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Wave Equation

$$\frac{d^2 q_j}{dt^2} = \frac{\tau}{md}(q_{j-1} - 2q_j + q_{j+1}), \quad \frac{m}{d}\frac{d^2 q_j}{dt^2} = \frac{\tau}{d}\frac{q_{j-1}-q_j}{d} - \frac{\tau}{d}\frac{q_j - q_{j+1}}{d}$$

$\frac{m}{d} = \rho$

as $d \to 0$ $\quad \frac{q_j - q_{j+1}}{d} \to \frac{q(x) - q(x+d)}{d} \to -\frac{\partial q}{\partial x}\Big|_{x+\frac{d}{2}}$

$$\frac{q_{j-1} - q_j}{d} \to \frac{q(x-d) - q(x)}{d} \to -\frac{\partial q}{\partial x}\Big|_{x-\frac{d}{2}}$$

$$\Rightarrow \lim_{d\to 0} \tau \left(\frac{\frac{\partial q}{\partial x}\Big|_{x+\frac{d}{2}} - \frac{\partial q}{\partial x}\Big|_{x-\frac{d}{2}}}{d} \right) = \tau \frac{\partial^2 q}{\partial x^2}\Big|_x = \tau \frac{\partial^2 q}{\partial x^2}$$

$$\Rightarrow \rho \frac{d^2 q}{dt^2} = \tau \frac{\partial^2 q}{\partial x^2}, \quad \frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2}$$

Forced and Damped Motion

$$L = E_k - U = \frac{\rho b}{4}\sum_r \left(\frac{d\eta_r}{dt}\right)^2 - \frac{\rho b}{4}\sum_r \omega_r^2 \eta_r^2 = \frac{\rho b}{4}\sum_r \left[\left(\frac{d\eta_r}{dt}\right)^2 - \omega_r^2\eta_r^2\right]$$

now we add a force per unit length $F(x,t)$ and a damping force $\propto$ velocity

$$\Rightarrow \rho \frac{d^2 q}{dt^2} + D\frac{\partial q}{\partial t} - \tau \frac{\partial^2 q}{\partial x^2} = F(x,t)$$ solved using normal coordinate

use a solution $q(x,t) = \sum_r \eta_r(t) \sin\left(\frac{r\pi x}{b}\right)$

$$\Rightarrow \sum_{r=1}^{\infty}\left[\left(\rho\frac{d^2\eta_r}{dt} + D\frac{d\eta_r}{dt} + \frac{r^2\pi^2\tau}{b^2}\eta_r\right)\sin\left(\frac{r\pi x}{b}\right)\right] = F(x,t)$$

$\times \sin\left(\frac{s\pi x}{b}\right)$ $\big\downarrow$ integrate

$$\sum_{r=1}^{\infty}\left(\rho\frac{d^2\eta_r}{dt^2} + D\frac{d\eta_r}{dt} + \frac{r^2\pi^2\tau}{b^2}\eta_r\right)\frac{b}{2}\delta_{rs} = \int_0^b F(x,t)\sin\left(\frac{s\pi x}{b}\right)dx$$

$$\frac{d^2\eta_s}{dt^2} + \frac{D}{\rho}\frac{d\eta_s}{dt} + \frac{s^2\pi^2\tau}{\rho b^2}\eta_s = \frac{2}{\rho b}\int_0^b F(x,t)\sin\left(\frac{s\pi x}{b}\right)dx$$

Fourier coefficient of $F(x,t)$

$$f_s(t) = \int_0^b F(x,t)\sin\left(\frac{s\pi x}{b}\right)dx$$

$$\Rightarrow \frac{d^2\eta_s}{dt^2} + \frac{D}{\rho}\frac{d\eta_s}{dt} + \frac{s^2\pi^2\tau}{\rho b^2}\eta_s = \frac{2}{\rho b}f_s(t)$$

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Separation of the Wave Equation

if we require a general solution of the wave equation that is harmonic

$$\Psi(x,t)=\psi(x)e^{i\omega t}$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\omega^2}{v^2}\psi=0$$

$$\Psi_r(x,t)=\psi_r(x)e^{i\omega_r t} \qquad \Psi(x,t)=\sum_r \Psi_r(x,t)=\sum_r \psi_r(x)e^{i\omega_r t}$$

$$\Psi(x,t)\equiv\psi(x)\cdot X(t), \quad X\frac{d^2\psi}{dx^2}-\frac{\psi}{v^2}\frac{d^2\psi}{dt^2}=0, \quad \underbrace{\frac{v^2}{\psi}\frac{d^2\psi}{dx^2}}_{x\text{ alone}}=\underbrace{\frac{1}{X}\frac{d^2X}{dt^2}}_{t\text{ alone}}$$

