A SURVEY OF THE NEGGERS-STANLEY CONJECTURE

XIFAN YU

Abstract. In this paper, we present a brief survey of the Neggers-Stanley conjecture. We introduce the (P, ω)-partition theory developed by Stanley [15], and motivate the formulation of the Neggers-Stanley conjecture [6, p. 21]. Although the Neggers-Stanley conjecture is disproved in its general form [2] [16], there are interesting open questions remaining for special cases. We collect some results related to the Neggers-Stanley conjecture, and list a number of open questions.

Contents

1. Introduction

The order polynomial of a finite labelled poset is a polynomial that contains rich information of the Jordan-Hölder set of the poset, and appears naturally in the enumeration of many combinatorial structures. One conjecture regarding the order polynomial that received a lot of attention was the Neggers-Stanley conjecture.

The Neggers-Stanley conjecture, or the Poset conjecture, asserts that two polynomials related to the order polynomial, the E-polynomial and W-polynomial, have only real roots for any finite labelled poset. The Neggers-Stanley conjecture has been verified for several classes of labelled posets, and a number of partial results have been proved. Although counterexamples to the Neggers-Stanley conjecture and Neggers original conjecture have been found, there are still interesting open questions regarding the E-polynomials and W-polynomials of special classes of labelled posets.

In Sections 2 and 3, we motivate the theory of (P, ω)-partitions, as well as the Neggers-Stanley conjecture. To remind ourselves that the general statement is false, we present two simple counterexamples to the Neggers-Stanley conjecture in Section

Date: August 29, 2020.

4. In Section 5, we discuss the effects of different operations on labelled posets on the order polynomials, E-polynomials, and W-polynomials, and we explore the interlacing properties of E-polynomials of labelled posets in Section 6. We shift our attention in Section 7, as we survey a recent result by Brändén on the unimodality of W-polynomials of sign-graded posets. Finally, we list a number of open questions on the Neggers-Stanley conjecture in Section 8.

For some results on the Neggers-Stanley conjecture and related problems not covered in this survey, we refer the reader to [6][8][11][12][13]. See [3] for a survey of techniques that can be used to prove real-rootedness, log-concavity, or unimodality.

2. Definitions

We start by giving relevant definitions to prepare further discussions.

Definition 2.1. A partially ordered set (poset) P is a set, together with a binary relation $\leq$ that satisfies

(1) For all $t \in P$, $t \leq t$ (reflexivity).
(2) If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry).
(3) If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity).

In this work, we will work with finite posets.

We will use $s < t$ to denote the relation $s \leq t$ and $s \neq t$. We say $s, t \in P$ are comparable if either $s \leq t$ or $t \leq s$, and we say $s, t \in P$ are incomparable otherwise, denoted $s \parallel t$. When there are multiple posets in the context, we may use subscripts to differentiate binary relations of different posets.

For $s, t \in P$, we say s covers t, denoted $t \prec s$, if $t < s$ and there does not exist $r \in P$ such that $t < r < s$. The Hasse diagram of a poset P is a diagram consisting of all the covering relations of the elements of P.

Definition 2.2. A poset P and a poset Q are isomorphic, denoted $P \cong Q$, if there exists an isomorphism $f : P \to Q$ such that $s \leq_P t$ if and only if $f(s) \leq_Q f(t)$.

A chain is a poset P whose elements are pairwise comparable. Therefore, a chain is a set with total order. We denote a chain of n elements by $\boldsymbol{n}$, and every chain of n elements is isomorphic to $\{1, 2, \ldots, n\}$ equipped with the natural order of integers.

An antichain is a poset P whose elements are pairwise incomparable.

Definition 2.3. A poset Q is a subposet of a poset P if

(1) Q is a subset of elements of P.
(2) For $s, t \in Q$, if $s \leq_Q t$, then $s \leq_P t$.

A poset Q is an induced subposet of a poset P if

(1) Q is a subset of elements of P.
(2) For $s, t \in Q$, $s \leq_Q t$ if and only if $s \leq_P t$.

Therefore, an induced subposet of P can be specified by a subset of elements of P, with the order induced from the partial order on P.

Definition 2.4. A subset C of a poset P is a chain if it is a chain when regarded as an induced subposet of P. The length of a finite chain C is $l(C) = |C| - 1$.

A maximal chain C of a poset P is a chain that is not contained in any larger chain of P.

A saturated chain C of a poset P is a chain $s_0 \prec s_1 \prec \cdots \prec s_t$ in which s_i covers s_{i-1} for all $i = 1, \ldots, t$.

Definition 2.5. A poset P is graded if every maximal chain of P has the same length.

The length of a maximal chain of a graded poset P is called the rank of the graded poset. Let P be a graded poset of rank n. Then, there is a unique rank function $\rho : P \to \{0, 1, \dots, n\}$ such that

$$\rho(s) = \begin{cases} 0 & \text{if } s \text{ is a minimal element of } P, \\ \rho(t) + 1 & \text{otherwise, where } t \text{ is an element covered by s.} \end{cases}$$

Remark 2.6. A graded poset P can be decomposed into level sets, with each level set consisting of elements of the same rank. Moreover, each level set is an antichain, and any maximal chain of P contains exactly one element from each level set.

Given two posets P and Q, we can build new posets from them using various operations.

Definition 2.7. Let P and Q be two posets on disjoint elements.

- The **dual** of P, denoted P^*, is the poset on the same set P, with order defined by $s \leq_{P^*} t$ if
$$t \leq_P s.$$
- The **disjoint union** of P and Q, denoted $P+Q$, is the poset on the disjoint union $P \sqcup Q$, with order defined by $s \leq_{P+Q} t$ if
$$s \leq_P t \text{ or } s \leq_Q t.$$
- The **ordinal sum** of P and Q, denoted $P \oplus Q$, is the poset on the disjoint union $P \sqcup Q$, with order defined by $s \leq_{P \oplus Q} t$ if
$$s \leq_P t, s \leq_Q t, \text{ or both } s \in P \text{ and } t \in Q.$$
- The **direct product** of P and Q, denoted $P \times Q$, is the poset on the Cartesian product $P \times Q$, with order defined by $(s, s') \leq_{P \times Q} (t, t')$ if
$$s \leq_P t \text{ and } s' \leq_Q t'.$$
- The **ordinal product** of P and Q, denoted $P \otimes Q$, is the poset on the Cartesian product $P \times Q$, with order defined by $(s, s') \leq_{P \otimes Q} (t, t')$ if
$$s <_P t, \text{ or both } s =_P t \text{ and } s' \leq_Q t'.$$

Examples 2.8. Let $\mathbf{1}$ denote the singleton poset, and $\mathbf{2}$ denote the 2-element chain.

(1) Let C be a chain of n elements. Then, $C \cong \underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{n \text{ times}}$.

(2) Let A be an antichain of n elements. Then, $A \cong \underbrace{\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}}$.

(3) Let B_n denote the poset on the power set $P([n])$, with partial order given by inclusion. Then, $B_n \cong \underbrace{\mathbf{2} \times \mathbf{2} \times \cdots \times \mathbf{2}}_{n \text{ times}} = \mathbf{2}^{\times n}$.

Definition 2.9. A poset P is called series-parallel if P can be built up from $\mathbf{1}$ using the ordinal sum and the disjoint union operations.

3. Neggers-Stanley conjecture

Before we state the Neggers-Stanley conjecture, we need to review the (P, ω)-partition theory [15] developed by Stanley.

Definition 3.1. A labelling of a poset P is a bijection $\omega : P \to \{1, 2, \ldots, n\}$, where $n = |P|$. A labelled poset is a pair of (P, ω).

(P, ω) is naturally labelled if ω is order-preserving, i.e., $\omega(s) \leq \omega(t)$ if $s \leq t$.

Remark 3.2. Since $\boldsymbol{n} \cong \{1, 2, \ldots, n\}$, we can think of a labelling of P as a bijection $\omega : P \to \boldsymbol{n}$. Moreover, the set of natural labellings is in bijective correspondence with the set of linear extensions of poset P.

Definition 3.3. A (P, ω)-partition is a function $\phi : P \to \mathbb{N}^+$ such that for $s, t \in P$

(1) If $s \leq t$, then $\phi(s) \leq \phi(t)$.
(2) If $s \leq t$ and $\omega(s) > \omega(t)$, then $\phi(s) < \phi(t)$.

A (P, ω)-partition with largest part $\leq m \in \mathbb{N}$ is a (P, ω)-partition $\phi : P \to \{1, 2, \ldots, m\}$. By convention, there is no (P, ω)-partition with largest part ≤ 0, except when P is the empty poset.

Therefore, a (P, ω)-partition is an order-preserving function which satisfies the extra condition that if $\omega(s) > \omega(t)$, then the pair $s \leq t$ is sent to a strictly ordered pair $\phi(s) < \phi(t)$.

In the case of a naturally labelled (P, ω), a (P, ω)-partition is just an order preserving function.

Remark 3.4. In the theory of P-partitions originated in [15], Stanley defines (P, ω)-partitions using order-reversing functions. Here, we use order-preserving functions in the definition of (P, ω)-partitions. Since there is a natural correspondence between order-preserving functions and order-reversing functions, this change of definition will not affect the formulations of several polynomials of our interest and of the Neggers-Stanley conjecture.

Now, let $\Omega(P, \omega; m)$ denote the number of (P, ω)-partitions with largest part $\leq m$. Let $e_j(P, \omega)$ denote the number of surjective (P, ω)-partitions with largest part $= j$. Then, we have the following identity.

Proposition 3.5.

$$\Omega(P, \omega; x) = \sum_{j \in \mathbb{N}} e_j(P, \omega) \binom{x}{j}.$$

Proof. To prove the above identity, we only need to show

$$\Omega(P, \omega; m) = \sum_{j \in \mathbb{N}} e_j(P, \omega) \binom{m}{j}.$$

The set of (P, ω)-partitions with largest part $\leq m$ can be partitioned according to the cardinality of the range. For (P, ω)-partitions with largest part $\leq m$ and cardinality of range $= j$, there are exactly $e_j(P, \omega)\binom{m}{j}$ of them. The proposition then follows. $\square$

Therefore, $\Omega(P, \omega; x)$ is a polynomial of x, which is called the order polynomial of (P, ω). The order polynomial $\Omega(P, \omega)$ has degree $n = |P|$, since $e_n(P, \omega) > 0$ and $e_j(P, \omega) = 0$ for $j > n$.

Definition 3.6. The E-polynomial of (P, ω) is defined as

$$E(P, \omega; x) = \sum_{j \in \mathbb{N}} e_j(P, \omega) x^j.$$

Consider the linear operator $\mathscr{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $\mathscr{E}(\binom{x}{j}) = x^j$. Then, $E(P, \omega) = \mathscr{E}(\Omega(P, \omega))$.

Definition 3.7. The Jordan-Hölder set of (P, ω), denoted $\mathcal{L}(P, \omega)$, is the set of permutations

$$\mathcal{L}(P, \omega) = \{\omega\sigma^{-1} : \sigma \in \ell(P)\},$$

where $\ell(P)$ is the set of linear extensions $\sigma : P \to \{1, 2, \ldots, n\}$.

