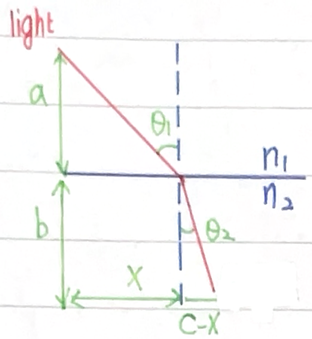
Some Methods in the Calculus of Variations

Introduction

Fermat's Principle

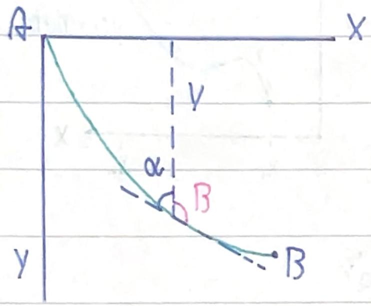


$$T(x) = \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{(c-x)^2+b^2}}{v_2} \quad \text{let } \frac{dT}{dx}=0$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+a^2}} - \frac{c-x}{v_2\sqrt{(c-x)^2+b^2}} = 0$$

$$\Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

Brachistochrone by Johann Bernoulli method



by Fermat's principle, we know $\frac{\sin\alpha}{v}$ = constant

$$\frac{1}{2}mv^2 = mgy, \quad v=\sqrt{2gy}$$

$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$$

since $\frac{\sin\alpha}{v}$ = constant, $\frac{1}{\sqrt{2gy(1+(\frac{dy}{dx})^2)}} = C$, $y(1+(\frac{dy}{dx})^2) = k$

$$\left(\frac{dy}{dx}\right)^2 = \frac{k}{y}-1, \quad \frac{dy}{dx} = \sqrt{\frac{k-y}{y}}, \quad dx = \sqrt{\frac{y}{k-y}}\,dy$$

$$\int dx = \int\sqrt{\frac{y}{k-y}}\,dy \qquad \text{let } y=k\sin^2\theta \quad dy = 2k\sin\theta\cos\theta\,d\theta$$

$$x = \int \tan\theta \cdot 2k\sin\theta\cos\theta\,d\theta$$

$$= \int 2k\sin^2\theta\,d\theta = k\int(1-\cos 2\theta)d\theta$$

$$= k(\theta - \tfrac{1}{2}\sin 2\theta) = \tfrac{1}{2}k(2\theta - \sin 2\theta)$$

$$y = k\sin^2\theta = k\left(\frac{1-\cos 2\theta}{2}\right) = \frac{k}{2}(1-\cos 2\theta)$$

let $a = \frac{k}{2}$, $\phi = 2\theta$

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$$

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Statement of the Problem

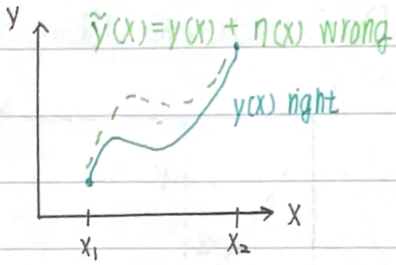
$$\tilde{F}(X)-F(X)=D(X)\ ,\quad \epsilon\frac{D(X)}{\epsilon}=\epsilon\eta(X)$$

$$\Rightarrow \underset{\text{所有集合}}{\tilde{F}(X)}=\underset{\text{最佳解}}{F(X)}+\epsilon\eta(X)$$

Euler's Equation

$$T(\epsilon)=\overset{\text{函数的函数}}{T(\tilde{y})}=\int_{X_1}^{X_2}\sqrt{\frac{1+(\tilde{y}')^2}{2y\tilde{y}}}\,dX=\int_{X_1}^{X_2}F(X,\tilde{y},\tilde{y}')dX$$

$$\tilde{y}=y+\epsilon\eta \qquad \tilde{y}'=y'+\epsilon\eta'$$



$$\Rightarrow \int_{X_1}^{X_2}F(X,y+\epsilon\eta,y'+\epsilon\eta')dX$$

$$I(\epsilon)=\int_{X_1}^{X_2}F(X,\underbrace{y+\epsilon\eta}_{u},\underbrace{y'+\epsilon\eta'}_{V})\,dX=\int_{X_1}^{X_2}F(X,u,v)dX$$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0}=0=\frac{\partial}{\partial\epsilon}\int_{X_1}^{X_2}F(X,u,v)dX=\int_{X_1}^{X_2}\frac{\partial F(X,u,v)}{\partial\epsilon}dX$$

