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The Uncertainty Principle

Generalized Uncertainty Principle

$$\sigma_A^2 = \langle (\hat{A} - \underset{a}{\langle A \rangle})^2 \rangle = \langle \Psi | (\hat{A} - a)^2 \Psi \rangle = \langle (\hat{A} - a)\Psi | (\hat{A} - a)\Psi \rangle = \langle f | f \rangle$$

likewise $\sigma_B^2 = \langle g | g \rangle \qquad g \equiv |(\hat{B} - b)\Psi\rangle$

$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2$ Schwarz inequality

$$|\langle f|g\rangle|^2 = |z|^2 = |\mathrm{Re}(z)|^2 + |\mathrm{Im}(z)|^2 \geq |\mathrm{Im}(z)|^2 = \left(\frac{z - z^*}{2i}\right)^2$$

$$\Rightarrow \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]\right)^2$$

$$\langle f|g\rangle = \langle (\hat{A} - \langle A\rangle)\Psi | (\hat{B} - \langle B\rangle)\Psi \rangle = \langle \Psi | (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi \rangle$$

$$= \langle \Psi | \hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle \Psi \rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle B\rangle \langle \Psi | \hat{A}\Psi \rangle - \langle A\rangle \langle \Psi | \hat{B}\Psi \rangle + \langle A\rangle\langle B\rangle \langle \Psi | \Psi \rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

similarly, $\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle B\rangle\langle A\rangle$

$$\Rightarrow \sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}(\langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle)\right]^2 = \left(\frac{1}{2i}\langle [\hat{A}, \hat{B}]\rangle\right)^2$$

if $\hat{A} = \hat{x} \quad \hat{B} = \hat{p} \quad [\hat{A}, \hat{B}] = [x, p] = i\hbar$

$$\Rightarrow \sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2, \quad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

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The Minimum-Uncertainty Wave Packet

when $\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$, $g(x)=cf(x)$

$$|z|^2 = [Re(z)]^2 + Im(z)^2 \geq Im(z)^2,$$

$\Rightarrow g(x) = iaf(x)$ where a is real

$$\Rightarrow \left(\frac{\hbar}{i}\frac{d}{dx} - \langle p\rangle\right)\Psi = ia(x-\langle x\rangle)\Psi$$

$$\frac{d\Psi}{dx} = \frac{i}{\hbar}(iax - ia\langle x\rangle + \langle p\rangle)\Psi = \frac{a}{\hbar}\left(-x+\langle x\rangle + \frac{i}{a}\langle p\rangle\right)\Psi$$

$$\frac{d\Psi}{\Psi} = \frac{a}{\hbar}\left(-x+\langle x\rangle+\frac{i\langle p\rangle}{a}\right)dx \ , \quad \ln\Psi = \frac{a}{\hbar}\left(-\frac{x^2}{2}+\langle x\rangle x + \frac{i\langle p\rangle}{a}x\right) + \text{constant}$$

let constant $= -\frac{\langle x\rangle^2 a}{2\hbar} + B \quad \Rightarrow \quad \ln\Psi = -\frac{a}{2\hbar}(x-\langle x\rangle)^2 + \frac{i\langle p\rangle}{\hbar}x + B$

$$\Rightarrow \Psi = e^{-\frac{a}{2\hbar}(x-\langle x\rangle)^2} e^{\frac{i\langle p\rangle x}{\hbar}} e^{B} = Ae^{-a(x-\langle x\rangle)^2/2\hbar} e^{\frac{i\langle p\rangle x}{\hbar}}$$

evidently the minimum-uncertainty wave packet is a gaussian

The Energy-Time Uncertainty Principle

measure of how fast the system is changing

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}\Psi\rangle = \left\langle\frac{\partial\Psi}{\partial t}\middle|\hat{Q}\Psi\right\rangle + \left\langle\Psi\middle|\frac{\partial\hat{Q}}{\partial t}\Psi\right\rangle + \left\langle\Psi\middle|\hat{Q}\frac{\partial\Psi}{\partial t}\right\rangle$$

$$= -\frac{1}{i\hbar}\langle\hat{H}\Psi|\hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}\Psi\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle \qquad i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

$\langle\hat{H}\Psi|\hat{Q}\Psi\rangle = \langle\Psi|\hat{H}\hat{Q}\Psi\rangle$

$$= -\frac{1}{i\hbar}\langle\Psi|(\hat{H}\hat{Q}-\hat{Q}\hat{H})\Psi\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

$$= \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle \qquad \text{generalized Ehrenfest theorem}$$

assume Q doesn't depend on t

$$\sigma_H^2\sigma_Q^2 \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right)^2 = \left(\frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}\right)^2 = \left(\frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|\right)^2$$

define $\Delta E \equiv \sigma_H$, $\Delta t \equiv \dfrac{\sigma_Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$ $\qquad \sigma_E\sigma_t \geq \frac{\hbar}{2}\frac{d\langle Q\rangle}{dt}$

$$\Rightarrow \Delta E\Delta t \geq \frac{\hbar}{2}$$