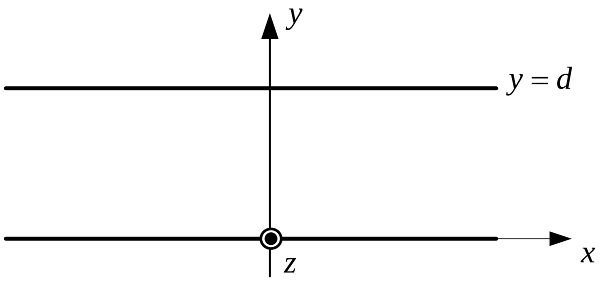
EE 523 Electromagnetic Wave Theory

HW2

Question 1

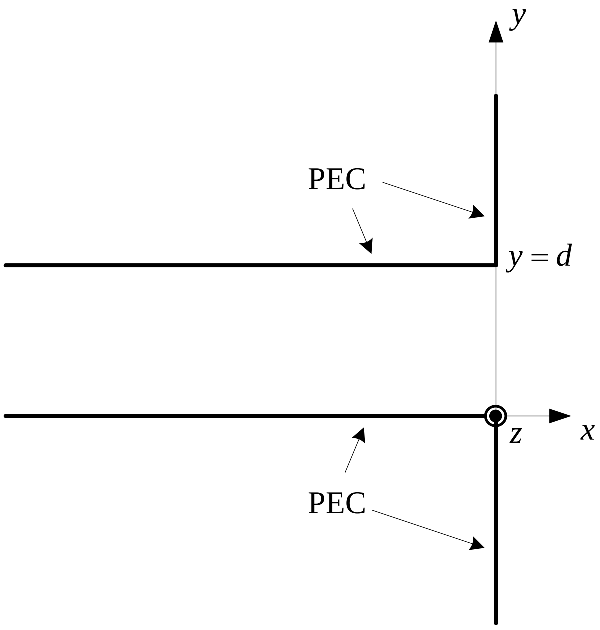
Consider the parallel plate waveguide shown below:



The PEC plates located at $y=0$ and $y=d$ are infinite in extent and the region $0<y<d$ is free space. We assume that guided waves propagate in $+x$-direction.

i. Find the expressions of $\bar{E}$ and $\bar{H}$ for TM_x modes. Hint: Start with the assumption $\bar{A}(x,y)=\psi(x,y)\hat{a}_x$.
ii. Find the expressions of $\bar{E}$ and $\bar{H}$ for TE_x modes. Hint: Start with the assumption $\bar{F}(x,y)=\psi(x,y)\hat{a}_x$.

Now consider the following figure related to radiation from a waveguide opening:



iii. By using Love's equivalence principle, show that the electromagnetic fields in the region $x > 0$ can be evaluated as the fields generated by the magnetic surface current over the waveguide opening (i.e. $x = 0$, $0 < y < d$) with density $\bar{J}_m = 2\bar{E} \times \hat{a}_x$.

iv. Assuming that the $\bar{E}$ field over the waveguide opening is approximately equal to the $\bar{E}$ field within the waveguide, evaluate the far field expressions of the $\bar{E}$ and $\bar{H}$ fields when the waveguide is supporting the fundamental TM_x mode.

v. Repeat part iv. when the waveguide is supporting the fundamental TE_x mode.

Hint for iv. and v.: You may use the radiation integral formulas given at the end of Lecture 8. Note that this problem is two-dimensional (i.e. no z-dependence), and the calculations are carried out on the xy-plane; i.e. $\theta = \frac{\pi}{2}$, and $\hat{a}_\theta = -\hat{a}_z$. The other alternative is deriving (and using) the radiation integral formulas in cylindrical coordinates.

Question 2

Consider $\bar{E}(x) = E_x(z,t)\hat{a}_x$, where $E_x(z,t) = a(z,t)e^{j(\omega t - k(\omega)z)}$. $k(\omega)$ is the (real-valued) wave number in a dispersive medium. Let us define an LTI system action as:

$$x(t) = E_x(z,t)\big|_{z=0} \text{ (input signal)}, \quad y(t) = E_x(z,t)\big|_{z=L} \text{ (Output signal)}$$

Let $a(0,t)$ be a low-pass signal, and assume that narrow-band assumption is valid for $E_x(0,t) = a(0,t)e^{j\omega_0 t}$, where ω_0 is the representation frequency of the band-pass signal.

