Angular Momentum

$$\vec{l} = \vec{r} \times \vec{p}$$
$$= (x\vec{i} + y\vec{j} + z\vec{k}) \times (p_x\vec{i} + p_y\vec{j} + p_z\vec{k}) = (yp_z - zp_y)\vec{i} + (zp_x - xp_z)\vec{j} + (xp_y - yp_x)\vec{k}$$

$\Rightarrow l_x = yp_z - zp_y \qquad l_y = zp_x - xp_z \qquad l_z = xp_y - yp_x$

the magnitude of $\vec{l}$, $l = \sqrt{l_x^2 + l_y^2 + l_z^2}$

$\Rightarrow l_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad l_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad l_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$[l_x, l_y] = [yp_z - zp_y, zp_x - xp_z] \qquad [A, B+C] = [A,B] + [A,C]$

$= [yp_z, zp_x] - [yp_z, xp_z]^{\to 0} - [zp_y, zp_x]^{\to 0} + [zp_y, xp_z]$

$= y[p_z, z]p_x + xp_y[z, p_z] \qquad = i\hbar(-yp_x + xp_y)$

$= i\hbar l_z$

similarly, $[l_x, l_y] = i\hbar l_z \qquad [l_y, l_z] = i\hbar l_x \qquad [l_z, l_x] = i\hbar l_y \qquad \vec{l}\times\vec{l} = i\hbar\vec{l}$

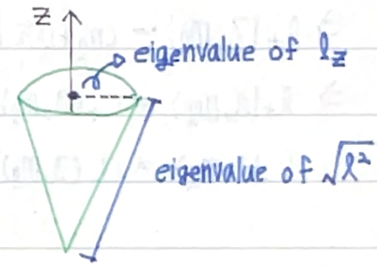
is a vector operator
not a classical operator

$[l^2, l_z] = [l_x^2 + l_y^2 + l_z^2, l_z] \qquad [AB, C] = A[B,C] + [A,C]B$

$= [l_x^2, l_x]^{\to 0} + [l_y^2, l_x] + [l_z^2, l_z] = l_y[l_y, l_x] + [l_y, l_x]l_y + l_z[l_z, l_x] + [l_z, l_x]l_z$

$= -i\hbar l_y l_z - i\hbar l_z l_y + i\hbar l_z l_y + i\hbar l_y l_z$

$= 0$



similarly, $[l^2, l_x] = [l^2, l_y] = [l^2, l_z] = 0$

or $[l^2, \vec{L}] = 0$

introduce linear combinations of the angular momentum operator, called shift operator

raising operator $l_+ = l_x + il_y \qquad l_x = \frac{l_+ + l_-}{2}$

lowering operator $l_- = l_x - il_y \qquad l_y = \frac{l_+ - l_-}{2i}$

$[l_z, l_+] = [l_z, l_x] + i[l_z, l_y]$

$= i\hbar l_y + \hbar l_x \qquad = \hbar l_+$

similarly, $[l_z, l_+] = \hbar l_+ \qquad [l_z, l_-] = -\hbar l_- \qquad [l_+, l_-] = 2\hbar l_z$

$[l^2, l_+] = [l^2, l_-] = 0$

$\ell_z |\lambda, m_\ell\rangle = \overbrace{m_\ell \hbar}^{\mu} |\lambda, m_\ell\rangle$

because ℓ^2 commutes with ℓ_z, the state $|\lambda, m_\ell\rangle$ is also an eigenstate of ℓ^2

$\ell^2 |\lambda, m_\ell\rangle = \overbrace{f(\lambda, m_\ell)\hbar^2}^{\lambda} |\lambda, m_\ell\rangle$

$\Rightarrow$ $\because$ ℓ^2 is hermitian, $\therefore f \in \mathbb{R}$

$\because$ ℓ^2 must have the same dimensions as $\hbar^2$, f is unitless

$\because$ ℓ^2 is the sum of squares of Hermitian operators, its eigenvalues are non-(−)

