Sinusoidal Driving Forces

$$F = -kx - b\frac{dx}{dt} + F_0\cos(\omega t)$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = A\cos(\omega t) \quad \text{where } A = \frac{F_0}{m}$$

the solution of its consists of two parts, a complementary function $x_h(t)$, and a particular solution $x_p(t)$

$$x_h(t) = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

for the $x_p(t)$, we try $x_p(t) = D\cos(\omega t - \delta)$

$$-\omega^2 D\cos(\omega t-\delta) - 2\beta D\omega\sin(\omega t-\delta) + \omega_0^2 D\cos(\omega t-\delta) = A\cos\omega t$$

$$-\omega^2 D(\cos\omega t\cos\delta + \sin\omega t\sin\delta) - 2\beta D\omega(\sin\omega t\cos\delta - \cos\omega t\sin\delta) + \omega_0^2 D(\cos\omega t\cos\delta + \sin\omega t\sin\delta) = A\cos\omega t$$

$$\Rightarrow \left\{A - D\left[(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta\right]\right\}\cos\omega t - \left\{D\left[(\omega_0^2-\omega^2)\sin\delta - 2\omega\beta\cos\delta\right]\right\}\sin\omega t = 0$$

$$\tan\delta = \frac{2\omega\beta}{\omega_0^2-\omega^2}, \quad \sin\delta = \frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} \quad \cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

$$D = \frac{A}{(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta} = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

$$\Rightarrow x_p(t) = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\cos(\omega t-\delta) \qquad \delta = \tan^{-1}\frac{2\omega\beta}{\omega_0^2-\omega^2}$$

$x_h(t)$ is transient effects, the terms contained in this solution damp out with time because of the factor $e^{-\beta t}$, the $x_p(t)$ represents the steady-state effects

$$x\left(t \gg \frac{1}{\beta}\right) = x_p(t)$$

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to find the angular frequency ω_R at which the amplitude D is a maximum (amplitude resonance frequency), we set

$$\frac{-2\omega(2\beta^2+\omega^2-\omega_0^2)}{[(\omega_0^2-\omega^2)^2+4\omega^2\beta^2]^{\frac{3}{2}}} \qquad \left.\frac{dD}{d\omega}\right|_{\omega=\omega_R}=0 \quad , \quad \omega_R=\sqrt{\omega_0^2-2\beta^2}\,!$$

$$2\beta^2+\omega_0^2-\omega_0^2=0$$

no resonance occurs if $\beta > \frac{\omega_0}{2}$

free oscillations, no damping > free oscillations, damping > driven oscillations

$$\omega_0^2=\frac{k}{m} \quad > \quad \omega_1^2=\omega_0^2-\beta^2 \quad > \quad \omega_R^2=\omega_0^2-2\beta^2$$

describe the degree of damping in an oscillating system in terms of the "quality factor"

$$Q \equiv \frac{\omega_R}{2\beta}$$

for a lightly damped oscillators

$$D=\frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} \qquad \text{at the } \omega=\omega_R=\sqrt{\omega_0^2-2\beta^2}$$

$$D_R=\frac{A}{2\beta\sqrt{\omega_0^2-\beta^2}}$$

let us find the frequency, $\omega=\omega'$ at which the amplitude is $\frac{1}{\sqrt{2}}D_R$

$$(\omega_0^2-\omega'^2)^2+(2\beta\omega)^2=(\sqrt{2})^2(2\beta\omega)^2 \quad , (\omega_0-\omega)^2(\omega_0+\omega)^2=4\omega^2\beta^2$$

$$(\omega')^2=\omega_0^2-2\beta^2\pm 2\beta\omega_0\sqrt{1-\frac{\beta^2}{\omega_0^2}} \cong \omega_0^2\pm 2\beta\omega_0$$

$$\omega' \cong \omega_0\left(1\pm\frac{\beta}{\omega_0}\right)$$

$$\Delta\omega=(\omega_0+\beta)-(\omega_0-\beta)=2\beta$$

$$\omega_R=\sqrt{\omega_0^2-2\beta^2}\cong\omega_0$$

$\Rightarrow Q \cong \frac{\omega_0}{\Delta\omega}$ where $\Delta\omega$ represents the frequency interval between the points on the amplitude resonance curve that are $\frac{1}{\sqrt{2}}$ of the maximum amplitude

