta)}$$

$$= -\frac{GmdM}{2Rr}\int \frac{d\ell(\theta)}{d\theta}d\theta$$

$$= \frac{GmdM}{2Rr}\ell$$

since $\ell^2(\theta) = R^2 + r^2 - 2Rr\cos\theta$

f' $2\ell(\theta)\frac{d\ell(\theta)}{d\theta} = 2Rr\sin\theta$

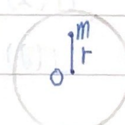
$\frac{1}{Rr}\ell(\theta)\frac{d\ell(\theta)}{d\theta}d\theta = \sin\theta\, d\theta$

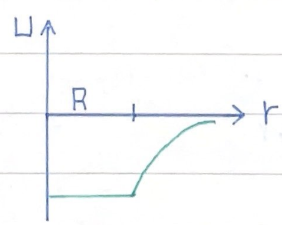
$r > R$

$$-\frac{GmdM}{2Rr}\ell\Big|_{r-R}^{r+R} = -\frac{GmdM}{r}$$

$r < R$

$$\frac{GmdM}{2Rr}\ell\Big|_{R-r}^{R+r} = -\frac{GmdM}{R}$$





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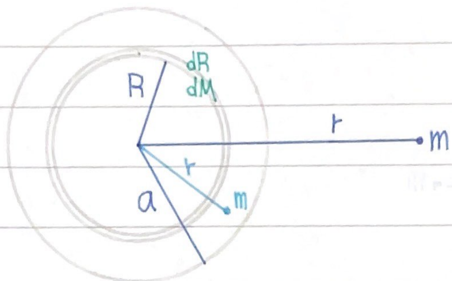


$$dM = \overset{\rho}{\boxed{\frac{M}{\frac{4}{3}\pi a^3}}}\, 4\pi R^2 dR$$

$r > a$

$$-\frac{Gm}{r}\int dM = -\frac{GMm}{r}$$

$r < a$

$0 < R < r$

$$-\frac{Gm}{r}\int dM = -\frac{Gm}{r}\frac{M}{\frac{4}{3}\pi a^3}\frac{4\pi}{3}r^3$$

$$= -Gm4\pi\rho\frac{r^3}{3}$$

$r < R < a$

$$-Gm\int\frac{dM}{R}$$

$$= -Gm\int\frac{\rho 4\pi R^2}{R}dR$$

$$= -Gm\rho 4\pi\int_r^a R\,dR$$

$$= -Gm\rho 4\pi\left(\frac{a^2}{2} - \frac{r^2}{2}\right)$$

$r > a$

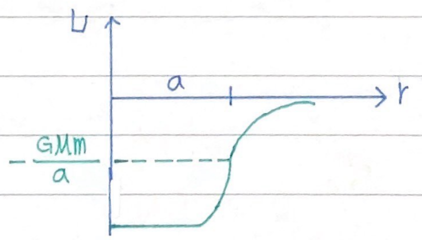
$$U(r) = -\frac{GMm}{r}$$

$r < a$

$$U(r) = -Gm4\pi\rho\frac{a^2}{2} + Gm4\pi\rho\frac{r^2}{6}$$

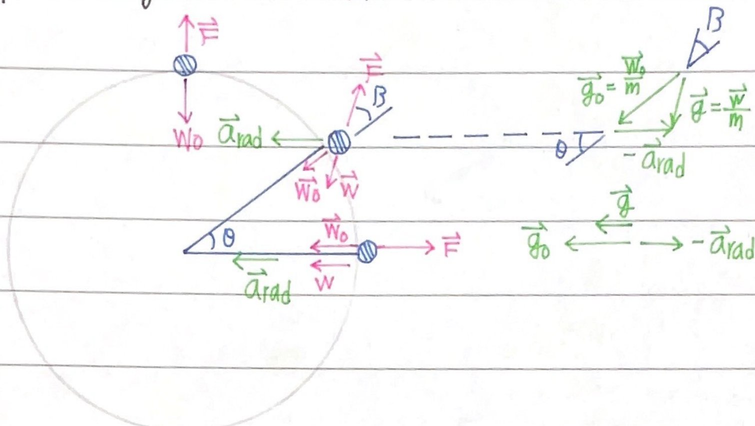
$$U(a) = -Gm4\pi\rho\frac{a^3}{3} = -\frac{GMm}{a}$$

$$U(0) = -Gm4\pi\rho\frac{a^2}{2}$$



SEASON

Apparent Weight and the Earth's Rotation



$\vec{W_0}$ ture weight
$\vec{F}$ force exerted by spring scale on object
$\vec{W}$ apparent weight opposite of $\vec{F}$

$W_0 - F = \frac{mv^2}{R_E}$ at the equator $\quad W = W_0 - \frac{mV^2}{R_E} \underset{\div m}{\Rightarrow} g = g_0 - \frac{V^2}{R_E}$

$\vec{W} = \vec{W_0} - m\vec{a}_{rad} = m\vec{g_0} - m\vec{a}_{rad}$

Black Holes

escape speed $= \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R > c$

$C = \sqrt{\frac{2GM}{R_S}}$ $\quad r < R_S \Rightarrow$ black holes

R_S → Schwarzschild radius

$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{Gm_X}}$