Decision Theory and Bayesian Analysis

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LECTURE 1
Bayesian Paradigm

1.1. Bayes theorem for distributions

If A and B are two events,

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}. \tag{1.1}$$

This is just a direct consequence of the multiplication law of probabilities that says we can express $P(A \mid B)$ as either $P(A)P(B \mid A)$ or $P(B)P(A \mid B)$. For discrete distributions, if Z, Y are discrete random variables

$$P(Z = z \mid Y = y) = \frac{P(Z = z)P(Y = y \mid Z = z)}{P(Y = y)}. \tag{1.2}$$

- How many distributions do we deal with here?

We can express the denominator in terms of the distribution in the numerator[**1**].

$$P(Y = y) = \sum_z P(Y = y, Z = z) = \sum_z P(Z = z)P(Y = y \mid Z = z). \tag{1.3}$$

- This is sometimes called the law of total probability

In this context, it is just an expression of the fact that as z ranges over the possible values of Z , the probabilities on the left hand-side of equation 1.2 make up the distribution of Z given $Y = y$, and so they must add up to one. The extension to continuous distribution is easy. If Z, Y are continuous random variable,

$$f(Z \mid Y) = \frac{f(Z)f(Y \mid Z)}{f(Y)}. \tag{1.4}$$

where the denominator is now expressed as an integral:

$$f(Y) = \int f(Z)f(Y \mid Z)\mathrm{d}Z. \tag{1.5}$$

$$f = \begin{cases} \textit{continous} \quad \textit{name}? \\ \textit{discrete} \quad \textit{name}? \end{cases} \tag{1.6}$$

1.2. How Bayesian Statistics Uses Bayes Theorem

Theorem 1.7 (Bayes' theorem)**.**

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$$

*$P(B)$=if we are interested in the event B, $P(B)$ is the initial or prior probability of the occurence of event B. Then we observe event A
$P(B \mid A)$ = How likely B is when A is known to have occurred is the posterior probability $P(B \mid A)$.*

Bayes' theorem can be understood as a formula for updating from prior to posterior probability, the updating consists of multiplying by the ratio $P(B \mid A)/P(A)$. It describes how a probability changes as we learn new information. Observing the occurrence of A will increase the probability of B if $P(B \mid A) > P(A)$. From the law of total probability,

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c). \tag{1.8}$$

where $P(B^c) = 1 - P(B)$.

Lemma 1.9.

$$P(A \mid B) - P(A) = \frac{P(A) - P(A \mid B^c)P(B^c)}{1 - P(B^c)} - P(A)$$

Proof.

$$P(A \mid B) - P(A) = \frac{P(A) - P(A \mid B^c)P(B^c) - P(A) + P(A)P(B^c)}{P(B)}$$

$$P(A \mid B) - P(A) = \frac{P(B^c)(P(A) - P(A \mid B^c))}{P(B)}$$

$$P(A \mid B) - P(A) = P(B^c)(\frac{P(B)P(A \mid B) + P(B^c)P(A \mid B^c)}{P(B)} - \frac{P(A \mid B^c)}{P(B)})$$

$$P(A \mid B) - P(A) = P(B^c)(P(A \mid B) - \frac{P(A \mid B^c)(1 - P(B^c))}{P(B)})$$

$$P(A \mid B) - P(A) = P(B^c)(P(A \mid B) - P(A \mid B^c))$$

□

1.2.1. Generalization of the Bayes' Theorem

Let $B_1, ..., B_n$ be a set of mutually exclusive events. Then

$$P(B_r \mid A) = \frac{P(B_r)P(A \mid B_r)}{P(A)} = \frac{P(B_r)P(A \mid B_r)}{\sum_{i=1}^{n} P(B_r)P(A \mid B_r)}. \tag{1.10}$$

- Assuming that $P(B_r) > 0$,$P(A \mid B) > P(A)$ if and only if $P(A \mid B) > P(A \mid B^c)$.

- In Bayesian inference we use Bayes' theorem in a particular way.
- Z is the parameter (vector) θ.
- Y is the data (vector) X.

So we have

$$f(\theta \mid X) = \frac{f(\theta) f(X \mid \theta)}{f(X)} \tag{1.11}$$

$$f(X) = \int f(\theta) f(X \mid \theta) \mathrm{d}\theta. \tag{1.12}$$

$$f(\theta) = \textit{prior}. \tag{1.13}$$

$$f(\theta \mid X) = \textit{posterior}. \tag{1.14}$$

$$f(X \mid \theta) = \textit{likelihood}. \tag{1.15}$$

1.2.2. Interpreting our sense

How do we interpret the things we see, hear, feel, taste or smell?

Example 1.2.1. I hear a song on the radio I identify the singer as Robbie Williams. Why do I think it's Robbie Williams?. Because he sounds like that. Formally, $P($ What I hear Robbie Williams $) >> P($What I hear someone else $)$

Example 1.2.2. I look out of the window and see what appears to be a tree. It has a big, dark coloured part sticking up out of the ground that branches into thinner sticks and on the ends of these are small green things. Clearly, $P(\textit{view} \mid \textit{tree})$ is high and $P(\textit{view} \mid \textit{car})$ or $P(\textit{view} \mid \textit{Robbie} \quad \textit{Williams})$ are very small. But $P(\textit{view} \mid \textit{carboard} \quad \textit{cutout} \quad \textit{cunningly} \quad \textit{painted} \quad \textit{to} \quad \textit{look} \quad \textit{like} \quad \textit{a} \quad \textit{tree})$ is also very high. Maybe even higher than $P(\textit{view} \mid \textit{tree})$ in the sense that what I see looks almost like a tree.
Does this mean I should now believe that I am seeing a cardboard cutout cunningly painted to look like a tree? No because it is much less likely to begin with than a red tree.

