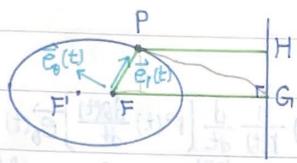


## Gravitation

### Kepler Problem

Newton's Law of Universal Gravitation to work & motion

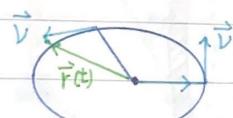


$$\frac{PF}{PG - PF \cos \theta} = e, \text{ assume } PF = r, FG = s$$

$$r = se - e \cos \theta, r(1 + e \cos \theta) = se$$

$$r = \frac{se}{1 + e \cos \theta}, \frac{1}{r} = \frac{1 + e \cos \theta}{se} = \frac{1}{se} + \frac{e \cos \theta}{s} = u$$

$$\vec{F} = m\vec{a} = m \left[ \left( \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right) \vec{e}_r(t) + \frac{1}{r(t)} \frac{d}{dt} \left[ r^2(t) \frac{d\theta(t)}{dt} \right] \vec{e}_\theta(t) \right]$$



$$\frac{\Delta \text{area}}{\Delta t} = \frac{1}{2} \vec{R}(t) \times \vec{V}(t), \vec{R}(t) = \vec{r}(t) \vec{e}_r(t), \vec{V}(t) = \frac{dr(t)}{dt} \vec{e}_r(t) + r(t) \frac{d\theta(t)}{dt} \vec{e}_\theta(t)$$

$$|v(t)| = \frac{1}{2} r^2(t) \frac{d\theta(t)}{dt} \vec{e}_r(t) \times \vec{e}_\theta(t)$$

$$= \frac{1}{2} r^2(t) \frac{d\theta(t)}{dt} \vec{e}_z(t) = A \vec{e}_z(t) \text{ is a constant}$$

$$\therefore \frac{d}{dt} \left[ r^2(t) \frac{d\theta(t)}{dt} \right] = \frac{d}{dt} (2A) = 0$$

$$\Rightarrow \vec{F} = m \left[ \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right] \vec{e}_r(t), \text{ where } r(t) \left( \frac{d\theta(t)}{dt} \right)^2 = \frac{4A^3}{r^3}$$

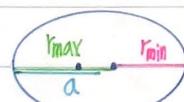
$$r(t) = \frac{se}{1 + e \cos \theta}, \frac{dr(t)}{dt} = \frac{dr}{du} \frac{du}{d\theta} \frac{d\theta}{dt} = \frac{-1}{u^2} (-\frac{\sin \theta}{s}) \frac{2A}{r^2(t)} = \frac{\sin \theta}{s} 2A$$

$$u(t) = \frac{1}{se} + \frac{1}{s} \cos \theta(t)$$

$$\frac{d^2 r(t)}{dt^2} = \frac{2A}{s^2} \cos \theta(t) \frac{d\theta(t)}{dt} = \frac{4A^2 \cos \theta(t)}{s^2 r^2(t)}$$

$$\Rightarrow \vec{F} = m \left[ \frac{4A^2 \cos \theta}{s r^2(t)} - \frac{4A^2}{r^3(t)} \right] = \frac{m 4A^2}{r^3(t) s} [r(t) \cos \theta - s] \quad \text{since } \frac{r(t) \cos \theta - s}{r(t)} = -\frac{s - r(t) \cos \theta}{r(t)} = \frac{1}{e}$$

$$= -\frac{4MA^2}{s e r^2(t)} \vec{e}_r(t) \leftarrow \text{universal gravitation}$$

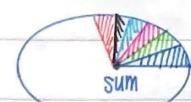


$$\theta = 0, r_{\min} = \frac{se}{1+e}; \theta = \pi, r_{\max} = \frac{se}{1-e}$$

$$2a = r_{\min} + r_{\max} = \frac{2se}{1-e^2}, a = \frac{se}{1-e^2}, a(1-e^2) = se$$

$$2c = r_{\max} - r_{\min} = \frac{2se^2}{1-e^2}, c = ae$$

$$A = \frac{\pi ab}{T}, 4A^2 = \frac{4\pi^2 a^2 b^2}{T^2} = \frac{4\pi^2 a^2 a^2 (1-e^2)}{T^2} = \frac{4\pi^2 a^3 se}{T^2}$$



$$\Rightarrow \vec{F} = -\frac{m 4\pi^2 a^3 se}{T^2} \frac{1}{r^2(t)} \vec{e}_r(t) = -\frac{m 4\pi^2 a^3}{T^2} \frac{1}{r^2(t)} \vec{e}_r(t) \quad \text{and } \frac{4\pi^2 a^3}{T^2} = GM$$

## Newton's Law of Universal Gravitation

**problem:** the motion orbit of the particle under the force is inversely proportional to the square of the distance from the source

$$\vec{F} = \frac{K}{r^2(t)} \vec{e}_r(t) = m \vec{a}(t) = m \left[ \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right] \vec{e}_r(t) + m \frac{1}{r(t)} \frac{d}{dt} \left[ r^2(t) \frac{d\theta(t)}{dt} \right] \vec{e}_\theta(t)$$

since  $\frac{1}{2} r^2(t) \frac{d\theta(t)}{dt} = A$ ,  $\frac{d\theta(t)}{dt} = \frac{2A}{r^2(t)}$ , therefore  $\frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 = \frac{d^2 r(t)}{dt^2} - \frac{4A^2}{r^3(t)}$

$$\Rightarrow \vec{F} = \frac{K}{r^2(t)} \vec{e}_r(t) = m \left[ \frac{d^2 r(t)}{dt^2} - \frac{4A^2}{r^3(t)} \right], \quad \frac{d^2 r(t)}{dt^2} - \frac{4A^2}{r^3(t)} = \frac{K}{m} \frac{1}{r^2(t)}$$

