Dynamics of Rigid Bodies

Inertia Tensor

$$\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_r \equiv 0 \quad \Rightarrow \quad \vec{v}_\alpha = \vec{V} + \vec{\omega} \times \vec{r}_\alpha$$

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$$

$$T = \frac{1}{2}\sum_\alpha m_\alpha (\vec{V} + \vec{\omega}\times\vec{r}_\alpha)^2 = \frac{1}{2}\sum_\alpha m_\alpha \vec{V}^2 + \underbrace{\sum_\alpha m_\alpha \vec{V}\cdot\vec{\omega}\times\vec{r}_\alpha}_{=\vec{V}\cdot\vec{\omega}\times(\sum_\alpha m_\alpha \vec{r}_\alpha) = \vec{V}\cdot\vec{\omega}\times M\vec{R} = 0} + \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$$

$$= T_{trans} + T_{rot}$$

$$T_{trans} = \frac{1}{2}\sum_\alpha m_\alpha (\vec{V})^2 = \frac{1}{2}MV^2$$

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2 \qquad (\vec{A}\times\vec{B})^2 = (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\vec{r}_\alpha)^2\right] \qquad \vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left[\sum_i \omega_i^2 \sum_k x_{\alpha k}^2 - \sum_i \omega_i x_{\alpha i} \sum_j \omega_j x_{\alpha j}\right] \qquad \omega_i = \sum_j \omega_j \delta_{ij}$$

$$= \frac{1}{2}\sum_\alpha \sum_{ij} m_\alpha \left[\omega_i\omega_j \delta_{ij} \sum_k x_{\alpha k}^2 - \omega_i\omega_j x_{\alpha i} x_{\alpha j}\right]$$

$$= \frac{1}{2}\sum_{ij}\omega_i\omega_j \underbrace{\sum_\alpha m_\alpha \left[\delta_{ij}\sum_k x_{\alpha k}^2 - x_{\alpha i}x_{\alpha j}\right]}_{I_{ij}}$$

$$= \frac{1}{2}\sum_{ij}\omega_i\omega_j I_{ij} = \frac{1}{2}I\omega^2 \qquad I, \text{ rotational inertia, moment of inertia}$$

$$\text{inertia tensor } \{I\} = \begin{Bmatrix} \sum_\alpha m_\alpha (x_{\alpha 2}^2 + x_{\alpha 3}^2) & -\sum_\alpha m_\alpha x_{\alpha 1} x_{\alpha 2} & -\sum_\alpha m_\alpha x_{\alpha 1} x_{\alpha 3} \\ -\sum_\alpha m_\alpha x_{\alpha 2} x_{\alpha 1} & \sum_\alpha m_\alpha (x_{\alpha 1}^2 + x_{\alpha 3}^2) & -\sum_\alpha m_\alpha x_{\alpha 2} x_{\alpha 3} \\ -\sum_\alpha m_\alpha x_{\alpha 3} x_{\alpha 1} & -\sum_\alpha m_\alpha x_{\alpha 3} x_{\alpha 2} & \sum_\alpha m_\alpha (x_{\alpha 1}^2 + x_{\alpha 2}^2) \end{Bmatrix}$$

$$= \begin{Bmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ -\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{Bmatrix}$$

I_{11}, I_{22}, I_{33} : moments of inertia about x_i axes $\qquad I_{ij} = I_{ji}$

$$I_{ij} = \int_V \rho(r)\left(\delta_{ij}\sum_k x_k^2 - x_i x_j\right)dV$$

Angular Momentum

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \quad , \quad \vec{p}_\alpha = m_\alpha \vec{v}_\alpha = m_\alpha \vec{\omega} \times \vec{r}_\alpha$$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) \quad \vec{A} \times (\vec{B} \times \vec{A}) = A^2 \vec{B} - \vec{A}(\vec{A} \cdot \vec{B})$$

$$= \sum_\alpha m_\alpha \left[r_\alpha^2 \vec{\omega} - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega}) \right]$$

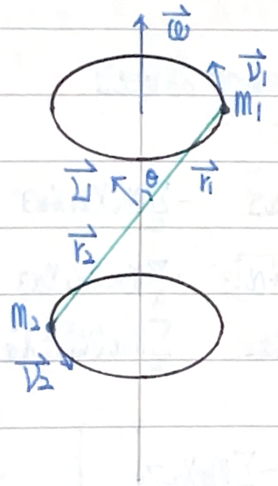
$$L_i = \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha k}^2 - x_{\alpha i} \sum_j x_{\alpha j} \omega_j \right)$$

$$= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha k}^2 - \omega_j x_{\alpha i} x_{\alpha j} \right)$$

$$= \sum_j \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j} \right) = \sum_j I_{ij} \omega_j$$

$\vec{L} = \{I\} \cdot \vec{\omega}$: $\vec{L}$ does not in general have the same direction as the $\vec{\omega}$

$$L_i = \sum_j I_{ij} \omega_j \xrightarrow{\times \frac{1}{2}\omega_i} \frac{1}{2} \sum_i \omega_i L_i = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j = T_{rot} = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot \{I\} \cdot \vec{\omega}$$



$$\vec{L} = \sum \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

let $m_1 = m_2$ $\quad \vec{r}_1 = -\vec{r}_2$, $\vec{v}_1 = -\vec{v}_2$

$$\vec{L} = 2 m_1 \vec{r}_1 \times \vec{v}_1$$

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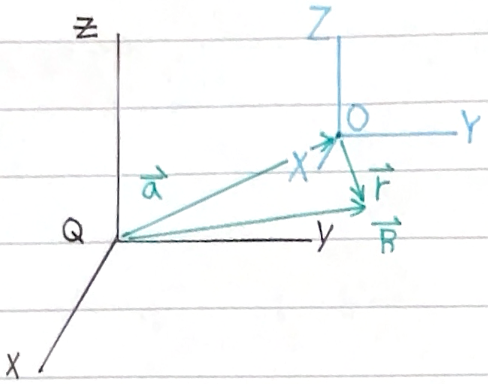
Moments of Interia for Different Body Coordinate System



