

$$\begin{aligned}
 \nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial S} \hat{S} + \frac{1}{S} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial Z} \hat{Z} \right) (A_S \hat{S} + A_\phi \hat{\phi} + A_Z \hat{Z}) = \frac{\partial A_S}{\partial S} \hat{S} + \frac{1}{S} \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_Z}{\partial Z} \hat{Z} \\
 &= \hat{S} \left(\frac{\partial A_S}{\partial S} \hat{S} + A_S \frac{\partial \hat{S}}{\partial S} + \frac{\partial A_\phi}{\partial S} \hat{\phi} + \frac{\partial \hat{\phi}}{\partial S} A_\phi + \frac{\partial A_Z}{\partial S} \hat{Z} + \frac{\partial \hat{Z}}{\partial S} A_Z \right) + \\
 &\quad \hat{\phi} \left(\frac{\partial A_S}{\partial \phi} \hat{S} + A_S \frac{\partial \hat{S}}{\partial \phi} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial \hat{\phi}}{\partial \phi} A_\phi + \frac{\partial A_Z}{\partial \phi} \hat{Z} + \frac{\partial \hat{Z}}{\partial \phi} A_Z \right) + \\
 &\quad \hat{Z} \left(\frac{\partial A_S}{\partial Z} \hat{S} + A_S \frac{\partial \hat{S}}{\partial Z} + \frac{\partial A_\phi}{\partial Z} \hat{\phi} + \frac{\partial \hat{\phi}}{\partial Z} A_\phi + \frac{\partial A_Z}{\partial Z} \hat{Z} + \frac{\partial \hat{Z}}{\partial Z} A_Z \right) \\
 &= \frac{\partial A_S}{\partial S} + \frac{1}{S} \frac{\partial A_\phi}{\partial \phi} + \frac{A_S}{S} + \frac{\partial A_Z}{\partial Z} = \left(\frac{\partial A_S}{\partial S} + \frac{A_S}{S} \right) + \frac{1}{S} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_Z}{\partial Z} \\
 &= \frac{1}{S} \frac{\partial}{\partial S} (A_S S) + \frac{1}{S} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_Z}{\partial Z}
 \end{aligned}$$

$$\begin{aligned}
 \nabla \times \vec{A} &= \left(\frac{\partial}{\partial S} \hat{S} + \frac{1}{S} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial Z} \hat{Z} \right) \times \vec{A} = \hat{S} \times \frac{\partial \vec{A}}{\partial S} + \frac{1}{S} \hat{\phi} \times \frac{\partial \vec{A}}{\partial \phi} + \hat{Z} \times \frac{\partial \vec{A}}{\partial Z} \\
 &= \hat{S} \times \left(\frac{\partial A_S}{\partial S} \hat{S} + \frac{\partial A_\phi}{\partial S} \hat{\phi} + \frac{\partial A_Z}{\partial S} \hat{Z} + A_S \frac{\partial \hat{S}}{\partial S} + A_\phi \frac{\partial \hat{\phi}}{\partial S} + A_Z \frac{\partial \hat{Z}}{\partial S} \right) + \\
 &\quad \hat{\phi} \times \left(\frac{\partial A_S}{\partial \phi} \hat{S} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + \frac{\partial A_Z}{\partial \phi} \hat{Z} + A_S \frac{\partial \hat{S}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} + A_Z \frac{\partial \hat{Z}}{\partial \phi} \right) + \\
 &\quad \hat{Z} \times \left(\frac{\partial A_S}{\partial Z} \hat{S} + \frac{\partial A_\phi}{\partial Z} \hat{\phi} + \frac{\partial A_Z}{\partial Z} \hat{Z} + A_S \frac{\partial \hat{S}}{\partial Z} + A_\phi \frac{\partial \hat{\phi}}{\partial Z} + A_Z \frac{\partial \hat{Z}}{\partial Z} \right) \\
 &= \frac{\partial A_\phi}{\partial S} \hat{Z} - \frac{\partial A_Z}{\partial S} \hat{\phi} - \frac{1}{S} \frac{\partial A_S}{\partial \phi} \hat{Z} + \frac{1}{S} \frac{\partial A_Z}{\partial \phi} \hat{S} + \frac{1}{S} A_\phi \hat{Z} + \frac{\partial A_S}{\partial Z} \hat{S} - \frac{\partial A_\phi}{\partial Z} \hat{S} \\
 &= \left(-\frac{1}{S} \frac{\partial A_Z}{\partial \phi} - \frac{\partial A_\phi}{\partial Z} \right) \hat{S} + \left(\frac{\partial A_S}{\partial Z} - \frac{\partial A_Z}{\partial S} \right) \hat{S} + \left(\frac{\partial A_\phi}{\partial S} + \frac{A_\phi}{S} - \frac{1}{S} \frac{\partial A_S}{\partial \phi} \right) \hat{Z} \\
 &= \left(-\frac{1}{S} \frac{\partial A_Z}{\partial \phi} - \frac{\partial A_\phi}{\partial Z} \right) \hat{S} + \left(\frac{\partial A_S}{\partial Z} - \frac{\partial A_Z}{\partial S} \right) \hat{S} + \frac{1}{S} \frac{\partial}{\partial S} (A_\phi S) - \frac{1}{S} \frac{\partial A_S}{\partial \phi}
 \end{aligned}$$