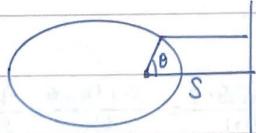
$$\frac{dr(t)}{dt} = \frac{dr}{du} \frac{du(\theta)}{d\theta} \frac{d\theta(t)}{dt} = -\frac{1}{u^2} \frac{du(\theta)}{d\theta} \frac{2A}{r^2(t)} = -2A \frac{du(\theta)}{d\theta} \Big|_{\theta=\theta(t)}$$

$$\frac{d^2 r(t)}{dt^2} = \frac{d}{dt} \left[ -2A \frac{du(\theta)}{d\theta} \right] = -2A \frac{d}{dt} \left[ \frac{du(\theta)}{d\theta} \frac{d\theta(t)}{dt} \right] = -\frac{4A^2}{r^2(t)} \frac{d^2 u(\theta)}{d\theta^2} \Big|_{\theta=\theta(t)}$$

$$\Rightarrow \vec{F} = -\frac{4A^2}{r^2(t)} \frac{d^2 u(\theta)}{d\theta^2} \Big|_{\theta=\theta(t)} = -\frac{4A^2}{r^3(t)} = \frac{K}{m} \frac{1}{r^2(t)}, \quad \frac{d^2 u(\theta)}{d\theta^2} + \frac{1}{r(t)} = -\frac{K}{4A^2 m}, \quad \frac{d^2 u(\theta)}{d\theta^2} = -\left[ \frac{K}{4A^2 m} + \frac{1}{r(t)} \right]$$

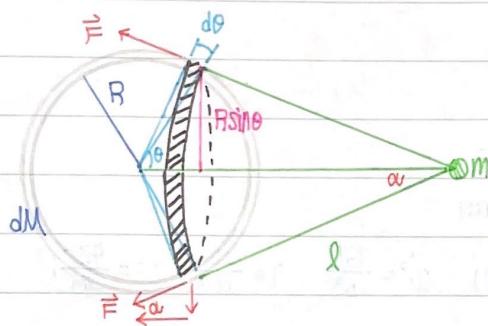
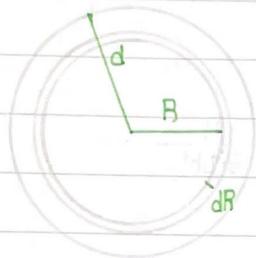
$$\Rightarrow w(\theta) = U(\theta) + \frac{K}{4A^2 m} \quad \frac{d^2 W(\theta)}{d\theta^2} = -W(\theta) \Rightarrow W(\theta) = \frac{1}{C} \cos(\theta - B) = \frac{1}{r} + \frac{K}{4A^2 m}$$

$$C = s \frac{K}{4A^2 m} = \frac{1}{se}$$



$$\frac{r}{s - r \cos \theta} = e, \quad r = se - er \cos \theta, \quad r(1 + e \cos \theta) = se, \quad \frac{1}{r} = \frac{1}{se} + \frac{1}{s} \cos \theta$$

 $e < 1$  $e = 1$  $e > 1$

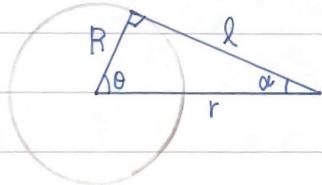


$$\vec{F} = \frac{dM}{4\pi R^2} \frac{(2\pi R \sin\theta)(R d\theta)}{\ell^2} m G \cos\alpha$$

$$= -G dM \int \frac{2\pi R \sin\theta}{4\pi R^2} m \frac{1}{\ell^2} \cos\alpha d\theta$$

$$= -\frac{G dM m}{2} \int \frac{\cos\alpha}{\ell^2} \sin\theta d\theta$$

since

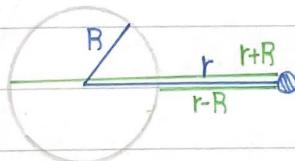
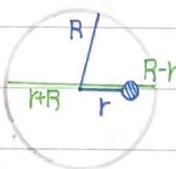


$$\ell^2 = R^2 + r^2 - 2Rr \cos\alpha$$

$$2\ell(\theta) \frac{d\ell(\theta)}{d\theta} = 2Rr \sin\theta, \sin\theta = \frac{Rr}{\ell} \frac{d\ell(\theta)}{d\theta}$$

$$R^2 = \ell^2 + r^2 - 2\ell \cos\alpha, \cos\alpha = \frac{\ell^2 + r^2 - R^2}{2\ell r}$$

$$\therefore \vec{F} = -\frac{G(dM)m}{2} \int \frac{\ell^2(\theta) + r^2 - R^2}{2\ell(\theta)} \frac{\ell(\theta)}{Br} \frac{d\ell(\theta)}{d\theta} \frac{1}{m(\theta)} d\theta = -\frac{G dM m}{4} \frac{1}{Br^2} \int (1 + \frac{r^2 - R^2}{\ell^2}) d\ell = -\frac{G(dM)m}{4Br^2} (\ell - \frac{R^2 - r^2}{\ell})$$