In statistical terms, consider some data X and some unknown parameter θ. The first step in any statistical analysis is to build a model that links the data to unknown parameters and the main function of this model is to allow us to state the probability of observing any data given any specified values of the parameters. That is the model defines $f(x \mid \theta)$.

When we think of $f(x \mid \theta)$ as a function of θ for fixed observed data X, we call it likelihood function and it by $L(\theta, X)$.

- So how can we combine this with our example?

This perspective underlies the differences between the two main theories of statistical inference.

- Frequentist inference essentially uses only the likelihood, it does not recognize $f(\theta)$.
- Bayesian inference uses both likelihood and $f(\theta)$.

The principal distinguishing feature of Bayesian inference as opposed to frequentist inference is its use of $f(\theta)$.

1.3. Prior to Posterior

We refer to $f(\theta)$ as the prior distribution of θ. It represents knowledge about θ prior to observing the data X. We refer to $f(\theta \mid X)$ as the posterior distribution of θ and it represents knowledge about θ after observing X.

- So we have two sources of information about θ.
- Here $f(x)$ does not depend on θ. Thus $\int f(\theta \mid x)\mathrm{d}\theta = 1$. Since $f(x)$ is a constant within the integral, we can take it outside to get $1 = f^{-1}(x) \int f(\theta) f(x \mid \theta)\mathrm{d}\theta$.
- $f(\theta \mid x) =\propto f(\theta) f(x \mid \theta) \propto f(\theta) L(\theta; x)$ (the posterior is proportional to the prior times the likelihood).
- The constant that we require to scale the right hand side to integrate to 1 is usually called the normalizing constant. If we haven't dropped any constants form $f(\theta)$ or $f(x \mid \theta)$, then the normalising constant is just $f^{-1}(x)$, otherwise it also restores any dropped constants.

1.4. Triplot

If for any value of θ, we have either $f(\theta) = 0$ or $f(x \mid \theta) = 0$, then we will also have $f(\theta \mid x) = 0$. This is called the property of zero preservation. So if either:

- the prior information says that this θ value is impossible
- the data say that this value of θ is impossible because if it were the true value, then the observed data would have been impossible, then the posterior distribution confirms that this value of θ is impossible.

Definition 1.16. Crowwell's Rule: If either information source completely rules out a specific θ, then the posterior must rule it out too.

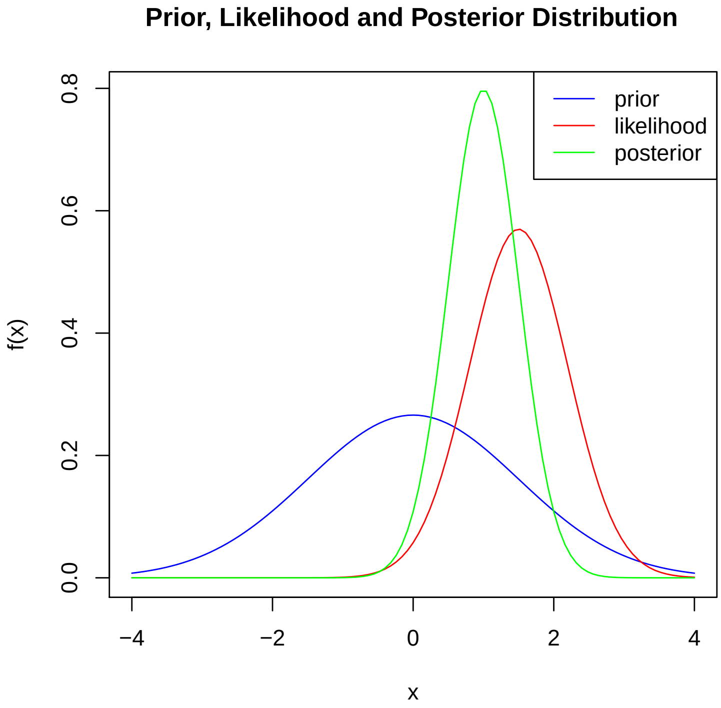


Figure 1. Triplot of prior, likelihood and posterior.

This means that we should be very careful about giving zero probability to something unless it is genuinely impossible. Once something has zero probability then no amount of further evidence can cause it to have a non-zero posterior probability.

- More generally, $f(\theta \mid x)$ will be low if either $f(\theta)$ is very small. We will tend to find that $f(x \mid \theta)$ is large when both $f(\theta)$ and $f(x \mid \theta)$ are relatively large, so that this θ value is given support by both information sources.

When θ is a scalar parameter, a useful diagram is the triplot, which shows the prior, likelihood and posterior on the same graph. An example is in Figure 1.[1]

A strong information source in the triplot is indicated by a curve that is narrow (and therefore, because it integrates to one, also has a high peak). A narrow curves concentrates on a small range of θ values, and thereby "rules out" all values of θ outside that range.

