EE 523: TAKE HOME MIDTERM

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Part 1

A gyroelectric medium at frequency ω has the constitutive relations: $\bar{D} = \bar{\bar{\varepsilon}} \cdot \bar{E}$, $\bar{B} = \mu_0 \bar{H}$. The permeability μ is a scalar, whereas the dyadic permittivity $\bar{\bar{\varepsilon}}$ is represented by the matrix:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & j\varepsilon_2 & 0 \\ -j\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

where $\varepsilon_1 > 0$, $|\varepsilon_2| < \varepsilon_1$, and $\varepsilon_3 > 0$.

a. Our main aim is to characterize circularly polarized plane waves propagating in z-direction within this medium. For this purpose, we consider the two (complex-valued) unit vectors denoting the circular directions $\hat{e}_+ = \frac{1}{2}(\hat{a}_x - j\hat{a}_y)$ and $\hat{e}_- = \frac{1}{2}(\hat{a}_x + j\hat{a}_y)$ for right and left polarizations. Show that any $\hat{E}$ field in the form $\hat{E} = E_x\hat{a}_x + E_y\hat{a}_y$ can be converted to the form $\hat{E} = E_+\hat{a}_+ + E_-\hat{a}_-$, where $E_+ = E_x + jE_y$, and $E_- = E_x - jE_y$.
Remark: Note that this operation is a basis change from the linear basis $\{\hat{a}_x, \hat{a}_y\}$ to the circular basis $\{\hat{e}_+, \hat{e}_-\}$
Solution:
The circular coordinate transformation defined in the problem can be given as:

$$\begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \end{bmatrix} = \begin{bmatrix} \hat{e}_+ \\ \hat{e}_- \end{bmatrix} \tag{1}$$

And also,

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \end{bmatrix} = \begin{bmatrix} E_+ & E_- \end{bmatrix} \begin{bmatrix} \hat{e}_+ \\ \hat{e}_- \end{bmatrix} \tag{2}$$

After replacing equation 1 into equation 2, the following relation can be obtained:

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \end{bmatrix} = \begin{bmatrix} E_+ & E_- \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \end{bmatrix} \tag{3}$$

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Therefore,

$$\begin{bmatrix} E_x & E_y \end{bmatrix} = \begin{bmatrix} E_+ & E_- \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \tag{4}$$

After inverting the matrix in equation 4 and multiplying both side of equation 4, the following system can be obtained:

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} = \begin{bmatrix} E_+ & E_- \end{bmatrix} \tag{5}$$

As a result,

$$E_+ = E_x + jE_y, \tag{6}$$

$$E_- = E_x - jE_y. \tag{7}$$

b. Show that under this transformation the permittivity matrix is diagonalized. That is:

$$\begin{bmatrix} D_+ \\ D_- \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 + \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_1 - \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \\ E_z \end{bmatrix}$$

where $D_\pm = D_x \pm jD_y$.

Solution:

By indicial notation, the relation $\bar{D} = \bar{\bar{\varepsilon}} \cdot \bar{E}$ can be given as:

$$D_i = \varepsilon_{ij} E_j. \tag{8}$$

For given basis transformation, the linear coordinate transformation matrix is found as:

$$a_{ij} = \begin{bmatrix} 1 & j & 0 \\ 1 & -j & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

After applying the coordinate transformation to the equation 8, it can be given as:

$$a_{ki} D_i = a_{ki} a_{lj} \varepsilon_{ij} a_{lj} E_j. \tag{10}$$

Note that

$$a_{ki} D_i = \begin{bmatrix} D_+ \\ D_- \\ D_z \end{bmatrix}, \tag{11}$$

$$a_{lj} E_j = \begin{bmatrix} E_+ \\ E_- \\ E_z \end{bmatrix}, \tag{12}$$

As a result,
(13)
$$a_{ki}a_{lj}\varepsilon_{ij} = \begin{bmatrix}1 & j & 0\\ 1 & -j & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon_1 & j\varepsilon_2 & 0\\ -j\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ j & -j & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\varepsilon_1+\varepsilon_2 & 0 & 0\\ 0 & \varepsilon_1-\varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix}$$

c. Show that the wave vectors for right and left circularly polarized plane waves propagating in z-direction are given as: $k_+ = \omega\sqrt{\mu\varepsilon_+}$ and $k_- = \omega\sqrt{\mu\varepsilon_-}$, where $\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2$. Note that the permittivity ε_3 has no role in this formulation.
Solution:
In a source free region, the governing equations for the electric field can be reduced to the following single differential equation:

$$(\mathbf{k}\cdot\mathbf{E})\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\mathbf{E} = -\omega^2\mu_0\bar{\bar{\varepsilon}}\cdot\mathbf{E}, \tag{14}$$