$\vec{R} = \vec{a} + \vec{r}$ $\quad\quad$ $x_i = a_i + X_i$

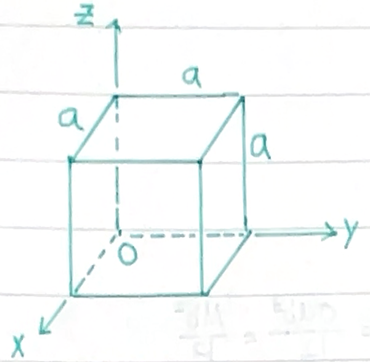
$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j}\right) = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (X_{\alpha k} + a_k)^2 - (X_{\alpha i} + a_i)(X_{\alpha j} + a_j)\right]$$

$$= \underbrace{\sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha k}^2 - X_{\alpha i} X_{\alpha j}\right)}_{I_{ij}} + \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (2X_{\alpha k} a_k + a_k^2) - X_{\alpha i} a_j - X_{\alpha j} a_i - a_i a_j\right]$$

$$= I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j\right) + \sum_\alpha m_\alpha \left(2\delta_{ij} \sum_k X_{\alpha k} a_k - a_i X_{\alpha j} - a_j X_{\alpha i}\right)$$

$$= I_{ij} + M(a^2 \delta_{ij} - a_i a_j) \quad\quad = \sum_\alpha m_\alpha \vec{r}_\alpha = 0 = O \text{ point}$$

Steiner's parallel-axis theorem



density ρ

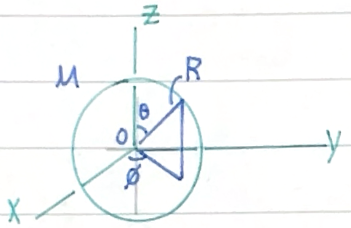
mass M

$$I_{11} = \int \rho \left[\delta_{11}(x_1^2+x_2^2+x_3^2) - x_1x_1\right] dv = \int_V \rho(x_2^2+x_3^2)\,dv = I_{22} = I_{33}$$

$$= \rho\int_0^a dx_3 \int_0^a dx_2\,(x_2^2+x_3^2)\int_0^b dx_1 = \frac{2}{3}\rho b^5 = \frac{2}{3}Mb^2$$

$$I_{12} = -\rho\int_0^b x_1dx_1\int_0^b x_2dx_2\int_0^b dx_3 = -\frac{1}{4}\rho b^5 = -\frac{1}{4}Mb^2 = I_{13} = I_{23}$$

$$\{I\} = \begin{bmatrix} \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 \end{bmatrix}$$



$$I_{33} = \rho\int_V \left[r^2 - x_3^2\right] dV = \rho\iiint (r^2 - r^2\cos^2\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \rho\int_0^R r^4dr\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{2}{5}\pi\rho R^5\int_0^{\pi}(\cos^2\theta - 1)\,d\cos\theta$$

$$= \frac{2}{5}\pi\rho R^5\,\frac{4}{3} = \frac{2}{5}\,\frac{4}{3}\pi R^3\rho R^2 = \frac{2}{5}MR^2$$

$$I_{12} = \rho\int_V (0 - x_1x_2)\,dV = -\rho\iiint r\sin\theta\cos\phi\; r\sin\theta\sin\phi\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\rho\int_0^R r^4dr\int_0^{\pi}\sin^3\theta\,d\theta\int_0^{2\pi}\sin\phi\cos\phi\,d\phi$$

$$= -\rho\,\frac{R^5}{5}\,\frac{4}{3}\,0 = 0$$

$$\{I\} = \begin{bmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{bmatrix}$$

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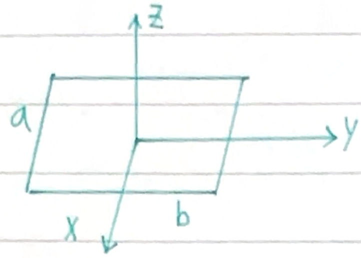

plane density σ

mass M

$$I_{11} = \sigma \iint \left[(x^2+y^2)-x^2\right] dx\,dy = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dy = \frac{\sigma a b^3}{12} = \frac{Mb^2}{12}$$

$$I_{22} = \sigma \iint \left[(x^2+y^2)-y^2\right] dx\,dy = \sigma \int_{-\frac{a}{2}}^{a/2} x^2 dx \int_{-\frac{b}{2}}^{b/2} dy = \frac{\sigma a^3 b}{12} = \frac{Ma^2}{12}$$

$$I_{33} = \sigma \iint (x^2+y^2)\, dx\, dy = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx \int_{-\frac{b}{2}}^{\frac{b}{2}} dy + \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dy$$
$$= \frac{Ma^2}{12} + \frac{Mb^2}{12} = \frac{M}{12}(a^2+b^2)$$

$$I_{12} = \sigma \iint (0-xy)\, dx\, dy = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} x\,dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y\,dy = 0$$

$$\{I\} = \begin{bmatrix} \frac{1}{12}Mb^2 & 0 & 0 \\ 0 & \frac{1}{12}Ma^2 & 0 \\ 0 & 0 & \frac{M}{12}(a^2+b^2) \end{bmatrix}$$