HOMEWORK 2

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1. Problem 1

Given a connection ∇ with torsion T and curvature R, prove that

A.

$$\wp_{X,Y,Z}\left(R(X,Y,Z) + T(T(X,Y),Z) + \nabla_Z T(X,Y)\right) = 0.$$

Solution:
Let's start the solution by defining torsion T and curvature R. The torsion of ∇ is the tensor field $T = T_\nabla$ of bidegree $(1,2)$ defined for each $X, Y, Z \in TM$ by

$$T(X,Y) = \nabla(X,Y) - \nabla(Y,X) - [X,Y].$$

And also the curvature of ∇ is defined as:

$$R(X,Y,Z) = \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) - \nabla([X,Y],Z).$$

Let's write the relation between torsion and curvature explicitly,

(1)
$$\begin{aligned} R(X,Y,Z) + T(T(X,Y),Z) + \nabla_Z T(X,Y) &= \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) \\ &\quad -\nabla([X,Y],Z) + \nabla(\nabla(X,Y),Z) - \nabla(Z,\nabla(X,Y)) - [\nabla(X,Y),Z] \\ &\quad +\nabla(Z,\nabla(X,Y)) - \nabla(Z,\nabla(Y,X)) - \nabla(Z,[X,Y]). \end{aligned}$$

The term $\nabla(Z,\nabla(X,Y))$ cancels out and it reduces to:

(2)
$$\begin{aligned} R(X,Y,Z) + T(T(X,Y),Z) + \nabla_Z T(X,Y) = \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) \\ -\nabla([X,Y],Z) + \nabla(\nabla(X,Y),Z) - [\nabla(X,Y),Z] - \nabla(Z,\nabla(Y,X)) - \nabla(Z,[X,Y]). \end{aligned}$$

Moreover, the next term in cycle with order (Z,X,Y) can be given as:

(3)
$$\begin{aligned} R(Z,X,Y) + T(T(Z,X),Y) + \nabla_Y T(Z,X) = \nabla(Z,\nabla(X,Y)) - \nabla(X,\nabla(Z,Y)) \\ -\nabla([Z,X],Y) + \nabla(\nabla(Z,X),Y) - [\nabla(Z,X),Y] - \nabla(Y,\nabla(X,Z)) - \nabla(Y,[Z,X]). \end{aligned}$$

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Finally, the term (Y, Z, X) is given as:

$$R(Y,Z,X) + T(T(Y,Z),X) + \nabla_X T(Y,Z) = \nabla(Y,\nabla(Z,X)) - \nabla(Z,\nabla(Y,X)) \\ -\nabla([Y,Z],X) + \nabla(\nabla(Y,Z),X) - [\nabla(Y,Z),X] - \nabla(X,\nabla(Z,Y)) - \nabla(X,[Y,Z]). \tag{4}$$

Note that

$$\nabla(\nabla(Y,Z),X) - [\nabla(Y,Z),X] - \nabla(X,\nabla(Z,Y)) = \nabla(X,\nabla(Y,Z)) - \nabla(X,\nabla(Z,Y)) \\ = \nabla(X,[Y,Z]). \tag{5}$$

After replacing this result into equation 4, it reduces to:

$$R(Y,Z,X) + T(T(Y,Z),X) + \nabla_X T(Y,Z) = \nabla(Y,\nabla(Z,X)) - \nabla(Z,\nabla(Y,X)) \\ -\nabla([Y,Z],X). \tag{6}$$

It also means that last four terms of equation 2 and 3 drops out and if they are all added together, the following summation is obtained:

$$\nabla(Y,\nabla(Z,X)) - \nabla(Z,\nabla(Y,X)) - \nabla([Y,Z],X) + \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) \\ -\nabla([X,Y],Z) + \nabla(Z,\nabla(X,Y)) - \nabla(X,\nabla(Z,Y)) - \nabla([Z,X],Y) \tag{7}$$

In summation 7,

$$\nabla(Y,\nabla(Z,X)) - \nabla(Y,\nabla(X,Z)) - \nabla([Z,X],Y) = \nabla(Y,[Z,X]) - \nabla([Z,X],Y) = [Y,[Z,X]], \tag{8}$$

$$\nabla(Z,\nabla(X,Y)) - \nabla(Z,\nabla(Y,X)) - \nabla([X,Y],Z) = \nabla(Z,[X,Y]) - \nabla([X,Y],Z) = [Z,[X,Y]], \tag{9}$$

$$\nabla(X,\nabla(Y,Z)) - \nabla(X,\nabla(Z,Y)) - \nabla([Y,Z],X) = \nabla(X,[Y,Z]) - \nabla([Y,Z],X) = [X,[Y,Z]]. \tag{10}$$

Therefore, the summation 7 can be written more compactly as:

$$[Y,[Z,X]] + [Z,[X,Y]] + [X,[Y,Z]] = 0. \tag{11}$$