[1] All plots are generated in R, relevant codes are provided in Appendix R Codes

- Over the range $\theta < -1$, the likelihood: low
- Over the range $\theta > 3$,the likelihood: low
- Values of θ between -1 and 3, the likelihood: high
- The maximum value of the posterior at: 1
- The Maximum likelihood estimation (MLE) of θ is ≈ 2

1.4.1. Normal Mean

For example, suppose that $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\mu,\, \sigma^2)$ and σ^2 is known. Then the likelihood is :

$$\begin{aligned} f(x \mid \mu) = \prod_{i=1}^{n} f(x_i \mid \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right) \\ \propto \exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right). \end{aligned} \tag{1.17}$$

As,

$$\begin{aligned} \sum(x_i - \bar{x} + \bar{x} - \mu)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2 + 2(\bar{x}-\mu)\sum(x_i-\bar{x}) \\ = \sum(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2 \\ \propto \exp\left(-\frac{1}{2\sigma^2}n(\bar{x}-\mu)^2\right). \end{aligned} \tag{1.18}$$

Note that $2(\bar{x}-\mu)\sum(x_i-\bar{x}) = 0$ as $\sum(x_i-\bar{x}) = 0$. Suppose the prior distribution for μ is normal:

$$\mu \sim \mathcal{N}(m, v). \tag{1.19}$$

Then applying Bayes' theorem we have:

$$\begin{aligned} f(\mu \mid x) \propto \underbrace{\exp\left(-\frac{1}{2\sigma^2}n(\bar{x}-\mu)^2\right)}_{f(x|\mu)} \underbrace{\exp\left(-\frac{1}{2v}n(\mu-m)^2\right)}_{f(\mu)} \\ = \exp\left(-\frac{\theta}{2}\right). \end{aligned} \tag{1.20}$$

Note that

$$\theta = n\sigma^{-2}(\bar{x}-\mu) + v^{-1}(\mu-m)^2 = (v^*)^{-1}(\mu-m^*)^2 + R \tag{1.21}$$

and

$$v^* = (n\sigma^{-2} + v^{-1})^{-1} \tag{1.22}$$

$$m^* = (n\sigma^{-2} + v^{-1})^{-1}(n\sigma^{-2}\bar{x} + v^{-1}m) = a\bar{x} + (1-a)m \tag{1.23}$$

where $a = n\sigma^{-}2/(n\sigma^{-2} + v^{-1})$

$$R = (n^{-1}\sigma^2 + v)(\bar{x} - m)^2 \tag{1.24}$$

Therefore,

$$f(\mu \mid x) \propto \exp\left(-\frac{1}{2v}n(\mu - m)^2\right) \tag{1.25}$$

and we have shown that the posterior distribution is normal too: $\mu \mid x \sim \mathcal{N}(m^*, v^*)$

- m^* = weighted average of the mean m and the usual frequentist data-only estimate $\bar{x}$.
  The weights $\propto$:
- Bayes' theorem typically works in this way. We usually find that posterior estimates are compromises between prior estimates and data based estimates and tend to be closer whichever information source is stronger. And we usually find that the posterior variance is smaller than the prior variance.

1.4.2. Weak Prior Information

It is the case where the prior information is much weaker that the data. This will occur, for instance, if we do not have strong information about Q before seeing the data, and if there are lots of data. Then in triplot, the prior distribution will be much broader and flatter that the likelihood. So the posterior is approximately proportional to the likelihood.

Example 1.4.1. In the normal mean analysis, we get weak prior information by letting the prior precision of v^{-1} become small. Then $m^* \to \bar{x}$ and $v^* \to \sigma^2/n$ so that the posterior distribution of μ corresponds very closely with standard frequentist theory.

LECTURE 2
Some Common Probability Distributions

Binomial on $Y \in \{0, 1, ..., n\}$ with parameters $n \in \{1, 2, 3, ...\}$ and $p \in (0, 1)$ is denoted by $Bi(n, p)$ and

$$f(y \mid n, p) = \binom{n}{y} p^y (1-p)^{n-y} \tag{2.1}$$

for $y = 0, 1, ..., n$. The mean is given as:

$$E(y) = np. \tag{2.2}$$

Also the variance is given as:

$$v(y) = np(1-p). \tag{2.3}$$

Beta on $Y \in \{0, 1\}$ with parameters $a, b > 0$ is denoted by $Beta(p, q)$ and the density function is:

$$f(y \mid p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1} \tag{2.4}$$

for $y \in (0, 1)$. The mean is given as:

$$E(y) = \frac{p}{p+q}, \tag{2.5}$$

Also the variance is given as:

$$v(y) = \frac{pq}{(p+q)^2(p+q+1)}. \tag{2.6}$$

$B(p, q) = \int_0^1 y^{p-1}(1-y)^{q-1}\mathrm{d}y$ is the beta function and defined to be the normalizing constant for this density.

- In beta distribution, p and q change the shape of the distribution. Discuss!

