HOMEWORK 4

ANIL AKSU

Problem 1

Prove relationship :

$$\tan\frac{\theta}{2} = \frac{Ze^2}{4\pi\epsilon_0 m v^2 b}.$$

for the deflection of an electron of mass m in a Coulomb collision with a much heavier ion, charge Ze. For $\theta/2$ find the impact parameter b_0 and scattering cross-section σ.

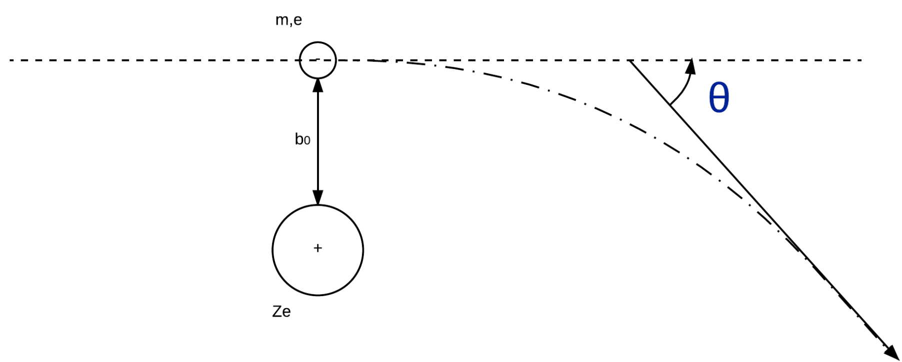

Figure 1. Electron scattering in a Coulomb collision with a much heavier ion.

Solution:
The equation of motion under electric field generated by a heavier ion in polar coordinates can be given as:

$$m(\ddot{r} - \dot{\alpha}^2 r) = -\frac{Ze^2}{4\pi\epsilon_0 r^2}, \tag{1}$$

where α is the radial angle and r is the distance between ion and electron. Equation 1 is the equation of motion in radial direction, however it is not sufficient. Equation of motion in radial direction can be given as:

$$2\dot{r}\dot{\alpha} + r\ddot{\alpha} = 0. \tag{2}$$

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The solution to equation 2 allows us to relate r with $\dot{\alpha}$ as follows:

$$\frac{\ddot{\alpha}}{\dot{\alpha}} = -2\frac{\dot{r}}{r}. \tag{3}$$

As a result,

$$\dot{\alpha} = \dot{\alpha}_0 \frac{r_0^2}{r^2}. \tag{4}$$

Let's define a new variable $\eta = 1/r$ to manipulate equations much easier. Therefore the angular velocity can be written in terms of new variable η as:

$$\dot{\alpha} = \dot{\alpha}_0 r_0^2 \eta^2. \tag{5}$$

Under this formulation, the radius r can be written as a function of α. Thus the time derivative with respect to time can be re-expressed as:

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\eta} = -\frac{1}{\eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}\alpha}\dot{\alpha} = -\dot{\alpha}_0 r_0^2 \frac{\mathrm{d}\eta}{\mathrm{d}\alpha}. \tag{6}$$

And also the second time derivative can be written as:

$$\ddot{r} = -(\dot{\alpha}_0 r_0^2)^2 \eta^2 \frac{\mathrm{d}^2\eta}{\mathrm{d}\alpha^2}. \tag{7}$$

If the relations in equations 7 and 5 are replaced back into equation 1, the equation of motion in radial direction can be re-expressed as:

$$-m(\dot{\alpha}_0 r_0^2)^2 \eta^2 \frac{\mathrm{d}^2\eta}{\mathrm{d}\alpha^2} - m(\dot{\alpha}_0 r_0^2)^2 \eta^3 = -\frac{Ze^2}{4\pi\epsilon_0}\eta^2. \tag{8}$$

After dividing both sides by $-m(\dot{\alpha}_0 r_0^2)^2\eta^2$, equation 8 can be re-written as:

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\alpha^2} + \eta = \frac{Ze^2}{4\pi\epsilon_0 m(\dot{\alpha}_0 r_0^2)^2}. \tag{9}$$

The solution to equation 9 can be given as:

$$\eta = c_1 \cos\alpha + \frac{Ze^2}{4\pi\epsilon_0 m(\dot{\alpha}_0 r_0^2)^2}. \tag{10}$$