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$$\frac{dx}{dt} = \frac{-A\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} \sin(\omega t-\delta)$$

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{mA^2}{2}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}\sin^2(\omega t-\delta)$$

to obtain a value of E_k independent of the time, we compute the average of E_k over one complete period of oscillation

$$\langle E_k\rangle = \frac{mA^2}{2}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}\langle\sin^2(\omega t-\delta)\rangle$$

the average value of the square of the sine function taken over one period

$$\langle\sin^2(\omega t-\delta)\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t-\delta)dt = \frac{1}{2}$$

$$\langle E_k\rangle = \frac{mA^2}{4}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}$$

the value ω for $\langle E_k\rangle$ a maximum is labeled ω_E and is obtained from

$$\left.\frac{d\langle E_k\rangle}{d\omega}\right|_{\omega=\omega_E} = 0 \quad , \quad \omega_E = \omega_0$$

amplitude resonance at $\sqrt{\omega_0^2-2\beta^2}$ ← potential energy resonance

kinetic energy resonance occurs at ω_0

⇒ damped oscillator is not a conservative system

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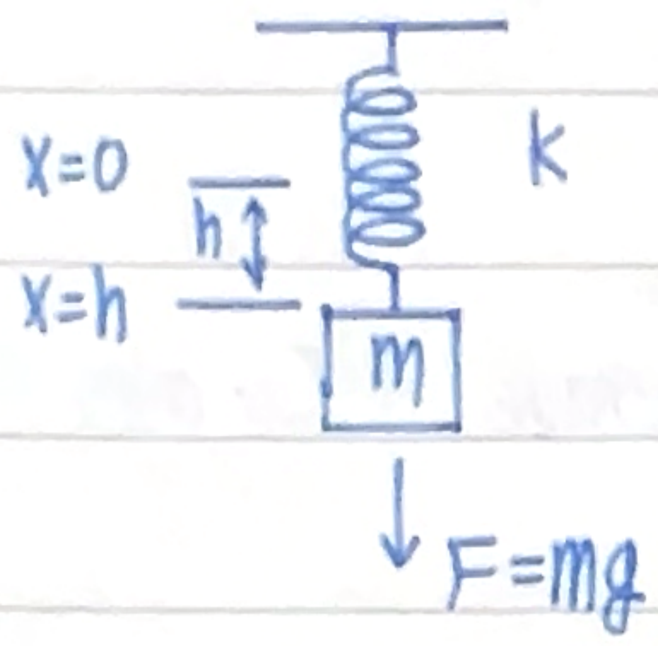
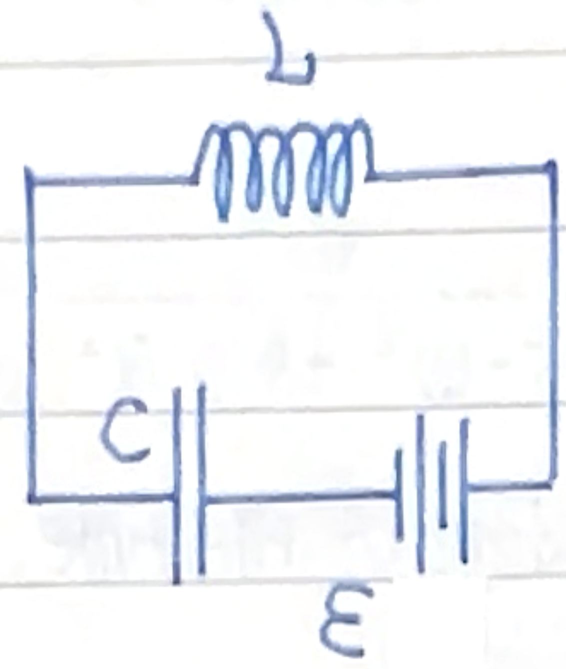
Physical Systems