Uniform distribution on $Y \in \{l, r\}$ where $-\infty < l < r < \infty$ is denoted by uniform (l, r) and its pdf is:

$$f(y \mid l, r) = \frac{1}{r - l} \tag{2.7}$$

for $y \in \{l, r\}$. The mean is given as:

$$E(y) = \frac{l + r}{2}, \tag{2.8}$$

Also the variance is given as:

$$v(y) = \frac{(r - l)^2}{12}. \tag{2.9}$$

Poisson distribution on $Y \in \{0, 1, 2, ..., \}$ with parameter $\theta > 0$ is denoted by $Poisson(\theta)$ and its pdf is:

$$f(y \mid \theta) = \frac{\exp(-\theta)\theta^y}{y!} \tag{2.10}$$

for $y = 0, 1, 2, ...$ The mean and the variance are given as[**2**]:

$$E(y) = v(y) = \theta. \tag{2.11}$$

Gamma distribution on $Y > 0$ with shape parameter $\alpha > 0$ and rate parameter $\lambda > 0$ is denoted by $Gamma(\alpha, \lambda)$ and the corresponding density is:

$$f(y \mid \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha - 1} \exp(-\lambda y) \tag{2.12}$$

for $y > 0$. The mean is given as:

$$E(y) = \frac{\alpha}{\lambda}, \tag{2.13}$$

Also the variance is given as:

$$v(y) = \frac{\alpha}{\lambda^2}. \tag{2.14}$$

Note that

$$\exp(\lambda) = Gamma(1, \lambda). \tag{2.15}$$

Univariate normal distribution on $Y \in \mathbb{R}$ with $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is denoted by $\mathcal{N}(\mu, \sigma^2)$ and its pdf is:

$$f(y \mid \mu, \sigma^2) = \frac{1}{\sigma}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(y - \mu)^2\right\}. \tag{2.16}$$

The mean is given as:

$$E(y) = \mu, \tag{2.17}$$

Also the variance is given as:

$$v(y) = \sigma^2. \tag{2.18}$$

K-variate normal distribution on $Y \in \mathbb{R}^k$ with vector $\mathbf{b} \in \mathbb{R}^k$ and positive definite symmetric (PDS) covariance matrix $\mathbf{C}$ is denoted by $\mathcal{N}_k(\mathbf{b}, \mathbf{C})$ and the corresponding density function is:

$$f(y \mid \mathbf{b}, \mathbf{C}) = \frac{1}{\underbrace{|\mathbf{C}|^{1/2}}_{determinant}} \frac{1}{(2\pi)^{k/2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mathbf{b})^T\mathbf{C}^{-1}(y-\mathbf{b})\right\}. \tag{2.19}$$

The mean is given as:

$$E(y) = \mathbf{b}, \tag{2.20}$$

And the covariance matrix is given as:

$$Cov(y) = \mathbf{C}. \tag{2.21}$$

2.1. Posterior

Not only the beta distributions are the simplest and the most convenient distributions for a random variable confined to $[0,1]$, they also work very nicely as prior distribution for a binomial observation. If $\theta \sim Be(p,q)$ then

$$f(x \mid \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}. \tag{2.22}$$

for $x = 1, 2, ..., n$.

$$f(\theta) = \frac{1}{Be(p,q)}\theta^{p-1}(1-\theta)^{q-1} \tag{2.23}$$

where $0 \leq \theta \leq 1$ and $p, q > 0$.

$$\begin{aligned} f(x) = \int f(\theta)f(x \mid \theta)\mathrm{d}\theta = \binom{n}{r}\frac{1}{Be(p,q)}\int_0^1 \theta^{p+x-1}(1-\theta)^{q+n-x-1}\mathrm{d}\theta \\ = \binom{n}{r}\frac{Be(p+x, q+n-x)}{Be(p,q)}. \end{aligned} \tag{2.24}$$

From

$$f(\theta \mid x) = \frac{f(\theta)f(x \mid \theta)}{f(x)}. \tag{2.25}$$

(2.26)

$$f(\theta \mid x) = \frac{\theta^{p+x-1}(1-\theta)^{q+n-x-1}}{Be(p+x, q+n-x)} \propto \underbrace{\theta^{p-1}(1-\theta)^{q-1}}_{Beta \quad part}\underbrace{\theta^{x}(1-\theta)^{n-x}}_{Binomial \quad part}.$$

So $(\theta \mid x) \propto Beta(p+x, q+n-x)$. The posterior mean is:

$$E(\theta \mid x) = \frac{p+x}{p+q+n} = \frac{p+q}{p+q+n}E(\theta) + \frac{n}{p+q+n}\hat{\theta} \tag{2.27}$$

where $\hat{\theta} = x/n$. The posterior variance is:

$$\begin{aligned} v(\theta \mid x) &= \frac{(p+x)(q+n-x)}{(p+q+n)^2(p+q+n+1)} \\ &= \frac{E(\theta)(1-E(\theta \mid x))}{p+q+n+1} \end{aligned} \tag{2.28}$$

But,

$$v(\theta) = \frac{E(\theta)(1-\theta)}{p+q+1} \tag{2.29}$$

So the posterior has higher relative precision that the prior.

<u>SPECIAL NOTE:</u>

The classical theory of estimation regards an estimator as good if it is unbiased and has small variance, or more generally if its mean-square-error is small. The MSE is an average squared error where the error is the difference between θ, i.e. y in previous notation , and the estimate t. In accordance with classical theory, the average is taken with respect to the sampling distribution of the estimator.