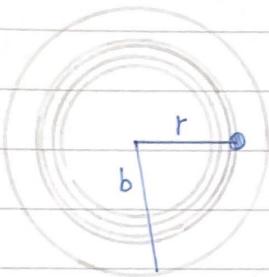
 $r > R$  $r < R$ 

$$-\frac{G(dM)m}{4Br^2} \left[ \ell - \frac{r^2 - R^2}{\ell} \right]_{R-r}^{r+R} = r+R - \frac{r^2 - R^2}{R-r} - (R-r) + \frac{R-R^2}{R-r}$$

$$= r+R - (r-R) - (R-r) + R+r - \frac{G(dM)m}{4Br^2} \left[ \ell - \frac{r^2 - R^2}{\ell} \right]_{R-r}^{R+r}$$

$$= r+R - R + R - R + R - \frac{G(dM)m}{4Br^2} (2r - 2R)$$

$$= 4R - \frac{G(dM)m}{4Br^2} (2r - 2R) = 0$$

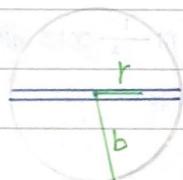


$$0 < r < b$$

$$-\frac{Gm \int dM}{r^2} = -\frac{GM \frac{4}{3}\pi r^3 \rho}{r^2}$$

$$= \frac{-GMmr}{b^3}$$

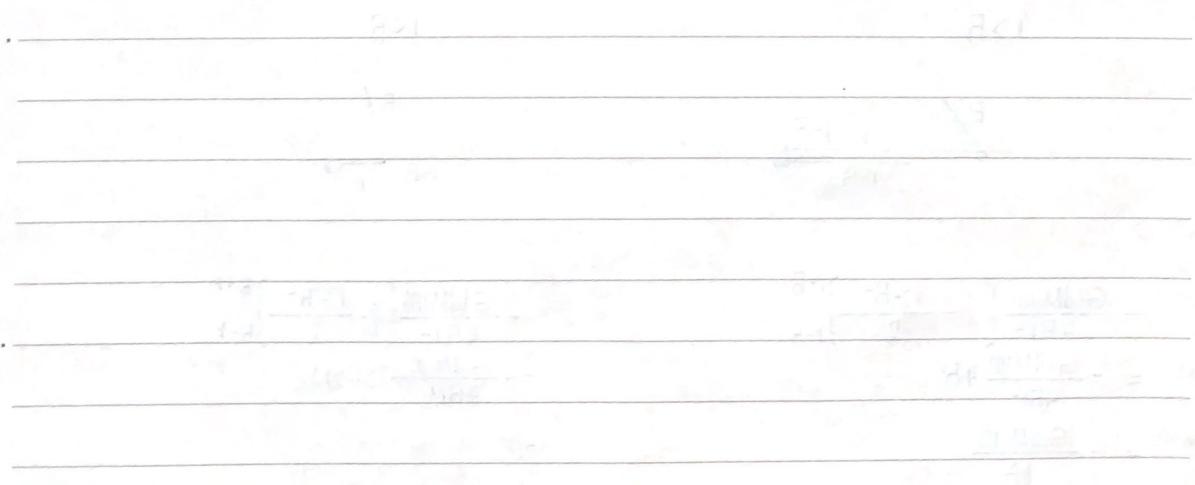
$$\rho = \frac{M}{\frac{4}{3}\pi b^3}$$



$$m \frac{d^2r(t)}{dt^2} = -\frac{GMmr}{b^3}$$

$$\Rightarrow \frac{d^2r(t)}{dt^2} = -\frac{GM}{b^3} r(t)$$

$$\Rightarrow r(t) = b \sin(\omega t + \beta) \quad \omega^2 = \frac{GM}{b^3} \quad T = \frac{2\pi}{\omega} \quad T^2 = \frac{4\pi^2}{GM} b^3$$



## Laplace-Lenz-Ruge Vector

$$m\vec{v}(t) \times 2A\vec{e}_z$$

$$\begin{aligned} \frac{d}{dt} [m\vec{v}(t) \times 2A\vec{e}_z] &= \frac{d}{dt} m\vec{v}(t) \times 2A\vec{e}_z + m\vec{v}(t) \times \frac{d}{dt} 2A\vec{e}_z = \vec{F} \times 2A\vec{e}_z \\ &= \frac{K}{r^2(t)} \vec{e}_r(t) \times r^2(t) \frac{d\theta(t)}{dt} \vec{e}_z \\ &= K \frac{d\theta(t)}{dt} \vec{e}_r(t) \times \vec{e}_z = -K \frac{d\theta(t)}{dt} \vec{e}_z = -K \frac{d}{dt} \vec{e}_r(t) \\ \Rightarrow \frac{d}{dt} \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-K} &= \frac{d}{dt} \vec{e}_r(t), \quad \frac{d}{dt} \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-K} - \frac{d}{dt} \vec{e}_r(t) = 0 \end{aligned}$$

$$\therefore \underline{\underline{L}}\underline{\underline{R}} = \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-K} - \vec{e}_r(t) \text{ along the major axes}$$

$$\begin{aligned} \vec{R}(t) \cdot \underline{\underline{L}}\underline{\underline{R}} &= |\vec{R}(t)| |\underline{\underline{L}}\underline{\underline{R}}| \cos \theta = \vec{R}(t) \cdot \left( \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-K} - \vec{e}_r(t) \right) \text{ since } \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \\ &= \frac{m(\vec{R}(t) \times \vec{v}(t)) \cdot 2A\vec{e}_z}{-K} - r(t) \quad \text{conic section} \\ &= -\frac{m4A^2}{K} - r(t) \quad \Leftrightarrow r(t)(1 + |\underline{\underline{L}}\underline{\underline{R}}| \cos \theta) = -\frac{4mA^2}{K} = se \end{aligned}$$

