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Gravitation

Gravitational Potential

$\vec{F} = -G\frac{Mm}{r^2}\vec{e}_r$ for a continuous distribution of matter $= -Gm\int_V \frac{\rho(\vec{r'})\vec{e}_r}{r^2}dV'$

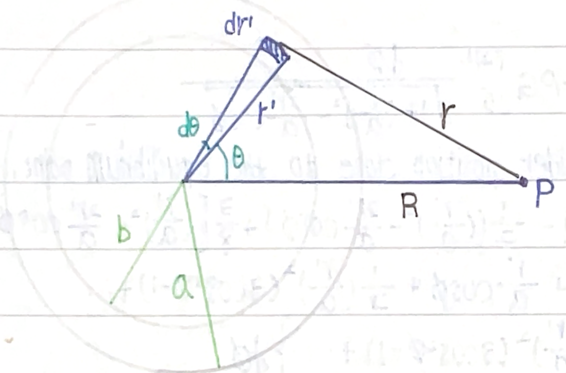
$\vec{g} = \frac{\vec{F}}{m} = -G\frac{M}{r^2}\vec{e}_r \qquad \vec{g} = -G\int_V \frac{\rho(\vec{r'})\vec{e}_r}{r^2}dV'$

$\vec{g} = -\nabla\Phi \qquad \nabla\Phi = \frac{d\Phi}{dr}\vec{e}_r = G\frac{M}{r^2}\vec{e}_r \qquad \Phi = -G\frac{M}{r}$

the work per unit mass $dW = -\vec{g}\cdot d\vec{r}$

$= (\nabla\Phi)\cdot d\vec{r} = \sum_i \frac{\partial\Phi}{\partial x_i}dx_i = d\Phi$

$U = m\Phi$, $\vec{F} = -\nabla U$



$\Phi = -G\int_V \frac{\rho(r')}{r}dV' = -2\pi\rho G\int_b^a (r')^2dr'\int_0^\pi \frac{\sin\theta}{r}d\theta \qquad r^2 = (r')^2 + R^2 - 2r'R\cos\theta$

$2rdr = 2r'R\sin\theta d\theta$

$= -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{r_{min}}^{r_{max}} dr \qquad \frac{\sin\theta}{r}d\theta = \frac{dr}{r'R}$

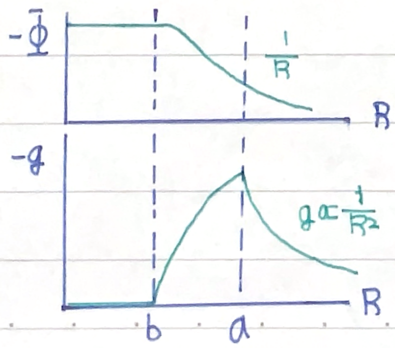
if P is outside the shell $\Phi(R>a) = -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{R-r'}^{R+r'} dr$

$= -\frac{4\pi\rho G}{R}\int_b^a (r')^2dr' = -\frac{4}{3}\frac{\pi\rho G}{R}(a^3-b^3)$

$\because M = \frac{4}{3}\pi\rho(a^3-b^3) \qquad = -\frac{GM}{R}$

$\Phi(R<b) = -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{r'-R}^{r'+R} dr = -4\pi\rho G\int_b^a r'dr' = -2\pi\rho G(a^2-b^2)$

$\Phi(b<R<a) = -\frac{4\pi\rho G}{3R}(R^3-b^3) - 2\pi\rho G(a^2-R^2) \qquad = -4\pi\rho G\left(\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right)$



