conservation of linear momentum : translationally invariant

$\vec{r}_\alpha \to \vec{r}_\alpha + \delta\vec{r}$ $\quad U(\vec{r}_1+\epsilon \cdots, t) = U(\vec{r}_1 \cdots \vec{r}_N, t)$, $\delta U=0$, $\delta E_k=0$, $\delta L=0$

let us write the Lagrangian in terms of rectangular coordinates $L = L(x_i, \frac{dx_i}{dt})$

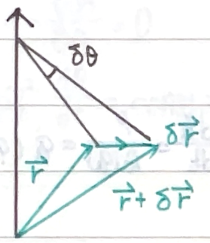
$\delta\vec{r} = \sum_i \delta x_i \vec{e}_i$

$\delta \frac{dx_i}{dt} = \frac{d}{dt}\delta x_i \equiv 0$ because we consider only a varied displacement

$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0$ $\quad \frac{\partial L}{\partial x_i}=0$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}=0$, $\frac{\partial L}{\partial \dot{x}_i}$ = constant

$\frac{\partial(E_k-U)}{\partial \dot{x}_i} = \frac{\partial E_k}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2} m \sum_i \left(\frac{dx_i}{dt}\right)^2\right) = m\frac{dx_i}{dt} = p_i$ = constant

conservation of angular momentum



$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$ $\quad \delta\frac{d\vec{r}}{dt} = \delta\frac{d\theta}{dt}\times\frac{d\vec{r}}{dt}$

$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$ and $p_i = \frac{\partial L}{\partial \dot{x}_i}$ $\quad \frac{dp_i}{dt} = \frac{\partial L}{\partial x_i}$

$= \sum_i \frac{dp_i}{dt}\delta x_i + \sum_i p_i \delta \dot{x}_i = 0$, $\vec{p}_i\cdot\delta\vec{r} + \vec{p}\cdot\delta\vec{r} = 0$

$\to \frac{d\vec{p}}{dt}\cdot(\delta\vec{\theta}\times\vec{r}) + \vec{p}\cdot(\delta\vec{\theta}\times\frac{d\vec{r}}{dt}) = 0$, $\delta\vec{\theta}\cdot(\vec{r}\times\frac{d\vec{p}}{dt}) + \delta\vec{\theta}\cdot(\frac{d\vec{r}}{dt}\times\vec{p}) = 0$

$\delta\vec{\theta}\cdot\frac{d}{dt}(\vec{r}\times\vec{p}) = 0$, $\frac{d}{dt}(\vec{r}\times\vec{p}) = 0$, $\vec{r}\times\vec{p}$ = constant

$$\vec{r}_1 \to \vec{r}_1 + \epsilon \quad \cdots \quad \vec{r}_N \to \vec{r}_N + \epsilon \qquad N \text{ particles}$$

$$U(\vec{r}_1+\epsilon \dots \vec{r}_N+\epsilon, t) = U(\vec{r}_1 \cdots \vec{r}_N, t)$$

$$\delta U = 0 \quad \delta L = 0 \quad \delta E_k = 0$$

$$\delta L = \epsilon \frac{\partial L}{\partial x_1} + \cdots \epsilon \frac{\partial L}{\partial x_N} = 0, \quad \sum_{\alpha=1}^{N} \frac{\partial L}{\partial x_\alpha} = 0$$

$$\frac{\partial L}{\partial x_\alpha} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\alpha} = \frac{d}{dt} P_{\alpha x}, \quad \sum_{\alpha=1}^{N} \frac{d}{dt} P_x = 0$$

conservation of l

$\vec{r}_\alpha \to \vec{r}_\alpha + \delta$ $\quad$), $\delta L = 0$

let us write $\quad$ $\frac{dx_i}{dt}$) $\quad$ $\frac{dz}{dt}$

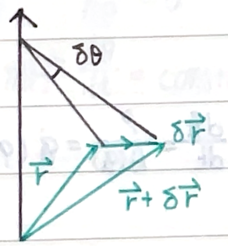
$\delta \vec{r} = \sum_i \epsilon$

$$\delta \frac{dx_i}{dt} = \frac{d}{dt}\delta x_i \equiv 0 \quad \text{because we consider only a varied displacement}$$