$$\begin{aligned} \underline{\underline{L}}\underline{\underline{R}} \cdot \underline{\underline{L}}\underline{\underline{R}} &= \left[ -\frac{m\vec{v}(t) \times 2A\vec{e}_z}{K} - \vec{e}_r(t) \right]^2 = 1 + \frac{m^2 \vec{v}(t) \cdot \vec{v}(t) 4A^2}{K^2} + 2 \frac{\vec{e}_r(t) \cdot (m\vec{v}(t) \times 2A\vec{e}_z)}{K r(t)} \\ &= 1 + m^2 \frac{|\vec{v}(t)|^2 4A^2}{K^2} + 2 \frac{m4A^2}{K} (m|\vec{v}(t)|^2 + 2 \frac{K}{r(t)}) \quad \text{assume } E = \frac{1}{2} m |\vec{v}(t)|^2 + \frac{K}{r(t)} \\ &= 1 + \frac{4A^2 m}{K^2} 2E \end{aligned}$$

$$\Rightarrow |\underline{\underline{L}}\underline{\underline{R}}| = 1 + \frac{4A^2 m}{K^2} 2E = 1 + \frac{4A^2 m}{K} \frac{2E}{K} = 1 - se \frac{2E}{K} = e^2, \quad 1 - e^2 = se \frac{2E}{K}, \quad \frac{K}{2E} = \frac{se}{1 - e^2} = a$$

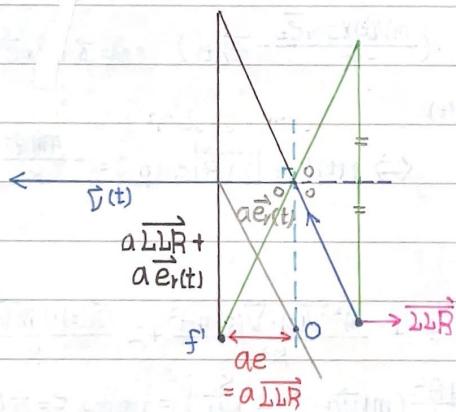
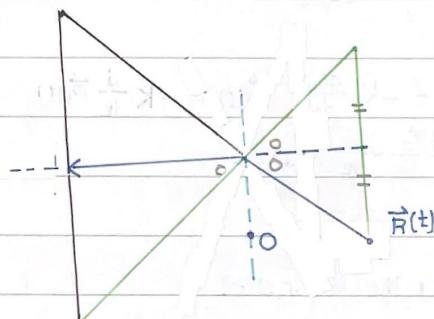
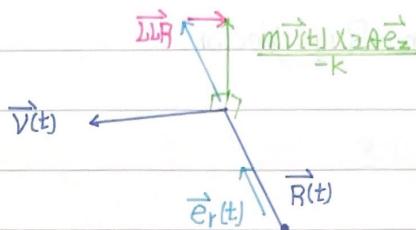
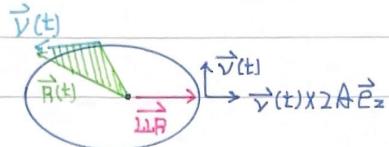
$$E = -\frac{K}{2a}, \quad K = -GMM$$

when  $\theta = 0$  or  $\pi$ ,  $\vec{v}(t) = r(t) \frac{d\theta(t)}{dt} \vec{e}_\theta(t)$

$$\underline{\underline{L}}\underline{\underline{R}} = \left[ -\frac{m r^3(t) \left( \frac{d\theta(t)}{dt} \right)^2}{K} - 1 \right] \vec{e}_r(t) = ?$$

NO.

DATE / /



$$a \overline{LUR} = a \frac{m \vec{v}(t) \times 2A \vec{e}_z}{-k} - a \vec{e}_r(t)$$

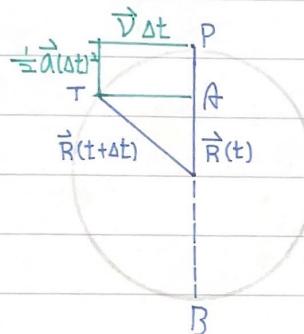
$$a \overline{LUR} + a \vec{e}_r(t) = a \frac{m \vec{v}(t) \times 2A \vec{e}_z}{-k}$$

$$\vec{F} = \frac{k}{r^2(t)} \vec{e}_r(t) = m \vec{a}(t) = m \left( \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right) \vec{e}_r(t) + m \frac{1}{r(t)} \frac{d}{dt} \left( r^2(t) \frac{d\theta(t)}{dt} \right)$$

$$\Rightarrow \frac{l^2(t)}{K} m \left[ \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right] \vec{e}_r(t) = \vec{e}_r(t)$$

$$\Rightarrow \frac{m r^3(t) \left( \frac{d\theta(t)}{dt} \right)^2}{-k} - 1 = \frac{m r^2(t)}{-k} \frac{d^2 r(t)}{dt^2} \quad K = -GMm$$

