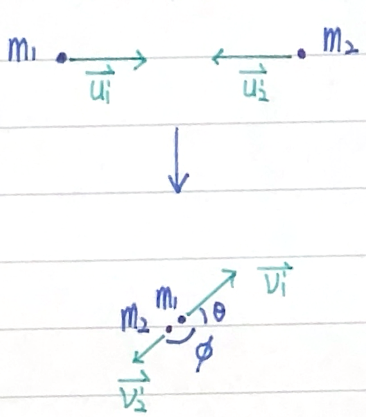
Elastic Collisions of Two Particles

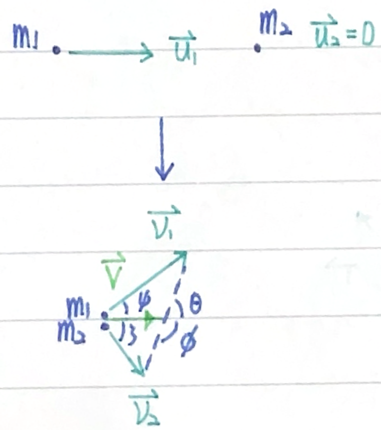
center of mass system	initial		final	
	particle 1	particle 2	particle 1	particle 2
mass	m_1	m_2	m_1	m_2
velocity	$\vec{u}_1'$	$\vec{u}_2'$	$\vec{v}_1'$	$\vec{v}_2'$
angle			θ	$\phi(\pi-\theta)$
kinetic energy	K_1'	K_2'	T_1'	T_2'

laboratory coordinate system	initial		final	
	particle 1	particle 2	particle 1	particle 2
mass	m_1	m_2	m_1	m_2
velocity	$\vec{u}_1$	$\vec{u}_2$	$\vec{v}_1$	$\vec{v}_2$
angle			ψ	ζ
kinetic energy	K_1	K_2	T_1	T_2

CM system



LAB system



$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{M} \qquad \vec{V} = \frac{m_1\vec{u_1} + m_2\vec{u_2}}{M} = \frac{m_1\vec{u_1}}{m_1+m_2} = -\vec{u_2'}$$

but $\vec{u_2} = 0$

if the collision is elastic $\quad \vec{u_1} = \vec{u_1'} + \vec{u_2'}$

initial $\quad \vec{V} + \vec{u_1'} = \vec{u_1} \qquad \vec{V} + \vec{u_2'} = \vec{u_2} = 0$

$$\Rightarrow \vec{u_1'} = \vec{u_1} - \vec{V} \qquad \vec{u_2'} = -\vec{V} = -\frac{m_1\vec{u_1}}{m_1+m_2}$$

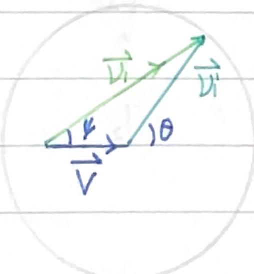
$$= \vec{u_1} - \frac{m_1\vec{u_1}}{m_1+m_2}$$

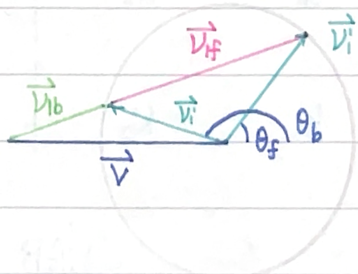
$$= \frac{m_2}{m_1+m_2}\vec{u_1}$$

final $\quad \vec{V} + \vec{v_1'} = \vec{v_1} \qquad \vec{V} + \vec{v_2'} = \vec{v_2}$

