EE 523 Electromagnetic Wave Theory

Take-Home Exam

(30% of the MT Exam)

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Part 1 A gyroelectric medium at frequency ω has the constitutive relations:

$\bar{D} = \bar{\bar{\varepsilon}}.\bar{E}$, $\bar{B} = \mu\bar{H}$. The permeability μ is a scalar, whereas the dyadic permittivity $\bar{\bar{\varepsilon}}$ is represented by the matrix:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & j\varepsilon_2 & 0 \\ -j\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

where $\varepsilon_1 > 0$, $|\varepsilon_2| < \varepsilon_1$, and $\varepsilon_3 > 0$.

i. Our main aim is to characterize circularly polarized plane waves propagating in *z*-direction within this medium. For this purpose, we consider the two (complex-valued) unit vectors denoting the "circular" directions $\hat{e}_+ = \frac{1}{2}\left(\hat{a}_x - j\hat{a}_y\right)$ and $\hat{e}_- = \frac{1}{2}\left(\hat{a}_x + j\hat{a}_y\right)$ for right and left polarizations. Show that any $\bar{E}$ field in the form $\bar{E} = E_x\hat{a}_x + E_y\hat{a}_y$ can be converted to the form $\bar{E} = E_+\hat{e}_+ + E_-\hat{e}_-$, where $E_+ = E_x + jE_y$, and $E_- = E_x - jE_y$.

   Remark: Note that this operation is a basis change from the linear basis $\left\{\hat{a}_x, \hat{a}_y\right\}$ to the circular basis $\left\{\hat{e}_+, \hat{e}_-\right\}$.

ii. Show that under this transformation the permittivity matrix is diagonalized. That is:
$$\begin{bmatrix} D_+ \\ D_- \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 + \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_1 - \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \\ E_z \end{bmatrix}$$, where $D_\pm = D_x \pm jD_y$.

iii. Show that the wave vectors for right and left circularly polarized plane waves propagating in *z*-direction are given as:
$k_+ = \omega\sqrt{\mu\varepsilon_+}$ and $k_- = \omega\sqrt{\mu\varepsilon_-}$, where $\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2$. Note that the permittivity ε_3 has no role in this formulation.

iv. Assume that the region $0 < z < d$ is filled with this gyroelectric medium, whereas the regions $z < 0$ and $z > d$ are free space. Assume that a linearly polarized plane wave $\bar{E}^i = E_0 e^{-jk_0 z}\hat{a}_x$ in the free space region $z < 0$ is incident to the slab. Show that the transmitted wave in the region $z > d$ will be a linearly polarized plane wave $\bar{E}^t = \alpha E_0 \left[\cos\psi \hat{a}_x + \sin\psi \hat{a}_y\right] e^{-jk_0 z}$, where α is a constant of proportionality (i.e. we are not concerned with the reflected waves from the boundaries of the slab), and ψ is the angle of rotation of the polarization direction which depends on k_+, k_- and d. Evaluate ψ.

This phenomenon is known as Faraday rotation, and it was experimentally discovered by Michael Faraday in 1845.

Hint: Decompose the wave propagating in the slab into its right and left circular components, and relate the rotation angle to the phase difference between the circular components at $z = d$, where the wave leaves the medium.

Part 2 Using the guidelines given below, show that a plasma medium acts as a gyroelectric medium when a constant external magnetic field $\bar{B} = B_0 \hat{a}_z$ is applied.

i. The equation of motion of a free electron in this medium is governed by the following differential equation $m_e \dfrac{d\bar{v}}{dt} = e\left(\bar{E} + \bar{v} \times \bar{B}\right)$, where m_e and e are the electron mass and charge respectively. Show that the components of the velocity vector $\bar{v}$ satisfy

$$\frac{dv_x}{dt} = \frac{e}{m_e} E_x + \omega_B v_y$$

$$\frac{dv_y}{dt} = \frac{e}{m_e} E_y - \omega_B v_x$$

where $\omega_B = \dfrac{eB_0}{m_e}$ is called the cyclotron frequency.

ii. In order to solve these coupled differential equations, search (in phasor domain) solutions in the form $v_\pm = v_x \pm jv_y$. Show that:

$$v_\pm = v_x \pm jv_y = \frac{\frac{e}{m_e}\left(E_x \pm jE_y\right)}{j\left(\omega \pm \omega_B\right)}$$

iii. Finally let N be the number of electrons per unit volume. Show that

$$\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2 = \varepsilon_0 \left[1 - \frac{\omega_p^2}{\omega\left(\omega \pm \omega_B\right)}\right], \text{ where } \omega_p = \sqrt{\frac{Ne^2}{m_e \varepsilon_0}} \text{ is the plasma frequency.}$$

Hint: Integrate the velocity to find the displacement of a single electron and obtain the susceptibility of the bulk material in phasor domain.