General Problem of Coupled Oscillations

$q_k = q_{k0} \qquad \frac{dq_k}{dt} = 0 \qquad \frac{d^2 q_k}{dt^2} = 0 \qquad k = 1, 2 \cdots n$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left(\frac{dx_{\alpha i}}{dt}\right)^2 = \frac{1}{2}\sum_\alpha m_\alpha \sum_j \frac{\partial x_{\alpha i}}{\partial q_j}\frac{dq_j}{dt}\sum_k \frac{\partial x_{\alpha i}}{\partial q_k}\frac{dq_k}{dt}$$

$$= \frac{1}{2}\sum_{jk}\sum_\alpha m \frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}\frac{dq_j}{dt}\frac{dq_k}{dt} = \frac{1}{2}\sum_{jk} m_{jk}\frac{dq_j}{dt}\frac{dq_k}{dt} \qquad \text{let } m_{jk} = \sum_\alpha m_\alpha \frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}$$

$$U = \underbrace{U_0}_{\text{let } U_0 = 0} + \sum_k \underbrace{\left.\frac{\partial U}{\partial q_k}\right|_0}_{=0} q_k + \frac{1}{2}\sum_{jk}\underbrace{\left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0}_{A_{jk}} q_j q_k + \cdots = \frac{1}{2}\sum_{jk} A_{jk} q_j q_k$$

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = 0, \quad \underbrace{\frac{\partial U}{\partial q_k}}_{\sum_j A_{jk} q_j} + \frac{d}{dt}\underbrace{\frac{\partial T}{\partial \dot q_k}}_{\sum_j m_{jk}\frac{dq_j}{dt}} = 0, \quad \sum_j \left(A_{jk} q_j + m_{jk}\frac{d^2 q_j}{dt^2}\right) = 0$$

except a solution $q_j(t) = a_j e^{i(\omega t - \delta)}$

$$\Rightarrow \sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0, \quad \left|A_{jk} - \omega^2 m_{jk}\right| = 0$$

characteristic equation or secular

ω_r : characteristic frequencies

$$q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)}$$

because it is only the real part of $q_j(t)$ that is physically meaningful

$$q_j(t) = \mathrm{Re}\sum_r a_{jr} e^{i(\omega_r t - \delta_r)} = \sum_r a_{jr}\cos(\omega_r t - \delta_r)$$

Orthogonality of the Eigenvectors

$$\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0 \quad , \quad \omega_s^2 \sum_k m_{jk} a_{ks} = \sum_k A_{jk} a_{ks} \qquad \omega_r^2 \sum_j m_{jk} a_{jr} = \sum_j A_{jk} a_{jr}$$

$$\downarrow \times a_{jr} \qquad\qquad \downarrow \times a_{ks}$$

$$\begin{cases} \omega_s^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} a_{jr} a_{ks} \\ \omega_r^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} a_{jr} a_{ks} \end{cases}$$

$$\Rightarrow (\omega_r^2 - \omega_s^2) \sum_{jk} m_{jk} a_{jr} a_{ks} = 0 \qquad \begin{cases} \text{if } r \neq s & \sum_{jk} m_{jk} a_{jr} a_{ks} = 0 \\ \text{if } r = s & \omega_r^2 - \omega_s^2 = 0 \end{cases}$$

$$\overset{>0}{T} = \frac{1}{2} \sum_{jk} m_{jk} \frac{dq_j}{dt} \frac{dq_k}{dt} = \frac{1}{2} \sum_{jk} m_{jk} \left[\sum_r \omega_r a_{jr} \sin(\omega_r t - \delta_r) \right] \left[\sum_s \omega_s a_{ks} \sin(\omega_s t - \delta_s) \right]$$

$$= \frac{1}{2} \sum_{rs} \omega_r \omega_s \sin(\omega_r t - \delta_r) \sin(\omega_s t - \delta_s) \sum_{jk} m_{jk} a_{jr} a_{ks}$$

$$\downarrow r = s$$

$$= \frac{1}{2} \sum_r \underbrace{\omega_r^2 \sin^2(\omega_r t - \delta_r)}_{\geq 0} \sum_{jk} \underbrace{m_{jk} a_{jr} a_{kr}}_{\geq 0}$$

if $\sum_{jk} m_{jk} a_{jr} a_{kr} = 1$ the a_{jr} are then said to be normalized

$$\sum_{jk} m_{jk} a_{jr} a_{ks} = \delta_{rs} \qquad \vec{a_r} = \sum_j a_{jr} \vec{e_j}$$