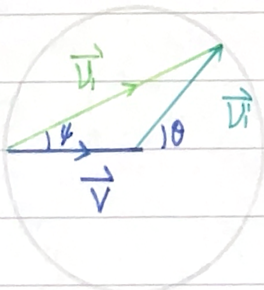
$$\vec{v_1'} = \vec{u_1'}$$

if $m_1 < m_2$, then $\vec{v_1'} > \vec{V}$ $\qquad$ if $m_1 > m_2$, then $\vec{V} > \vec{v_1'}$





if $m_1 = m_2$



from figure, we have $\vec{v_1'}\sin\theta = \vec{v_1}\sin\psi$ —(1) $\quad \vec{v_1'}\cos\theta + \vec{V} = \vec{v_1}\cos\psi$ —(2)

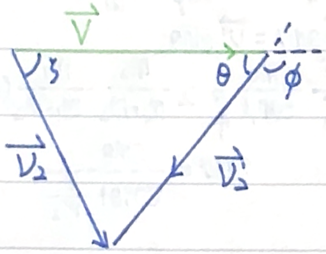
$$\frac{(1)}{(2)} = \tan\psi = \frac{\vec{v_1'}\sin\theta}{\vec{v_1'}\cos\theta + \vec{V}} = \frac{\sin\theta}{\cos\theta + \frac{\vec{V}}{\vec{v_1'}}} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

$$\frac{\vec{V}}{\vec{v_1'}} = \frac{\frac{m_1}{m_1+m_2}\vec{u_1}}{\frac{m_2}{m_1+m_2}\vec{u_1}} = \frac{m_1}{m_2}$$

when $m_2 \gg m_1$ $\quad \tan\psi \approx \tan\theta$, $\psi \approx \theta$

when $m_2 = m_1$ $\quad \tan\psi = \dfrac{\sin\theta}{\cos\theta + 1} = \dfrac{\sqrt{1-\cos^2\theta}}{(\sqrt{1+\cos\theta})^2} = \dfrac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$

$$= \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan\frac{\theta}{2} \qquad \Rightarrow \psi = \frac{\theta}{2}$$



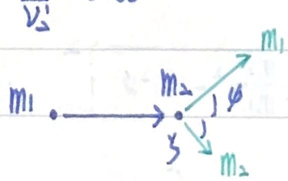
$\vec{v_2}\sin\zeta = \vec{v_2'}\sin\theta$ —(1)

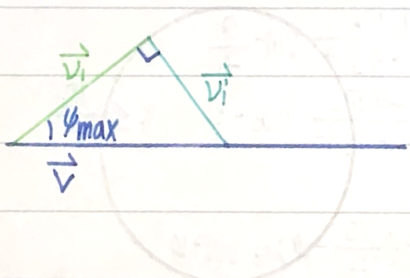
$\vec{v_2}\cos\zeta = \vec{V} - \vec{v_2'}\cos\theta$ —(2)

$$\frac{(1)}{(2)} = \tan\zeta = \frac{\vec{v_2'}\sin\theta}{\vec{V} - \vec{v_2'}\cos\theta} = \frac{\sin\theta}{\frac{\vec{V}}{\vec{v_2'}} - \cos\theta} = \frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$\Rightarrow 2\zeta = \pi - \theta = \phi$

for $m_1 = m_2$ $\quad \zeta + \psi = \dfrac{\pi}{2}$





for $m_1 > m_2$ (mean $\vec{v_1'} < \vec{V}$)

$$\sin\psi_{max} = \frac{\vec{v_1'}}{\vec{V}} = \frac{m_2}{m_1}, \quad \psi_{max} = \sin^{-1}\frac{m_2}{m_1}$$

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$$T_0 = \frac{1}{2} m_1 u_1^2 = K_1 \qquad T_0' = \frac{1}{2}\left[m_1(\vec{u_1'})^2 + m_2(\vec{u_2'})^2\right] = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\vec{u_1}^2 = \frac{m_2}{m_1+m_2}T_0$$

$$T_1' = \frac{1}{2} m_1 (\vec{v_1'})^2 = \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right)^2 m_1 \vec{u_1}^2 = \left(\frac{m_2}{m_1+m_2}\right)^2 T_0 = \left(\frac{m_2}{m_1+m_2}\right)^2 K_1$$