Given a labelled poset (P, ω), we can build another poset Q on $\{1, 2, \ldots, n\}$ by identifying $s \in P$ with $\omega(s) \in Q$. Then, the Jordan-Hölder set of (P, ω) can be viewed as the set of inverses of linear extensions of this poset Q on $\{1, 2, \ldots, n\}$.

Definition 3.8. The W-polynomial of (P, ω), also known as the (P, ω)-Eulerian polynomial [3], is defined as

$$W(P, \omega; x) = \sum_{\tau \in \mathcal{L}(P, \omega)} x^{des(\tau)+1},$$

where $des(\tau)$ is the number of descents of τ, i.e., $des(\tau) = \#\{i : \tau(i) > \tau(i+1)\}$.

Definition 3.9. Two labelled posets (P, ω) and (Q, ν) are similar if there exists an isomorphism $f : P \to Q$ such that $\omega(s) \leq \omega(t)$ if and only if $\nu(f(s)) \leq \nu(f(t))$.

It is not hard to see that if two labelled posets are similar, they share identical order polynomial, E-polynomial, and W-polynomial.

Examples 3.10. Let P be a chain of n elements. Let ω be an arbitrary labelling of P. Then, the Jordan-Hölder set of (P, ω) contains only one permutation $\sigma \in S_n$ corresponding to the unique linear extension of the chain. Therefore, the W-polynomial of (P, ω) is

$$W(P, \omega; x) = x^{des(\sigma)+1},$$

which clearly has only real roots.

Let Q be an antichain of n elements. Let ν be an arbitrary labelling of Q. Then, the Jordan-Hölder set of (Q, ν) is $\mathcal{L}(Q, \nu) = S_n$. Therefore, the W-polynomial of (Q, ν) is

$$W(Q, \nu; x) = \sum_{\tau \in S_n} x^{des(\tau)+1} = x A_n(x),$$

where $A_n(x)$ is the n-th Eulerian polynomial. It is known that Eulerian polynomials have only real roots [7].

The two minimal examples above lead to interesting questions about properties of W-polynomials of labelled posets. Now we are ready to state the Neggers-Stanley conjecture [6, p. 21].

Conjecture 3.11. *For any finite labelled poset (P, ω), the polynomial $W(P, \omega)$ has only real roots.*

This conjecture first appeared in [9], where Neggers conjectured the real-rootedness of W-polynomial for naturally labelled posets in 1978. Stanley later made the generalized conjecture above for all labelled posets in 1986.

Recall that $E(P,\omega;x) = \sum_j e_j(P,\omega)x^j$, where $e_j(P,\omega)$ is the number of surjective (P,ω)-partitions with largest part $\leq j$. Let us write the W-polynomial of (P,ω) as $W(P,\omega;x) = \sum_j w_j(P,\omega)x^j$. By definition of W-polynomial, the coefficients $w_j(P,\omega)$ have the following combinatorial interpretation

$$w_j(P,\omega) = \#\{\tau \in \mathcal{L}(P,\omega) : des(\tau)+1 = j\}.$$

The following proposition [15] connects the polynomials $E(P,\omega)$ and $W(P,\omega)$.

Proposition 3.12. *Given a labelled poset (P,ω), let $n = |P|$. Then,*

$$(1-x)^n E(P,\omega;\frac{x}{1-x}) = W(P,\omega;x).$$

Proof. By a change of variable, we may as well show

$$\begin{aligned} E(P,\omega;y) &= (1+y)^n W(P,\omega;\frac{y}{1+y}) \\ &= \sum_{i\in\mathbb{N}} w_i(P,\omega)y^i(1+y)^{n-i}. \end{aligned}$$

We only need to show

$$e_j(P,\omega) = \sum_{i=0}^{j} w_i(P,\omega)\binom{n-i}{j-i}.$$

Let $\mathcal{F}_j(P,\omega)$ denote the set of surjective (P,ω)-partitions with largest part $=$ j. Consider the mapping $T : \mathcal{F}_j(P,\omega) \to \mathcal{L}(P,\omega)$ where $T(f)$ is defined as the permutation written as word by writing down the labels of $f^{-1}(1)$ in ascending order, then writing down the labels of $f^{-1}(2)$ in ascending order, and so on up to writing down the labels of $f^{-1}(j)$ in ascending order.

For simplicity, we identify the elements of P with the labels assigned to them. For example, if $f \in \mathcal{F}_j(P,\omega)$ is given by $f(1) = 1, f(3) = 1, f(5) = 1, f(4) = 2, f(2) = 3$, then $T(f) = 13542 \in \mathcal{L}(P,\omega)$. It is not hard to check for any (P,ω)-partition $f \in \mathcal{F}_j(P,\omega)$, we have $T(f) \in \mathcal{L}(P,\omega)$.

For $f \in \mathcal{F}_j(P,\omega)$, the number of descents in $T(f)$ is $\leq j-1$. Now consider an element σ of the Jordan-Hölder set $\mathcal{L}(P,\omega)$ with d descents for $0 \leq d \leq j-1$. It is not hard to see that $T^{-1}(\sigma)$ consists of exactly $\binom{n-d-1}{j-d-1}$ (P,ω)-partitions of $\mathcal{F}_j(P,\omega)$. Thus, we have

$$e_j(P,\omega) = \sum_{i=0}^{j} w_i(P,\omega)\binom{n-i}{j-i}$$

as desired.

□

Hence, either both or none of $E(P,\omega)$ and $W(P,\omega)$ are real-rooted. Therefore, an equivalent formulation of Conjecture 3.11 is

Conjecture 3.13. *For any finite labelled poset (P,ω), the polynomial $E(P,\omega)$ has only real roots.*

The study of real-rooted polynomials has a long history that can be dated back to Newton. The following theorem is a classic result due to Newton. Recall that a sequence $\{a_0, a_1, \ldots, a_n\}$ is unimodal if there exists an index $k \in \mathbb{N}$ such that $a_i \leq a_{i+1}$ for $0 \leq i \leq k-1$ and $a_i \geq a_{i+1}$ for $k \leq i \leq n-1$. A nonnegative sequence $\{a_0, a_1, \ldots, a_n\}$ is log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$.

Theorem 3.14. *Let $\sum_{i=0}^n a_i x^i$ be a polynomial with nonnegative coefficients that has only real roots. Then, the sequence $\{a_0, a_1, \ldots, a_n\}$ is log-concave and unimodal. In fact, the sequence $\{\frac{a_i}{\binom{n}{i}}\}$ is log-concave and unimodal.*

Consequently, the real-rootedness of a polynomial with nonnegative coefficients implies the log-concavity and unimodality of the coefficients. Therefore, the following conjecture would be a consequence of the Neggers-Stanley conjecture.

Conjecture 3.15. *For any finite labelled poset (P, ω), the sequences $\{e_j(P, \omega)\}$ and $\{w_j(P, \omega)\}$ are log-concave and unimodal.*

Note that although the Neggers-Stanley conjecture implies conjecture 3.15, the converse is not true. Besides questions about real-rootedness of the E-polynomials and W-polynomials, there are also other interesting questions that can be asked about them.

Given a real-rooted polynomial, it is a natural question to ask for the locations of the roots. Clearly, for a polynomial with nonnegative coefficients, its real roots are nonpositive. The following proposition indicates that the interval $[-1, 0]$ has some connection with the roots of $E(P, \omega)$.

Proposition 3.16. *The real roots of $E(P, \omega)$ are in the interval $[-1, 0]$.*

To prove the proposition above, we will need the following reciprocity theorems of Stanley [15, Prop 13.2]. Let $\omega : P \to \{1, 2, \ldots, n\}$ be a labelling of P. The complement of ω, denoted $\overline{\omega} : P \to \{1, 2, \ldots, n\}$, is given by $\overline{\omega}(s) = n+1-\omega(s)$. In other words, the complement $\overline{\omega}$ is the unique labelling of P such that $\omega(s) < \omega(t)$ if and only if $\overline{\omega}(s) > \overline{\omega}(t)$.

Proposition 3.17. *Given a labelled poset (P, ω), let $n = |P|$. Then,*
(a) $\Omega(P, \overline{\omega}; x) = (-1)^n \Omega(P, \omega; -x)$.
(b) $E(P, \overline{\omega}; x) = (-1)^n \frac{x}{x+1} E(P, \omega; -x-1)$.
(c) $W(P, \overline{\omega}; x) = x^{n+1} W(P, \omega; \frac{1}{x})$.

The proof of Proposition 3.16 then follows directly from Proposition 3.17. If α is a real root of $E(P, \omega)$, then $-\alpha - 1$ is a real root of $E(P, \overline{\omega})$. Since both $E(P, \omega)$ and $E(P, \overline{\omega})$ are polynomials with nonnegative coefficients, we have $\alpha \leq 0$ and $-\alpha - 1 \leq 0$, from which we arrive at $-1 \leq \alpha \leq 0$.

As a consequence of Proposition 3.16, the Neggers-Stanley conjecture has the following equivalent formulation:

Conjecture 3.18. *For any finite labelled poset (P, ω), the polynomial $E(P, \omega)$ is $[-1, 0]$-rooted.*

Moreover, the multiplicities of -1 and 0 as roots of $E(P, \omega)$ have combinatorial interpretations. Simon [11] proved the following proposition in the case of naturally labelled posets, and Wagner [17] extended the proposition to all labelled posets.

Proposition 3.19. *Let (P, ω) be a nonempty labelled poset. Then,* *(i) The multiplicity of* 0 *as a root of* $E(P,\omega)$ *is one greater than the maximum number of descents of* (P,ω) *in a maximal chain of* P*.*

(ii) The multiplicity of -1 *as a root of* $E(P,\omega)$ *is the maximum number of ascents of* (P,ω) *in a maximal chain of* P*.*

Proof. Let (P,ω) be a nonempty labelled poset.

(i) Let $x_0 \prec x_1 \prec \cdots \prec x_k$ be a maximal chain of P that achieves the maximum number of descents. Suppose this chain contains m descents. Then, any surjective (P,ω)-partition will have largest part $\geq m+1$, and we have

$$e_0(P,\omega) = e_1(P,\omega) = \cdots = e_m(P,\omega) = 0.$$

To show that the multiplicity of 0 as a root of $E(P,\omega)$ is $m+1$, we need to show $e_{m+1}(P,\omega) \neq 0$. Consider the function $f : P \to \{1,2,\ldots,m+1\}$ where $f(s)$ is defined as the maximum number of descents in a saturated chain $x_0 \prec x_1 \prec \cdots \prec x_t = s$. Clearly, this is a surjective (P,ω)-partition with largest part $= m+1$. Thus, $e_{m+1}(P,\omega) \neq 0$.