Let the envelope be given as $a(0,t) = \exp\left(-\frac{t^2}{2\tau_0^2}\right)$. We can derive an expression for the scaling (i.e. distortion) in the Gaussian envelope, after the wave travels in the dispersive medium. That is, assuming that τ_0 is an approximation for the duration of $a(0,t)$, we can find how the duration of the envelope changes at $x = L$. The derivation of this result is as follows:

$$E_x(0,t) = a(0,t)e^{j\omega_0 t}$$

Evaluate the Fourier transform of both sides:

$$\hat{E}_x(0,\omega) = \hat{a}(0,\omega) * 2\pi\delta(\omega - \omega_0) = 2\pi\hat{a}(0,\omega - \omega_0)$$

Now, the wave can be propagated in frequency domain:

$$\hat{E}_x(z,\omega) = 2\pi\hat{a}(0,\omega - \omega_0)e^{-jk(\omega)z}$$

By using the inverse Fourier transform:

$$E_x(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\hat{a}(0,\omega-\omega_0)e^{-jk(\omega)z}e^{j\omega t}d\omega$$

Let $\omega'=\omega-\omega_0$. Then:

$$E_x(z,t)=\int_{-\infty}^{\infty}\hat{a}(0,\omega')e^{-jk(\omega'+\omega_0)z}e^{j(\omega'+\omega_0)t}d\omega'$$

The Taylor series expansion (with three terms) of $k(\omega'+\omega_0)$ around ω_0 is:

$$k(\omega'+\omega_0)=k(\omega_0)+k'(\omega_0)\omega'+\frac{1}{2}k''(\omega_0)\omega'^2$$

$$E_x(z,t)=e^{j(\omega_0 t-k(\omega_0)z)}\int_{-\infty}^{\infty}\hat{a}(0,\omega')e^{j\omega'(t-k'(\omega_0)z)}e^{-j\frac{k'(\omega_0)\omega'^2 z}{2}}d\omega'$$

$$FT\left\{a(0,t)=\exp\left(-\frac{t^2}{2\tau_0^2}\right)\right\}=\hat{a}(0,\omega)=\sqrt{2\pi}\tau_0\exp\left(-\frac{\tau_0^2\omega^2}{2}\right)$$

So:

$$E_x(z,t)=e^{j(\omega_0 t-k(\omega_0)z)}\int_{-\infty}^{\infty}\sqrt{2\pi}\tau_0\exp\left(-\frac{\tau_0^2\omega'^2}{2}\right)e^{j\omega'(t-k'(\omega_0)z)}e^{-j\frac{k''(\omega_0)\omega'^2 z}{2}}d\omega'$$

$$E_x(z,t)=e^{j(\omega_0 t-k(\omega_0)z)}\int_{-\infty}^{\infty}\sqrt{2\pi}\tau_0 e^{-\frac{\omega'^2(\tau_0^2+jk''(\omega_0)z)}{2}}e^{j\omega'(t-k'(\omega_0)z)}d\omega'$$

$$E_x(z,t)=e^{j(\omega_0 t-k(\omega_0)z)}\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\sqrt{2\pi}\tau_0 e^{-\frac{\omega'^2(\tau_0^2+jk''(\omega_0)z)}{2}}e^{j\omega'(t-k'(\omega_0)z)}d\omega'$$

$$E_x(z,t)=e^{j(\omega_0 t-k(\omega_0)z)}\frac{\tau_0}{\sqrt{\tau_0^2+jk''(\omega_0)z}}e^{-\frac{(t-k'(\omega_0)z)^2}{2(\tau_0^2+jk''(\omega_0)z)}}$$

In the exponent, we have the term $\frac{1}{\tau_0^2+jk''(\omega_0)z}=\frac{\tau_0^2}{\tau_0^4+\left(k''(\omega_0)z\right)^2}-j\frac{k''(\omega_0)z}{\tau_0^4+\left(k''(\omega_0)z\right)^2}$, real part of which can be interpreted as the inverse of the square of pulse duration as the pulse propagates. Why?