$> 0 \quad \oplus \quad > 0$



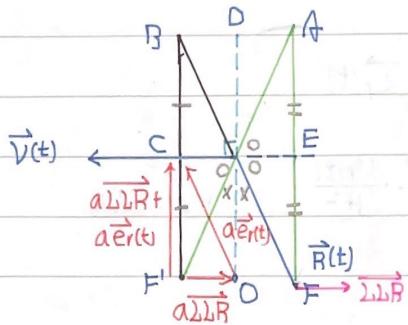
$$\vec{R}(t+\Delta t) = \vec{R}(t) + \underbrace{\frac{d\vec{R}(t)}{dt} \Delta t}_{\vec{v}(t) \Delta t} + \underbrace{\frac{d^2 \vec{R}(t)}{dt^2} \frac{\Delta t^2}{2!}}_{\vec{a} \frac{(\Delta t)^2}{2}}$$

$$(\overline{AT})^2 = \overline{PA} \cdot \overline{AB}$$

$$\overline{PA} = \frac{|\vec{v}|^2 \Delta t^2}{\overline{AB}}, \quad \Delta t \rightarrow 0, \quad \overline{PA} \rightarrow 0, \quad \overline{AB} \rightarrow 2r, \quad |\vec{a}| = \frac{|\vec{v}|^2}{r}$$

$$|\vec{a}| = \frac{|\vec{v}|^2}{r}, \quad |\vec{v}| = \frac{2\pi r}{T}, \quad |\vec{v}| = \frac{4\pi^2 r^2}{T^2} \text{ and } T^2 = kr^3$$

$$= \frac{4\pi^2 r^2}{kr^3}$$



$$a \cdot \overrightarrow{LLR} = a \cdot \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-k} - \vec{a}_{er}(t) = \vec{F}'O = ae$$

$$a \cdot \overrightarrow{LRL} + \vec{a}_{er}(t) = a \frac{m\vec{v}(t) \times 2A\vec{e}_z}{-k} = \vec{F}'C$$

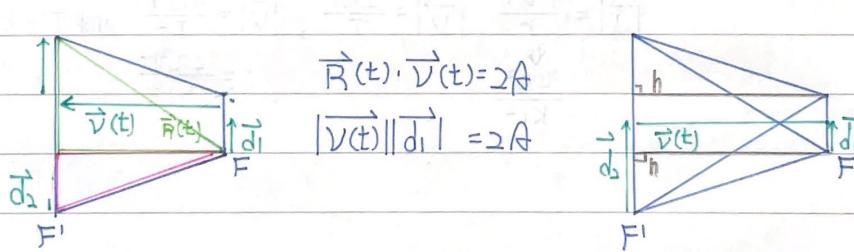
$$2\overline{FE} = \overline{AF}$$

$\triangle AFF'B$  is isosceles trapezium

$$\overline{AF} \approx \overline{BF}$$

$$\overline{AF} = \overline{BF} = 2a$$

$$\overline{BF} \parallel \overline{CO}$$



$$\triangle (|\vec{d}_1| + |\vec{d}_2|)^2 + h^2 = (2a)^2 \quad 4|\vec{d}_1||\vec{d}_2| = 4a^2(1-e^2) = 4b^2$$

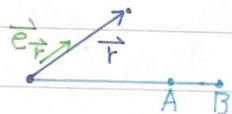
$$\triangle (|\vec{d}_1| - |\vec{d}_2|)^2 + h^2 = (2ae)^2$$

$$\frac{\vec{d}_1 \cdot a \cdot m\vec{v}(t) \times 2A\vec{e}_z}{|\vec{d}_1| - k} = \frac{a m (\vec{d}_1 \times \vec{v}(t)) \cdot 2A\vec{e}_z}{-k |\vec{d}_1|} = \frac{a m 2A\vec{e}_z \cdot 2A\vec{e}_z}{-k |\vec{d}_1|}$$

$$= \frac{ase}{|\vec{d}_1|} = \frac{a \cdot a(1-e^2)}{|\vec{d}_1|} = \frac{b^2}{|\vec{d}_1|} = |\vec{d}_2|$$

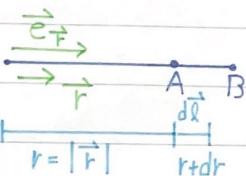
- The Potential Energy of the Force is Inversely Proportional to the Square of the Distance From the Source

$$\vec{F} = \frac{K}{r^2}$$



if O, A, B collinearity

$$\vec{F} = \frac{K}{r^2} \vec{e}_r, \quad dL = dr \vec{e}_r, \quad \vec{F} \cdot d\vec{L} = \frac{K}{r^2} dr$$

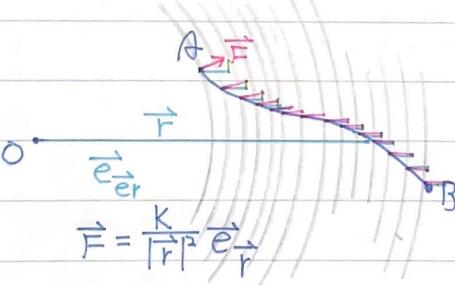


$$U(A) - U(B) = \int_A^B \vec{F} \cdot d\vec{L} = \int_{r_A}^{r_B} \frac{K}{r^2} dr = -\frac{K}{r} \Big|_{r=r_A}^{r=r_B} = \frac{K}{r_A} - \frac{K}{r_B}$$

assume  $r_B \rightarrow \infty, U(B)=0$

$$\Rightarrow U(A) = \frac{K}{r_A}$$

if O, A, B not collinearity



$$dL = dr \vec{e}_r + dL_{\perp}$$

$$\vec{F} \cdot dL = \frac{K}{r^2} dr$$

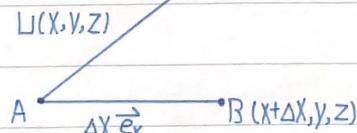
$$\vec{F} = \frac{K}{r^2} \vec{e}_r$$

$$\int_A^B \vec{F} \cdot d\vec{L} = \int_{r_A}^{r_B} \frac{K}{r^2} dr = \frac{K}{r_A} - \frac{K}{r_B} = U(A) - U(B)$$