(ii) The multiplicity of -1 as a root of $E(P,\omega)$ is the maximum number of ascents of (P,ω) in a maximal chain of P following (i) and the reciprocity theorem in Proposition 3.17(b):

$$E(P,\omega;x) = (-1)^n \frac{x}{x+1} E(P,\overline{\omega};-x-1)$$

$\square$

4. Counterexamples to the Neggers-Stanley conjecture

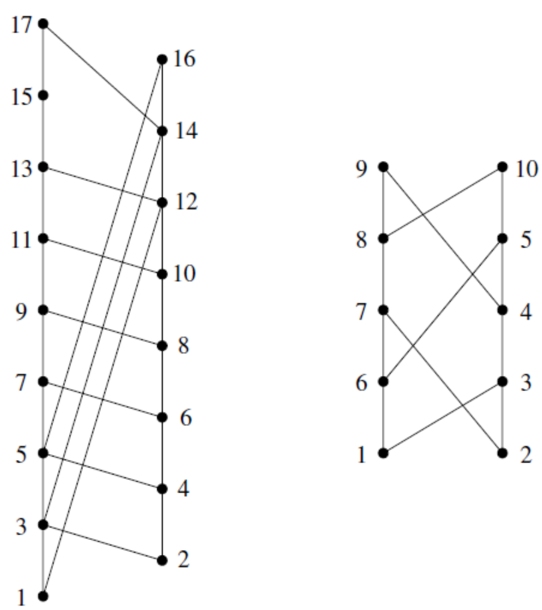

Figure 1. Counterexamples to Neggers conjecture (left) and the Neggers-Stanley conjecture (right), taken from [16].

A counterexample to the Neggers-Stanley conjecture was discovered by Brändén [2] in 2004, which is a poset of 22 elements with an unnatural labelling.

Soon after, a counterexample to Neggers original conjecture was discovered by Stembridge [16] as shown on the left of Figure 1, which is a naturally labelled poset of 17 elements. A minimal counterexample to the Neggers-Stanley conjecture was also provided by Stembridge in [16], as shown on the right of Figure 1.

However, this is not the end of the story. The Neggers-Stanley conjecture turns out to be true over a number of interesting families of labelled posets. It is again an interesting question to ask what properties of a labelled poset would ensure the conditions in the Neggers-Stanley conjecture.

Moreover, Conjecture 3.15 is still open. It is an interesting question to ask what other properties of a polynomial would ensure the unimodality or the log-concavity of the coefficients.

In the next three sections, we present a brief survey of the past work on the Neggers-Stanley conjecture.

5. Operations on labelled posets

The operations on posets can be generalized to labelled posets. We only need to define the operations on the labellings, while the operations on the posets will stay unchanged.

Definition 5.1. Let (P, ω) and (Q, ν) be two finite labelled posets on disjoint elements. Let $n = |P|$, and $m = |Q|$.

- The **dual** of (P, ω) is $(P^*, \overline{\omega})$, with the labelling $\overline{\omega} : P \to \{1, 2, \ldots, n\}$ given by the complement of ω.
- The **disjoint union** of (P, ω) and (Q, ν), denoted $(P, \omega) + (Q, \nu)$, is defined as $(P + Q, \omega + \nu)$, with the labelling

  $$\omega + \nu : P \sqcup Q \to \{1, 2, \ldots, n + m\}$$

  given by any bijection such that there are order-preserving injections

  $$\begin{aligned}\sigma :&\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n + m\},\\ \tau :&\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n + m\},\end{aligned}$$

  with $\mathrm{Im}(\sigma) \cap \mathrm{Im}(\tau) = \varnothing$, $(\omega + \nu)|_P = \sigma\omega$, and $(\omega + \nu)|_Q = \tau\nu$.
- There are two types of ordinal sums for labelled posets.
  - The **natural ordinal sum** of (P, ω) and (Q, ν), denoted $(P, \omega) \oplus_0 (Q, \nu)$, is defined as $(P \oplus Q, \omega \oplus_0 \nu)$, with the labelling

    $$\omega \oplus_0 \nu : P \sqcup Q \to \{1, 2, \ldots, n + m\}$$

    given by $(\omega \oplus_0 \nu)|_P = \omega$ and $(\omega \oplus_0 \nu)|_Q = \nu + n$.
  - The **strict ordinal sum** of (P, ω) and (Q, ν), denoted $(P, \omega) \oplus_1 (Q, \nu)$, is defined as $(P \oplus Q, \omega \oplus_1 \nu)$, with the labelling

    $$\omega \oplus_1 \nu : P \sqcup Q \to \{1, 2, \ldots, n + m\}$$

    given by $\omega \oplus_1 \nu = \nu \oplus_0 \omega$.
- The **direct product** of (P, ω) and (Q, ν), denoted $(P, \omega) \times (Q, \nu)$, is defined as $(P \times Q, \omega \times \nu)$, with the labelling

  $$\omega \times \nu : P \times Q \to \{1, 2, \ldots, nm\}$$

  given by $(\omega \times \nu)(s, t) = (\omega(s) - 1)m + \nu(t)$.

- The **ordinal product** of (P,ω) and (Q,ν), denoted $(P,\omega)\otimes(Q,\nu)$, is defined as $(P\otimes Q,\omega\otimes\nu)$, with the labelling
$$\omega\otimes\nu : P\times Q\to\{1,2,\dots,nm\}$$
given by $\omega\otimes\nu=\omega\times\nu$.

Remark 5.2. Although the labelling $\omega+\nu$ of the disjoint union $(P+Q,\omega+\nu)$ in the above definition is given as an arbitrary labelling among a set of labellings that satisfy the definition, this will not result in ambiguity of the order polynomial $\Omega(P+Q,\omega+\nu)$, the formula of which is given in Proposition 5.7. Any two such labellings produce two similar labelled posets. Therefore, this disjoint union is well-defined with respect to the order polynomials, and hence the E-polynomials and W-polynomials.

It is natural to ask about the behavior of E-polynomials and W-polynomials under the operations above. If E-polynomials and W-polynomials behave well under certain operations, we might expect the real-rootedness of E-polynomials and W-polynomials, or the log-concavity and unimodality of their coefficients to be preserved under those operations. Ideally, we would hope the E-polynomial (W-polynomial) of a binary operation of two posets to depend only on the E-polynomials (W-polynomials) of those two posets and the cardinality of the two posets. For example, we want a function $F_{\oplus_0}:\mathbb{R}[x]\times\mathbb{R}[x]\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}[x]$ that describes the effect of the natural ordinal sum on E-polynomials, such that for all labelled posets (P,ω) and (Q,ν),
$$F_{\oplus_0}(E(P,\omega),E(Q,\nu),|P|,|Q|)=E(P\oplus Q,w\oplus_0\nu).$$
For unary operation of taking dual, we want a function $F_*:\mathbb{R}[x]\times\mathbb{N}\to\mathbb{R}[x]$, such that for all labelled poset (P,ω),
$$F_*(E(P,\omega),|P|)=E(P^*,\overline{\omega}).$$

We first start with the simple unary operation. Consider the dual $(P^*,\overline{\omega})$. The following proposition gives a simple formula of $E(P^*,\overline{\omega})$ in terms of $E(P,\omega)$.

Proposition 5.3. *Let (P,ω) be a labelled poset. Then,*
$$E(P^*,\overline{\omega})=E(P,\omega)$$

Proof. Let $\mathcal{F}_j(P,\omega)$ denote the set of surjective (P,ω)-partitions with largest part $=j$, and similarly we have $\mathcal{F}_j(P^*,\overline{\omega})$. Define $T:\mathcal{F}_j(P,\omega)\to\mathcal{F}_j(P^*,\overline{\omega})$ by $T(f)(s)=j+1-f(s)$. It is easy to see that T is a bijection between $\mathcal{F}_j(P,\omega)$ and $\mathcal{F}_j(P^*,\overline{\omega})$, and $e_j(P,\omega)=e_j(P^*,\overline{\omega})$. Hence, we have $E(P^*,\overline{\omega})=E(P,\omega)$. □

The natural ordinal sum and the strict ordinal sum behave well with respect to E-polynomials. Stanley proved the following proposition in [15, Prop 12.2].

Proposition 5.4. *Let (P,ω) and (Q,ν) be two labelled posets. Then,*
(a) $E(P\oplus Q,\omega\oplus_0\nu)=\frac{x+1}{x}E(P,\omega)E(Q,\nu)$, if P and Q are non-empty.
(b) $E(P\oplus Q,\omega\oplus_1\nu)=E(P,\omega)E(Q,\nu)$.

Proof. To prove (a), we need to show
$$e_j(P\oplus Q,\omega\oplus_0\nu)=\sum_{i\in\mathbb{N}}e_i(P,\omega)(e_{j-i}(Q,\nu)+e_{j-i+1}(Q,\nu)).$$

Given any surjective $(P \oplus Q, \omega \oplus_0 \nu)$-partition $f : P \sqcup Q \to \{1, 2, \dots, j\}$, its images of P and of Q are either disjoint or sharing exactly one element. In the former case, there are $\sum_{i \in \mathbb{N}} e_i(P, \omega) e_{j-i}(Q, \nu)$ such $(P \oplus Q, \omega \oplus_0 \nu)$-partitions. In the latter case, there are $\sum_{i \in \mathbb{N}} e_i(P, \omega) e_{j-i+1}(Q, \nu)$ such $(P \oplus Q, \omega \oplus_0 \nu)$-partitions.

(b) follows from the same reasoning.

□

Consequently, if $E(P, \omega)$ and $E(Q, \nu)$ have only real roots, so do $E(P \oplus Q, \omega \oplus_0 \nu)$ and $E(P \oplus Q, \omega \oplus_1 \nu)$. This shows that the set of labelled posets that satisfy the Neggers-Stanley conjecture is closed under the natural ordinal sum and the strict ordinal sum.

Another operation that behaves well with respect to E-polynomials is the disjoint union operation. Before we give the complete statement, we need to define the diamond product of polynomials.

Definition 5.5. Given $f, g \in \mathbb{R}[x]$, the diamond product of f and g is defined as

$$f \diamondsuit g = \sum_{k \in \mathbb{N}} \frac{x^k (x+1)^k}{k! k!} (D^k f)(D^k g),$$

where $D = \frac{d}{dx}$ is the differential operator.

Recall that $\mathscr{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ is the linear operator defined by $\mathscr{E}(\binom{x}{j}) = x^j$. Wagner [17] proved the following theorem.

Theorem 5.6. *Let $f, g \in \mathbb{R}[x]$ be two polynomials. Then,*

$$\mathscr{E}(fg) = \mathscr{E}(f) \diamondsuit \mathscr{E}(g)$$

Proof. We will need the following two identities:

$$\binom{x}{i}\binom{x}{j} = \sum_{k \in \mathbb{N}} \binom{k}{k-i, k-j, i+j-k} \binom{x}{k},$$
$$D^i x^j = \sum_{k \in \mathbb{N}} \binom{i}{k} (j)_k x^{j-k} D^{i-k},$$

where D is the differential operator, and $(t)_k = t(t-1) \dots (t-k+1)$ is the k-th falling factorial. The first identity counts the number of ways of choosing two possibly overlapping sets of size i and size j out of x elements. The second identity follows from applying the product rule of differentiation.