Before imposing a boundary condition, let's physically re-evaluate the problem, as $r \to \infty$, $\dot{\alpha} r \to 0$. Therefore, all the kinetic energy can be given in terms of radial velocity. Therefore, the kinetic energy can be written as:

$$\frac{m}{2}\dot{r}^2 = \frac{m}{2}(\dot{\alpha}_0 r_0)^2 - \frac{Ze^2}{2\pi\epsilon_0 r_0} \tag{11}$$

Therefore,

$$\dot{r} = \sqrt{(\dot{\alpha}_0 r_0)^2 - \frac{Ze^2}{4\pi\epsilon_0 m r_0}} \tag{12}$$

This can also be expressed by equation 10 as:

$$\dot{r} = \dot{\alpha}_0 r_0^2 c_1 \sin\alpha. \tag{13}$$

Also at $r \to \infty$,

$$c_1 \cos\alpha = -\frac{Ze^2}{4\pi\epsilon_0 m(\dot{\alpha}_0 r_0^2)^2}. \tag{14}$$

Problem 2

Start with the electromagnetic wave equation,

$$(c^2k^2 - \omega^2)\vec{E}_1 = i\omega\vec{j}_1/\epsilon_0 = (-n_0e^2/m\epsilon_0)\vec{E}_1.$$

and substitute $\vec{j}_1$ in terms of $\vec{E}_1$ by using electron fluid equation of motion:

$$m_e n_e \left[\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla)\,\mathbf{u}_e\right] = -\nabla p_e - en_e\mathbf{E}.$$

(including the electron pressure) and then the Poisson's equation. BY seperately dotting and crossing the resulting equation with $\vec{k}$, show how to generate the dispersion relation for longitudinal plane waves and for high-frequency electro-magnetic waves, and also show that one dispersion relation $\omega(\vec{k})$ must hold if $\vec{E}_1 \cdot \vec{k} \neq 0$ and the other must hold if $\vec{E}_1 \times \vec{k} \neq 0$. This implies that there is no class of waves that propagates with $\vec{k}$ at intermediate angle to $\vec{E}_1$.

Solution:

the Poisson's equation can be given as:

$$\epsilon_0 \nabla \cdot \mathbf{E} = -en_e. \tag{15}$$

The pressure can be associated with the number of electron within a infinitesimal volume as:

$$p = \gamma T n_e. \tag{16}$$

The linearized momentum equation can also be given as:

$$m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} = -\nabla p_e - en_e\mathbf{E}. \tag{17}$$

After taking gradient of the pressure field 16 and replacing in equation 17, the momentum equation can be re-expressed as:

$$m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} = -\gamma T \nabla n_e - en_e\mathbf{E}. \tag{18}$$

After Fourier transforming equation 18, the following relation can be obtained:

$$i\omega m_e n_e \mathbf{u}_e = -\mathbf{k}\gamma T n_e - en_e\mathbf{E}. \tag{19}$$

Note that the current density $\vec{j}_1 = -en_e\mathbf{u}_e$ and also $\epsilon_0(c^2k^2 - \omega^2)\vec{E}_1 = i\omega\vec{j}_1$. After replacing these into equation 19, it can be given as:

$$m_e\epsilon_0(c^2k^2 - \omega^2)\mathbf{E} = -\mathbf{k}\gamma T n_e - en_e\mathbf{E}. \tag{20}$$

After taking the product of the equation 21 and $\mathbf{k}$, a single equation can be obtained as:

$$m_e\epsilon_0(c^2k^2 - \omega^2)\mathbf{k}\cdot\mathbf{E} = -k^2\gamma T n_e - en_e\mathbf{k}\cdot\mathbf{E}. \tag{21}$$

By using equation 15:

$$n_e = -\epsilon_0\mathbf{k}\cdot\mathbf{E}/e. \tag{22}$$

and also assuming $n_e \approx n_0$ where n_0 is constant, this replaced in the term $en_e\mathbf{E}$. The dispersion relation can be obtained as:

$$\omega^2 = c^2k^2 + \frac{k^2\gamma T}{m_e e} + \frac{en_0}{m_e\epsilon_0}. \tag{23}$$