$$\begin{aligned}
 \nabla^2 U &= \hat{S} \cdot \left(\hat{S} \frac{\partial^2 U}{\partial S^2} - \hat{\phi} \frac{\partial U}{\partial \phi} + \hat{Z} \frac{\partial^2 U}{\partial Z^2} \right) + \\
 &\quad \frac{\hat{\phi}}{S} \cdot \left(\hat{\phi} \frac{\partial U}{\partial S} + \hat{S} \frac{\partial^2 U}{\partial S \partial \phi} - \hat{S} \frac{\partial U}{\partial \phi} + \hat{\phi} \frac{\partial^2 U}{\partial \phi^2} + \hat{Z} \frac{\partial^2 U}{\partial Z \partial \phi} \right) + \\
 &\quad \hat{Z} \cdot \left(\hat{S} \frac{\partial^2 U}{\partial S \partial Z} + \hat{\phi} \frac{\partial^2 U}{\partial \phi \partial Z} + \hat{Z} \frac{\partial^2 U}{\partial Z^2} \right) \\
 &= \frac{\partial^2 U}{\partial S^2} + \frac{1}{S} \frac{\partial U}{\partial S} + \frac{1}{S^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial Z^2} \\
 &= \frac{1}{S} \frac{\partial}{\partial S} (S \frac{\partial U}{\partial S}) + \frac{1}{S^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial Z^2} \\
 \nabla^2 &\equiv \frac{1}{S} \frac{\partial}{\partial S} (S \frac{\partial U}{\partial S}) + \frac{1}{S^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial Z^2}
 \end{aligned}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$= r \sin\theta \cos\phi \hat{x} + r \sin\theta \sin\phi \hat{y} + r \cos\theta \hat{z}$$

$$\hat{r} = \frac{\partial \vec{r}}{\partial r} =$$

$$\frac{\partial \hat{r}}{\partial r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$$

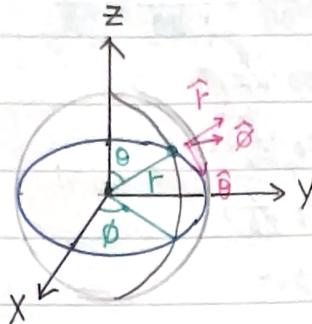
$$|\frac{\partial \hat{r}}{\partial r}|^2 = 1$$

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$$dV = dr \, r d\theta \, r \sin\theta d\phi$$

