HOMEWORK 3

ANIL AKSU

Problem 1

Compute the sectional curvature in Poincaré Plane.
Solution:
Poincaré metric g on a manifold M is given as:

$$g = \frac{\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y}{y^2}. \tag{1}$$

And also the sectional curvature is defined as:

$$K(u, v) = -\frac{< R(u, w)u, w >}{< u, u >< v, v > - < u, v >^2}. \tag{2}$$

where

$$R^{\alpha}_{\beta jk} = \frac{\partial \Gamma^{\alpha}_{\beta k}}{\partial x_j} - \frac{\partial \Gamma^{\alpha}_{\beta j}}{\partial x_k} + \Gamma^{\alpha}_{\gamma j}\Gamma^{\gamma}_{\beta k} - \Gamma^{\alpha}_{\gamma k}\Gamma^{\gamma}_{\beta j}.$$

Christoffel symbol of second kind $\Gamma^{\gamma}_{\beta j}$ can be computed using Euler-Lagrange equation. First, let's define the energy functional as:

$$\int_a^b F(\dot{\vec{x}}, \vec{x}, t)\mathrm{d}t = \int_a^b \dot{x}^i \dot{x}^j g_{ij} \mathrm{d}t. \tag{3}$$

The energy functional above has to satisfy Euler-Lagrange equations which are given as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x_i} = 0. \tag{4}$$

Equation 4 will result in the following set of differential equations:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\dot{x}^j \dot{x}^k = 0. \tag{5}$$

In Poincaré Plane, the energy functional is defined as:

$$F(\dot{\vec{x}}, \vec{x}, t) = \frac{\dot{x}^2 + \dot{y}^2}{y^2}. \tag{6}$$

Therefore, Euler-Lagrange equations give the following set of ordinary differential equations:

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0, \tag{7}$$

Date: April 13, 2017.

$$\ddot{y} - \frac{\dot{y}^2}{y} + \frac{\dot{x}^2}{y} = 0. \tag{8}$$

Therefore, Christoffel symbols are obtained as:

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \tag{9}$$

$$\Gamma^y_{yy} = -\frac{1}{y}, \tag{10}$$

$$\Gamma^y_{xx} = \frac{1}{y}. \tag{11}$$

By using Christoffel symbols derived above, let's write down the components of the curvature $R^\alpha_{\beta jk}$,

$$R^x_{xxx} = \frac{\partial \Gamma^x_{xx}}{\partial x} - \frac{\partial \Gamma^x_{xx}}{\partial x} + \Gamma^x_{\gamma x}\Gamma^\gamma_{xx} - \Gamma^x_{\gamma x}\Gamma^\gamma_{xx} = 0, \tag{12}$$

and

$$R^x_{xxy} = \frac{\partial \Gamma^x_{xy}}{\partial x} - \frac{\partial \Gamma^x_{xx}}{\partial y} + \Gamma^x_{\gamma x}\Gamma^\gamma_{xy} - \Gamma^x_{\gamma y}\Gamma^\gamma_{xx} = -R^x_{xyx} = 0, \tag{13}$$

and

$$R^x_{xyy} = \frac{\partial \Gamma^x_{xy}}{\partial y} - \frac{\partial \Gamma^x_{xy}}{\partial y} + \Gamma^x_{\gamma y}\Gamma^\gamma_{xy} - \Gamma^x_{\gamma y}\Gamma^\gamma_{xy} = 0, \tag{14}$$

and

$$R^x_{yxx} = \frac{\partial \Gamma^x_{yx}}{\partial x} - \frac{\partial \Gamma^x_{yx}}{\partial x} + \Gamma^x_{\gamma x}\Gamma^\gamma_{yx} - \Gamma^x_{\gamma x}\Gamma^\gamma_{yx} = 0, \tag{15}$$

and

$$R^x_{yxy} = \frac{\partial \Gamma^x_{yy}}{\partial x} - \frac{\partial \Gamma^x_{yx}}{\partial y} + \Gamma^x_{\gamma x}\Gamma^\gamma_{yy} - \Gamma^x_{\gamma y}\Gamma^\gamma_{yx} = -R^x_{yyx} = -\frac{1}{y^2}, \tag{16}$$

and

$$R^x_{yyy} = \frac{\partial \Gamma^x_{yy}}{\partial y} - \frac{\partial \Gamma^x_{yy}}{\partial y} + \Gamma^x_{\gamma y}\Gamma^\gamma_{yy} - \Gamma^x_{\gamma y}\Gamma^\gamma_{yy} = 0, \tag{17}$$

Furthermore,

$$R^y_{xxx} = \frac{\partial \Gamma^y_{xx}}{\partial x} - \frac{\partial \Gamma^y_{xx}}{\partial x} + \Gamma^y_{\gamma x}\Gamma^\gamma_{xx} - \Gamma^y_{\gamma x}\Gamma^\gamma_{xx} = 0, \tag{18}$$

and

$$R^y_{xxy} = \frac{\partial \Gamma^y_{xy}}{\partial x} - \frac{\partial \Gamma^y_{xx}}{\partial y} + \Gamma^y_{\gamma x}\Gamma^\gamma_{xy} - \Gamma^y_{\gamma y}\Gamma^\gamma_{xx} = -R^y_{xyx} = \frac{1}{y^2}, \tag{19}$$

and

$$R^y_{xyy} = \frac{\partial \Gamma^y_{xy}}{\partial y} - \frac{\partial \Gamma^y_{xy}}{\partial y} + \Gamma^y_{\gamma y}\Gamma^\gamma_{xy} - \Gamma^y_{\gamma y}\Gamma^\gamma_{xy} = 0, \tag{20}$$

and

$$R^y_{yxx} = \frac{\partial \Gamma^y_{yx}}{\partial x} - \frac{\partial \Gamma^y_{yx}}{\partial x} + \Gamma^y_{\gamma x}\Gamma^\gamma_{yx} - \Gamma^y_{\gamma x}\Gamma^\gamma_{yx} = 0, \tag{21}$$

and

$$R^y_{yxy} = \frac{\partial \Gamma^y_{yy}}{\partial x} - \frac{\partial \Gamma^y_{yx}}{\partial y} + \Gamma^y_{\gamma x}\Gamma^\gamma_{yy} - \Gamma^y_{\gamma y}\Gamma^\gamma_{yx} = -R^y_{yyx} = 0, \tag{22}$$

and

$$R^y_{yyy} = \frac{\partial \Gamma^y_{yy}}{\partial y} - \frac{\partial \Gamma^y_{yy}}{\partial y} + \Gamma^y_{\gamma y}\Gamma^\gamma_{yy} - \Gamma^y_{\gamma y}\Gamma^\gamma_{yy} = 0. \tag{23}$$

After expanding curvature $R^\alpha_{\beta jk}$, let's write down vectors u and v explicitly,

$$u = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}, \tag{24}$$

and

$$v = b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y}. \tag{25}$$

As a result,

$$R(u,v)u = (b_1 a_2 - a_1 b_2)\frac{a_2}{y^2}\frac{\partial}{\partial x} + (a_1 b_2 - b_1 a_2)\frac{a_1}{y^2}\frac{\partial}{\partial y} \tag{26}$$

Therefore,

$$< R(u,v)u, v >= \frac{(a_1 b_2 - b_1 a_2)^2}{y^4}. \tag{27}$$

and also

$$< u,u >< v,v > - < u,v >^2 = \frac{(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2}{y^4} = \frac{(a_1 b_2 - b_1 a_2)^2}{y^4} \tag{28}$$

Finally,

$$K(u,v) = -1. \tag{29}$$