NO.

DATE / /

The Second Form of the Euler Equation

$$\frac{dF(x,y,y')}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx}$$
$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

also $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\frac{\partial F}{\partial y'}$

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{dF}{dx} - \frac{\partial F}{\partial x} - y'\frac{\partial F}{\partial y} + y'\frac{d}{dx}\frac{\partial F}{\partial y'}$$
$$= \frac{dF}{dx} - \frac{\partial F}{\partial x} + y'\underbrace{\left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right)}_{0}$$

$$\Rightarrow \frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

if $\frac{\partial F}{\partial x} = 0$, then $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$, $F - y'\frac{\partial F}{\partial y'} = \text{constant}$

geodesic problem : find the geodesic on a sphere

$ds = r\sqrt{(d\theta^2 + \sin^2\theta\, d\phi^2)}$ $\quad S = r\int_{①}^{②}\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\; d\phi$

$F = \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}$

$\frac{\partial F}{\partial \phi} = 0 \quad \Rightarrow \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta} - \frac{d\theta}{d\phi}\frac{\partial}{\partial\theta'}\left(\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\right) = \text{constant} \equiv a$

$\Rightarrow \sin^2\theta = a\sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta} \qquad \frac{d\phi}{d\theta} = \frac{a\csc^2\theta}{\sqrt{1 - a^2\csc^2\theta}}$

$\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right) + \alpha \qquad \beta^2 \equiv \frac{1-a^2}{a^2}$

$\Rightarrow \cot\theta = \beta\sin(\phi - \alpha)$

$(\beta\cos\alpha) r\sin\theta\sin\phi - (\beta\sin\alpha) r\sin\theta\cos\phi = r\cos\theta$

$\beta\cos\alpha \equiv A \qquad \beta\sin\alpha \equiv B$

$\Rightarrow A(r\sin\theta\sin\phi) - B(r\sin\theta\cos\phi) = r\cos\theta$

$\Rightarrow \quad Ay - Bx = z \quad$ the equation of a plane passing through the center of the sphere

NO.

DATE

Functions with Several Dependent Variables

$$F = F\{y_1(x), y_1'(x), y_2(x), y_2'(x), \cdots ; x\}$$
$$= F\{y_i(x), y_i'(x); x\} \qquad i = 1, 2, \cdots, n$$

$$\hat{y}_i = y_i + \epsilon\eta_i$$
$$\Rightarrow \frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} \right)\eta_i \, dx$$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0, \quad i = 1, 2, \cdots, n$$

Euler's Equations When Auxiliary Conditions Are Imposed

$$F = F\{y_i, y_i'; x\} = F\{y, y'; z, z'; x\}$$

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \overbrace{\frac{\partial F}{\partial \epsilon}}^{\eta_1} + \left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \right) \overbrace{\frac{\partial z}{\partial \epsilon}}^{\eta_2} \right] dx$$

an equation of constraint $g\{y_i; x\} = g\{y, z; x\} = 0$

$$dg = \left(\frac{\partial g}{\partial y}\overbrace{\frac{\partial y}{\partial \epsilon}}^{\eta_1} + \frac{\partial g}{\partial z}\overbrace{\frac{\partial z}{\partial \epsilon}}^{\eta_2} \right) d\epsilon = 0 \quad \text{since} \quad \frac{\partial x}{\partial \epsilon} = 0$$

$$\tilde{y} = y + \epsilon\eta_1 \qquad \tilde{z} = z + \epsilon\eta_2 \qquad \frac{\partial y}{\partial \epsilon} = \eta_1 \quad \frac{\partial z}{\partial \epsilon} = \eta_2$$

$$\Rightarrow \frac{\partial g}{\partial y}\eta_1 = -\frac{\partial g}{\partial z}\eta_2 \Rightarrow \frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right)\eta_1 + \left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \right)\eta_2 \right] dx$$

$$\frac{\eta_2}{\eta_1} = -\frac{\partial g/\partial y}{\partial g/\partial z} \Rightarrow \frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) - \left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \right)\frac{\partial g/\partial y}{\partial g/\partial z} \right]\eta_1 \, dx = 0$$

$$\Rightarrow \underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right)\frac{\partial y}{\partial g}}_{-\lambda(x)} = \underbrace{\left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \right)\frac{\partial z}{\partial g}}_{-\lambda(x)} = -\lambda(x)$$