this situation is possible only if … equal to the same constant, we choose be $-\omega^2$

$$\Rightarrow \frac{d^2\psi}{dx^2}+\frac{\omega^2}{v^2}\psi=0 \qquad \frac{d^2\psi}{dt^2}+\omega^2\psi=0$$

the solution are $\psi(x)=Ae^{i\frac{\omega}{v}x}+Be^{-i\frac{\omega}{v}x}$

$$X(t)=Ce^{i\omega t}+De^{-i\omega t}$$

$$\Psi(x,t)=\psi(x)X(t)\sim e^{\pm i\frac{\omega}{v}x}e^{\pm i\omega t}$$

$$\sim e^{\pm i\frac{\omega}{v}(x\pm vt)}$$

we select a wave propagating in a particular direction and phase

$$\Psi_r(x,t)\sim e^{-i\frac{\omega_r}{v}(x-vt)}$$

$$\frac{d^2\psi}{dx^2}+k^2\psi=0$$ time-dependent form of the 1-D wave eq. also called Helmholtz

$$\underline{k^2}=\frac{\omega^2}{v^2} \qquad \lambda=\frac{v}{\nu}=\frac{2\pi v}{\omega}, \quad k=\frac{2\pi}{\lambda}$$

propagation constant, wave number

$$\Rightarrow \Psi_r(x,t)\sim e^{\pm ik_r(x\pm vt)} \qquad \Psi(x,t)\sim e^{-ik(x-vt)}=e^{i(\omega t-kx)}$$

assumption two traveling wave equal magnitude, moving in opposite direction

$$\Psi=\Psi_++\Psi_-=Ae^{-ik(x+vt)}+Ae^{-ik(x-vt)}$$

$$=Ae^{-ikx}(e^{i\omega t}+e^{-i\omega t})=2Ae^{-ikx}\cos\omega t$$

real part is $2A\cos kx\cos\omega t$

Phase Velocity, Dispersion, and Attenuation

we restrict our attention to the particular combination $\Psi(x,t) = A e^{i(\omega t - kx)}$

if the exponential remains constants, then $\Psi(x,t)$ also remains constant

and $\phi \equiv \omega t - kx$ phase

to ensure ϕ = constant, we set $d\phi = 0$ or $\omega dt = k dx$

$$V = \frac{dx}{dt} = \frac{\omega}{k} = \nu$$

frequency for the rth mode of the loaded string when terminated at both ends

$$\omega_r = 2\sqrt{\frac{\tau}{md}} \sin\left[\frac{r\pi}{2(n+1)}\right] \quad 1 \le r \le n \quad k_{max} = \frac{\pi}{d}$$

when $r=2$, then $L = \lambda$ $\quad \lambda_r = \frac{2L}{r}$

$$\frac{r\pi}{2(n+1)} = \frac{r\pi d}{2d(n+1)} = \frac{r\pi d}{2L} = \frac{\pi d}{\lambda_r} = \frac{k_r d}{2} \qquad \omega_r = 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{k_r d}{2}\right)$$

to study the propagation of a wave in the loaded string

initiate a disturbance by forcing one of the particles $q_0(t) = A e^{i\omega t}$

$$\omega < 2\sqrt{\frac{\tau}{md}}$$

$$V = \frac{\omega}{k} = \sqrt{\frac{\tau d}{m}} \frac{\left|\sin\left(\frac{kd}{2}\right)\right|}{\frac{kd}{2}} = V(k) \quad \text{frequency-dependent}$$

$\lambda \to \infty$ or $k \to 0$ $\quad V(\lambda \to \infty) = \sqrt{\frac{\tau d}{m}}$

$$V_{continuous} = \nu = \sqrt{\frac{\tau}{\rho}}$$

if forcing the string to vibrate at a $f > 2\sqrt{\frac{\tau}{md}}$, we allow k become complex

$k \equiv \kappa - i\beta \quad \kappa, \beta > 0$

$$\Rightarrow \omega = 2\sqrt{\frac{\tau}{md}} \sin\left[\frac{d}{2}(\kappa - i\beta)\right]$$

$$= 2\sqrt{\frac{\tau}{md}} \left(\sin\frac{dk}{2}\cos\frac{i\beta d}{2} - \cos\frac{kd}{2}\sin\frac{i\beta d}{2}\right)$$

$$= 2\sqrt{\frac{\tau}{md}} \left(\sin\frac{kd}{2}\cosh\frac{\beta d}{2} - i\cos\frac{kd}{2}\sinh\frac{\beta d}{2}\right)$$

if the frequency is to be a real quanity, the imaginary part must vanish

may have $\cos\left(\frac{kd}{2}\right) = 0$ or $\sinh\left(\frac{\beta d}{2}\right) = 0$

$\quad \hookrightarrow$ require $\beta = 0$