$$=\int_{X_1}^{X_2}\frac{\partial F}{\partial X}\underbrace{\frac{\partial X}{\partial\epsilon}}_{0}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial\epsilon}+\frac{\partial F}{\partial V}\frac{\partial V}{\partial\epsilon}\,dX \qquad u=y+\epsilon\eta \quad \frac{\partial u}{\partial\epsilon}=\eta$$

$$V=y'+\epsilon\eta' \quad \frac{\partial V}{\partial\epsilon}=\eta'$$

$$=\int_{X_1}^{X_2}\frac{\partial F}{\partial u}\eta+\frac{\partial F}{\partial V}\eta'\,dX$$

$$\because \epsilon=0 \qquad =\int_{X_1}^{X_2}\frac{\partial F}{\partial y}\eta+\underbrace{\frac{\partial F}{\partial y'}}_{u}\underbrace{\frac{d\eta}{dX}}_{dv}dX=0 \quad \text{and} \quad \int u\,dv=uv-\int v\,du$$

$$=\int_{X_1}^{X_2}\frac{\partial F}{\partial y}\eta+\underbrace{\left.\frac{\partial F}{\partial y'}\eta\right|_{X_1}^{X_2}}_{0}-\int\frac{d}{dX}\left(\frac{\partial F}{\partial y'}\right)\eta\,dX$$

$$=\int_{X_1}^{X_2}\frac{\partial F}{\partial y}\eta-\frac{d}{dX}\frac{\partial F}{\partial y'}\eta\,dX$$

$$=\int_{X_1}^{X_2}\left(\frac{\partial F}{\partial y}-\frac{d}{dX}\frac{\partial F}{\partial y'}\right)\eta\,dX=0$$

$$\Rightarrow \frac{\partial F}{\partial y}-\frac{d}{dX}\frac{\partial F}{\partial y'}=0$$ Euler-Lagrange equation

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Brachistochrone

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$t_{AB} = \int_{x_2}^{x_1} \frac{ds}{v} = \int_{x_2}^{x_1} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} \quad \text{and} \quad F = \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0$$

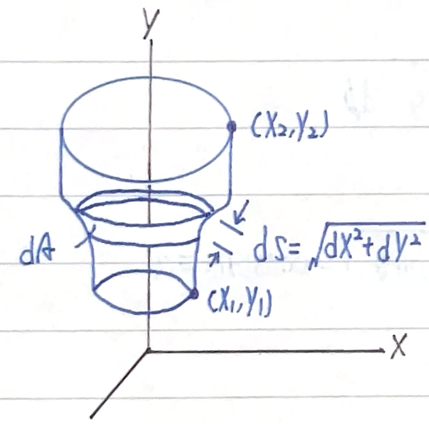
$$\Rightarrow \frac{d}{dx}\frac{\partial F}{\partial y'} = 0, \quad \frac{\partial F}{\partial y'} = \text{constant} = \frac{1}{\sqrt{2a}}$$

$$\Rightarrow \frac{\left(\frac{dy}{dx}\right)^2}{x\left(1+\left(\frac{dy}{dx}\right)^2\right)} = 2a, \quad y = \int \frac{x\,dx}{\sqrt{2ax-x^2}} \quad \text{let } \begin{array}{l} x = a(1-\cos\theta) \\ dx = a\sin\theta\,d\theta \end{array}$$

$$\frac{dy}{dx} = \sqrt{\frac{x}{2a-x}}$$

$$\Rightarrow y = \int a(1-\cos\theta)\,d\theta, \quad \begin{array}{l} y = a(\theta-\sin\theta)+C \\ x = a(1-\cos\theta) \end{array} \quad : \text{cycloid}$$

minimal surface



$$dA = 2\pi x\,ds = 2\pi x\sqrt{dx^2+dy^2}$$

$$A = 2\pi\int_{x_1}^{x_2} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\text{let } f = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right] = 0$$

$$\frac{x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \text{constant} \equiv a$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}} \qquad y = \int \frac{a\,dx}{\sqrt{x^2-a^2}}$$

$$y = a\cosh^{-1}\left(\frac{x}{a}\right) + b$$

$$x = a\cosh\left(\frac{y-b}{a}\right)$$