assume  $r_B \rightarrow \infty, U(B)=0$

$$\Rightarrow U(A) = \frac{K}{r_A}$$

$C(x, y + \Delta y, z)$

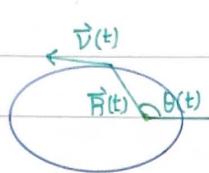


$$U(x, y, z) - U(x + \Delta x, y, z) = \int_A^B \vec{F} \cdot d\vec{L} = F_x \Delta x$$

$$F_x = \lim_{\Delta x \rightarrow 0} -\frac{U(x + \Delta x, y, z) - U(x, y, z)}{\Delta x} = -\frac{dU}{dx} \Big|_{y, z \text{ fixed}} = -\frac{\partial U(x, y, z)}{\partial x}$$

$$\Rightarrow F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x} \vec{e}_x + \frac{\partial U}{\partial y} \vec{e}_y + \frac{\partial U}{\partial z} \vec{e}_z\right) = -\left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}\right) U = -\nabla U$$



$$E = \frac{1}{2} m \vec{V}(t) \cdot \vec{V}(t) + \frac{K}{r(t)}$$

and  $\vec{V} = \frac{d\vec{R}(t)}{dt} = \frac{dr(t)}{dt} \vec{e}_r(t) + r(t) \frac{d\theta}{dt} \vec{e}_\theta(t)$

$$\vec{V} \cdot \vec{V} = \left(\frac{dr(t)}{dt}\right)^2 + r^2(t) \left(\frac{d\theta(t)}{dt}\right)^2$$

$$E = \frac{m}{2} \left[ \left(\frac{dr(t)}{dt}\right)^2 + r^2(t) \left(\frac{d\theta(t)}{dt}\right)^2 \right] + \frac{K}{r(t)}$$

when  $r = \begin{cases} r_{\max} & \frac{dr(t)}{dt} = 0 \\ r_{\min} & \end{cases}$  and  $r^2(t) \frac{d\theta(t)}{dt} = 2A$

$$\Rightarrow E = \frac{m}{2} \frac{4A^2}{r^2(t)} + \frac{K}{r(t)} \quad \text{when } r = r_{\max} \text{ or } r_{\min}$$

$$\Rightarrow Er^2(t) - Kr(t) - 2mA^2 = 0$$

$$\Rightarrow r_{\pm} = \frac{K \pm \sqrt{K^2 + 4E2mA^2}}{2E} \quad r_{\max} + r_{\min} = r_+ + r_- = \frac{K}{E}$$

if  $K < 0$ ,  $E = \frac{K}{2a}$ ,  $K = -GMm$

$$\text{and } r_{\max} - r_{\min} = 2ae = \frac{\sqrt{K^2 + 8EMA^2}}{E} = \frac{\sqrt{K^2 + 8EMA^2}}{\frac{K}{2a}} \Rightarrow e = \sqrt{1 + \frac{8EMA^2}{K^2}}$$

$$\Rightarrow e = \frac{r_{\max} - r_{\min}}{2a}$$

$$\vec{F} = \frac{K}{r^2(t)} \vec{e}_r(t) = m \left\{ \left[ \frac{d^2 r(t)}{dt^2} - r(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right] \vec{e}_r(t) + \frac{1}{r(t)} \frac{d}{dt} \left[ r^2(t) \frac{d\theta(t)}{dt} \right] \vec{e}_\theta(t) \right\}$$

$$r^2(t) \frac{d\theta(t)}{dt} = 2A$$

$$\Rightarrow \frac{d^2 r(t)}{dt^2} - \frac{4A^2}{r^2(t)} = \frac{K}{m r^2(t)}$$

$$E = \frac{m}{2} \left[ \left( \frac{dr(t)}{dt} \right)^2 + r^2(t) \left( \frac{d\theta(t)}{dt} \right)^2 \right] + \frac{K}{r(t)}$$

$$\Rightarrow \frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)} = \left( \frac{dr(t)}{dt} \right)^2$$

$$\Rightarrow \sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}} = \frac{dr(t)}{dt}$$

$$\Rightarrow I = \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} \frac{dr(t)}{dt} = \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} \frac{dr}{d\theta(t)} \frac{d\theta(t)}{dt} = \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} \frac{2A}{r^2(t)} d\theta$$

$$\int_{\theta(0)}^{\theta(t)} I d\theta = \int_{\theta(0)}^{\theta(t)} \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} \frac{2A}{r^2} \frac{dr(t)}{d\theta} d\theta = \int_{\theta(0)}^{\theta(t)} \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} \frac{2A}{r^2} dr(t)$$

$$\boxed{U = \frac{1}{r(t)}}, \quad du = -\frac{1}{r^2} dr$$

$$= \int_{\theta(0)}^{\theta(t)} \frac{1}{\sqrt{\frac{2}{m} \left( E - \frac{K}{r(t)} \right) - \frac{4A^2}{r^2(t)}}} - 2A du = \int_{\theta(0)}^{\theta(t)} \frac{1}{\sqrt{\frac{2E}{m} - \frac{2K}{m} U - 4A^2 U^2}} - 2A du$$