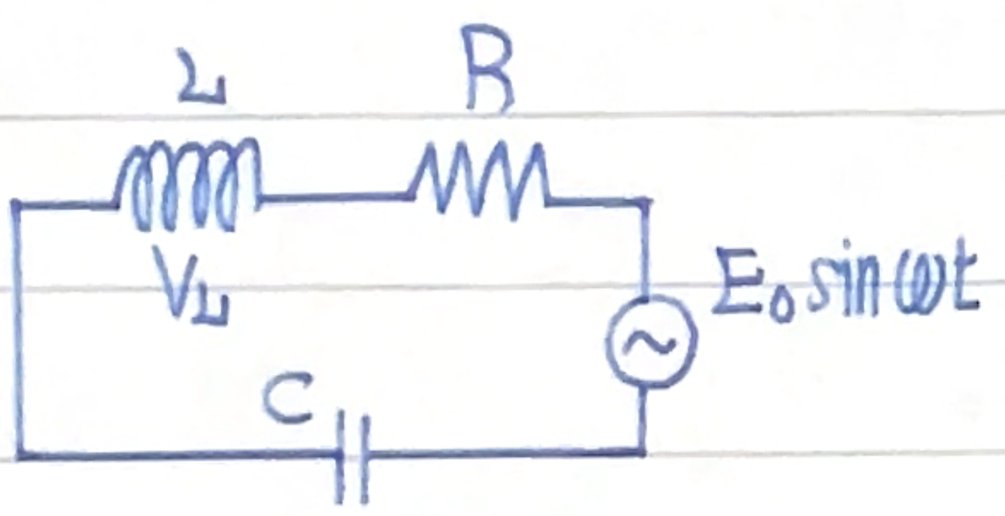




$$m\frac{d^2x}{dt^2} + k(x-h) = 0 \quad \Leftrightarrow \quad L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = \varepsilon = \frac{q_1}{C}$$

$$x(t) = h + A\cos\omega_0 t \qquad L\frac{d^2q}{dt^2} + \frac{1}{C}q = \frac{q_1}{C}$$

$$q(t) = q_1 + (q_0 - q_1)\cos\omega_0 t$$



$$V_L = L\frac{dI}{dt} = L\frac{d^2q}{dt^2} \qquad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0\sin\omega t$$

$$V_R = LI = L\frac{dq}{dt} \qquad \beta = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad A = \frac{E_0}{L}$$

$$V_C = \frac{q}{C}$$

$$I = \frac{-E_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}\sin(\omega t - \delta)$$

$$V_L = \frac{-\omega L E_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}\cos(\omega t - \delta) = V(\omega)\cos(\omega t - \delta)$$

$$\frac{dV(\omega)}{d\omega} = \frac{LE_0\left(R^2 - \frac{2L}{C} + \frac{2}{\omega^2 C^2}\right)}{\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]^{\frac{3}{2}}} \qquad \omega_{max} = \frac{1}{\sqrt{LC - \frac{R^2C^2}{2}}}$$

$$\omega_R = \sqrt{\frac{1}{LC} - \frac{2R^2}{L^2}}$$

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Principle of Superposition - Fourier Series

$$\left(\frac{d^2}{dt^2}+a\frac{d}{dt}+b\right)x(t) = A\cos\omega t$$

$$\Rightarrow \quad \mathbf{L}\, x(t) = F(t)$$

linear operator

linear operator obey the principle of superposition

$$\mathbf{L}(x_1+x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

$$\mathbf{L}x_1 = F_1(t) \qquad \mathbf{L}x_2 = F_2(t)$$

$$\Rightarrow \mathbf{L}(a_1x_1+a_2x_2) = a_1F_1(t)+a_2F_2(t)$$

$$\Rightarrow \mathbf{L}\left(\sum_{n=1}^{N} a_n x_n(t)\right) = \sum_{n=1}^{N} a_n F_n(t)$$

$$x(t) = \sum_{n=1}^{N} a_n x_n(t) \quad F(t) = \sum_{n=1}^{N} a_n F_n(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t - \phi_n)$$

the steady-state solution is $x(t) = \frac{1}{m}\sum_n \frac{a_n}{\sqrt{(\omega_0^2-\omega_n)^2+4\omega_n^2\beta^2}}\cos(\omega_n t-\phi_n-\delta_n)$

$$\delta_n = \tan^{-1}\left(\frac{2\omega_n\beta}{\omega_0^2-\omega_n^2}\right)$$

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n \sin n\omega t)$$

$$a_n = \frac{2}{T}\int_0^T F(t')\cos n\omega t'\, dt' = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} F(t')\cos n\omega t'\, dt'$$

$$b_n = \frac{2}{T}\int_0^T F(t')\sin n\omega t'\, dt' = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} F(t')\sin n\omega t'\, dt'$$