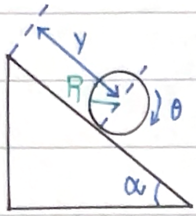
Lagrange undetermined multiplier

$$\begin{cases} \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\dfrac{\partial F}{\partial y'} + \lambda(x)\dfrac{\partial g}{\partial y} = 0 \\ \dfrac{\partial F}{\partial z} - \dfrac{d}{dx}\dfrac{\partial F}{\partial z'} + \lambda(x)\dfrac{\partial g}{\partial z} = 0 \end{cases}$$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0 \qquad \sum_i \frac{\partial g_j}{\partial y_i} dy_i = 0$$

$$g_j\{y_i ; x\} = 0$$

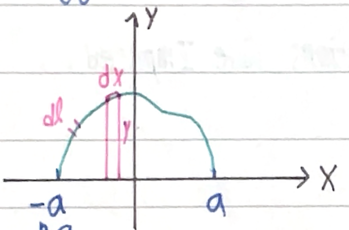
Example



$y = R\theta$ equation of constraint $g(y,\theta) = y - R\theta = 0$

$$\frac{\partial g}{\partial y} = 1 \qquad \frac{\partial g}{\partial \theta} = -R$$

Example



length l

largest area

$dA = y\,dx \qquad I = \int_{-a}^{a} y\,dx$

constraint equation $y(x)$: $y(-a)=0$, $y(a)=0$ $\quad \int dl = l$

$$dl = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \int_{-a}^{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = l \qquad y(x)=y, \quad g(x) = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{\partial f}{\partial y} = 1 \qquad \frac{\partial f}{\partial y'} = 0 \qquad \frac{\partial g}{\partial y} = 0 \qquad \frac{\partial g}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) = 0 \Rightarrow 1 - \lambda\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0, \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = \frac{1}{\lambda}$$

$$\Rightarrow \frac{\lambda y'}{\sqrt{1+(y')^2}} = x - C_1, \quad dy = \frac{\pm(x-C_1)\,dx}{\sqrt{\lambda^2-(x-C_1)^2}}, \quad y = \mp\sqrt{\lambda^2-(x-C_1)^2} + C_2$$

$$(x-C_1)^2 + (y-C_2)^2 = \lambda^2, \quad C_1 = C_2 = 0 \quad a = \lambda = \frac{l}{\pi}$$

The δ Notation

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta(x)\,dx$$ can be written as

$$\frac{\partial I}{\partial \epsilon} d\epsilon = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\frac{\partial y}{\partial \epsilon} d\epsilon\, dx$$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y\, dx$$ where $\frac{\partial I}{\partial \epsilon} d\epsilon \equiv \delta I$ $\quad \frac{\partial y}{\partial \epsilon} d\epsilon \equiv \delta y$

$$= \delta \int_{x_1}^{x_2} F\{y, y'; x\}\, dx = 0$$

by hypothesis, the limits of integration are not affected by the variation

$$\delta I = \int_{x_1}^{x_2} \delta F\, dx$$ $= F(y+\epsilon\eta \ldots) - F(x, y, y')$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) dx$$ but $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d(\delta y)}{dx}$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\frac{d}{dx}\delta y\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y\, dx$$

$$\delta F = F(x, y+\epsilon\eta, y'+\epsilon\eta') - F(x, y, y')$$

$\tilde{y} - y = \delta y \equiv \epsilon\eta(x)$

$$= \left(F(x, y, y') + \frac{\partial F}{\partial y}\underbrace{\epsilon\eta}_{\delta y} + \frac{\partial F}{\partial y'}\underbrace{\epsilon\eta'}_{\delta y'}\right) - F(x, y, y')$$

SEASON