Recall that $\{\binom{x}{i}\}$ is a basis of the space $\mathbb{R}[x]$. Let $f = \sum_{i\in\mathbb{N}} \alpha_i \binom{x}{i}$, and $g = \sum_{i\in\mathbb{N}} \beta_i \binom{x}{i}$. Then,

$$
\begin{aligned}
\mathscr{E}(fg) &= \mathscr{E}\left(\sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \binom{x}{i}\binom{x}{j}\right) \\
&= \mathscr{E}\left(\sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \sum_{k\in\mathbb{N}} \binom{k}{k-i,k-j,i+j-k}\binom{x}{k}\right) \\
&= \sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \sum_{k\in\mathbb{N}} \binom{k}{k-i,k-j,i+j-k} x^k \\
&= \sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \sum_{k\in\mathbb{N}} \frac{k!k!}{(k-i)!(k-j)!(i+j-k)!k!} x^k \\
&= \sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \sum_{k\in\mathbb{N}} \frac{(k)_i(k)_j}{(i+j-k)!k!} x^k \\
&= \sum_{i,j\in\mathbb{N}} \frac{\alpha_i\beta_j}{(i+j)!} \sum_{k\in\mathbb{N}} (k)_i(k)_j \binom{i+j}{k} x^k \\
&= \sum_{i,j\in\mathbb{N}} \frac{\alpha_i\beta_j}{(i+j)!} \sum_{k\in\mathbb{N}} (k)_j \binom{i+j}{k} x^i D^i x^k \\
&= \sum_{i,j\in\mathbb{N}} \frac{\alpha_i\beta_j}{(i+j)!} \sum_{k\in\mathbb{N}} \binom{i+j}{k} x^i D^i x^j D^j x^k \\
&= \sum_{i,j\in\mathbb{N}} \frac{\alpha_i\beta_j}{(i+j)!} x^i D^i x^j D^j \sum_{k\in\mathbb{N}} \binom{i+j}{k} x^k \\
&= \sum_{i,j\in\mathbb{N}} \frac{\alpha_i\beta_j}{(i+j)!} x^i D^i x^j D^j (x+1)^{i+j}.
\end{aligned}
$$

Now, each general term in the summation can be further expressed as

$$
\begin{aligned}
\frac{1}{(i+j)!} x^i D^i x^j D^j (x+1)^{i+j} &= \frac{1}{i!} x^i D^i x^j (x+1)^i \\
&= \frac{1}{i!} x^i \sum_{k\in\mathbb{N}} \binom{i}{k} (j)_k x^{j-k} D^{i-k} (x+1)^i \\
&= \frac{1}{i!} x^i \sum_{k\in\mathbb{N}} \binom{i}{k} (i)_{i-k} (j)_k x^{j-k} (x+1)^k \\
&= \sum_{k\in\mathbb{N}} \binom{i}{k} \frac{(j)_k}{k!} x^{i+j-k} (x+1)^k \\
&= \sum_{k\in\mathbb{N}} \binom{i}{k}\binom{j}{k} x^{i+j-k} (x+1)^k.
\end{aligned}
$$

Substituting this into $\mathscr{E}(fg)$, we get

$$\begin{aligned}
\mathscr{E}(fg) &= \sum_{i,j\in\mathbb{N}} \alpha_i\beta_j \sum_{k\in\mathbb{N}} \binom{i}{k}\binom{j}{k} x^{i+j-k}(x+1)^k \\
&= \sum_{k\in\mathbb{N}} \frac{x^k(x+1)^k}{k!k!} \sum_{i,j\in\mathbb{N}} \alpha_i\beta_j (i)_k (j)_k x^{i+j-2k} \\
&= \sum_{k\in\mathbb{N}} \frac{x^k(x+1)^k}{k!k!} \left(\sum_{i\in\mathbb{N}} \alpha_i (i)_k x^{i-k}\right)\left(\sum_{j\in\mathbb{N}} \beta_j (j)_k x^{j-k}\right) \\
&= \sum_{k\in\mathbb{N}} \frac{x^k(x+1)^k}{k!k!} \left(D^k\mathscr{E}(f)\right)\left(D^k\mathscr{E}(g)\right) \\
&= \mathscr{E}(f) \lozenge \mathscr{E}(g)
\end{aligned}$$

as desired.

□

It is easy to see that the disjoint union has a simple effect on the order polynomials, as shown in the next proposition.

Proposition 5.7. *Let (P,ω) and (Q,ν) be two labelled posets. Then,*

$$\Omega(P+Q,\omega+\nu) = \Omega(P,\omega)\Omega(Q,\nu).$$

Recall that the E-polynomial $E(P,\omega)$ and the order polynomial $\Omega(P,\omega)$ are related in the following way:

$$E(P,\omega) = \mathscr{E}(\Omega(P,\omega)).$$

Therefore, combining Theorem 5.6 and Proposition 5.7 immediately gives the description of effect of the disjoint union on the E-polynomials.

Theorem 5.8. *Let (P,ω) and (Q,ν) be two labelled posets. Then,*

$$E(P+Q,\omega+\nu) = E(P,\omega) \lozenge E(Q,\nu).$$

Moreover, the diamond product has a surprising effect on $[-1,0]$-rooted polynomials. Specifically, Wagner proved the following theorem. The proof of the theorem is quite difficult, and we will not present its proof here. See [18].

Theorem 5.9. *If $f,g \in \mathbb{R}[x]$ are both $[-1,0]$-rooted, then $f\lozenge g$ is also $[-1,0]$-rooted.*

Recall from Proposition 3.16 that $E(P,\omega)$ have only real roots if and only if $E(P,\omega)$ is $[-1,0]$-rooted. As a consequence, Theorem 5.9 shows that if $E(P,\omega)$ and $E(Q,\nu)$ both have only real roots, so does $E(P+Q,\omega+\nu)$. Thus, the set of labelled posets that satisfy the Neggers-Stanley conjecture is also closed under the disjoint union.

Therefore, the set of labelled posets that satisfy the Neggers-Stanley conjecture is closed under the ordinal sum and the disjoint union. In particular, we deduce that the family of series-parallel labelled posets satisfies the Neggers-Stanley conjecture [17]. A series-parallel labelled poset is different from a series-parallel poset with an arbitrary labelling, but the labelling of a series-parallel labelled poset is produced from the ordinal sum and the disjoin union operations. In other words, a series-parallel labelled poset is a labelled poset that can be built up from labelled singletons $(\mathbf{1},\iota)$ using the ordinal sum and the disjoint union operations.

Theorem 5.10. *Let (P, ω) be a series-parallel labelled poset. Then, $E(P, \omega)$ has only real roots.*

Proof. We prove the statement by induction on $|P|$.

If $|P| = 1$, then $E(P, \omega) = x$, which has only real roots.

If $|P| > 1$, by definition of series-parallel labelled poset, there exist series-parallel labelled posets $(Q, \nu), (S, \mu)$ such that (P, ω) can be obtained by taking the ordinal sum or the disjoint union of (Q, ν) and (S, μ), and by induction the E-polynomials of both posets have only real roots. Since the ordinal sum and the disjoint union preserve the real-rootedness of E-polynomials, we conclude that $E(P, \omega)$ has only real roots. □

Contrary to those of the ordinal sum and the disjoint union, the behaviors of the ordinal product and the direct product on the E-polynomials and W-polynomials are much harder to understand. As a result, if (P, ω) and (Q, ν) are two labelled posets that satisfy the Neggers-Stanley conjecture, we do not have a general statement regarding whether the Neggers-Stanley conjecture holds for $(P \otimes Q, \omega \otimes \nu)$ or $(P \times Q, \omega \times \nu)$. In this work, we give some evidence on the difficulty of obtaining a general statement of the behaviors of the ordinal product and the direct product with respect to the E-polynomials or W-polynomials.

The following proposition shows that the effect of the direct product on W-polynomials cannot be described by a function $H_\times : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$, and similarly, the effect of the ordinal product on W-polynomials cannot be descibed by a function $H_\otimes : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$.

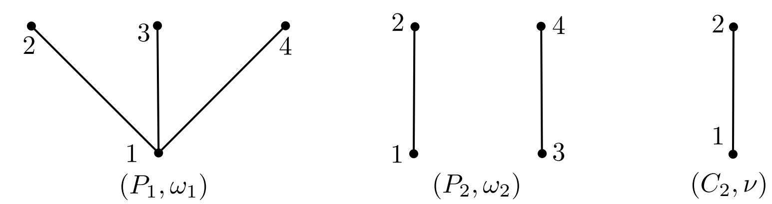

FIGURE 2. Two labelled posets (P_1, ω_1) and (P_2, ω_2) of cardinality 4 sharing the same W-polynomial (left), and a naturally labelled chain of cardinality 2 (right).

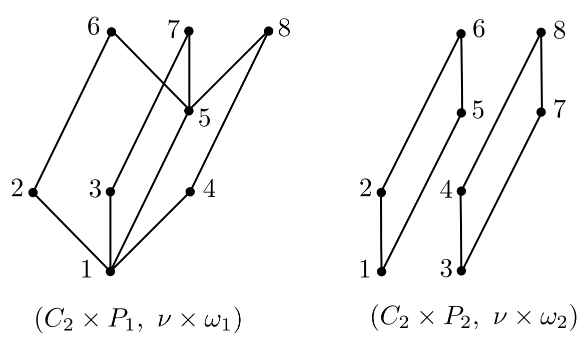

FIGURE 3. The direct products $(C_2 \times P_1, \nu \times \omega_1)$ and $(C_2 \times P_2, \nu \times \omega_2)$.

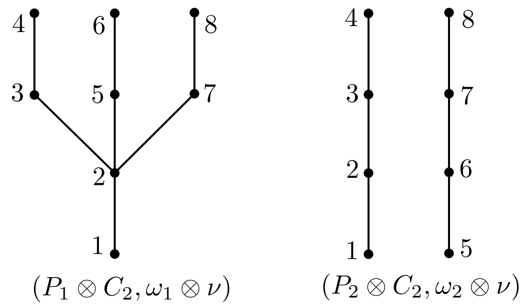

FIGURE 4. The ordinal products $(P_1 \otimes C_2, \omega_1 \otimes \nu)$ and $(P_2 \otimes C_2, \omega_2 \otimes \nu)$.

Proposition 5.11. *The following functions do not exist:*

(a) A function $H_\times : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$ *such that for all labelled posets* (P, ω) *and* (Q, ν),

$$H_\times(W(P, \omega), W(Q, \nu), |P|, |Q|) = W(P \times Q, \omega \times \nu).$$

(b) A function $H_\otimes : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$ *such that for all labelled posets* (P, ω) *and* (Q, ν),

$$H_\otimes(W(P, \omega), W(Q, \nu), |P|, |Q|) = W(P \otimes Q, \omega \otimes \nu).$$

Proof. Consider the two labelled posets (P_1, ω_1) and (P_2, ω_2) in Figure 2, both of cardinality 4. They have identical W-polynomial:

$$W(P_1, \omega_1; x) = W(P_2, \omega_2; x) = x + 4x^2 + x^3.$$

Let (C_2, ν) denote a naturally labelled 2-element chain, as shown in Figure 2.