In Bayesian inferenc, θ is a random variable and it is therefore appropriate to average the squared error with respect to the posterior distribution of θ. Consider

$$\begin{aligned} E\left\{(t-\theta)^2 \mid x\right\} &= E(t^2 \mid x) - E(2t\theta \mid x) + E(\theta^2 \mid x) \\ &= t^2 - E(2t\theta \mid x) + E(\theta^2 \mid x) \\ &= \{t - E(\theta \mid x)\}^2 + v(\theta \mid x). \end{aligned} \tag{2.30}$$

Therefore the estimate t which minimizes posterior expected square error is $t = E(\theta \mid x)$, the posterior mean. The posterior mean can therefore be seen as an estimate of θ which is the best in the sense of minimizing expected squared error. This is distinct from, but clearly related to, its more natural role as a useful summary of location of the posterior distribution.

2.2. Weak Prior

If we reduce the prior relative precision to zero by setting $p = q = 0$, we get $\theta \mid x \sim Be(x, n - x)$. Then $E(\theta \mid x) = \hat{\theta}$ and $v(\theta \mid x) = \hat{\theta}(1-\hat{\theta})/(n+1)$ results which nicely parallel standard frequentist theory.

- Notice that we are not really allowed to let either parameter of the beta distribution be zero. However, by making p and q extremely small, we get as close to these results as we like. We can think of $p = q = 0$ as a defining limiting (if strictly improper) case of weak prior information.

Example 2.2.1. A doctor proposes a new treatment protocol for a certain kind of cancer. With current methods about 40% of patients with this cancer survive six months after diagnosis. After one year of using the new protocol, 15 patients with diagnosis, of whom 6 survived. After two years a further 55 patients have been followed to teh six months mark, of whom 28 survived. So in total we have 34 patients surviving out of 70.

Let θ be the true success rate of the new treatment protocol, i.e. the true proportion of patients who survive 6 months and we wish to make comparison of θ with the current survival rate 40%.

Suppose that the doctor in charge has prior information leading her to assign a prior distribution with expectation $E(\theta) = 0.45$, i.e. expects a slight improvement over the existing protocol, form 40% to 45%, however her prior standard deviation is 0.07, $v(\theta) = 0.07^2 = 0.0049$

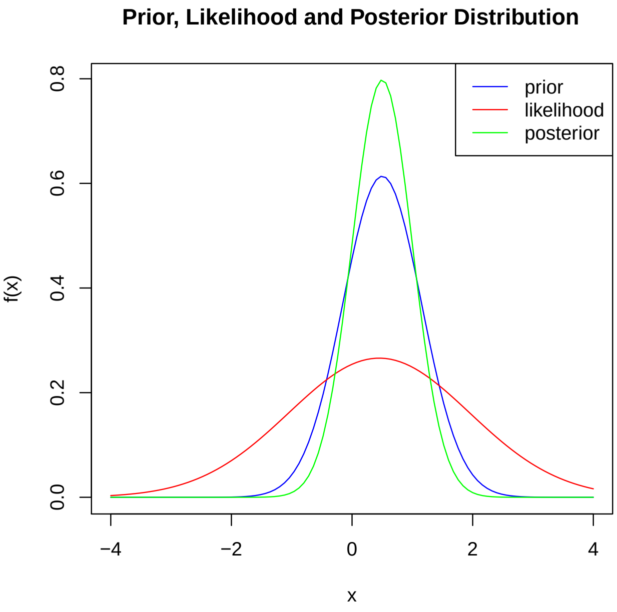


Figure 1. The triplot from first year's data.

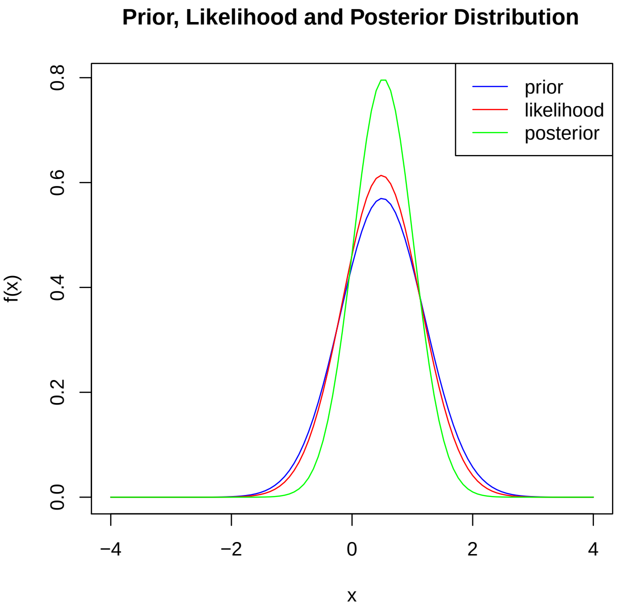


Figure 2. The triplot from two years' data.

2.3. Sequential Updating

In the last example we pooled the data from the two years and went back to the original prior distribution to use Bayes' theorem. We did not need to do this. A nice feature of Bayes' theorem is the possibility of updating sequentially, incorporating data as they arrive. In this case, consider the data to be just the new patients observed to a six months follow-up during the second year. These comprise 55 patients, of whom 28 had survived. The doctor could consider these as the data x with $n = 55$ and $r = 28$. What would the prior information be?
Clearly, the prior distribution should express her information prior to obtaining these new data, i.e. after the first years' data, so her prior for this second analysis is her <u>posterior</u> distribution form the first. This was $Be(28.28, 36.23)$. Combining this prior with the new data gives the same posterior $Be(28.28+28, 36.23+27) = Be(56.28, 63.23)$ as before. This simply confirms that we can get to the posterior distribution.

