Second derivatives

divergence of gradient $\nabla\cdot(\nabla T)$ divergence of curl $\nabla\cdot(\nabla\times\vec{V})$

curl of gradient $\nabla\times(\nabla T)$ curl of curl $\nabla\times(\nabla\times\vec{V})$

gradient of divergent $\nabla\cdot(\nabla\cdot\vec{V})$

$\nabla\cdot(\nabla T)$

$$=\nabla\left(\frac{\partial T}{\partial x}\hat{i}+\frac{\partial T}{\partial y}\hat{j}+\frac{\partial T}{\partial z}\hat{k}\right)=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial T}{\partial x}\hat{i}+\frac{\partial T}{\partial y}\hat{j}+\frac{\partial T}{\partial z}\hat{k}\right)$$

$$=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} \quad =\nabla^2 T$$ Laplaction of T

$(\nabla\cdot\nabla)\vec{V}$

$$=\left[\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\right]\vec{V}$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\hat{i}V_x+\hat{j}V_y+\hat{k}V_z)$$

$$=\nabla^2 V_x\hat{i}+\nabla^2 V_y\hat{j}+\nabla^2 V_z\hat{k} \qquad \nabla^2\vec{V}=(\nabla\cdot\nabla)\vec{V}\neq\nabla(\nabla\cdot\vec{V})$$

$\nabla\cdot(\nabla\times\vec{V})$

$\because\ \vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$

$$\Rightarrow (\nabla\times\nabla)\cdot\vec{V}=0\cdot\vec{V}=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}=0$$

means if $\nabla\cdot\vec{A}=0$, then $\vec{A}=\nabla\times\vec{B}$

EX: $\nabla\cdot\vec{B}=0$ $\vec{B}=\nabla\times\vec{A}$ vector potential

$\nabla\times(\nabla T)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z}\end{vmatrix}=\left(\frac{\partial^2 T}{\partial y\partial z}-\frac{\partial^2 T}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2 T}{\partial z\partial x}-\frac{\partial^2 T}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2 T}{\partial x\partial y}-\frac{\partial^2 T}{\partial y\partial x}\right)\hat{k}=0$$

means if $\nabla\times\vec{A}=0$ then $\vec{A}=\pm\nabla V$

EX: $\nabla\times\vec{E}=0$ for static E, then $\vec{E}=-\nabla V$

$\nabla\times(\nabla\times\vec{V})$

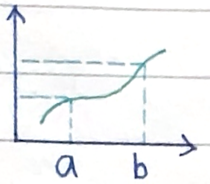
$\because\ \vec{A}\times\vec{B}\times\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\cdot\vec{C}$

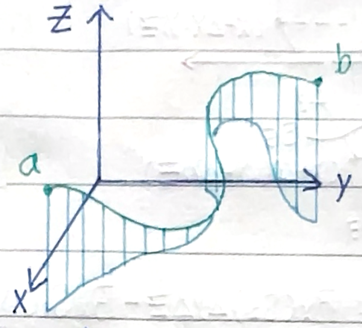
$\Rightarrow \nabla(\nabla\cdot\vec{V})-(\nabla\cdot\nabla)\cdot\vec{V}$

Integral Calculus

$$\int_a^b \frac{dF(x)}{dx} = -\int_b^a \frac{dF(x)}{dx} = \int_a^b f(x) = F(b)-F(a)$$

$$\int_a^b (\nabla g)\cdot d\vec{l} = \int \frac{\partial g}{\partial x}dx + \int \frac{\partial g}{\partial y}dy + \int \frac{\partial g}{\partial z}dz = g(b_x, b_y, b_z) - g(a_x, a_y, a_z)$$



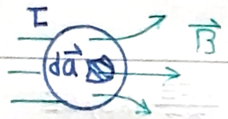


corollary 1: $\int_a^b (\nabla g)\cdot d\vec{l} = g(\vec{b}) - g(\vec{a})$

corollary 2: $\oint (\nabla g)\cdot d\vec{l} = 0 = \int_a^a (\nabla g)\cdot d\vec{l}$

Gauss'theorem, Green's theorem, divergence theorem

$\int$ faucets within the volume $= \int$ flow out through the surface $\int_V (\nabla\cdot\vec{B})d\tau = \oint_S \vec{B}\cdot d\vec{a}$