$$\boxed{W^2 = \left( U + \frac{K}{4A^2 m} \right)^2}$$

$$= U^2 + \frac{K}{2A^2 m} U + \frac{K^2}{16A^4 m^2}$$

$$dW = du$$

$$= \int_{\theta(0)}^{\theta(t)} \frac{-1}{\sqrt{\frac{2E}{4A^2 m} - \frac{2K}{4A^2 m} U - U^2}} du = \int_{\theta(0)}^{\theta(t)} \frac{-1}{\sqrt{\frac{2E}{4A^2 m} + \frac{K^2}{16A^4 m^2} - W^2}} dW$$

$$\boxed{B^2 = \frac{2E}{4A^2 m} + \frac{K^2}{16A^4 m^2}}$$

$$\boxed{B = \sqrt{K^2 + 8EA^2 m}}$$

$$\boxed{= \frac{4A^2 m}{-K \sqrt{1 + \frac{8EA^2 m}{K^2}}}}$$

$$\boxed{= \int_{\sqrt{B^2 - W^2}}^{-1} \frac{1}{\sqrt{1 - \frac{W^2}{B^2}}} dW = \int_{\sqrt{1 - P^2}}^{-1} \frac{1}{\sqrt{1 - \frac{P^2}{B^2}}} dW}$$

$$= \int_{\theta(0)}^{\theta(t)} \cos^{-1} \frac{W(t)}{B} d\theta = \cos^{-1} \frac{W(t)}{B} - \cos^{-1} \frac{W(0)}{B}$$

$$\boxed{P = \frac{W}{B}}$$

$$\Rightarrow \theta(t) - \theta(0) + C = \cos^{-1} \frac{W(t)}{B}, \quad \cos[\theta(t) - \theta(0) + C] = \frac{W(t)}{B} = \frac{\frac{1}{r(t)} + \frac{K}{4A^2 m}}{-\frac{K}{4A^2 m} \sqrt{1 + \frac{8EA^2 m}{K^2}}} = \cos \phi(t)$$

$$\Rightarrow \frac{1}{r(t)} + \frac{K}{4A^2 m} = -\frac{K}{4A^2 m} \sqrt{1 + \frac{8EA^2 m}{K^2}} \cos \phi(t), \quad \frac{1}{r(t)} = -\frac{K}{4A^2 m} \left( \sqrt{1 + \frac{8EA^2 m}{K^2}} + 1 \right) \cos \phi(t)$$

$$\frac{1}{se} = -\frac{K}{4A^2 m} \quad e^2 = 1 + \frac{8EA^2 m}{K^2}$$

$$\Rightarrow r(t) = \frac{se}{1 + e \cos \phi(t)} \quad \text{conic section}$$

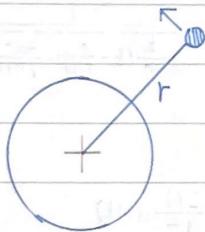
### Gravitational Potential Energy

$$\begin{aligned} W_{\text{grav}} &= \int_{r_1}^{r_2} F dr \\ &= \int_{r_1}^{r_2} -\frac{GM_E m}{r^2} dr = -GM_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{GM_E m}{r_2} - \frac{GM_E m}{r_1} \\ &= GM_E m \frac{r_1 - r_2}{r_1 r_2} \end{aligned}$$

if the body stays close to the earth

$$= GM_E m \frac{r_1 - r_2}{R_E^2} \quad \text{and} \quad g = \frac{GM_E}{R_E^2}$$

$$= mg(r_1 - r_2)$$



$$\begin{aligned} \frac{GM_E m}{r^2} &= m \frac{v^2}{r}, \quad v = \sqrt{\frac{GM_E}{r}} \\ T &= \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{r}{GM_E}} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM_E}} \end{aligned}$$

$$E = K + U$$

$$\begin{aligned} &= \frac{1}{2} m \vec{v}^2 + \left( -\frac{GM_E m}{r} \right) \\ &= \frac{1}{2} m \frac{GM_E}{r} - \frac{GM_E m}{r} \\ &= -\frac{GM_E m}{2r} \quad (\text{circular orbits}) \\ &= -\frac{GM_E m}{2a} \quad \text{where } a \text{ is semimajor axis length elliptical orbits} \end{aligned}$$

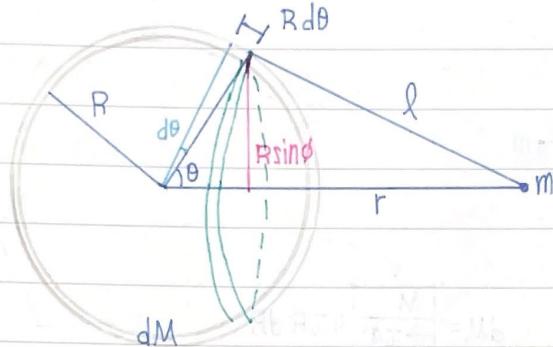
$$\frac{1}{2} m \vec{v}^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{\max}}$$

$$\vec{v}^2 = 2GM_E \left( \frac{1}{R_E} - \frac{1}{r_{\max}} \right)$$

let  $r_{\max} \rightarrow \infty$

$$\Rightarrow \text{espace space } \vec{v} = \sqrt{\frac{2GM_E}{R_E}}$$

## Spherical Mass Distributions



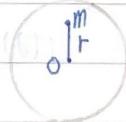
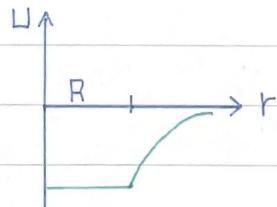
$$\begin{aligned}
 & -\frac{GM}{l} \int \frac{R d\theta \cdot 2\pi R \sin\theta}{4\pi R^2} dM = -G(m dM) \int \frac{\sin\theta d\theta}{2l} \\
 & = -\frac{Gm dM}{2Rr} \int \frac{l(\theta) \frac{dl(\theta)}{d\theta} d\theta}{l(\theta)} \\
 & = -\frac{Gm dM}{2Rr} \int \frac{dl(\theta)}{d\theta} d\theta \\
 & = \frac{Gm dM}{2Rr} l
 \end{aligned}$$