$(\ell^2 - \ell_z^2)|\lambda, m_\ell\rangle = (\ell_x^2 + \ell_y^2)|\lambda, m_\ell\rangle \geq 0$

$= [f(\lambda, m_\ell) - m_\ell^2]\hbar^2 |\lambda, m_\ell\rangle$

If $|\omega\rangle$ is an eigenstate of Ω
$\Omega^2|\omega\rangle = \Omega\,\omega|\omega\rangle$
$= \omega\,\Omega|\omega\rangle = \omega^2|\omega\rangle$
$\ell_z^2|\lambda, m_\ell\rangle = m_\ell^2\hbar^2|\lambda, m_\ell\rangle$

$\Rightarrow f(\lambda, m_\ell) \geq m_\ell^2$

$\ell^2 \ell_+ |\lambda, m_\ell\rangle = \ell_+ \ell^2 |\lambda, m_\ell\rangle = \ell_+ f(\lambda, m_\ell)\hbar^2 |\lambda, m_\ell\rangle = \overbrace{f(\lambda, m_\ell)\hbar^2}^{\lambda} \ell_+ |\lambda, m_\ell\rangle$

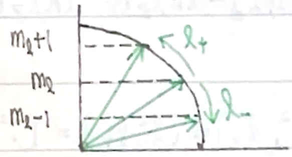
$\ell_z \ell_+ |\lambda, m_\ell\rangle = (\ell_+ \ell_z + [\ell_z, \ell_+])|\lambda, m_\ell\rangle = (\ell_+ \ell_z + \hbar \ell_+)|\lambda, m_\ell\rangle$

$= (\ell_+ m_\ell \hbar + \hbar \ell_+) \quad |\lambda, m_\ell\rangle = \overbrace{(m_\ell + 1)\hbar}^{\mu} \ell_+ |\lambda, m_\ell\rangle$

$\Rightarrow \ell_z |\lambda, m_\ell + 1\rangle = (m_\ell + 1)\hbar |\lambda, m_\ell + 1\rangle$

$\Rightarrow \ell_+ |\lambda, m_\ell\rangle = C_+(\lambda, m_\ell)\hbar |\lambda, m_\ell + 1\rangle$

$\ell_- |\lambda, m_\ell\rangle = C_-(\lambda, m_\ell)\hbar |\lambda, m_\ell - 1\rangle$



m_ℓ must have a maximum value which we shall denote ℓ

$\ell_+ |\lambda, \ell\rangle = 0$ $\quad$ $\ell_z |\lambda, \ell\rangle = \hbar \ell |\lambda, \ell\rangle$ $\quad$ $\ell^2 |\lambda, \ell\rangle = \lambda |\lambda, \ell\rangle$

$\underline{\ell_- \ell_+} |\lambda, \ell\rangle = 0 = (\ell^2 - \ell_z^2 - \hbar \ell_z)|\lambda, \ell\rangle$

$(\ell_x - i\ell_y)(\ell_+ + i\ell_y) = \ell_x^2 + \ell_y^2 + i\ell_x\ell_y - i\ell_y\ell_x$
$= \ell_x^2 + \ell_y^2 + i[\ell_x, \ell_y] = \ell^2 - \ell_z^2 + i\,i\hbar\ell_z$

$\ell^2 |\lambda, \ell\rangle = (\ell_z^2 + \hbar \ell_z)|\lambda, \ell\rangle = (\ell^2 + \ell)\hbar^2 |\lambda, \ell\rangle \Rightarrow f(\lambda, \ell) = \ell(\ell+1)$

