EE 523: HOMEWORK 1

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Problem 1

Maxwell equations are given as[2]:

$$\nabla \cdot \mathbf{D} = \rho, \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{4}$$

Note that $\mathbf{D} = \epsilon \mathbf{E}$ where ϵ is the permittivity of free space, $\mathbf{H} = \mu \mathbf{B}$ where μ is permeability of free space and $J = \rho \mathbf{v}$ where $\mathbf{v}$ is the charge velocity. By using Maxwell equations given above, show that proper boundary conditions between two media with permittivities ϵ_1, ϵ_2, permeabilities μ_1, μ_2 separated with an interface with unit normal $\mathbf{a}_n$ can be given as[2, 3]:

$$D_{1n} - D_{2n} = \rho_s,$$
$$E_{1t} = E_{2t},$$
$$B_{1n} = B_{2n},$$
$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

where n and t subscripts stand for normal and tangential components of the vector field, s subscript stands for quantities at the surface.

Solution:

The volume integral of equation 1 covering the interface between two media can be given as:

$$\int_{V_1} \nabla \cdot \mathbf{D} \mathrm{d}V + \int_{V_2} \nabla \cdot \mathbf{D} \mathrm{d}V = \int_{V_1+V_2} \rho \mathrm{d}V. \tag{5}$$

As the volume V_1 and the volume V_2 shrinks to the interface[1], the volume integrals at the left hand-side of equation 5 can be given as surface integrals, but the right hand-side of equation 5 still stay as volume integral and converges to the surface charge density as follows:

$$\int_S (\mathbf{D}_1 - \mathbf{D}_2)\mathbf{a}_n \mathrm{d}S = \int_S \rho_s \mathrm{d}S. \tag{6}$$

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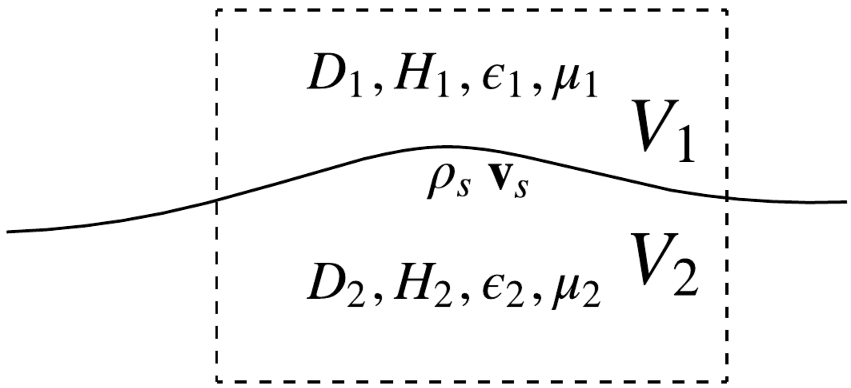

FIGURE 1. Jump surface separating media with permittivities ϵ_1, ϵ_2, permeabilities μ_1, μ_2.

Therefore, the product of $\mathbf{D}$ with the unit normal of the interface between two media results in the following boundary (interface) condition:

$$D_{1n} - D_{2n} = \rho_s. \tag{7}$$

Similarly, equation 2 can be integrated in a volume covering the interface as follows:

$$\int_{V_1} \nabla \times \mathbf{E} \mathrm{d}V + \int_{V_2} \nabla \times \mathbf{E} \mathrm{d}V = -\int_{V_1+V_2} \frac{\partial \mathbf{B}}{\partial t} \mathrm{d}V. \tag{8}$$

As the volume V_1 and the volume V_2 shrinks to the interface, the volume integrals at the left hand-side of equation 8 can be given as surface integrals, but the right hand-side of equation 8 still stay as volume integral:

$$\oint_{\partial V_1} \mathbf{a} \times \mathbf{E} \mathrm{d}S + \oint_{\partial V_2} \mathbf{a} \times \mathbf{E} \mathrm{d}S = -\int_{V_1+V_2} \frac{\partial \mathbf{B}}{\partial t} \mathrm{d}V. \tag{9}$$

The right hand-side of equation 9 becomes negligible as the volume shrinks to the interface, therefore the integral 9 reduces to the following relation:

$$\oint_{\partial V_1} \mathbf{a} \times \mathbf{E} \mathrm{d}S + \oint_{\partial V_2} \mathbf{a} \times \mathbf{E} \mathrm{d}S = 0 \tag{10}$$

Equivalently,

$$E_{1t} = E_{2t}, \tag{11}$$

The volume integral of equation 3 covering the interface between two media can be given as:

$$\int_{V_1} \nabla \cdot \mathbf{B} \mathrm{d}V + \int_{V_2} \nabla \cdot \mathbf{B} \mathrm{d}V = 0. \tag{12}$$

As the volume V_1 and the volume V_2 shrinks to the interface[1], the volume integrals at the left hand-side of equation 12 can be given as surface integrals:

$$\int_S (\mathbf{B}_1 - \mathbf{B}_2)\mathbf{a}_n \mathrm{d}S = 0. \tag{13}$$

Therefore, the product of $\mathbf{D}$ with the unit normal of the interface between two media results in the following boundary (interface) condition:

$$B_{1n} = B_{2n}. \tag{14}$$

Finally, equation 4 can be integrated in a volume covering the interface as follows:

$$\int_{V_1} \nabla \times \mathbf{H} \mathrm{d}V + \int_{V_2} \nabla \times \mathbf{H} \mathrm{d}V = \int_{V_1+V_2} \mathbf{J} \mathrm{d}V + \int_{V_1+V_2} \frac{\partial \mathbf{D}}{\partial t} \mathrm{d}V. \tag{15}$$

As the volume V_1 and the volume V_2 shrinks to the interface, the volume integrals at the left hand-side of equation 15 can be given as surface integrals, but the right hand-side of equation 15 still stay as volume integral:

$$\oint_{\partial V_1} \mathbf{a} \times \mathbf{H} \mathrm{d}S + \oint_{\partial V_2} \mathbf{a} \times \mathbf{H} \mathrm{d}S = \int_{V_1+V_2} \mathbf{J} \mathrm{d}V + \int_{V_1+V_2} \frac{\partial \mathbf{D}}{\partial t} \mathrm{d}V. \tag{16}$$

The right hand-side of equation 16 becomes negligible as the volume shrinks to the interface, therefore the integral 16 reduces to the following relation:

$$\oint_{\partial V_1} \mathbf{a} \times \mathbf{H} \mathrm{d}S + \oint_{\partial V_2} \mathbf{a} \times \mathbf{H} \mathrm{d}S = \int_{V_1+V_2} \mathbf{J} \mathrm{d}V. \tag{17}$$

As a result,

$$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

References

[1] *Continuum Mechanics: Concise Theory and Problems.* Dover, 1999.
[2] *Fundamentals of Engineering Electromagnetics.* Addison-Wesley, 2007.
[3] J S Hesthaven and T Warburton. High-order / Spectral Methods on Unstructured Grids I . Time-domain Solution of Maxwell ' s Equations Form. 2001.