So: $\tau^2(x)=\tau_0^2+\frac{\left(k''(\omega_0)z\right)^2}{\tau_0^2}$.

Table 1. Rectangular Waveguide Specifications

Waveguide Size	JAN WG Desig	MIL-W-85 Dash #	Material	Freq Range (GHz)	Freq Cutoff (GHz)	Power (at 1 Atm)		Insertion Loss (dB/100ft)	Dimensions (Inches)	
						CW	Peak		Outside	Wall Thickness
WR284	RG48/U RG75/U	1-039 1-042	Copper Aluminum	2.60 - 3.95	2.08	45 36	7650	.742-.508 1.116-.764	3.000x1.500	0.08
WR229	RG340/U RG341/U	1-045 1-048	Copper Aluminum	3.30 - 4.90	2.577	30 24	5480	.946-.671 1.422-1.009	2.418x1.273	0.064
WR187	RG49/U RG95/U	1-051 1-054	Copper Aluminum	3.95 - 5.85	3.156	18 14.5	3300	1.395-.967 2.097-1.454	1.000x1.000	0.064
WR159	RG343/U RG344/U	1-057 1-060	Copper Aluminum	4.90 - 7.05	3.705	15 12	2790	1.533-1.160 2.334-1.744	1.718x0.923	0.064
WR137	RG50/U RG106/U	1-063 1-066	Copper Aluminum	5.85 - 8.20	4.285	10 8	1980	1.987-1.562 2.955-2.348	1.500x0.750	0.064
WR112	RG51/U RG68/U	1-069 1-072	Copper Aluminum	7.05 - 10.0	5.26	6 4.8	1280	2.776-2.154 4.173-3.238	1.250x0.625	0.064
WR90	RG52/U RG67/U	1-075 1-078	Copper Aluminum	8.2 - 12.4	6.56	3 2.4	760	4.238-2.995 6.506-4.502	1.000x0.500	0.05
WR75	RG346/U RG347/U	1-081 1-084	Copper Aluminum	10.0 - 15.0	7.847	2.8 2.2	620	5.121-3.577 7.698-5.377	0.850x0.475	0.05
WR62	RG91/U RG349/U	1-087 1-091	Copper Aluminum	12.4 - 18.0	9.49	1.8 1.4	460	6.451-4.743 9.700-7.131	0.702x0.391	0.04
WR51	RG352/U RG351/U	1-094 1-098	Copper Aluminum	15.0 - 22.0	11.54	1.2 1	310	8.812-6.384 13.250-9.598	0.590x0.335	0.04
WR42	RG53/U	1-100	Copper	18.0 - 26.5	14.08	0.8	170	13.80-10.13	0.500x0.250	0.04
WR34	RG354/U	1-107	Copper	2.0 - 33.0	17.28	0.6	140	16.86-11.73	0.420x0.250	0.04
WR28	RG271/U	3-007	Copper	26.5 - 40.0	21.1	0.5	100	23.02-15.77	0.360x0.220	0.04

i. Suppose that a wave in the form $E_x(y,z,t) = A\sin\left(\frac{\pi y}{b}\right)a(z,t)e^{j(\omega t - k_z(\omega)z)}$ propagates in the WR90 waveguide, whose specifications are given in Table 1. Let $\omega_0 = 2\pi\times10^{10}$ rad/s and assume that the envelope is a Gaussian pulse with "approximate" duration $\tau_0 = 1\,\mu$s at $z=0$. A is a constant, $b = 2a$, and the waveguide is empty (no material).
   a. Identify the waveguide mode given above. Is it the fundamental mode?
   b. Find the phase and group velocities of the wave and determine the phase and group delays when the wave propagates a distance $L = 1\,\text{m}$. Since there is no material medium in the waveguide, how can you explain the dispersion in this case?
   c. Find the change in pulse duration at $L = 1\,\text{m}$.

ii. Repeat i. for $\omega_0 = 2\pi\times6.6\times10^9$ rad/s. Comment on your results.

Remarks:

i. In Table 1, commercially available waveguide configurations are given. Note that the WR90 waveguide, <u>whose dimensions are given in inches</u>, is used in the X-band (8-12 GHz).

ii. Note that the term $\sin\left(\frac{\pi y}{b}\right)$ has no effect in the calculations.