$= (\ell_-\ell_+ + \ell_z^2 + \hbar \ell_z)|\lambda, \ell\rangle$

$= (0 + \hbar^2 \ell^2 + \hbar^2 \ell)|\lambda, \ell\rangle$

$= \hbar^2 \ell(\ell+1) \; |\lambda, \ell\rangle$

$f(\lambda, m_\ell) = \ell(\ell+1)$

m_ℓ must have a min. value which we shall denote $k = -\ell$

$l_-|\lambda, k\rangle = 0 \Rightarrow f(\lambda, k) = k(k-1)$

$l(l+1) = k(k-1)$, $k = \underset{\text{acceptable}}{\underline{-l}}$ or $l+1$

therefore, $f(\lambda, m_l) = l(l+1)$ for $m_l = l, l-1, \cdots, -l$

$l^2|l, m_l\rangle = l(l+1)\hbar^2|l, m_l\rangle$

$l_z|l, m_l\rangle = m_l\hbar\ |l, m_l\rangle$

$l-(-l) = N$

$l = \frac{N}{2}$

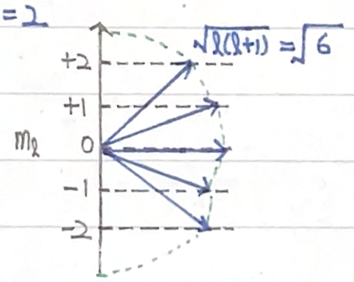
* the magnitude of l is $\sqrt{l(l+1)}\hbar$ with $l = 0, \frac{1}{2}, 1, \cdots$
* the component on z-axis is limited to the $2l+1$ values $m_l\hbar$ with $m_l = +l, l-1, \cdots, -l$
* for orbital angular momenta, l only integral $|l, m_l\rangle$
* for internal angular momentum (spin), l possibly half-integral $|s, m_s\rangle$
* consider both of angular momentum $|j, m_j\rangle$

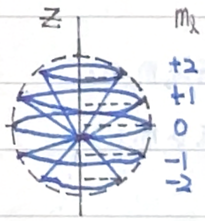
$\Rightarrow j^2|j, m_j\rangle = j(j+1)\hbar^2|j, m_j\rangle$

$m_j = j, j-1, \cdots, -j$

$j_z|j, m_j\rangle = m_j\hbar\ |j, m_j\rangle$

if $l = 2$





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$j_\pm |j, m_j\rangle = C_\pm(j, m_j)\hbar\, |j, m_j \pm 1\rangle$ $\langle j', m_j' | j, m_j\rangle = \delta_{j'j}\,\delta_{m_j' m_j}$

$\langle j, m_j+1 | j_+ | j, m_j\rangle = C_+(j, m_j)\hbar$ find the coefficients

$\langle j, m_j-1 | j_- | j, m_j\rangle = C_-(j, m_j)\hbar$

$j_- j_+ |j, m_j\rangle = (j^2 - j_z^2 - \hbar j_z)|j, m_j\rangle = [j(j+1) - m_j(m_j+1)]\hbar^2 |j, m_j\rangle$
$= j_- C_+(j, m_j)\hbar |j, m_j+1\rangle = C_+(j, m_j) C_-(j, m_j+1)\hbar^2 |j, m_j\rangle$

$C_+(j, m_j) C_-(j, m_j+1) = j(j+1) - m_j(m_j+1)$

similarly, $\langle j, m_j | j_- | j, m_j+1\rangle = \langle j, m_j | j_x - i j_y | j, m_j+1\rangle$
$= \langle j, m_j | j_x | j, m_j\rangle - i\langle j, m_j | j_y | j, m_j+1\rangle$
$= \langle j, m_j+1 | j_x | j, m_j\rangle^* - i\langle j, m_j+1 | j_y | j, m_j\rangle^*$
$= [\langle j, m_j+1 | j_x | j, m_j\rangle + i\langle j, m_j+1 | j_y | j, m_j\rangle]^*$
$= \langle j, m_j+1 | j_+ | j, m_j\rangle^*$

two operators A and B are each other's Hermitian conjugate if $\langle a|A|b\rangle = \langle b|B|a\rangle^*$

