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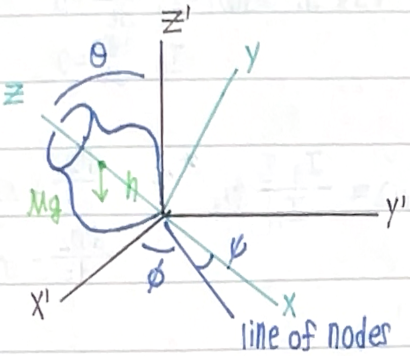
Motion of a Symmetric Top with One Point Fixed

choose the origins of the fixed and the body coordinate systems to coincide



$\omega_1 = \frac{d\phi}{dt}\sin\theta\sin\psi + \frac{d\theta}{dt}\cos\psi \quad \omega_2 = \frac{d\phi}{dt}\sin\theta\cos\psi - \frac{d\theta}{dt}\sin\psi \quad \omega_3 = \frac{d\phi}{dt}\cos\theta + \frac{d\psi}{dt}$

assume $I_1 = I_2 \neq I_3$

$$T = \frac{1}{2}\sum I_i\omega_i^2 = \frac{1}{2}I_1(\omega_1^2+\omega_2^2) + \frac{1}{2}I_3\omega_3^2$$
$$= \frac{1}{2}I_1\left[\left(\frac{d\phi}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2\right] + \frac{1}{2}I_3\left[\frac{d\phi}{dt}\cos\theta + \left(\frac{d\psi}{dt}\right)\right]^2$$

$$\boxed{\begin{aligned}\omega_1^2 &= \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\sin^2\psi + 2\frac{d\phi}{dt}\frac{d\theta}{dt}\sin\theta\sin\psi\cos\psi + \left(\frac{d\theta}{dt}\right)^2\cos^2\psi \\ \omega_2^2 &= \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\cos^2\psi - 2\frac{d\phi}{dt}\frac{d\theta}{dt}\sin\theta\sin\psi\cos\psi + \left(\frac{d\theta}{dt}\right)^2\sin^2\psi \\ \omega_1^2+\omega_2^2 &= \left(\frac{d\phi}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2\end{aligned}}$$

$U = Mgh\cos\theta$

$$L = \frac{1}{2}I_1\left[\left(\frac{d\phi}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2\right] + \frac{1}{2}I_3\left[\frac{d\phi}{dt}\cos\theta + \frac{d\psi}{dt}\right]^2 - Mgh\cos\theta$$

$$P_\phi = \frac{\partial L}{\partial\dot\phi} = (I_1\sin^2\theta + I_3\cos^2\theta)\frac{d\phi}{dt} + I_3\frac{d\psi}{dt}\cos\theta = \text{constant} \quad \text{——} (1)$$

$$P_\psi = \frac{\partial L}{\partial\dot\psi} = I_3\left(\frac{d\psi}{dt} + \frac{d\phi}{dt}\cos\theta\right) = \text{constant} \quad \text{——} (2)$$

from (2) $\frac{d\psi}{dt} = \frac{P_\psi - I_3 \frac{d\phi}{dt}\cos\theta}{I_3}$

substituting into (1), $(I_1 \sin^2\theta + I_3\cos^2\theta)\frac{d\phi}{dt} + (P_\psi - I_3\frac{d\phi}{dt}\cos\theta)\cos\theta = P_\phi$

$(I_1\sin^2\theta)\frac{d\phi}{dt} + P_\psi\cos\theta = P_\phi$ $\quad \frac{d\phi}{dt} = \frac{P_\phi - P_\psi\cos\theta}{I_1\sin^2\theta}$

$\frac{d\psi}{dt} = \frac{P_\psi}{I_3} - \frac{(P_\phi - P_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}$

by hypothesis, the system we are considering is conservative

$$E = \frac{1}{2}I_1\left[\left(\frac{d\phi}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2\right] + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta = \text{constant}$$

and $I_3\omega_3^2 = \frac{(I_3\omega_3)^2}{I_3} = \frac{P_\psi^2}{I_3} = \text{constant}$

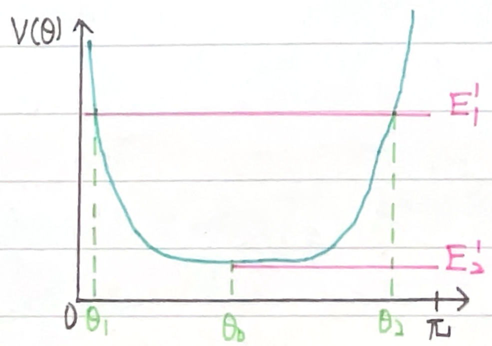
so $E' = E - \frac{1}{2}I_3\omega_3^2 = \text{constant} = \frac{1}{2}I_1\left[\left(\frac{d\phi}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2\right] + Mgh\cos\theta$