and for the wave field $\bar{E} = E_0 e^{-jk_+z}\hat{e}_+$, the equation 14 can be given as:

$$-k_+^2 E_0 e^{-jk_+z} = -\omega^2\mu_0\varepsilon_+ E_0 e^{-jk_+z}. \tag{15}$$

For non-trivial electric field, the following relation must hold:

$$k_+ = \omega\sqrt{\mu\varepsilon_+} \tag{16}$$

Similarly, for the wave field $\bar{E} = E_0 e^{-jk_-z}\hat{e}_-$, the equation 14 can be given as:

$$-k_-^2 E_0 e^{-jk_+z} = -\omega^2\mu_0\varepsilon_- E_0 e^{-jk_+z}. \tag{17}$$

For non-trivial electric field, the following relation must hold:

$$k_- = \omega\sqrt{\mu\varepsilon_-} \tag{18}$$

d. Assume that the region $0 < z < d$ is filled with this gyroelectric medium, whereas the regions $z < 0$ and $z > d$ are free space. Assume that a linearly polarized plane wave $\bar{E}^i = E_0 e^{-jk_0z}\hat{a}_x$ in the free space region $z < 0$ is incident to the slab. Show that the transmitted wave in the region $z > d$ will be a linearly polarized plane wave $\bar{E}^t = \alpha E_0\left[\cos\psi\hat{a}_x + \sin\psi\hat{a}_y\right]e^{-jk_0z}$, where α is a constant of proportionality (i.e. we are not concerned with the reflected waves from the boundaries of the slab), and ψ is the angle of rotation of the polarization direction which depends on k_+, k_- and d. Evaluate ψ.
This phenomenon is known as Faraday rotation, and it was experimentally discovered by Michael Faraday in 1845.
Hint: Decompose the wave propagating in the slab into its right and left circular components, and relate the rotation angle to the phase difference between the circular components at $z = d$, where the wave leaves the medium.
Solution:

The interface conditions at $z = 0$ can be given as:

$$E_{1t} = E_{2t}, \tag{19}$$

$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = 0. \tag{20}$$

where n and t subscripts stand for normal and tangential components of the vector field.

$$\bar{E}_{tra} = E_- e^{-jk_- z}\hat{e}_- + E_+ e^{-jk_+ z}\hat{e}_+ = T_\perp^1 E_0 e^{-jk_0 z}\hat{a}_x. \tag{21}$$

which has to be satisfied at $z = 0$ where $T_\perp^1$ is the transmission coefficients, therefore,

$$E_- = E_+ = T_\perp^1 E_0. \tag{22}$$

At $z = d$, the electric field can be given as:

$$T_\perp^1 E_0 (e^{-jk_- d}\hat{e}_- + e^{-jk_+ d}\hat{e}_+) = \frac{T_\perp^1 E_0}{2}(\hat{a}_x(e^{-jk_- d} + e^{-jk_+ d}) + j\hat{a}_y(e^{-jk_+ d} - e^{-jk_- d})). \tag{23}$$

In the equation above, the sums and the differences of complex exponentials can be expanded as:

$$e^{-jk_- d} + e^{-jk_+ d} = e^{-j\frac{k_+ d + k_- d}{2}}(e^{-j\frac{k_+ d - k_- d}{2}} + e^{-j\frac{k_- d - k_+ d}{2}}) = 2\cos(\frac{k_+ - k_-}{2}d)e^{-j\frac{k_+ + k_-}{2}d} \tag{24}$$

and

$$e^{-jk_+ d} - e^{-jk_- d} = e^{-j\frac{k_+ d + k_- d}{2}}(e^{-j\frac{k_+ d - k_- d}{2}} - e^{-j\frac{k_- d - k_+ d}{2}}) = -2j\sin(\frac{k_+ - k_-}{2}d)e^{-j\frac{k_+ + k_-}{2}d} \tag{25}$$

As a result, the field can be expressed as:

$$T_\perp^1 E_0 e^{-j\frac{k_+ + k_-}{2}d}(\cos(\frac{k_+ - k_-}{2}d)\hat{a}_x + \sin(\frac{k_+ - k_-}{2}d)\hat{a}_y). \tag{26}$$

Since at the second transmission, the transmitted wave field after $z > d$ has to match the electric field above due to interface conditions. The rotation angle of the polarization direction can be found as:

$$\psi = \frac{k_+ - k_-}{2}d. \tag{27}$$

Part 2

Using the guidelines given below, show that a plasma medium acts as a gyroelectric medium when a constant external magnetic field $\bar{B} = B_0\hat{a}_z$ is applied.