Spherical Coordinates



$$x = r \sin\theta \cos\phi \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r \sin\theta \sin\phi \quad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$z = r \cos\theta \quad \tan\phi = \frac{y}{x}$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$$

$$\hat{\phi} = \frac{\hat{z} \times \hat{r}}{\sin\theta} = -\hat{x} \sin\phi + \hat{y} \cos\phi$$

$$\hat{\theta} = \hat{\phi} \times \hat{r} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z} \sin\theta$$

$$\frac{\partial \hat{r}}{\partial r} = 0$$

$$\frac{\partial \hat{\phi}}{\partial r} = 0$$

$$\frac{\partial \hat{\theta}}{\partial r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z} \sin\theta = \hat{\theta}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{x} \sin\theta - \hat{y} \sin\theta \sin\phi - \hat{z} \cos\theta = -\hat{r}$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\hat{x} \sin\theta \sin\phi + \hat{y} \sin\theta \cos\phi$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{x} \cos\phi - \hat{y} \sin\phi$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = -\hat{x} \cos\theta \sin\phi + \hat{y} \cos\theta \cos\phi$$

$$= (-\hat{x} \sin\phi + \hat{y} \cos\phi) \sin\theta$$

$$= -(\hat{r} \sin\theta + \hat{\theta} \cos\theta)$$

$$= \hat{\phi} \sin\theta$$

$$d\vec{F} = d(r\hat{r}) = \hat{r} dr + r d\hat{r} = \hat{r} dr + r \left(\frac{\partial \hat{r}}{\partial r} dr + \frac{\partial \hat{r}}{\partial \theta} d\theta + \frac{\partial \hat{r}}{\partial \phi} d\phi \right) = \hat{r} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin\theta d\phi$$

$$du = \nabla u \cdot d\vec{r} = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta + \frac{\partial u}{\partial \phi} d\phi = (\nabla u)_r dr + (\nabla u)_\theta r d\theta + (\nabla u)_\phi r \sin\theta d\phi$$

$$\nabla \equiv \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\nabla \cdot \vec{A} = \left(\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \cdot \vec{A} = \hat{r} \frac{\partial \vec{A}}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial \vec{A}}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial \vec{A}}{\partial \phi}$$

$$= \hat{r} \left(\frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial r} + A_\theta \frac{\partial \hat{\theta}}{\partial r} + A_\phi \frac{\partial \hat{\phi}}{\partial r} \right) +$$

$$\frac{\hat{\theta}}{r} \left(\frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \theta} + A_\theta \frac{\partial \hat{\theta}}{\partial \theta} + A_\phi \frac{\partial \hat{\phi}}{\partial \theta} \right) +$$

$$\frac{\hat{\phi}}{r \sin\theta} \left(\frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \phi} + A_\theta \frac{\partial \hat{\theta}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} \right) - (r \sin\theta + \hat{\theta} \cos\theta)$$

$$= \frac{\partial A_r}{\partial r} + \left(\frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r} \right) + \left(\frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} + \frac{A_r}{r} + \frac{A_\theta \cos\theta}{r \sin\theta} \right)$$