(a) Consider the direct products $(C_2 \times P_1, \nu \times \omega_1)$ and $(C_2 \times P_2, \nu \times \omega_2)$ as shown in Figure 3. The corresponding W-polynomials are:

$$W(C_2 \times P_1, \nu \times \omega_1; x) = x + 27x^2 + 116x^3 + 116x^4 + 27x^5 + x^6$$
$$W(C_2 \times P_2, \nu \times \omega_2; x) = x + 27x^2 + 112x^3 + 112x^4 + 27x^5 + x^6$$

Thus, there does not exist a function $H_\times : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$ such that for all labelled posets (P, ω) and (Q, ν),

$$H_\times(W(P, \omega), W(Q, \nu), |P|, |Q|) = W(P \times Q, \omega \times \nu).$$

(b) Consider the ordinal products $(P_1 \otimes C_2, \omega_1 \otimes \nu)$ and $(P_2 \otimes C_2, \omega_2 \otimes \nu)$ as shown in Figure 4. The corresponding W-polynomials are:

$$W(P_1 \otimes C_2, \omega_1 \otimes \nu; x) = x + 20x^2 + 48x^3 + 20x^4 + x^5$$
$$W(P_2 \otimes C_2, \omega_2 \otimes \nu; x) = x + 16x^2 + 36x^3 + 16x^4 + x^5$$

Thus, there does not exist a function $H_\otimes : \mathbb{R}[x] \times \mathbb{R}[x] \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}[x]$ such that for all labelled posets (P, ω) and (Q, ν),

$$H_\otimes(W(P, \omega), W(Q, \nu), |P|, |Q|) = W(P \otimes Q, \omega \otimes \nu).$$

□

Note that the E-polynomials are explicitly related to the W-polynomials by Proposition 3.12. Hence, the effect of the direct product or the ordinal product on the E-polynomials can neither be described by functions $\mathbb{R}[x]\times\mathbb{R}[x]\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}[x]$.

Nevertheless, the ordinal product, when the first operand is restricted to series-parallel labelled posets, behaves well with respect to the E-polynomials. The following proposition follows directly from the effects of the ordinal sum and the disjoint union on E-polynomials given in Proposition 5.4 and Theorem 5.8.

Proposition 5.12. *Let (P,ω) be a series-parallel labelled poset. Then, there exists an explicit function $T_\otimes(P,\omega):\mathbb{R}[x]\to\mathbb{R}[x]$ such that for all labelled poset (Q,ν),*

$$T_\otimes(P,\omega)(E(Q,\nu))=E(P\otimes Q,\omega\otimes\nu).$$

Moreover, if $E(Q,\nu)$ has only real roots, so does $E(P\otimes Q,\omega\otimes\nu)$.

Proof. We prove the statement by induction on $|P|$.

If $|P|=1$, then we have $(P\otimes Q,\omega\otimes\nu)\cong(Q,\nu)$. Thus, we have $T_\otimes(P,\omega)=\mathrm{Id}$, which preserves the real-rootedness of polynomials.

If $|P|>1$, by definition of series-parallel labelled posets, there exist nonemepty (P_1,ω_1) and (P_2,ω_2) such that P can be obtained by applying the natural ordinal sum, the strict ordinal sum, or the disjoint union on (P_1,ω_1) and (P_2,ω_2). By induction, there exist $T_\otimes(P_1,\omega_1),T_\otimes(P_2,\omega_2):\mathbb{R}[x]\to\mathbb{R}[x]$ such that

$$\begin{aligned}T_\otimes(P_1,\omega_1)(E(Q,\nu))&=E(P_1\otimes Q,\omega_1\otimes\nu),\\ T_\otimes(P_2,\omega_2)(E(Q,\nu))&=E(P_2\otimes Q,\omega_2\otimes\nu),\end{aligned}$$

for all labelled poset (Q,ν).

For the ordinal product, we have the following distributive properties:

$$\begin{aligned}(P_1+P_2,\omega_1+\omega_2)\otimes(Q,\nu)&\cong(P_1\otimes Q,\omega_1\otimes\nu)+(P_2\otimes Q,\omega_2\otimes\nu),\\ (P_1\oplus P_2,\omega_1\oplus_0\omega_2)\otimes(Q,\nu)&\cong(P_1\otimes Q,\omega_1\otimes\nu)\oplus_0(P_2\otimes Q,\omega_2\otimes\nu),\\ (P_1\oplus P_2,\omega_1\oplus_1\omega_2)\otimes(Q,\nu)&\cong(P_1\otimes Q,\omega_1\otimes\nu)\oplus_1(P_2\otimes Q,\omega_2\otimes\nu).\end{aligned}$$

If $(P,\omega)\cong(P_1+P_2,\omega_1+\omega_2)$, by Theorem 4.8, we have

$$\begin{aligned}T_\otimes(P,\omega)(E(Q,\nu))&=E((P_1+P_2,\omega_1+\omega_2)\otimes(Q,\nu))\\ &=E((P_1\otimes Q,\omega_1\otimes\nu)+(P_2\otimes Q,\omega_2\otimes\nu))\\ &=E(P_1\otimes Q,\omega_1\otimes\nu)\lozenge E(P_2\otimes Q,\omega_2\otimes\nu)\\ &=T_\otimes(P_1,\omega_1)(E(Q,\nu))\lozenge T_\otimes(P_2,\omega_2)(E(Q,\nu)).\\ T_\otimes(P,\omega)&=T_\otimes(P_1,\omega_1)\lozenge T_\otimes(P_2,\omega_2),\end{aligned}$$

which preserves the real-rootedness of E-polynomials by Theorem 5.9.

Similarly, if $(P,\omega)\cong(P_1\oplus P_2,\omega_1\oplus_0\omega_2)$ or $(P,\omega)\cong(P_1\oplus P_2,\omega_1\oplus_1\omega_2)$, we can obtain $T_\otimes(P,\omega)$ by Proposition 5.4, which preserves the real-rootedness of E-polynomials. □

Next, we introduce the notion of block posets and the composition of posets.

Definition 5.13. A block poset is a poset P together with a partition of elements $P=B_1\sqcup B_2\sqcup\cdots\sqcup B_k$ such that for all pairs B_i,B_j with $i\neq j$, if $s\leq t$ for some $s\in B_i,t\in B_j$, then $u\leq v$ for all $u\in B_i,v\in B_j$.

The blocks of the block poset P are the parts in the partition of elements $P = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$. The skeleton of the block poset P is the poset T on the blocks $\{B_i\}_{i=1}^k$, with the order defined by $B_i \leq_T B_j$ if

$$s \leq_P t \text{ for some } s \in B_i, t \in B_j.$$

A block labelled poset is a block poset obtained from a composition operation in the following definition.

Definition 5.14. Given a labelled poset (P, ω), let $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$ be a sequence of nonempty labelled posets indexed by elements of P.

The block labelled poset of composing (Q, ν) into (P, ω), denoted $(P[Q], \omega[\nu])$, is the following labelled poset on the disjoint union $\bigsqcup_{r \in P} Q_r$,

(i) with the labelled blocks given by (Q_r, ν_r) of (Q, ν),

(ii) the order defined by $s \leq_{P[Q]} t$ if

$$s \leq_{Q_r} t \text{ for some } (Q_r, \nu_r) \text{ of } (Q, \nu), \text{ or}$$
$$s \in Q_u, t \in Q_v, \text{ and } u \leq_P v \text{ for two different blocks } (Q_u, \nu_u) \text{ and } (Q_v, \nu_v).$$

(iii) and the labelling $\omega[\nu]$ given by the unique bijection such that

(a) If $s, t \in Q_r$ and $\nu_r(s) \leq \nu_r(t)$ for some (Q_r, ν_r) of (Q, ν), then

$$(\omega[\nu])(s) \leq (\omega[\nu])(t).$$

(b) If $s \in Q_u, t \in Q_v$ for two different blocks (Q_u, ν_u) and (Q_v, ν_v) of (Q, ν), and $\omega(u) \leq \omega(v)$, then

$$(\omega[\nu])(s) \leq (\omega[\nu])(t).$$

Remark 5.15. Let (P, ω) be a labelled poset, and $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$ be a sequence of nonempty labelled posets indexed by elements of P. Then, P is exactly the skeleton of the block labelled poset $(P[Q], \omega[\nu])$.

The skeleton of a block poset can be viewed as the glue to the blocks, with the order of elements from different blocks given by the order of the skeleton.

Note that when all the blocks (Q_r, ν_r) of (Q, ν) are identical, the block labelled poset $(P[Q], \omega[\nu])$ is exactly the ordinal product $(P \otimes Q_r, \omega \otimes \nu_r)$.

Immediately, we can generalize Proposition 5.12 to the following proposition, the proof of which is essentially the same.

Proposition 5.16. *Let (P, ω) be a series-parallel labelled poset. Let $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$ be a sequence of nonempty labelled posets such that $E(Q_r, \nu_r)$ is real-rooted for all (Q_r, ν_r) in (Q, ν). Then, $E(P[Q], \omega[\nu])$ has only real roots.*

Wagner studied the composition $(P[Q], \omega[\nu])$ when the cardinality of P is at most 3. He proved the following proposition [17].

Proposition 5.17. *Let (P_1, ω_1), (P_2, ω_2), and (P_3, ω_3) be three nonempty labelled posets. Let (T, μ) be a labelled poset of cardinality* 3.

If $E(P_1, \omega_1)$, $E(P_2, \omega_2)$, and $E(P_3, \omega_3)$ are all real-rooted, then

$$E(T[P_1, P_2, P_3], \mu[\omega_1, \omega_2, \omega_3])$$

is also real-rooted.

By further proving that every nonempty labelled forest is similar to a recursively labelled forest (see [1, Section 4] for the notion of recursive labelling), Wagner was able to prove the following proposition [17] using Proposition 5.17.

Proposition 5.18. *Let (P, ω) be a nonempty labelled forest. Then, $E(P, \omega)$ is real-rooted.*

6. Interlacing property

In this section, we consider the interlacing properties of E-polynomials.

Definition 6.1. Let $f, g \in \mathbb{R}[x]$ be polynomials such that $\deg(f) = \deg(g) + 1 = d$. We say g interlaces f, denoted $g \preceq f$, if f and g are real-rooted, and

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d,$$

where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{d-1}$ are roots of f and g respectively.

Moreover, if the inequalities are strict, i.e.,

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{d-1} < \alpha_d,$$

we say g strictly interlaces f, denoted $g \prec f$.

Brändén proved that the interlacing properties of polynomials are preserved under taking diamond product with $[-1, 0]$-rooted polynomials, as stated in the following theorem [4].

Theorem 6.2. *Let $h \in \mathbb{R}[x]$ be $[-1, 0]$-rooted, and let $f \in \mathbb{R}[x]$ be real-rooted.*

(a) Then, $f \diamondsuit h$ is $[-1, 0]$-rooted, and if $g \preceq f$, then $g \diamondsuit h \preceq f \diamondsuit h$.

(b) If h is $(-1, 0)$-rooted and simple-rooted, and f is simple-rooted, then $f \diamondsuit h$ is simple-rooted, and for all $g \prec f$, $g \diamondsuit h \prec f \diamondsuit h$.

Recall that for a labelled poset (P, ω), $E(P, \omega)$ is a degree n polynomial where $n = |P|$. For each $s \in P$, let $(P \setminus \{s\}, \omega_s)$ be the induced labelled poset on $P \setminus \{s\}$ with the labelling ω_s given by $\omega_s = \sigma_s \omega$, where

$$\sigma_s : \{1, 2, \ldots, n\} \setminus \{\omega(s)\} \to \{1, 2, \ldots, n-1\}$$

is the unique order-preserving bijection. Then, $E(P \setminus \{s\}, \omega_s)$ is a degree $n - 1$ polynomial. Without causing confusion, we will drop the subscript of ω_s and simply write $(P \setminus \{s\}, \omega)$.