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0 \qquad \frac{\partial L}{\partial x_i} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial x_i'} = 0, \quad \frac{\partial L}{\partial x_i'} = \text{constant}$$

$$\frac{\partial(E_k - U)}{\partial x_i'} = \frac{\partial E_k}{\partial x_i'} = \frac{\partial}{\partial x_i'}\left(\frac{1}{2}m\sum_i\left(\frac{dx_i}{dt}\right)^2\right) = m\frac{dx_i}{dt} = p_i = \text{constant}$$

conservation of angular momentum



$$\delta\vec{r} = \delta\vec{\theta}\times\vec{r} \qquad \delta\frac{d\vec{r}}{dt} = \delta\frac{d\vec{\theta}}{dt}\times\frac{d\vec{r}}{dt}$$

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial x_i'}\delta x_i' = 0 \quad \text{and} \quad p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \frac{dp_i}{dt} = \frac{\partial L}{\partial x_i}$$

$$= \sum_i \frac{dp_i}{dt}\delta x_i + \sum_i p_i \delta\dot{x}_i = 0, \quad \dot{\vec{p}}_i\cdot\delta\vec{r} + \vec{p}\cdot\delta\dot{\vec{r}} = 0$$

$$\Rightarrow \frac{d\vec{p}}{dt}\cdot(\delta\vec{\theta}\times\vec{r}) + \vec{p}\cdot\left(\delta\vec{\theta}\times\frac{d\vec{r}}{dt}\right) = 0, \quad \delta\vec{\theta}\cdot\left(\vec{r}\times\frac{d\vec{p}}{dt}\right) + \delta\vec{\theta}\cdot\left(\frac{d\vec{r}}{dt}\times\vec{p}\right) = 0$$

$$\delta\vec{\theta}\cdot\frac{d}{dt}(\vec{r}\times\vec{p}) = 0, \quad \frac{d}{dt}(\vec{r}\times\vec{p}) = 0, \quad \vec{r}\times\vec{p} = \text{constant}$$

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Canonical Equations of Motion - Hamilotonian Dynamics

generalized momenta $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ $\qquad \frac{dp_j}{dt} = \frac{\partial L}{\partial q_j}$

$H = \sum_j p_j \frac{dq_j}{dt} - L \qquad \dot{q}_j = \frac{dq_j}{dt}(q_k, p_k, t)$

$H(q_k, p_k, t) = \sum_j p_j \frac{dq_j}{dt} - L(q_k, \dot{q}_k, t)$

$dH = \sum_k \left(\frac{dq_k}{dt} dp_k + p_k \frac{d^2 q_k}{dt^2} - \frac{\partial L}{\partial q_k} dq_k - \cancel{\frac{\partial L}{\partial \dot{q}_k} \frac{d^2 q_k}{dt^2}} \right) - \frac{\partial L}{\partial t} dt$

$= \sum_k \left(\frac{dq_k}{dt} dp_k - \frac{dp_k}{dt} dq_k \right) - \frac{\partial L}{\partial t} dt$

$\Rightarrow \underbrace{\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \qquad -\frac{dp_k}{dt} = \frac{\partial H}{\partial q_k}}_{\text{Hamilton's equations of motion}}, \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}, \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t}$

Lagrange $(q_i, \dot{q}_i)$ $\quad$ Hamilton (q_i, p_i)

$L = E_k - U = \frac{1}{2} A(q) \left(\frac{dq}{dt}\right)^2 - U(q), \quad p = \frac{\partial L}{\partial \dot{q}} = A(q) \frac{dq}{dt} \qquad \frac{dq}{dt} = \frac{p}{A(q)} = \dot{q}(q, p)$

$H = \frac{dq}{dt} p_i - L = A(q) \left(\frac{dq}{dt}\right)^2 - L = 2E_k - (E_k - U) = E_k + U$

$\frac{\partial H}{\partial q} = p \cancel{\frac{\partial \dot{q}}{\partial q}} - \left[\frac{\partial L}{\partial q} + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} \cancel{\frac{\partial \dot{q}}{\partial q}} \right] = -\frac{\partial L}{\partial q} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = -\frac{d}{dt} p$

$\frac{\partial H}{\partial p} = \left(\frac{dq}{dt} + p \cancel{\frac{\partial \dot{q}}{\partial p}} \right) - \cancel{\frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial p}} = \frac{dq}{dt}$

$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\underbrace{\frac{\partial H}{\partial q_i}}_{-\frac{dp_i}{dt}} \frac{dq_i}{dt} + \underbrace{\frac{\partial H}{\partial p_i}}_{\frac{dq_i}{dt}} \frac{dp_i}{dt} \right) + \frac{\partial H}{\partial t}$