- In a single step, combining all the data with a prior distribution representing information available before any of the data were obtained.
- Sequentially, combining each item or block of new data with a prior distribution representing information available just before the new data were obtained (but after getting data previously received).

2.4. Normal Sample

Let $X_1, X_2, ..., X_n$ be from $\mathcal{N}(\mu, \sigma^2)$. $\theta = (\mu, \sigma^2) \rightarrow$ unknown parameters. The likelihood is:

$$\begin{aligned} f(x \mid \mu, \sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\} \\ &\propto \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2}\left\{n(\bar{x}-\mu)^2 + S^2\right\}\right] \end{aligned} \tag{2.31}$$

where $S^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$

2.5. NIC distributions

For the prior distribution, we now need a <u>joint</u> distribution for μ and σ^2

Definition 2.32. The normal-inverse-chi-squared distribution(NIC) has density:

$$f(x \mid \mu, \sigma^2) \propto \sigma^{-(d+3)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{v^{-1}(\mu - m)^2 + a\right\}\right] \tag{2.33}$$

where $a > 0$, $d > 0$ and $v > 0$.

The following facts are easy to derive about $NIC(m, v, a, d)$ distribution.

(a) The conditional distribution of μ given σ^2 is $\mathcal{N}(\mu, v\sigma^2)$ so $E(\mu \mid \sigma^2) = m$, $v(\mu \mid \sigma^2) = v\sigma^2$.

(b) The marginal distribution of σ^2 is such that $a\sigma^{-2} \sim \chi^2_d$. We say that σ^2 has the inverse-chi-square distribution $IC(a, d)$. We have $E(\sigma^2) = a/(d{-}2)$ if $d > 2$ and $v(\sigma^2) = 2a^2/\left\{(d-2)^2(d-4)\right\}$ if $d > 4$.

(c) The conditional distribution of σ^2 given μ is $IC(v^{-1}(\mu - m)^2 + a, d{+}1)$ and in particular $E(\sigma^2 \mid \mu) = (v^{-1}(\mu - m)^2 + a)/(d - 1)$ provided $d > 1$.

(d) The marginal distribution of μ is such that $(\mu - m)\sqrt{d}/\sqrt{av}\mu + d$. We say that μ has t-distribution $t_d(m, av/d)$. We have $E(\mu) = m$ if $d > 1$, and $v(\mu) = av/(d-2)$ if $d > 2$.

2.6. Posterior

Supposing then that the prior distribution is $NIC(m, v, a, d)$, we find

$$f(\mu, \sigma^2 \mid x) \propto \sigma^{d+n+3} \exp\left[-\frac{1}{2\sigma^2}\theta\right] \tag{2.34}$$

where $\theta = v^{-1}(\mu - m)^2 + a + n(\bar{x} - \mu) + s^2$ is a quadratic expression in μ. After completing the square, we see that $\mu, \sigma^2 \mid x \propto NIC(m^*, v^*, a^*, d^*)$ where $m^* = (v^{-1}m + n\bar{x})/(v^{-1} + n)$, $v^* = (v^{-1} + n)^{-1}$, $a^* = a + S^2 + (\bar{x} - m)^2/(n^{-1} + v)$, $d^* = d + n$. To interpret these results, note first that the posterior mean of μ is m^* which is a weighted average of the prior mean m and the usual data only-estimate $\bar{x}$ with weights v^{-1} and n.

The posterior mean of σ^2 is $a^*/(d^* - 2)$ which is a weighted average of three terms: the prior mean $a/(d-2)$ with weight $(d-2)$, the usual data-only estimate $S^2/(n-1)$ with weight $(n-1)$ and $(\bar{x} - m)/(n^{-1} + v)$ with weight 1.

2.7. Weak prior

We clearly obtain weak prior information about μ by letting v go to infinity or $v^{-1} \to 0$. Then $m^* = \bar{x}$, $v^* = 1/n$, $a^* = a + S^2$, because the third term disappears.

To obtain weak prior information also about σ^2, if it is usual to set $a = 0$ and $d = 1$. Then $a^* = S^2$ and $d^* = n - 1$. The resulting inference match the standard frequentist results very closely with these parameters, since we have:

$$\frac{(\mu - \bar{x})\sqrt{n}}{S/\sqrt{n-1}} \propto t_{n-1}, \tag{2.35}$$

$$\frac{S^2}{\sigma^2} \propto \chi^2_{n-1} \tag{2.36}$$

Exactly the same distribution statements underlie standard frequentist inference in this problem.

LECTURE 3
Inference

Summarisation: Here all inferences are derived form the posterior distribution. In frequentist statistics, there are three kinds of inference:

- Point estimation: unbiasedness, minimum variance estimation
- Interval estimation: Credible intervals in Bayesian inference, confidence interval in frequentist approach
- Hypothesis testing: Signifcance test.

In Bayesian inference, the posterior distribution expresses all that is known about θ. So it uses an appropriate summaries of the posterior distribution to describe the main features of what we now know about θ.