Let $\mathcal{I}$ denote the family of finite labelled posets (P, ω) such that for all $s \in P$,

$$E(P \setminus \{s\}, \omega) \preceq E(P, \omega).$$

Brändén [4] proved the following theorem.

Theorem 6.3. *The family $\mathcal{I}$ is closed under the ordinal sum and the disjoint union.*

Proof. We will prove closure under each operation separately. Let $(P, \omega), (Q, \nu) \in \mathcal{I}$ be two labelled posets.

- *(Ordinal Sum)* Let $(S, \mu) = (P \oplus Q, \omega \oplus_0 \nu)$. Let $s \in P$.
  If $|P| = 1$, then
  $$\begin{aligned} E(S \setminus \{s\}, \mu) &= E((P \setminus \{s\}) \oplus Q, \omega \oplus_0 \nu) \\ &= E(Q, \nu) \\ &\preceq (x+1)E(Q, \nu) \\ &= E(P \oplus Q, \omega \oplus_0 \nu). \end{aligned}$$
  If $|P| > 1$, then

$$\begin{aligned}E(S\setminus\{s\},\mu) &= E((P\setminus\{s\})\oplus Q,\omega\oplus_0\nu)\\ &= \frac{x+1}{x}E(P\setminus\{s\},\omega)E(Q,\nu)\\ &\preceq \frac{x+1}{x}E(P,\omega)E(Q,\nu)\\ &= E(P\oplus Q,\omega\oplus_0\nu).\end{aligned}$$

Therefore, we have $(S,\mu)=(P\oplus Q,\omega\oplus_0\nu)\in\mathcal{I}$.
Similarly, we can show that $(P\oplus Q,\omega\oplus_1\nu)\in\mathcal{I}$.

- *(Disjoint Union)* Let $(S,\mu)=(P+Q,\omega+\nu)$. Let $s\in P$.
  By Theorem 5.8, we have

$$\begin{aligned}E(S\setminus\{s\},\mu) &= E((P\setminus\{s\})+Q,\omega+\nu)\\ &= E(P\setminus\{s\},\omega)\lozenge E(Q,\nu).\end{aligned}$$

By Proposition 3.16, $E(Q,\nu)$ is $[-1,0]$-rooted. By Theorem 6.2, we have

$$\begin{aligned}E(S\setminus\{s\},\mu) &= E(P\setminus\{s\},\omega)\lozenge E(Q,\nu)\\ &\preceq E(P,\omega)\lozenge E(Q,\nu)\\ &= E(P+Q,\omega+\nu).\end{aligned}$$

Therefore, we have $(S,\mu)=(P+Q,\omega+\nu)\in\mathcal{I}$.

□

Corollary 6.4. *Let (P,ω) be a series-parallel labelled poset. Then, for all $s\in P$,*

$$E(P\setminus\{s\},\omega)\preceq E(P,\omega).$$

We can further generalize Theorem 6.3 by considering the interlacing properties of polynomials whose degrees differ by more than one.

Definition 6.5. Let $f,g\in\mathbb{R}[x]$ be polynomials such that $\deg(f)=\deg(g)+k=d$ for some $k>0$. We say g k-interlaces f, denoted $g\preceq_k f$, if f and g are real-rooted, and for $1\le i\le d-k$,

$$\alpha_i\le\beta_i\le\alpha_{i+k},$$

where $\alpha_1\le\alpha_2\le\cdots\le\alpha_d$ and $\beta_1\le\beta_2\le\cdots\le\beta_{d-k}$ are roots of f and g respectively.

Moreover, if the inequalities are strict, i.e., for $1\le i\le d-k$,

$$\alpha_i<\beta_i<\alpha_{i+k},$$

we say g strictly k-interlaces f, denoted $g\prec_k f$.

Lemma 6.6. *Given $k\in\mathbb{Z}^+$, let $f_0,f_k\in\mathbb{R}[x]$ be polynomials such that deg(f_k) = deg(f_0) + $k=d$. Then, $f_0\preceq_k f_k$ if and only if there exist $f_1,f_2,\ldots,f_{k-1}\in\mathbb{R}[x]$ such that*

$$f_0\preceq f_1\preceq\cdots\preceq f_{k-1}\preceq f_k.$$

Proof. If there exist $f_1,f_2,\ldots,,f_{k-1}\in\mathbb{R}[x]$ such that

$$f_0\preceq f_1\preceq\cdots\preceq f_{k-1}\preceq f_k,$$

then clearly $f_0\preceq_k f_k$.

Now suppose $f_0 \preceq_k f_k$. We prove the statement by induction on k. If $k = 1$, then we are done. If $k > 1$, we will show that there exists $f_{k-1} \in \mathbb{R}[x]$ such that

$$f_0 \preceq_{k-1} f_{k-1} \preceq f_k.$$

Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$ be the roots of f_k, and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{d-k}$ be the roots of f_0. Then, for $1 \leq i \leq d-k$,

$$\alpha_i \leq \beta_i \leq \alpha_{i+k}.$$

Since $\beta_{i-k+1} \leq \alpha_{i+1}$ and $\alpha_i \leq \beta_i$, we have

$$\alpha_i \leq \beta_{i-k+1} \leq \beta_i \leq \alpha_{i+1}.$$

Then, we can choose $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{d-k}$ such that

$$\alpha_i \leq \beta_{i-k+1} \leq \gamma_i \leq \beta_i \leq \alpha_{i+1}.$$

For $d-k < i \leq d-1$, we simply choose $\gamma_i = \alpha_{i+1}$. Then, we get a sequence $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{d-1}$ such that for $1 \leq i \leq d-k$, $\gamma_i \leq \beta_i \leq \gamma_{i+k-1}$, and for $1 \leq i \leq d-1$, $\alpha_i \leq \gamma_i \leq \alpha_{i+1}$. Let $f_{k-1} \in \mathbb{R}[x]$ be the polynomial with roots $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{d-1}$. Then,

$$f_0 \preceq_{k-1} f_{k-1} \preceq f_k.$$

By induction, there exist $f_1, f_2, \ldots, f_{k-1} \in \mathbb{R}[x]$ such that

$$f_0 \preceq f_1 \preceq \cdots \preceq f_{k-1} \preceq f_k.$$

□

Now we consider the interlacing properties of the E-polynomials of block labelled posets. Let $(P[Q], \omega[\nu])$ be a block labelled poset with the set of labelled blocks $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$. For each (Q_r, ν_r), let $(P[Q] \setminus Q_r, \omega[\nu]_{Q_r})$ be the induced block labelled poset on $P[Q] \setminus Q_r$ with the labelling $\omega[\nu]_{Q_r}$ given by $\omega[\nu]_{Q_r} = \sigma_{Q_r}\omega[\nu]$, where

$$\sigma_{Q_r} : \{1, 2, \ldots, n\} \setminus \omega(Q_r) \to \{1, 2, \ldots, n - |Q_r|\}$$

is the unique order-preserving bijection. Without causing confusion, we will drop the subscript of $\omega[\nu]_{Q_r}$ and simply write $(P[Q] \setminus Q_r, \omega[\nu])$.

Let $\mathcal{B}$ denote the family of finite block labelled posets $(P[Q], \omega[\nu])$ with the set of blocks $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$ such that for all (Q_r, ν_r),

$$E(P[Q] \setminus Q_r, \omega[\nu]) \preceq_k E(P[Q], \omega[\nu]),$$

where $k = |Q_r|$.

When we talk about the ordinal sum or the disjoint union of two block labelled posets $(P[Q], \omega[\nu])$ and $(S[T], \mu[\tau])$, the resulting block labelled poset has the set of labelled blocks as the disjoint union of the blocks of (Q, ν) and (T, τ), and the ordinal sum or the disjoint union operation is applied on (P, ω) and (S, μ).

Theorem 6.7. *The family $\mathcal{B}$ is closed under the ordinal sum and the disjoint union.*

Proof. By Theorem 6.2 and Lemma 6.6, we can deduce that the k-interlacing properties of polynomials are preserved under taking diamond product with $[-1, 0]$-rooted polynomials:

If $h \in R[x]$ is $[-1, 0]$-rooted, and $f, g \in \mathbb{R}[x]$ are real-rooted polynomials such that $g \preceq_k f$, then $g \diamond h \preceq_k f \diamond h$.

The rest of the proof is essentially the same as the proof of Theorem 6.3. □

We can then deduce from Theorem 6.7 the interlacing properties of the E-polynomials of block series-parallel labelled posets with labelled blocks that satisfy the Neggers-Stanley conjecture.

Corollary 6.8. *Let (P, ω) be a series-parallel labelled poset. Let $(Q, \nu) = \{(Q_r, \nu_r)\}_{r \in P}$ be a sequence of nonempty labelled posets such that $E(Q_r, \nu_r)$ is real rooted for all (Q_r, ν_r). Then, for all (Q_r, ν_r) of (Q, ν),*

$$E(P[Q] \setminus Q_r, \omega[\nu]) \preceq_k E(P[Q], \omega[\nu]),$$

where $k = |Q_r|$.

7. Sign-graded poset

A recent progress on the Neggers-Stanley conjecture is the unimodality of the W-polynomails of sign-graded posets due to Brändén [5]. Brändén extended the result by Reiner and Welker [10] that proved the unimodality of the W-polynomials of naturally labelled graded posets.

Recall that a poset P is graded if all maximal chains of P have the same length. A sign-graded poset is a generalization of the above notion for labelled posets.

Let (P, ω) be a labelled poset. Let $C(P) = \{s \prec t : s, t \in P\}$ denote the set of covering relations. Associate a function $\epsilon_\omega : C(P) \to \{-1, 1\}$ to the covering relations of the labelled poset (P, ω) by

$$\epsilon_\omega(s,t) = \begin{cases} 1 & \text{if } \omega(s) < \omega(t) \\ -1 & \text{if } \omega(s) > \omega(t) \end{cases}$$

Definition 7.1. Let (P, ω) be a labelled poset. Let ϵ_ω be the function of the covering relations of P as given above. (P, ω) is sign-graded if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_t$, the sum

$$\sum_{i=1}^{t} \epsilon_\omega(x_{i-1}, x_i)$$

is the same. This common value is called the rank of (P, ω), denoted $r(\omega)$. In this case, P is said to be ω-graded with rank $r(\omega)$.

The rank function of a sign-graded poset (P, ω) is $\rho_\omega : P \to \mathbb{Z}$ defined by

$$\rho_\omega(s) = \begin{cases} 0 & \text{if } s \text{ is a minimal element of } P \\ \rho_\omega(t) + \epsilon_\omega(t,s) & \text{otherwise, where } t \text{ is an element covered by } s \end{cases}$$

With this definition of the rank function, $\rho_\omega(s) = \sum_{i=1}^{a} \epsilon_\omega(x_{i-1}, x_i)$ for any saturated chain $x_0 \prec x_1 \prec \cdots \prec x_a = s$, where x_0 is a minimal element of P. For any saturated chain $s = y_0 \prec y_1 \prec \cdots \prec y_b$ where y_b is a maximal element of P, we have $r(\omega) - \rho_\omega(s) = \sum_{i=1}^{b} \epsilon_\omega(y_{i-1}, y_i)$. Figure 5 is an example of a sign-graded poset together with the associated rank function.