$$T_2' = \frac{1}{2} m_2 (\vec{v_2'})^2 = \frac{1}{2}\left(\frac{m_1}{m_1+m_2}\right)^2 m_2 \vec{u_1}^2 = \frac{m_1 m_2}{(m_1+m_2)^2} T_0 = \frac{m_1 m_2}{(m_1+m_2)^2} K_1$$

$$\frac{T_1}{T_0} = \frac{T_1}{K_1} = \frac{\frac{1}{2} m_1 \vec{v_1}^2}{\frac{1}{2} m_1 \vec{u_1}^2} = \frac{\vec{v_1}^2}{\vec{u_1}^2}$$

$$\boxed{\begin{aligned}(\vec{v_1'})^2 &= \vec{v_1}^2 + \vec{V}^2 - 2\vec{v_1}\vec{V}\cos\psi \\ \vec{v_1}^2 &= \vec{v_1'}^2 - \vec{V}^2 + 2\vec{v_1}\vec{V}\cos\psi\end{aligned}}$$

$$= \left(\frac{\vec{v_1'}}{\vec{u_1}}\right)^2 - \left(\frac{\vec{V}}{\vec{u_1}}\right)^2 + 2\frac{\vec{v_1}\vec{V}}{\vec{u_1}^2}\cos\psi$$

$$= \left(\frac{m_2}{m_1+m_2}\right)^2 - \left(\frac{m_1}{m_1+m_2}\right)^2 + 2\frac{m_2}{m_1+m_2}\frac{m_1}{m_1+m_2}\left(\cos\theta + \frac{m_1}{m_2}\right)$$

$$= 1 - \frac{2m_1 m_2}{(m_1+m_2)^2}(1-\cos\theta)$$

$$\frac{\vec{v_1'}}{\vec{u_1}} = \frac{m_2}{m_1+m_2} \qquad \frac{\vec{V}}{\vec{u_1}} = \frac{m_1}{m_1+m_2}$$

$$2\frac{\vec{v_1}\vec{V}}{\vec{u_1}^2}\cos\psi = 2\vec{v_1'}\frac{\sin\theta}{\sin\psi}\frac{\vec{V}}{\vec{u_1}^2}\cos\psi$$

$$\vec{v_1}\sin\psi = \vec{v_1'}\sin\theta$$

$$= 2\frac{\vec{v_1'}}{\vec{u_1}}\frac{\vec{V}}{\vec{u_1}}\frac{\sin\theta}{\tan\psi} = 2\frac{m_2}{m_1+m_2}\frac{m_1}{m_1+m_2}\left(\cos\theta + \frac{m_1}{m_2}\right)$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

$$\Rightarrow \cos\theta = \frac{m_1}{m_2}\left[-\sin^2\psi \pm \cos\psi\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\right]$$

$$\Rightarrow \frac{T_1}{K_1} = \left(\frac{m_1}{m_1+m_2}\right)^2\left[\cos\psi \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\psi}\right]^2 \qquad \text{if } m_1 = m_2 \quad \frac{T_1}{K_1} = \cos^2\psi$$

$$K_1 = T_1 + T_2, \quad \frac{T_2}{K_1} = 1 - \frac{T_1}{K_1} = 1 - \left[1 - \frac{2m_1 m_2}{(m_1+m_2)^2}(1-\cos\theta)\right]$$

$$= \frac{4m_1 m_2}{(m_1+m_2)^2}\frac{1-\cos\theta}{2} = \frac{4m_1 m_2}{(m_1+m_2)^2}\sin^2\frac{\theta}{2}$$

$$= \frac{4m_1 m_2}{(m_1+m_2)^2}\cos^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{4m_1 m_2}{(m_1+m_2)^2}\cos^2\zeta \qquad \zeta \le \frac{\pi}{2}$$

$$\text{if } m_1 = m_2, \quad \frac{T_2}{K_1} = \cos^2\zeta = \cos^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos^2\left(\frac{\pi}{2} - \psi\right) = \sin^2\psi$$