$g(R<b) = 0$

$g(b<R<a) = \frac{4\pi\rho G}{3}\left(\frac{b^3}{R^2} - R\right)$

$g(R>a) = -\frac{GM}{R^2}$

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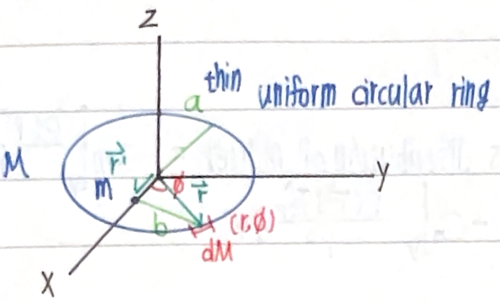

$$d\Phi = -G\frac{dM}{b} = -\frac{Ga\rho}{b}d\phi \qquad \rho = \frac{M}{2\pi a}, \; dM = \rho a\, d\phi$$

$$b = |\vec{r}-\vec{r}'| = |a\cos\phi\,\vec{e}_1 + a\sin\phi\,\vec{e}_2 - r'\vec{e}_1| = |(a\cos\phi - r')\vec{e}_1 + a\sin\phi\,\vec{e}_2|$$

$$= \sqrt{(a\cos\phi - r')^2 + a^2\sin^2\phi} = \sqrt{a^2+(r')^2-2ar'\cos\phi} = a\sqrt{1+\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi}$$

$$\Phi(r') = -G\int\frac{dM}{b} = -\rho a G\int_0^{2\pi}\frac{d\phi}{b} = -\rho G\int_0^{2\pi}\frac{d\phi}{\sqrt{1+\left(\frac{r'}{a}\right)^2-\frac{2r'}{a}\cos\phi}}$$

the integral is difficult, so let us consider position close to the equilibrium point, $r'=0$

if $r' \ll a$, $\sqrt{1+\left(\frac{r'}{a}\right)^2-\frac{2r'}{a}\cos\phi} = 1 - \frac{1}{2}\left(\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi\right) + \frac{3}{8}\left[\left(\frac{r'}{a}\right)^2 - \frac{2r'}{a}\cos\phi\right]^2 + \cdots$

$$= 1 + \frac{r'}{a}\cos\phi + \frac{1}{2}\left(\frac{r'}{a}\right)^2(3\cos^2\phi - 1) + \cdots$$

$$\Phi(r') = -\rho G\int_0^{2\pi}\left\{1 + \frac{r'}{a}\cos\phi + \frac{1}{2}\left(\frac{r'}{a}\right)^2(3\cos^2\phi-1) + \cdots\right\}d\phi$$

$$= -\frac{MG}{a}\left[1 + \frac{1}{4}\left(\frac{r'}{a}\right)^2 + \cdots\right]$$

$$U(r') = m\Phi(r') = -\frac{MmG}{a}\left[1 + \frac{1}{4}\left(\frac{r'}{a}\right)^2 + \cdots\right]$$

$$\frac{dU(r')}{dr'} = 0 = -\frac{MmG}{a}\,\frac{1}{2}\,\frac{r'}{a^2} + \cdots \Rightarrow r' = 0$$

$$\frac{d^2U(r')}{d(r')^2} = -\frac{mMG}{2a^3} + \cdots < 0 \quad \Rightarrow \text{unstable}$$

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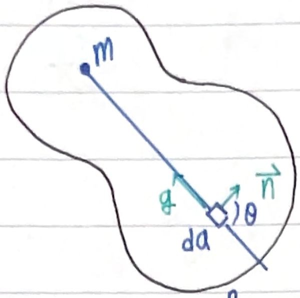

gravitational flux $\Phi_m = \int_S \vec{n}\cdot\vec{g}\, da$ $\qquad \vec{n}\cdot\vec{g} = -Gm\dfrac{\cos\theta}{r^2}$

$$= -Gm\int_S \frac{\cos\theta}{r^2}\, da$$

$$= -4\pi Gm$$

$$= -4\pi G \sum_i m_i = -4\pi G\int_V \rho\, dv$$

$$= \int_V \nabla\cdot\vec{g}\, dv$$

$$\Rightarrow \int_V (-4\pi G)\rho\, dv = \int_V \nabla\cdot\vec{g}\, dv \Rightarrow \nabla\cdot\vec{g} = -4\pi G\rho$$

$$\nabla\cdot\vec{g} = -\nabla\cdot\nabla\Phi$$

$$-\nabla\cdot\nabla\Phi = -4\pi G\rho$$

$$\nabla^2\Phi = 4\pi G\rho$$ Poisson's equation