Note that a naturally labelled poset (P, ω) is sign-graded if and only if P is graded.

Brändén proved that the order polynomial of a sign-graded poset (P, ω) depends only on the underlying poset and the rank $r(\omega)$, as shown in the following proposition [5].

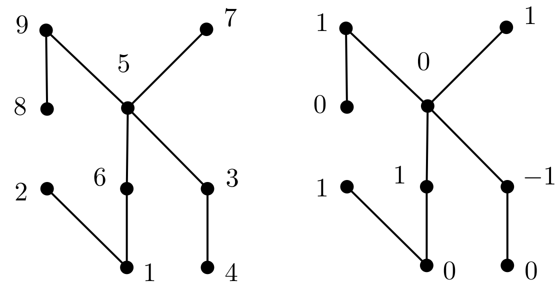

FIGURE 5. An example of a sign-graded poset (left), and the associated rank function (right).

$\epsilon_\omega(x,y)$	$\epsilon_\nu(x,y)$	f	Δ	$T(f)$
1	1	$f(x) \leq f(y)$	$\Delta(x) = \Delta(y)$	$T(f)(x) \leq T(f)(y)$
1	-1	$f(x) \leq f(y)$	$\Delta(x) = \Delta(y) - 1$	$T(f)(x) < T(f)(y)$
-1	1	$f(x) < f(y)$	$\Delta(x) = \Delta(y) + 1$	$T(f)(x) \leq T(f)(y)$
-1	-1	$f(x) < f(y)$	$\Delta(x) = \Delta(y)$	$T(f)(x) < T(f)(y)$

FIGURE 6. Possible combinations of ϵ_ω and ϵ_ν, taken from [5].

Proposition 7.2. *Let P be an ω-graded and ν-graded poset. Then,*

$$\Omega(P,\omega; x - \frac{r(\omega)}{2}) = \Omega(P,\nu; x - \frac{r(\nu)}{2}).$$

Proof. Let $\mathcal{O}(P,\omega)$ and $\mathcal{O}(P,\nu)$ denote the set of (P,ω)-partitions and the set of (P,ν)-partitions respectively. Define a mapping $T : \mathcal{O}(P,\omega) \to \mathcal{O}(P,\nu)$ as

$$T(f)(x) = f(x) + \Delta(x), \text{ where}$$
$$\Delta(x) = \frac{\rho_\omega(x) - \rho_\nu(x)}{2}.$$

First we show that for any (P,ω)-partition $f \in \mathcal{O}(P,\omega)$, we have $T(f) \in \mathcal{O}(P,\nu)$. Observe that $\rho_\omega(x)$ and $\rho_\nu(x)$ have the same parity. Therefore, $\Delta(x)$ is an integer. Now consider a pair of elements $x \prec y$. Figure 6 shows that $T(f)$ is a (P,ν)-partition, provided that $T(f) > 0$. Note that $T(f)$ achieves minimum on some minimal element $s \in P$, where we have $\Delta(s) = 0$. Hence, $T(f)(s) = f(s) > 0$ for minimal element $s \in P$, proving that $T(f)$ is indeed a (P,ν)-partition.

Moreover, T is a bijection between the (P,ω)-partitions and the (P,ν)-partitions, since T^{-1} is given by

$$T^{-1}(g)(x) = g(x) - \Delta(x).$$

Now consider a (P,ω)-partition $f \in \mathcal{O}(P,\omega)$ with largest part $\leq m$. Then, for any maximal element $t \in P$, $f(t) \leq m$. Note that for maximal element $t \in P$, we have

$$\Delta(t) = \frac{\epsilon_\omega(t) - \epsilon_\nu(t)}{2} = \frac{r(\omega) - r(\nu)}{2},$$
$$T(f)(t) = f(t) + \Delta(t) \leq m + \frac{r(\omega) - r(\nu)}{2}.$$

The same observation holds for T^{-1}. Thus, there is a bijection between the (P,ω)-partitions with largest part $\leq m$ and the (P,ν)-partitions with largest part $\leq m + \frac{r(\omega)-r(\nu)}{2}$, which gives

$$\Omega(P,\omega;x) = \Omega(P,\nu;x + \frac{r(\omega)-r(\nu)}{2}).$$

□

Proposition 7.3. *Given an ω-graded poset P, let $n = |P|$. Then,*

$$\Omega(P,\omega;x) = (-1)^n \Omega(P,\omega;-x-r(\omega)).$$

Proof. Note that for the complement labelling $\overline{\omega}$, P is also $\overline{\omega}$-graded, and

$$\begin{aligned} \epsilon_{\overline{\omega}}(s,t) &= -\epsilon_\omega(s,t) \\ r(\omega) &= -r(\overline{\omega}) \end{aligned}$$

By the reciprocity theorem for the order polynomials given in Proposition 3.17,

$$\Omega(P,\omega;x) = (-1)^n \Omega(P,\overline{\omega};-x).$$

Combined with Proposition 7.2,

$$\begin{aligned} \Omega(P,\overline{\omega};-x) &= \Omega(P,\omega;-x + \frac{r(\overline{\omega})-r(\omega)}{2}) \\ &= \Omega(P,\omega;-x-r(\omega)) \\ \Omega(P,\omega;x) &= (-1)^n \Omega(P,\omega;-x-r(\omega)). \end{aligned}$$

□

Corollary 7.4. *Let P be an ω-graded poset. Then, $W(P,\omega;x)$ is symmetric with center of symmetry $\frac{n+1-r(\omega)}{2}$. If P is also ν-graded, then*

$$W(P,\omega;x) = x^{r(\nu)-r(\omega)} W(P,\nu;x).$$

Proof. By combinatorial reasoning similar to the one in Proposition 3.12, we can obtain

$$\Omega(P,\omega;x) = \sum_{j\in\mathbb{N}} w_j(P,\omega) \binom{x+n-j}{n}.$$

By Proposition 7.3, we have

$$\begin{aligned} \Omega(P,\omega;x) &= (-1)^n \Omega(P,\omega;-x-r(\omega)) \\ &= \sum_{j\in\mathbb{N}} w_j(P,\omega)(-1)^n \binom{-x-r(\omega)+n-j}{n} \\ &= \sum_{j\in\mathbb{N}} w_j(P,\omega) \binom{x+r(\omega)+j-1}{n} \\ &= \sum_{j\in\mathbb{N}} w_{n+1-r(\omega)-j}(P,\omega) \binom{x+n-j}{n} \end{aligned}$$

Therefore, $w_j(P,\omega) = w_{n+1-r(\omega)-j}$, and $W(P,\omega;x)$ is symmetric with center of symmetry $\frac{n+1-r(\omega)}{2}$. The relation of W-polynomials of (P,ω) and (P,ν) follows from Proposition 7.2.

□

Therefore, when studying the W-polynomial of a sign-graded poset, we may choose a labelling ω of our own choice such that P is ω-graded for the poset P, since choosing different consistent labellings of a sign-graded poset only results in a shift of the coefficients as shown in Corollary 7.4. The following canonical labelling is a good choice for sign-graded posets.

Note that if P is ω-graded, then the length of every maximal chain of P has the same parity. Define the canonical labelling τ in the following way: we first define the associated function ϵ_τ on the covering relations, and then shows that there exists a labelling τ that is consistent with ϵ_τ.

Define $\epsilon_\tau(s,t) = (-1)^{l(C)}$, where C is a saturated chain $x_0 \prec x_1 \prec \cdots \prec x_k = s$ and x_0 is a minimal element of P. The corresponding rank function ρ_τ takes value in $\{0,1\}$. Now pick a labelling τ by assigning the larger labels to the set of elements with rank 1, and the smaller labels to the set of elements with rank 0. It is easy to verify that this labelling is consistent with ϵ_τ, as rank-0 elements are covered by rank-1 elements and vice versa.

Any such labelling τ of a sign-graded poset P is called the canonical labelling of P. P is τ-graded with rank function ρ_τ taking values in $\{0,1\}$. Moreover, for any covering relation $s \prec t$, if $\rho_\tau(s) < \rho_\tau(t)$, then $\tau(s) < \tau(t)$.

Next we turn the attention to the Jordan-Hölder set of sign-graded posets. The intuition behind the definition of sign-gradedness lies in the following observation. Recall that the Jordan-Hölder set $\mathcal{L}(P,\omega)$ of a labelled poset (P,ω) is the set of permutations of the labels of P corresponding to the set of linear extensions of P. Let $x, y \in P$ be a pair of incomparable elements. Let P_x^y be the poset obtained from P by adding the covering relation $x \prec y$, and similarly we have P_y^x. Since the label of x comes either before or after that of y in any element of the Jordan-Hölder set $\mathcal{L}(P,\omega)$, we have

$$\mathcal{L}(P,\omega) = \mathcal{L}(P_x^y,\omega) \sqcup \mathcal{L}(P_y^x,\omega).$$

Let P be a τ-graded poset, where τ is the canonical labelling. Let $x, y \in P$ be a pair of incomparable elements such that $\rho_\tau(y) = \rho_\tau(x) + 1$. Note that τ being the canonical labelling implies that $\tau(x) < \tau(y)$. Let P_x^y and P_y^x be the two posets as defined above. Then, they are also τ-graded as shown in the next proposition [5].

Proposition 7.5. *P_x^y and P_y^x are τ-graded with the same rank as (P,τ).*

Proof. First, we show P_x^y is τ-graded.

Let $s_0 \prec s_1 \prec \cdots \prec s_t$ be a maximal chain in P_x^y such that $s_k = x$ and $s_{k+1} = y$ for some k. Then,

$$\begin{aligned}
\sum_{i=1}^{t} \epsilon_\tau(s_{i-1},s_i) &= \sum_{i=1}^{k} \epsilon_\tau(s_{i-1},s_i) + \epsilon_\tau(x,y) + \sum_{i=k+2}^{t} \epsilon_\tau(s_{i-1},s_i) \\
&= \rho_\tau(x) + \epsilon_\tau(x,y) + (r(\tau) - \rho_\tau(y)) \\
&= \rho_\tau(x) + 1 + (r(\tau) - \rho_\tau(y)) \\
&= r(\tau)
\end{aligned}$$

Next, we show P_y^x is τ-graded.

Let $s_0 \prec s_1 \prec \cdots \prec s_t$ be a maximal chain in P_y^x such that $s_k = y$ and $s_{k+1} = x$ for some k. Then,

$$\begin{aligned}\sum_{i=1}^{t} \epsilon_\tau(s_{i-1}, s_i) &= \sum_{i=1}^{k} \epsilon_\tau(s_{i-1}, s_i) + \epsilon_\tau(y, x) + \sum_{i=k+2}^{t} \epsilon_\tau(s_{i-1}, s_i) \\ &= \rho_\tau(y) + \epsilon_\tau(y, x) + (r(\tau) - \rho_\tau(x)) \\ &= \rho_\tau(y) - 1 + (r(\tau) - \rho_\tau(x)) \\ &= r(\tau)\end{aligned}$$

□

Let Q be a poset on the same set of elements as P. We say Q extends P if $s \leq_Q t$ whenever $s \leq_P t$. In other words, Q extends P if P is a spanning subposet of Q. In the proposition above, P_x^y and P_y^x both extend P and share the same rank function as (P, τ).