$= \frac{1}{2}I_1\left(\frac{d\theta}{dt}\right)^2 + \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$

$= \frac{1}{2}I_1\left(\frac{d\theta}{dt}\right)^2 + V(\theta)$

effective potential $V(\theta) = \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}[E' - V(\theta)]}}$$



$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos\theta_0 (P_\phi - P_\psi \cos\theta_0)^2 + P_\phi \sin^2\theta_0 (P_\phi - P_\psi \cos\theta_0)}{I_1 \sin^3\theta_0} - Mgh\sin\theta_0 = 0$$

let $B \equiv P_\phi - P_\psi \cos\theta_0$

$$\Rightarrow \cos\theta_0 B^2 - P_\psi \sin^2\theta_0 \, B + (Mgh I_1 \sin^4\theta_0) = 0$$

$$\Rightarrow B = \frac{P_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \underbrace{\sqrt{1 - \frac{4MghI_1\cos\theta_0}{P_\psi^2}}}_{\text{must } >0}\right)$$

if $\theta_0 < \frac{\pi}{2}$, $P_\psi^2 \geq 4MghI_1\cos\theta_0$, $P_\psi = I_3\omega_3 \geq \sqrt{4MghI_1\cos\theta_0}$

$$\omega_3 \geq \frac{2}{I_3}\sqrt{mghI_1\cos\theta_0}$$

steady precession can occur at the

$$\frac{d\phi_0}{dt} = \frac{B}{I_1 \sin^2\theta_0} \quad \begin{cases} B_+ : \dot{\phi}_0(+) & \text{fast precession} \\ B_- : \dot{\phi}_0(-) & \text{slow precession} \end{cases}$$

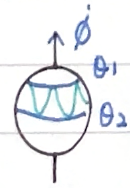
if ω_3 (or P_ψ) is large, then

$$\dot{\phi}_0(+) \cong \frac{I_3\omega_3}{I_1\cos\theta_0}$$

$$\dot{\phi}_0(-) \cong \frac{Mgh}{I_3\omega_3}$$

in general, $\theta_1 < \theta < \theta_2$

(a) $\frac{d\phi}{dt} > 0$ or < 0



(b) $\frac{d\phi}{dt}$ sometime >0 or <0

(c) $\left.\frac{d\phi}{dt}\right|_{\theta=\theta_1} = 0$ or $\left.(P_\phi - P_\psi\cos\theta)\right|_{\theta=\theta_1} = 0$

nutation

Stability of Rigid-Body Rotations

assume $I_3 > I_2 > I_1$, the body axes coincide with the principal axes, and we start with the body rotating around the X axis

$$\vec{\omega} = \omega_1 \vec{e}_x$$

if we apply a small perturbation $\vec{\omega} = \omega_1 \vec{e}_x + \lambda \vec{e}_y + \mu \vec{e}_z$

$$(I_2 - I_3)\lambda\mu - I_1 \frac{d\omega_1}{dt} = 0 \quad , \quad \because \lambda\mu = 0 \, , \quad \frac{d\omega_1}{dt} = 0 \qquad \omega_1 = \text{constant}$$

$$(I_3 - I_1)\mu\omega_1 - I_2 \frac{d\lambda}{dt} = 0 \qquad \frac{d\lambda}{dt} = \frac{I_3 - I_1}{I_2}\omega_1 \mu \quad \text{—— (1)}$$

$$(I_1 - I_2)\lambda\omega_1 - I_3 \frac{d\mu}{dt} = 0 \qquad \frac{d\mu}{dt} = \frac{I_1 - I_2}{I_3}\omega_1 \lambda \quad \text{—— (2)}$$

$$\frac{d(1)}{dt} = \frac{d^2\lambda}{dt^2} = \frac{I_3 - I_1}{I_2}\omega_1 \frac{d\mu}{dt} \quad \text{and (2):} \quad \frac{d^2\lambda}{dt^2} + \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2 \lambda = 0$$

$$\lambda(t) = A e^{i\Omega_{1\lambda} t} + B e^{-i\Omega_{1\lambda} t} \qquad \Omega_{1\lambda} = \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

oscillatory

the solution for λ when the rotation is around the x-axis

by hypothesis, $I_1 < I_3$ and $I_1 < I_2$, so $\Omega_{1\lambda}$ is real

$\Omega_{1\mu} = \Omega_{1\lambda} = \Omega$

if we consider rotations round the y and z-axes

$$\Omega_1 = \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \qquad \Omega_2 = \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \qquad \Omega_3 = \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

because $I_1 < I_2 < I_3$ $\quad \Omega_1, \Omega_3$ real $\quad \Omega_2$ imaginary $\rightarrow$ unstable