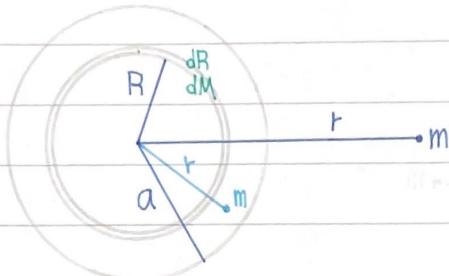
since  $l^2(\theta) = R^2 + r^2 - 2Rr \cos\theta$

$$\begin{aligned}
 & \text{f' } \frac{dl(\theta)}{d\theta} = 2Rr \sin\theta \\
 & \frac{1}{Rr} l(\theta) \frac{dl(\theta)}{d\theta} d\theta = \sin\theta d\theta
 \end{aligned}$$

$$-\frac{Gm dM}{2Rr} l \Big|_{r=R}^{r=R+r} = -\frac{Gm dM}{r}$$

$$\frac{Gm dM}{2Rr} l \Big|_{r=r}^{r=R+r} = -\frac{Gm dM}{R}$$





$$dM = \frac{\rho}{\frac{4}{3}\pi a^3} 4\pi R^2 dR$$

 $r > a$ 

$$-\frac{Gm}{r} \int dM = -\frac{GMm}{r}$$

 $r < a$ 

$$\begin{aligned} 0 < R < r \\ -\frac{Gm}{r} \int dM &= -\frac{Gm}{r} \frac{M}{\frac{4}{3}\pi a^3} \frac{4\pi}{3} r^3 \\ &= -Gm 4\pi \rho \frac{r^3}{3} \end{aligned}$$

$$\begin{aligned} r < R < a \\ -Gm \int \frac{dM}{R} &= -Gm \int \frac{\rho 4\pi R^2}{R} dR \\ &= -Gm \rho 4\pi \int_r^a R dR \\ &= -Gm \rho 4\pi \left(\frac{a^2}{2} - \frac{r^2}{2}\right) \end{aligned}$$

 $r > a$ 

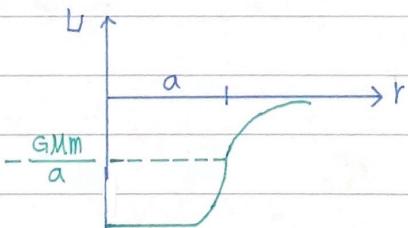
$$U(r) = -\frac{GMm}{r}$$

 $r < a$ 

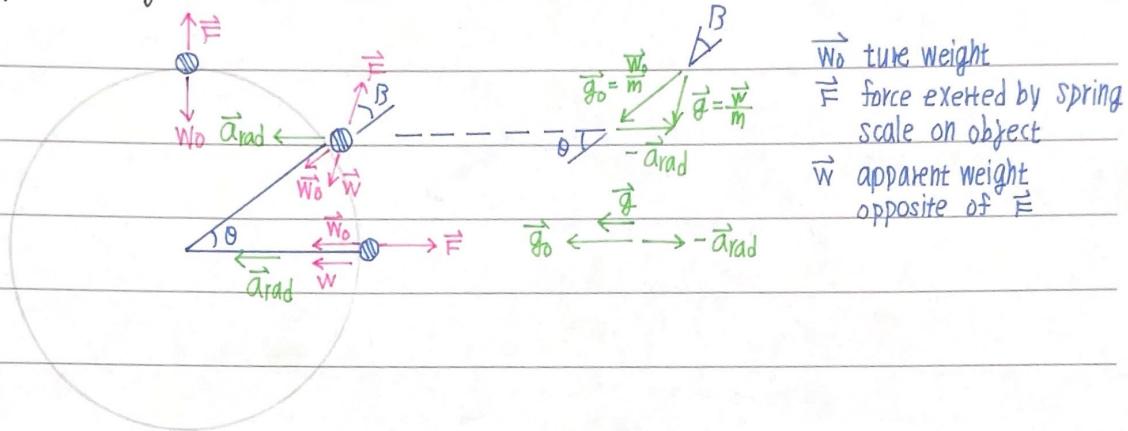
$$U(r) = -Gm 4\pi \rho \frac{a^2}{2} + Gm 4\pi \rho \frac{r^2}{6}$$

$$U(a) = -Gm 4\pi \rho \frac{a^3}{3} = -\frac{GMm}{a}$$

$$U(0) = -Gm 4\pi \rho \frac{a^2}{2}$$



### Apparent Weight and the Earth's Rotation



$$W_0 - F = \frac{mv^2}{R_E} \text{ at the equator} \quad W = W_0 - \frac{mv^2}{R_E} \Rightarrow g = g_0 - \frac{v^2}{R_E}$$

$$\vec{W} = \vec{W}_0 - m \vec{a}_{\text{rad}} = m \vec{g}_0 - m \vec{a}_{\text{rad}}$$

### Black Holes

$$\text{escape speed} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}} R > c$$

$$C = \sqrt{\frac{2GM}{R_S}}$$

$\downarrow$

$r < R_S \Rightarrow \text{black holes}$

$$T = \frac{2\pi a^{\frac{3}{2}}}{NGMx}$$