$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]$$

$$U = U(\vec{r}) = U(x,y,z)$$

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{\partial L}{\partial \dot{x}} = \frac{\partial E_k}{\partial \dot{x}} = m\frac{dx}{dt} = P_x$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \qquad \frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \qquad \frac{\partial L}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}}$$

for generalized coordinates

$$\underbrace{\frac{\partial L}{\partial q_i}}_{\text{ith component of generalized force}} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{\text{ith component of the generalized momentum}}$$

Equivalence of Lagrange's

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} = 0$$

$$\frac{\partial (T-U)}{\partial x_i} - \frac{d}{dt}\frac{\partial (T-U)}{\partial x_i'}$$

$$\Rightarrow -\frac{\partial U}{\partial x_i} = \frac{d}{dt}\frac{\partial E_k}{\partial x_i'}$$

$$\Rightarrow F_i = \frac{dp_i}{dt}$$

some coordinate depend on time

$$x_i = x_i(q_j, t) \qquad \frac{dx_i}{dt} = \sum_j \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial x_i}{\partial t} \quad \text{and} \quad \frac{\partial x_i'}{\partial q_j'} = \frac{\partial x_i}{\partial q_j} \qquad \frac{\frac{\partial x_i}{\partial t}}{\frac{\partial q}{\partial t}}$$

$$\text{generalized momentum } P_j = \frac{\partial E_k}{\partial q_j'}$$

example, for a particle moving in plane polar coordinates

$$E_k = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] \quad \text{where } P_r = m\frac{dr}{dt} \text{ linear momentum}$$

$$P_\theta = mr^2\frac{d\theta}{dt} \text{ angular momentum}$$

$$\text{virtual work } \delta W = \sum_i F_i \delta x_i = \sum_{ij} F_i \frac{\partial x_i}{\partial q_j}\delta q_j \equiv \sum_j Q_j \delta q_j$$

$$\text{generalized force } Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = -\frac{\partial U}{\partial q_j}$$

$$P_j = \frac{\partial E_k}{\partial q_j'} = \frac{\partial}{\partial q_j'}\left[\sum_i \frac{1}{2}m\left(\frac{dx_i}{dt}\right)^2\right] = \sum_i m\frac{dx_i}{dt}\frac{\partial x_i'}{\partial q_j'} \qquad \sum_i\left(m\frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial q_j} + m\frac{dx_i}{dt}\frac{d}{dt}\frac{\partial x_i}{\partial q_j}\right) = \frac{dP_j}{dt}$$

$$\text{and } \frac{d}{dt}\frac{\partial x_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j}\frac{dq_k}{dt} + \frac{\partial^2 x_i}{\partial q_j \partial t} \qquad \frac{d}{dt}\frac{\partial x_i}{\partial x} = \sum \frac{\partial^2 x_i}{\partial y\,\partial x}\frac{\partial y}{\partial t} + \frac{\partial}{\partial y}\frac{\partial x_i}{\partial t}$$

$$\frac{dP_j}{dt} = \underbrace{\sum_i m\frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial q_j}}_{Q_j} + \underbrace{\sum_{ik} m\frac{dx_i}{dt}\frac{\partial^2 x_i}{\partial q_k \partial q_j}\frac{dq_k}{dt} + \sum_i m\frac{dx_i}{dt}\frac{\partial^2 x_i}{\partial q_j \partial t}}_{\frac{\partial E_k}{\partial q_j} = \sum_i m\frac{dx_i}{dt}\frac{\partial x_i'}{\partial q_j} = \sum_i m\frac{dx_i}{dt}\frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial x_i}{\partial t}\right)}$$

$$= Q_j + \frac{\partial E_k}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial q_j'}\right) - \frac{\partial E_k}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}, \quad \frac{d}{dt}\frac{\partial (E_k - U)}{\partial q_j'} - \frac{\partial (E_k - U)}{\partial q_j} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j} = 0$$

SEASON

Equivalence of Lagrange's and Newton's Equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \qquad i=1,2,3$$

$$\frac{\partial (T-U)}{\partial x_i} - \frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{x}_i} = 0 \qquad \frac{\partial E_k}{\partial x_i} = 0 \qquad \frac{\partial U}{\partial \dot{x}_i} = 0$$

$$\Rightarrow -\frac{\partial U}{\partial x_i} = \frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}_i}$$

$$\Rightarrow F_i = \frac{dp_i}{dt}$$

for a conservative system $-\frac{\partial U}{\partial x_i} = F_i$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left[\sum_{j=1}^{3}\frac{1}{2}m\left(\frac{dx_j}{dt}\right)^2\right] = \frac{d}{dt}\left(m\frac{dx_i}{dt}\right) = \frac{dp_i}{dt}$$

some coordinate depend on time

$$x_i = x_i(q_j, t) \qquad \frac{dx_i}{dt} = \sum_j \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial x_i}{\partial t} \quad \text{and} \quad \frac{\partial \dot{x}_i}{\partial \dot{q}_i} = \frac{\partial x_i}{\partial q_i} \qquad \frac{\frac{\partial x_i}{\partial t}}{\frac{\partial q_i}{\partial t}}$$

generalized momentum $p_j = \frac{\partial E_k}{\partial \dot{q}_j}$

example, for a particle moving in plane polar coordinates

$$E_k = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$$

where $p_r = m\frac{dr}{dt}$ linear momentum

$p_\theta = mr^2\frac{d\theta}{dt}$ angular momentum

virtual work $\delta W = \sum_i F_i \delta x_i = \sum_{ij} F_i \frac{\partial x_i}{\partial q_j}\delta q_j \equiv \sum_j Q_j \delta q_j$

generalized force $Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = -\frac{\partial U}{\partial q_j}$

$$p_j = \frac{\partial E_k}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\sum_i \frac{1}{2}m\left(\frac{dx_i}{dt}\right)^2\right] = \sum_i m\frac{dx_i}{dt}\frac{\partial \dot{x}_i}{\partial \dot{q}_j} \qquad \sum_i\left(m\frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial q_j} + m\frac{dx_i}{dt}\frac{d}{dt}\frac{\partial x_i}{\partial q_j}\right) = \frac{dp_j}{dt}$$

$$\text{and} \quad \frac{d}{dt}\frac{\partial x_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j}\frac{dq_k}{dt} + \frac{\partial^2 x_i}{\partial q_j \partial t} \qquad \frac{d}{dt}\frac{\partial x_i}{\partial x} = \sum \frac{\partial^2 x_i}{\partial y \partial x}\frac{\partial y}{\partial t} + \frac{\partial}{\partial y}\frac{\partial x_i}{\partial t}$$

$$\frac{dp_j}{dt} = \underbrace{\sum_i m\frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial q_j}}_{Q_j} + \underbrace{\sum_{ik} m\frac{dx_i}{dt}\frac{\partial^2 x_i}{\partial q_k \partial q_j}\frac{dq_k}{dt} + \sum_i m\frac{dx_i}{dt}\frac{\partial^2 x_i}{\partial q_j \partial t}}_{\frac{\partial E_k}{\partial q_j} = \sum_i m\frac{dx_i}{dt}\frac{\partial \dot{x}_i}{\partial q_j} = \sum_i m\frac{dx_i}{dt}\frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial x_i}{\partial t}\right)}$$

$$= Q_j + \frac{\partial E_k}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_j}\right) - \frac{\partial E_k}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}, \quad \frac{d}{dt}\frac{\partial (E_k - U)}{\partial \dot{q}_j} - \frac{\partial (E_k - U)}{\partial q_j} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$