In the context of sign-graded posets with canonical labelling, we say (Q, τ) is saturated if s, t are comparable whenever $|\rho_\tau(s) - \rho_\tau(t)| = 1$. By repeatedly applying the decomposition

$$\mathcal{L}(P, \tau) = \mathcal{L}(P_x^y, \tau) \sqcup \mathcal{L}(P_y^x, \tau)$$

for a pair of incomparable elements $x, y \in P$ such that $\rho_\tau(y) = 1$ and $\rho_\tau(x) = 0$, we can uniquely decompose the Jordan-Hölder set of (P, τ) as

$$\mathcal{L}(P, \tau) = \bigsqcup_{Q} \mathcal{L}(Q, \tau),$$

where the disjoint union is taken over saturated τ-graded posets Q that extend P. By definition of the W-polynomial,

$$W(P, \tau) = \sum_{Q} W(Q, \tau).$$

Saturated τ-graded posets have especially nice structures as given in the following proposition.

Proposition 7.6. *Let (P, τ) be a saturated τ-graded poset with canonical labelling. Then, (P, τ) can be written as an alternating ordinal sum of antichains:*

$$(P, \tau) = A_0 \oplus_0 A_1 \oplus_1 A_2 \oplus_0 \cdots \oplus_a A_k,$$

where $a \in \{0, 1\}$, and each A_i is a labelled antichain of certain size.

Proof. Let A_0 be the set of minimal elements of P. Clearly, A_0 is an antichain, and every element of A_0 has rank 0.

Let A_1 be the set of elements of P that cover some elements of A_0. Then, every element of A_1 has rank 1, and A_1 is an antichain. Moreover, since (P, τ) is a saturated and canonically labelled, every element of A_1 covers every element of A_0.

Let A_2 be the set of elements of P that cover some elements of A_1. Similarly, every element of A_2 has rank 0, and every element of A_2 covers every element of A_1. Continuing in this fashion, we get the desired representation of (P, τ) as an alternating ordinal sum of antichains:

$$(P, \tau) = A_0 \oplus_0 A_1 \oplus_1 A_2 \oplus_0 \cdots \oplus_a A_k.$$

□

Now we are ready to deduce the unimodality of the W-polynomial of a sign-graded poset P. Let S^d denote the space of symmetry polynomials in $\mathbb{R}[x]$ with center of symmetry $\frac{d}{2}$. Then, S^d has a basis

$$B_d = \{x^i(1+x)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor}$$

Let S_+^d denote the nonnegative span of the basis B_d. Thus, S_+^d is a cone, and every polynomial in S_+^d has unimodal coefficients.

Lemma 7.7. *Let $c, d \in \mathbb{N}$. Then,*

$$S^c S^d \subset S^{c+d}$$
$$S_+^c S_+^d \subset S_+^{c+d}.$$

Suppose $h \in S^d$ has positive leading coefficient and has only real and nonpositive roots. Then, $h \in S_+^d$.

Proof. Observing that

$$\begin{aligned} B_c B_d &= \{x^i(1+x)^{c-2i} \cdot x^j(1+x)^{d-2j} : 0 \le i \le \lfloor \frac{c}{2} \rfloor, 0 \le j \le \lfloor \frac{d}{2} \rfloor\} \\ &= \{x^k(1+x)^{c+d-2k} : 0 \le k \le \lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor\} \\ &\subset B_{c+d}, \end{aligned}$$

we deduce the inclusions

$$S^c S^d \subset S^{c+d}$$
$$S_+^c S_+^d \subset S_+^{c+d}.$$

Suppose $h \in S^d$ has positive leading coefficient and has only real and nonpositive roots. Assume the leading coefficient of h is 1. Let $\alpha \notin \{-1, 0\}$ be a root of h. Since $h \in S^d$, $h(x) = x^d h(\frac{1}{x})$, so $\frac{1}{\alpha}$ is also a root of h. Therefore, h can be written as a product of

$$\begin{aligned} h(x) &= x^a(x+1)^b(x-\alpha_1)(x-\frac{1}{\alpha_1})\dots(x-\alpha_k)(x-\frac{1}{\alpha_k}) \\ &= x^a(x+1)^b(x^2+(-\alpha_1-\frac{1}{\alpha_1})+1)\dots(x^2+(-\alpha_k-\frac{1}{\alpha_k})+1) \end{aligned}$$

Since $-\alpha_i - \frac{1}{\alpha_i} \ge 2$, we have $x^2+(-\alpha_i-\frac{1}{\alpha_i})+1 \in S_+^2$. Since $x \in S_+^2$ and $x+1 \in S_+^1$, by the inclusion $S_+^s S_+^t \subset S_+^{s+t}$,

$$h(x) = x^a(x+1)^b(x^2+(-\alpha_1-\frac{1}{\alpha_1})+1)\dots(x^2+(-\alpha_k-\frac{1}{\alpha_k})+1) \in S_+^d.$$

□

Theorem 7.8. *Given a sign-graded poset (P, τ) with canonical labelling, let $n = |P|$. Then, $W(P, \tau) \in S_+^{n+1-r(\tau)}$.*

Proof. Recall that we have

$$W(P,\tau) = \sum_Q W(Q,\tau),$$

where the sum is taken over saturated τ-graded posets Q that extend P.

By Proposition 7.6, each τ-graded poset Q that extends P can be written as an alternating ordinal sum of antichains:

$$(Q,\tau) = A_0 \oplus_0 A_1 \oplus_1 A_2 \oplus_0 \cdots \oplus_a A_k$$

By Proposition 3.12 and Proposition 5.4, for two nonempty labelled posets (P_1,ω_1) and (P_2,ω_2), we have

$$W(P_1 \oplus P_2, \omega_1 \oplus_0 \omega_2) = \frac{1}{x} W(P_1,\omega_1)W(P_2,\omega_2)$$
$$W(P_1 \oplus P_2, \omega_1 \oplus_1 \omega_2) = W(P_1,\omega_1)W(P_2,\omega_2)$$

Since the W-polynomial of an antichain of size t is the t-th Eulerian polynomial multiplied by x, $W(A_i)$ is real-rooted and symmetric. By Lemma 7.7 and Corollary 7.4, $W(Q,\tau)$ is real-rooted and belongs to the cone $S_+^{n+1-r(\tau)}$.

Therefore, $W(P,\tau)$, which is the sum of $W(Q,\tau) \in S_+^{n+1-r(\tau)}$, also belongs to $S_+^{n+1-r(\tau)}$.

□

8. Future directions

Many questions related to the Neggers-Stanley conjecture remain unsettled. Here we list some of these questions and encourage the reader to attempt to answer them in the future.

1. Are the coefficients of W-polynomials of sign-graded posets log-concave? Do sign-graded posets have real-rooted W-polynomials?
2. Are the coefficients of W-polynomials of labelled posets log-concave or unimodal in general?
3. Given a labelled poset, does the unimodality (log-concavity) of the coefficients of its W-polynomial imply the unimodality (log-concavity) of the coefficients of its E-polynomial, and vice versa?
4. Is the class of labelled posets that satisfy the Neggers-Stanley conjecture closed under the ordinal product (direct product)? Recall that the boolean poset on the powerset $P([n])$ can be written as the direct product of n 2-element chains.
5. Is real-rootedness of W-polynomials preserved under the composition operation (Definition 5.14)? Is the unimodality (log-concavity) of the coefficients of W-polynomials (E-polynomials) preserved under the composition operation?

Acknowledgments

I would like to give special thanks to my mentor Adán Medrano Martin Del Campo for all his efforts in helping and guiding me this summer. This work would not have been made possible without the support of my mentor. I would like to thank Prof. Peter May for providing this wonderful opportunity and organizing UChicago Math REU. I would like to thank Prof. László Babai for pointing me to helpful references on unimodal sequences and the Neggers-Stanley conjecture. I would also like to thank my advisor Prof. Lorenzo Orecchia for his wholehearted support in my participation in UChicago Math REU, and for all the time outside REU that we spent together on interesting algorithmic problems. I would like to thank all my friends at UChicago Math REU this summer, for giving me mental support and for having helpful conversations with me. Finally, I would like to express my deepest gratitude to my dear parents.

References

[1] Björner, Anders, and Michelle L. Wachs. "Permutation statistics and linear extensions of posets." Journal of Combinatorial Theory, Series A 58, no. 1 (1991): 85-114.

[2] Brändén, Petter. "Counterexamples to the Neggers-Stanley Conjecture." Electronic Research Announcements of the American Mathematical Society 10, no. 17 (2004): 155-158.

[3] Brändén, Petter. "Unimodality, log-concavity, real-rootedness and beyond." Handbook of enumerative combinatorics 87 (2015): 437.

[4] Brändén, Petter. "On operators on polynomials preserving real-rootedness and the Neggers-Stanley Conjecture." Journal of Algebraic Combinatorics 20, no. 2 (2004): 119-130.

[5] Brändén, Petter. "Sign-graded posets, unimodality of W-polynomials and the Charney-Davis Conjecture." arXiv preprint math/0406019 (2004).

[6] Brenti, Francesco. "Unimodal, log-concave and Pólya frequency sequences in combinatorics." PhD diss., Massachusetts Institute of Technology, 1988.

[7] Comtet, Louis. Advanced Combinatorics: The art of finite and infinite expansions. Springer Science & Business Media, 2012.

[8] Gasharov, Vesselin. "On the Neggers–Stanley conjecture and the Eulerian polynomials." Journal of Combinatorial Theory, Series A 82, no. 2 (1998): 134-146.

[9] Neggers, Joseph. Representations of finite partially ordered sets. Journal of Combinatorics, Information & System Sciences. 3, 1978.

[10] Reiner, Victor, and Volkmar Welker. "On the Charney–Davis and Neggers–Stanley conjectures." Journal of Combinatorial Theory, Series A 109, no. 2 (2005): 247-280.

[11] Simion, Rodica. "A multiindexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences." Journal of Combinatorial Theory, Series A 36, no. 1 (1984): 15-22.

[12] Stanley, Richard P. "Log-concave and unimodal sequences in algebra, combinatorics, and geometry." Ann. New York Acad. Sci 576, no. 1 (1989): 500-535.

[13] Stanley, Richard P. "Positivity problems and conjectures in algebraic combinatorics." Mathematics: frontiers and perspectives 295 (2000): 319.

[14] Stanley, Richard P. "Enumerative Combinatorics Volume 1 second edition." Cambridge studies in advanced mathematics (2011).

[15] Stanley, Richard P. *Ordered structures and partitions.* Vol. 119. American Mathematical Soc., 1972.

[16] Stembridge, John. "Counterexamples to the poset conjectures of Neggers, Stanley, and Stembridge." Transactions of the American Mathematical Society 359, no. 3 (2007): 1115-1128.

[17] Wagner, David G. "Enumeration of functions from posets to chains." European journal of combinatorics 13, no. 4 (1992): 313-324.

[18] Wagner, David G. "Total positivity of Hadamard products." Journal of mathematical analysis and applications 163, no. 2 (1992): 459-483.