a. The equation of motion of a free electron in this medium is governed by the following differential equation $m_e\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = e(\hat{E} + \hat{v} \times \hat{B})$, where m_e and e are the electron mass and charge respectively. Show that the components of the velocity vector $\hat{v}$ satisfy:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{e}{m_e}E_x + \omega_b v_y,$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{e}{m_e}E_x - \omega_b v_x.$$

where $\omega_b = \frac{eB_0}{m_e}$ is called the cyclotron frequency.
Solution:
The forcing due to magnetic field can be expanded as:

$$\hat{v} \times \hat{B} = det(\begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ v_x & v_y & 0 \\ 0 & 0 & B_0 \end{bmatrix}) = B_0 v_y \hat{a}_x - B_0 v_x \hat{a}_y. \tag{28}$$

After replacing this term into equation of motion, the set of governing equations can be obtained as:

$$m_e\frac{\mathrm{d}v_x}{\mathrm{d}t} = e(E_x + B_0 v_y), \tag{29}$$

$$m_e\frac{\mathrm{d}v_y}{\mathrm{d}t} = e(E_y - B_0 v_x). \tag{30}$$

Dividing both sides of equations 29 and 30 results in the following form:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{e}{m_e}E_x + \omega_b v_y, \tag{31}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{e}{m_e}E_y - \omega_b v_x. \tag{32}$$

where $\omega_b = \frac{eB_0}{m_e}$ is called the cyclotron frequency.

b. In order to solve these coupled differential equations, search (in phasor domain) solutions in the form $v_\pm = v_x \pm jv_y$. Show that:

$$v_\pm = v_x \pm jv_y = \frac{\frac{e}{m_e}(E_x \pm jE_y)}{j(\omega \pm \omega_b)}.$$

Solution:

Fourier transforming equations 31 and 32 in time results in the following set of algebraic equations:

$$j\omega v_x = \frac{e}{m_e}E_x + \omega_b v_y, \tag{33}$$

$$j\omega v_y = \frac{e}{m_e}E_y - \omega_b v_x. \tag{34}$$

In matrix form, equations 33 and 34 can be given as:

$$\begin{bmatrix} j\omega & -\omega_b \\ \omega_b & j\omega \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{e}{m_e}E_x \\ \frac{e}{m_e}E_y \end{bmatrix} \tag{35}$$

By inverting the matrix in equation 35, the velocity components can be obtained as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \frac{1}{\omega_b^2 - \omega^2} \begin{bmatrix} j\omega & \omega_b \\ -\omega_b & j\omega \end{bmatrix} \begin{bmatrix} \frac{e}{m_e}E_x \\ \frac{e}{m_e}E_y \end{bmatrix} \tag{36}$$

Therefore,

$$v_+ = v_x + jv_y = \frac{\frac{e}{m_e}(E_x + jE_y)j(\omega - \omega_b)}{\omega_b^2 - \omega^2} = \frac{\frac{e}{m_e}(E_x + jE_y)}{j(\omega_b + \omega)}, \tag{37}$$

and also

$$v_- = v_x - jv_y = \frac{\frac{e}{m_e}(E_x - jE_y)j(\omega + \omega_b)}{\omega_b^2 - \omega^2} = \frac{\frac{e}{m_e}(E_x - jE_y)}{j(\omega - \omega_b)}. \tag{38}$$

c. Finally let N be the number of electrons per unit volume. Show that $\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2 = \varepsilon_0 \left[1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_b)}\right]$, where $\omega_p = \sqrt{\frac{Ne^2}{m_e\varepsilon_0}}$ is the plasma frequency. Hint: Integrate the velocity to find the displacement of a single electron and obtain the susceptibility of the bulk material in phasor domain.

Solution:

The dipole moment is defined as:

$$\bar{p}(t) = q(-l(t))\hat{e}\pm \tag{39}$$

where $q = e$ and $l(t)$ can be obtained by integrating the velocity field $v_\pm$ as:

$$l(t) = \frac{v_\pm}{j\omega} = -\frac{\frac{e}{m_e}(E_x \pm jE_y)}{\omega(\omega \pm \omega_b)} \tag{40}$$

The macroscopic electric polarization vector $P\bar{(t)}$ is evaluated as:

$$\bar{P}(t) = Np\bar{(t)} \tag{41}$$

therefore,

$$\bar{P}(t) = -\frac{\frac{Ne^2}{m_e}}{\omega(\omega \pm \omega_b)}(E_x \pm jE_y). \tag{42}$$

Also

$$\bar{D} = \varepsilon_0\bar{E} + \bar{P}. \tag{43}$$

After replacing equation 42 into the relation 43, the following relation is obtained:

$$\bar{D} = \varepsilon_0 \bar{E}(1 - \frac{\frac{Ne^2}{m_e \varepsilon_0}}{\omega(\omega \pm \omega_b)}) \tag{44}$$

As a result,

$$\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2 = \varepsilon_0 \left[1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_b)}\right]. \tag{45}$$