HOMEWORK 4

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Problem 1

Prove that in "fake" Poincaré Plane where Poincaré metric g on a manifold M is given as:

$$g = \frac{\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y}{y}.$$

a).

$$\Gamma^{*}_{**} = 0,$$

except

$$\Gamma^{x}_{yx} = \Gamma^{x}_{xy} = -\frac{1}{y^2},$$

and

$$\Gamma^{y}_{xx} = \Gamma^{y}_{yy} = -\frac{1}{2y}.$$

b). The geodesic $\gamma(0) = (0, 1)$ where $\dot{\gamma}(0) = -\partial/\partial y$ is $x = 0$ and $y = (1 - t/2)^2$.

Solution(a):
Christoffel symbol of second kind $\Gamma^{\gamma}_{\beta j}$ can be computed using Euler-Lagrange equation. First, let's define the energy functional as:

$$\int_a^b F(\vec{\dot{x}}, \vec{x}, t)\mathrm{d}t = \int_a^b \dot{x}^i \dot{x}^j g_{ij} \mathrm{d}t. \tag{1}$$

The energy functional above has to satisfy Euler-Lagrange equations which are given as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial F}{\partial \dot{x}^i}) - \frac{\partial F}{\partial x_i} = 0. \tag{2}$$

Equation 2 will result in the following set of differential equations:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^{i}_{jk} \dot{x}^j \dot{x}^k = 0. \tag{3}$$

In "fake" Poincaré Plane, the energy functional is defined as:

$$F(\vec{\dot{x}}, \vec{x}, t) = \frac{\dot{x}^2 + \dot{y}^2}{y}. \tag{4}$$

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Therefore, Euler-Lagrange equations give the following set of ordinary differential equations:

$$\ddot{x} - \frac{\dot{x}\dot{y}}{y} = 0, \tag{5}$$

$$\ddot{y} - \frac{\dot{y}^2}{2y} + \frac{\dot{x}^2}{2y} = 0. \tag{6}$$

Therefore, Christoffel symbols are obtained as:

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{2y}, \tag{7}$$

$$\Gamma^y_{yy} = -\frac{1}{2y}, \tag{8}$$

$$\Gamma^y_{xx} = \frac{1}{2y}. \tag{9}$$

Solution(b):
Equation 5 can be re-arranged as:

$$\frac{\ddot{x}}{\dot{x}} = \frac{\dot{y}}{y}. \tag{10}$$

After integration both sides of equation 10, it leads to the following relation:

$$\dot{x} = cy \tag{11}$$

After imposing initial conditions $\dot{x} = 0$, if it is replaced into equation 6, it reduces to the following equation:

$$\ddot{y} - \frac{\dot{y}^2}{2y} = 0. \tag{12}$$

Equation 12 can be re-arranged as:

$$\frac{\ddot{y}}{\dot{y}} = \frac{\dot{y}}{2y}. \tag{13}$$

After integrating both sides of equation 13,

$$\dot{y}^2 = cy. \tag{14}$$

After imposing initial conditions, the integration constant $c = 1$. Therefore, the equation 14 can be re-expressed as integral:

$$-\frac{\mathrm{d}y}{\sqrt{y}} = \mathrm{d}t. \tag{15}$$

It results in the following equation:

$$y = (1 - \frac{t}{2})^2. \tag{16}$$