Stoke's theorem

corollary1: $\int(\nabla\times\vec{V})\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used

corollary2: $\oint(\nabla\times\vec{V})\cdot d\vec{a} = 0$ for any closed surface

$$\int_S (\nabla\times\vec{V})\cdot d\vec{a} = \oint_{\substack{\text{boundary}\\\text{line}}} \vec{V}\cdot d\vec{l}$$

let $\vec{V} = \nabla T$ and $a=b$, $\begin{cases} \text{gradient theorem} & \int_a^b \vec{V}\cdot d\vec{l} = \oint \vec{V}\cdot d\vec{l} = 0 \\ \text{Stoke's theorem} & \oint \vec{V}\cdot d\vec{l} = \int_S (\nabla\times\vec{V})\cdot d\vec{a} = \int_S \nabla\times(\nabla T)\cdot d\vec{a} = 0 \end{cases}$

means $\nabla\times(\nabla T) = 0$

let $\oint_S (\nabla\times\vec{V})\cdot d\vec{a} = 0$

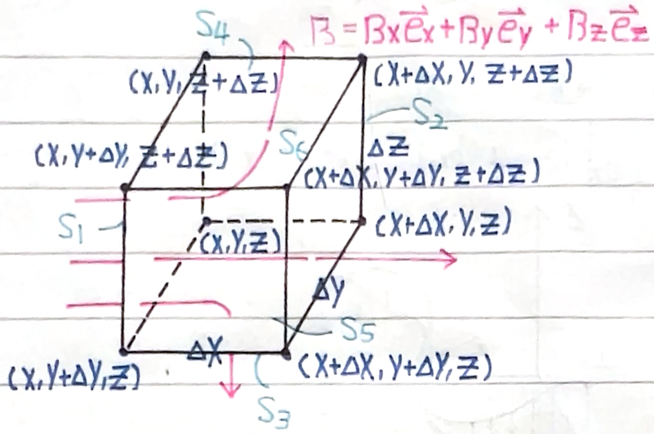
$\oint_S (\nabla\times\vec{V})\cdot d\vec{a} = \int_V \nabla\cdot(\nabla\times\vec{V})d\tau = 0$ means $\nabla\cdot(\nabla\times\vec{V}) = 0$

$$\int \nabla\cdot(f\vec{A})d\tau = \int f(\nabla\cdot\vec{A})d\tau + \int \vec{A}\cdot(\nabla f)d\tau = \oint_S f\vec{A}\cdot d\vec{a}$$

$$\Rightarrow \int_V f(\nabla\cdot\vec{A})d\tau = -\int_V \vec{A}\cdot(\nabla f)d\tau + \oint_S f\vec{A}\cdot d\vec{a}$$

Green's theorem $\int_V (\nabla\cdot\vec{B})d\tau = \oint_S \vec{B}\cdot d\vec{a}$

(1)



flux of S_1 and S_2: $B_x(S_2)\Delta y\Delta z - B_x(S_1)\Delta y\Delta z$ and $B_x(S_2) = B_x(S_1) + \frac{\partial B_x}{\partial x}\Delta x$

$\Rightarrow \left[B_x(S_1) + \frac{\partial B_x}{\partial x}\Delta x\right]\Delta y\Delta z - B_x(S_1)\Delta y\Delta z = \frac{\partial B_x}{\partial x}\Delta x\Delta y\Delta z$

similarly, flux$(S_3, S_4) = \frac{\partial B_z}{\partial z}\Delta x\Delta y\Delta z$, flux$(S_5, S_6) = \frac{\partial B_y}{\partial y}\Delta x\Delta z\Delta y$

$\Rightarrow \oint \vec{B}\cdot d\vec{a} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\Delta x\Delta y\Delta z = (\nabla\cdot\vec{B})\Delta V$

$\Rightarrow \lim_{\Delta x\Delta y\Delta z\to 0}\frac{\oint\vec{B}\cdot d\vec{a}}{\Delta x\Delta y\Delta z} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla\cdot\vec{B}$

(2)

$\int_V(\nabla\cdot\vec{B})d\tau = \int_V\frac{\partial A_x}{\partial x}dxdydz + \int_V\frac{\partial A_y}{\partial y}dxdydz + \int_V\frac{\partial A_z}{\partial z}dxdydz$

$= \iint dydz\int_{x_1}^{x_2}\frac{\partial A_x}{\partial x}dx = \iint\left[A_x(x_2, y_0, z_0) - A_x(x_1, y_0, z_0)\right]dydz$

$= A_x(P_2)da_{2x} - A_x(P_1)(-da_{1x}) = A_{x_2}da_{2x} + A_{x_1}da_{1x} = \oint_S A_x\, da_x$

$= \oint_S(A_x da_x + A_y da_y + A_z da_z) = \oint_S \vec{A}\cdot d\vec{a}$

consider a point P at the center of a small volume ΔV, if ΔV is very small

$\int_{\Delta V}\nabla\cdot\vec{B}\,d\tau = \langle\nabla\cdot\vec{B}\rangle_P\Delta V$, $\langle\nabla\cdot\vec{B}\rangle_P = \frac{1}{\Delta V}\oint_S\vec{B}\cdot d\vec{a}$

$\nabla\cdot\vec{B} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint_S\vec{B}\cdot d\vec{a}$