Plots: To draw the posterior density

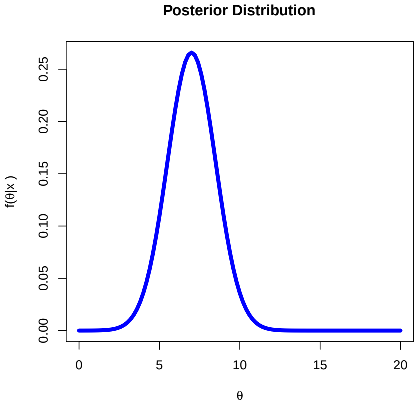
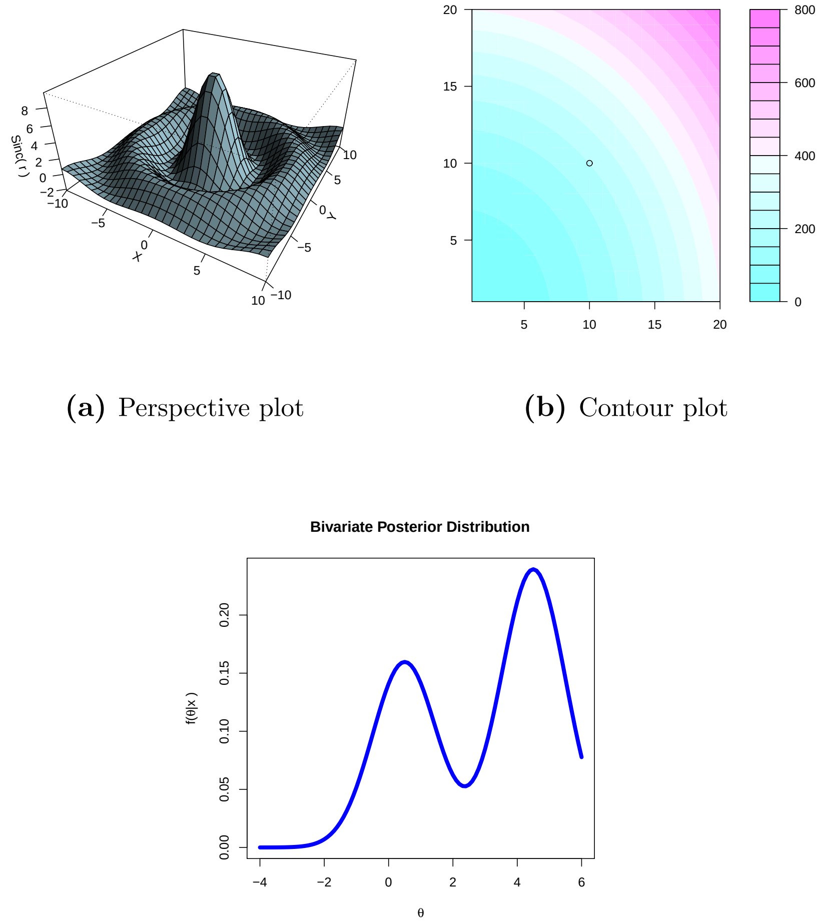


Figure 1. A posterior density plot.

For a bivariate parameter, we can still usefully draw the density as a perspective plot or a contour plot.



(a) Perspective plot

(b) Contour plot

Figure 3. Marginal Densities.

3.1. Shape

In general, plots illustrate the shape of the posterior distribution. Important features of shape are modes (and antimodes), skewness and kurtosis (peakedness or heavy tails). The quantitative summaries of shape are needed to supplement like the view of mode (antimode).
The first task is to identify turning points of the density, i.e. solutions of $f'(\theta) = 0$. Such points include local maxima and minima of $f(\theta)$ which we call mode and antimode, respectively.

A point θ_0 is characterized as a mode if $f'(\theta_0) = 0$ and $f''(\theta_0) < 0$ whereas it is an antimode if $f'(\theta_0) = 0$ and $f''(\theta_0) > 0$. Any point θ_0 for which $f''(\theta_0) = 0$ is a point of inflection of the density (whether or not $f'(\theta_0) = 0$).

Example 3.1.1. Consider the gamma density

$$f(\theta) = \frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta} \qquad ; \qquad \theta > 0 \tag{3.1}$$

where a, b are positive constants.

$$f'(\theta) = \frac{a^b}{\Gamma(b)}\left\{(b-1) - a\theta\right\}\theta^{b-2}e^{-a\theta} \tag{3.2}$$

$$f''(\theta) = \frac{a^b}{\Gamma(b)}\left\{a^2\theta^2 - 2a(b-1)\theta - (b-1)(b-2)\right\}\theta^{b-3}e^{-a\theta} \tag{3.3}$$

So from $f'(\theta)$, the turning point at $\theta = (b-1)/a$. For $b \leq 1$, $f'(\theta) < 0$ for all $\theta \geq 0$, so $f(\theta)$ is monotonic decreasing and the mode is at $\theta = 0$. For $\theta > 1$, $f(\theta) \to 0$ as $\theta \to 0$, so the turning point $\theta = 0$ is not a mode. In this case, $f'(\theta) > 0$ for $\theta < (b-1)/a$ and $f'(\theta) < 0$ for $\theta > (b-1)/a$. Therefore $\theta = (b-1)/a$ is the mode. Looking at $f''(\theta)$, the quadratic expression has roots at $\theta = \frac{b-1}{a} \mp \frac{(b-1)^{1/2}}{a}$. Therefore $b > 1$, these are the points of inflection.