$\Rightarrow C_-(j, m_j+1) = C_+^*(j, m_j)$

$C_+(j, m_j) C_-(j, m_j+1) = |C_+(j, m_j)|^2 = j(j+1) - m_j(m_j+1)$, $C_+(j, m_j) = \sqrt{j(j+1) - m_j(m_j+1)}$

similarly $C_-(j, m_j) = \sqrt{j(j+1) - m_j(m_j-1)}$

$j_+ |j, m_j\rangle = \sqrt{j(j+1) - m_j(m_j+1)}\,\hbar\, |j, m_j+1\rangle$

$j_- |j, m_j\rangle = \sqrt{j(j+1) - m_j(m_j-1)}\,\hbar\, |j, m_j+1\rangle$

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$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}(\vec{r} \times \nabla)$$

$$= \frac{\hbar}{i}\left[r(\hat{r} \times \hat{r})\frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta})\frac{\partial}{\partial \theta} + (\hat{r} \times \hat{\theta})\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right] = \frac{\hbar}{i}\left(\hat{\varphi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)$$

$\hat{\theta} = (\cos\theta\cos\phi)\hat{i} + (\cos\theta\sin\phi)\hat{j} - \sin\theta\,\hat{k}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$$= \frac{\hbar}{i}\left[(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\frac{\partial}{\partial \theta} - (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

$$\Rightarrow l_x = -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial \theta} + \cot\theta\cos\varphi\frac{\partial}{\partial \varphi}\right)$$

$$l_y = \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial \theta} - \cot\theta\sin\varphi\frac{\partial}{\partial \varphi}\right)$$

$$l_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}$$

$$l_+ = l_x + il_y = \frac{\hbar}{i}\left[(-\sin\phi + i\cos\phi)\frac{\partial}{\partial \theta} - (\cos\phi + i\sin\phi\cot\theta\frac{\partial}{\partial \phi})\right]$$

$\cos\phi + i\sin\phi = e^{i\phi}$

$$= \hbar e^{i\phi}\left(\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \phi}\right)$$

$$l_- = -\hbar e^{-i\phi}\left(\frac{\partial}{\partial \theta} - i\cot\theta\frac{\partial}{\partial \phi}\right)$$

then $l_+|l,l\rangle = 0 = \hbar e^{i\varphi}\left(\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \varphi}\right)\psi_{ll}(\theta,\varphi) = 0$, $\psi_{ll}(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$

$$\Rightarrow \frac{\tan\theta}{\Theta}\frac{d\Theta}{d\theta} = -\frac{i}{\Phi}\frac{d\Phi}{d\varphi} = c$$

$$\Rightarrow \tan\theta\frac{d\Theta}{d\theta} = c\Theta \qquad \frac{d\Phi}{d\varphi} = ic\Phi \ , \qquad \Theta \propto \sin^c\theta \qquad \Phi \propto e^{ic\varphi}$$

$\because\ l_z\psi_{ll} = l\hbar\psi_{ll} \qquad \therefore\ \psi_{ll} = \underline{N}\sin^l\theta\, e^{il\varphi}$

Normalization constant

Ex: $l_-|l,l\rangle = c_-(l,l)\hbar|l,l-1\rangle = \sqrt{l(l+1)-l(l-1)}\,\hbar|l,l-1\rangle = \sqrt{2l}\,\hbar|l,l-1\rangle$

$$= -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial \theta} - i\cot\theta\frac{\partial}{\partial \varphi}\right)N\sin^l e^{il\varphi}$$

$$= -N\hbar e^{-i\varphi}(l\sin^{l-1}\theta\cos\theta - i\,il\cot\theta\sin^l\theta)e^{il\varphi}$$

$$= -2Nl\hbar\sin^{l-1}\theta\cos\theta\, e^{i(l-1)\varphi}$$

$$\Rightarrow \psi_{ll-1} = -\sqrt{2l}\,N\sin^{l-1}\theta\cos\theta\, e^{i(l-1)\varphi}$$