$$= \left(\frac{\partial A_r}{\partial r} + 2 \frac{A_r}{r} \right) + \left(\frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_\theta \cos\theta}{r \sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \cdot \vec{A} \equiv \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned}
 \nabla \times \vec{A} &= (\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{\partial}{\partial \phi}) \times \vec{A} = \hat{r} \times \frac{\partial \vec{A}}{\partial r} + \hat{\theta} \times \frac{\partial \vec{A}}{\partial \theta} + \hat{\phi} \times \frac{\partial \vec{A}}{\partial \phi} \\
 &= \hat{r} \times \left(\frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial r} + A_\theta \frac{\partial \hat{\theta}}{\partial r} + A_\phi \frac{\partial \hat{\phi}}{\partial r} \right) + \\
 &\quad \frac{\hat{\theta}}{r} \times \left(\frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \theta} + A_\theta \frac{\partial \hat{\theta}}{\partial \theta} + A_\phi \frac{\partial \hat{\phi}}{\partial \theta} \right) + \\
 &\quad \frac{\hat{\phi}}{r \sin \theta} \times \left(\frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \phi} + A_\theta \frac{\partial \hat{\theta}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} \right) - (\hat{r} \sin \theta + \hat{\theta} \cos \theta) \\
 &= \left(\frac{\partial A_\theta}{\partial r} - \frac{\partial A_\phi}{\partial r} \right) \hat{\theta} + \left(-\frac{1}{r} \frac{\partial A_r}{\partial \theta} + \frac{1}{r} \frac{\partial A_\phi}{\partial \theta} \right) \hat{r} + \left(\frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial \phi} \right) \hat{\phi} + \\
 &\quad \left(\frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} - \frac{A_\phi \cos \theta}{r \sin \theta} \right) \\
 &= \hat{r} \left(\frac{1}{r} \frac{\partial A_\phi}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} + \frac{A_\phi \cos \theta}{r \sin \theta} \right) + \hat{\theta} \left(-\frac{\partial A_\phi}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} - \frac{A_\phi}{r} \right) + \hat{\phi} \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r} \right) \\
 \nabla \times \vec{A} &= \frac{\hat{r}}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\phi}{\partial \phi} \right] + \frac{\hat{\theta}}{r \sin \theta} \left[\frac{\partial A_r}{\partial \phi} - \sin \theta \frac{\partial}{\partial r} (r A_\phi) \right] + \frac{\hat{\phi}}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]
 \end{aligned}$$

$$\begin{aligned}
 \nabla^2 u &= \left(\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{\partial}{\partial \phi} \right) \cdot \left(\hat{r} \frac{\partial u}{\partial r} + \hat{\theta} \frac{\partial u}{\partial \theta} + \hat{\phi} \frac{\partial u}{\partial \phi} \right) \\
 &= \hat{r} \cdot \frac{\partial}{\partial r} \left(\hat{r} \frac{\partial u}{\partial r} + \hat{\theta} \frac{\partial u}{\partial \theta} + \hat{\phi} \frac{\partial u}{\partial \phi} \right) + \hat{\theta} \cdot \frac{\partial}{\partial \theta} \left(\hat{r} \frac{\partial u}{\partial r} + \hat{\theta} \frac{\partial u}{\partial \theta} + \hat{\phi} \frac{\partial u}{\partial \phi} \right) + \\
 &\quad \left(\frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \right) \hat{r} \frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \\
 &= \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u}{\partial r^2} - \frac{\hat{\theta}}{r^2} \frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{\hat{\phi}}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi^2} \right) + \\
 &\quad \frac{\hat{\theta}}{r} \cdot \left(\hat{\theta} \frac{\partial u}{\partial r} + \hat{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\hat{r}}{r} \frac{\partial u}{\partial \theta} + \hat{\theta} \frac{\partial^2 u}{\partial \theta^2} - \frac{\hat{\phi} \cos \theta}{r \sin^2 \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi \partial \theta} \right) + \\
 &\quad \frac{\hat{\phi}}{r \sin \theta} \cdot \left(\hat{\phi} \sin \theta \frac{\partial u}{\partial r} + \hat{r} \frac{\partial^2 u}{\partial r \partial \phi} + \frac{\hat{\phi} \cos \theta}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r} \frac{\partial^2 u}{\partial \theta \partial \phi} - \frac{\hat{r} \sin \theta + \hat{\theta} \cos \theta}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\
 &= \left(\frac{\partial^2 u}{\partial r^2} \right) + \left(-\frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) - \left(-\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\
 &= \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) + \left(\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \theta} \right) - \left(\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\
 &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \\
 \nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
 \end{aligned}$$