NO.

DATE / /

Formalism

Eigenfunction of a Hermitian Operator

Discrete Spectra

(i) eigenvalues are real

$$\hat{Q} f_n = a_n f_n, \quad \hat{Q}|f_n\rangle = a_n |f_n\rangle$$

$$\langle f_n|\hat{Q} f_n\rangle = \langle f_n|a_n f_n\rangle = a_n\langle f_n|f_n\rangle$$

$$= \langle \hat{Q} f_n|f_n\rangle$$
$$= \langle a_n f_n|f_n\rangle \quad \Rightarrow a_n^* = a_n$$
$$= a_n^* \langle f_n|f_n\rangle$$

(ii) eigenfunctions for distinct eigenvalues are orthogonal

$$\begin{cases}\hat{Q} f_1 = a_1 f_1 \\ \hat{Q} f_2 = a_2 f_2\end{cases} \quad \text{and } a_1 \neq a_2$$

$$\langle f_2|\hat{Q} f_1\rangle = \langle \hat{Q} f_2|f_1\rangle \qquad \because a_1 \neq a_2$$

$$\langle f_2|a_1 f_1\rangle \qquad \langle a_2 f_2|f_1\rangle \qquad \therefore \langle f_2|f_1\rangle = 0$$

$$a_1\langle f_2|f_1\rangle = a_2\langle f_2|f_1\rangle$$

if $a_1 = a_2$ $\quad \hat{Q}|f_1\rangle = a|f_1\rangle$, $\hat{Q}|f_2\rangle = a|f_2\rangle$ then $f = \alpha f_1 + \beta f_2$

$$\Rightarrow \hat{Q} f = af = \hat{Q}(\alpha|f_1\rangle + \beta|f_2\rangle) = \alpha a|f_1\rangle + \beta a|f_2\rangle$$
$$= a(\alpha|f_1\rangle + \beta|f_2\rangle) = a|f\rangle$$

Continuous Spectra

example momentum operator $\hat{p} f(x) = p f(x)$, $\frac{\hbar}{i}\frac{d}{dx} f(x) = p f(x)$, $f_p(x) = A e^{\frac{ipx}{\hbar}}$

define $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk$

$$\langle f_{p'}(x)|f_p(x)\rangle = |A|^2 \int_{-\infty}^{+\infty} e^{-\frac{ip'x}{\hbar}} e^{\frac{ipx}{\hbar}} dx = |A|^2 \int_{-\infty}^{+\infty} e^{i(p-p')\frac{x}{\hbar}} dx$$
$$= 2\pi |A|^2 \delta\left(\frac{p-p'}{\hbar}\right) = 2\pi\hbar |A|^2 \delta(p-p')$$

let $A = \frac{1}{\sqrt{2\pi\hbar}}$ $\quad \langle f_{p'}|f_p\rangle = \delta(p-p')$ Dirac orthonormality

$$|f\rangle = \sum_n C_n|f_n\rangle, \quad \langle f_m|f\rangle = \sum_n C_n\langle f_m|f_n\rangle = \sum_n C_n \delta_{mn} = C_m \qquad C_n = \langle f_n|f\rangle$$

$$f(x) = \int_{-\infty}^{+\infty} C_p f_p(x) dp = \int_{-\infty}^{+\infty} C_p \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

$$\langle f_{p'}|f\rangle = \int_{-\infty}^{+\infty} C_p \langle f_{p'}|f_p\rangle dp = C(p')$$

example position operator

$$\hat{x}\, g_y(x) = y\, g_y(x) \quad , \quad g_y(x) = A\,\delta(x-y)$$

$$\int_{-\infty}^{+\infty} g_{y'}^*(x)\, g_y(x)\, dx = |A|^2 \int_{-\infty}^{+\infty} \delta(x-y')\,\delta(x-y)\, dx = |A|^2 \delta(y-y')$$

if we set $A=1$ $\quad \langle g_{y'} | g_y \rangle = \delta(y-y')$

$$f(x) = \int_{-\infty}^{+\infty} c(y)\, g_y(x)\, dy = \int_{-\infty}^{+\infty} c(y)\,\delta(x-y)\, dy = c(x)$$

Generalized Statistical Interpretation

measurement of $\hat{Q} \Rightarrow$ one value of $\{q_n\}$

discrete $\quad P(q_n) = |c_n|^2 \quad$ if $|f\rangle = \sum_n c_n |f_n\rangle$

continuous $\quad |c(z)|^2 dz = P(z, z+dz)$

after measurement

the state collapses to $\begin{cases} q_n \\ [c(z), c(z+dz)] \end{cases}$

Example $\quad g_y(x) = \delta(x-y)$

$$c(y) = \langle g_y | \Psi \rangle = \int_{-\infty}^{+\infty} \delta(x-y)\, \Psi(x,t)\, dx = \Psi(y,t)$$

$$|c(y)|^2 dy = |\Psi(y,t)|^2 dy$$

Example $\quad f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$

$$c(p) = \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}}\, \Psi(x,t) = \Phi(p,t)$$

$$|c(p)|^2 dp = P(p, p+dp) \qquad \Rightarrow \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{\frac{ipx}{\hbar}}\, \Phi(p,t)$$