HOMEWORK 1

ANIL AKSU

1. Problem 1

Prove that $x_1, ..., x_q \in U$ are linearly independent iff $x_1 \wedge ... \wedge x_q \neq 0$.

Solution:
Let x_n where $1 \leq n \leq q$ be a linear combination of $x_i, x_j, x_k \in \bigcup_1^q x_i$ such that:

$$x_n = \alpha x_i + \beta x_j + \gamma x_k. \tag{1}$$

The wedge product can be arranged so that:

$$x_1 \wedge ... \wedge x_q = sign(\sigma) x_1 \wedge ... \wedge (x_i \wedge x_j \wedge x_k \wedge x_n) \wedge ... \wedge x_q. \tag{2}$$

Moreover,

$$x_i \wedge x_j \wedge x_k \wedge x_n = x_i \wedge x_j \wedge (x_k \wedge (\alpha x_i + \beta x_j + \gamma x_k)), \tag{3}$$

which is equivalent to:

$$x_i \wedge x_j \wedge (x_k \wedge (\alpha x_i + \beta x_j + \gamma x_k)) = x_i \wedge x_j \wedge (\alpha x_k \wedge x_i + \beta x_k \wedge x_j) \tag{4}$$

As the wedge product of x_k by itself is zero, the last term vanishes. Furthermore,

$$x_i \wedge x_j \wedge (\alpha x_k \wedge x_i + \beta x_k \wedge x_j) = \alpha x_i \wedge x_j \wedge x_k \wedge x_i + \beta x_i \wedge x_j \wedge x_k \wedge x_j. \tag{5}$$

Finally,

$$\alpha x_i \wedge x_j \wedge x_k \wedge x_i + \beta x_i \wedge x_j \wedge x_k \wedge x_j = \alpha x_k \wedge x_j \wedge (x_i \wedge x_i) - \beta x_i \wedge x_k \wedge (x_j \wedge x_j) = 0. \tag{6}$$

As a result,

$$x_1 \wedge ... \wedge x_q = 0. \tag{7}$$

Therefore, it is shown that $x_1, ..., x_q \in U$ must be linearly independent to obtain non-zero wedge product $x_1 \wedge ... \wedge x_q$.

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