Example 3.1.2. Consider the mixture of two normal distributions

$$f(\theta) = \underbrace{\frac{0.8}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\theta^2\right\}}_{\sim\mathcal{N} \quad with \quad weight} + \underbrace{\frac{0.2}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(\theta-4)^2\right\}}_{\sim\mathcal{N} \quad with \quad weight} \tag{3.4}$$

$$f'(\theta) = -\frac{0.8\theta}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\theta^2\right\} - \frac{0.2(\theta-4)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(\theta-4)^2\right\} \tag{3.5}$$

For $\theta \leq 0$, $f'(\theta) > 0$ and for $\theta \geq 0$, $f'(\theta) < 0$, the turning points at $\theta = 0.00034$, 2.46498 and 3.9945.

(3.6)

$$f''(\theta) = -\frac{0.8(\theta^2-1)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\theta^2\right\} - \frac{0.2((\theta^2-8\theta-15)}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(\theta-4)^2\right\}$$

This is positive for $\theta \leq -1$, for $1 \leq \theta \leq 3$ and $\theta \geq 5$, confirming that the middle turning point is an antimode. Calculating $f''(\theta)$ at the other points confirms them to be modes. Finally points of inflection are at $\theta = -0.99998$, $\theta = 0.98254$, $\theta = 3.17903$, $\theta = 4.99971$.

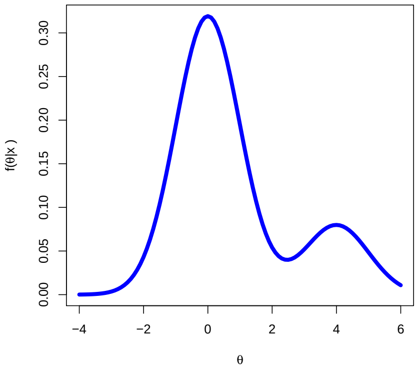


Figure 4. Plot of mixture of two normal distributions

3.2. Visualizing multivariate densities

Turning points:

At a mode, or at a turning pons generally, the gradients of function in all directions are zero. Therefore, the turning points are solutions of the simultaneous equations:

$$0 = \frac{\partial f(\theta,\phi)}{\partial \phi} = \frac{\partial f(\theta,\phi)}{\partial \theta}. \tag{3.7}$$

The turning points may be classified by examining the symmetric matrix $F''(\theta,\phi)$ of the second order partial derivatives. For instance, for a bivariate density $f(\theta,\phi)$,

$$F''(\theta,\phi) = \begin{bmatrix} \frac{\partial^2 f(\theta,\phi)}{\partial \theta^2} & \frac{\partial^2 f(\theta,\phi)}{\partial \theta \partial \phi} \\ \frac{\partial^2 f(\theta,\phi)}{\partial \phi \partial \theta} & \frac{\partial^2 f(\theta,\phi)}{\partial \phi^2} \end{bmatrix} \tag{3.8}$$

$F''(\theta,\phi)$ is known as the Hessian matrix. The second derivative of $f(\theta)$,m a direction t is after differentiating. At a mode, this must be negative in all directions. So that the Hessian matrix is negative definite. Similarly, at an antimode, it is positive definite. In the intermediate case, where $F''(\theta,\phi)$ is indefinite, i.e. has both positive and negative eigenvalues, we have a saddle point.

If $F''(\theta,\phi)$ is positive definite, regions are:

$F''(\theta,\phi)$ are negative definite, regions are:

$F''(\theta,\phi)$ is indefinite, indefinite curvature. On the boundaries between these regions, one eigenvalues of $F''(\theta,\phi)$ is zero, so all points on such boundaries are inflection points.

A point of inflection corresponds to the second derivative being zero in some direction t, therefore inflection points are characterized by $F''(\theta, \phi)$ being singular. In fact, in more than two dimensions, we can further subdivide the regions of indefinite curvature according to how many positive eigenvalues of $F''(\theta, \phi)$ has, and all these subregions are also separated by inflection boundaries.

Location:

A plot gives a good idea of location, but the conventional location measures for distributions are also useful. These include the mean, mode and median.

Dispersion:

The usual dispersion measure is the variance, or for a multivariate distribution the variance-covariance matrix.

Dependence:

It is important with multivariate distributions to summarize the dependence between individual parameters. This can be done with correlation coefficients, but plots of regression functions (conditional mean functions) can be more informative.

3.3. Informal Inferences

(a) Point estimation: The obvious posterior estimate of θ is its posterior mean $\hat{\theta} = E(\theta \mid x)$. Modes and medians are also natural point estimates, and they all have intuitively different interpretations. The mean is the expected value, the median is the central value and the mode is the most probable value.

(b) Interval estimation: If asked to provide an interval in which θ probably lies, we can readily derive such a thing from its posterior distribution. For instance, in the density shown on page 1, there is a probability 0.05 to the left of $\theta = 3.28$ and also 0.05 to the right of $\theta = 11.84$. So the interval $(3.28, 11.84)$ is a 90% posterior probability for θ. We call such an interval a credible interval.

- If a frequentist had found this interval, it means that it would say that if we repeatedly draw samples of data from the same population, and applied the rule that was used to derive this particular interval to each off those datasets, then 90% of those intervals would contain θ.

If a Bayesian approach, there is a posterior probability 0.9 that θ lies between 3.28 between 11.84.

Definition 3.9. A $100(1-\alpha)\%$ credible set for θ is a subset C such that:

$$1-\alpha \leq P(C \mid y) = \int_C p(\theta \mid y) \mathrm{d}\theta. \tag{3.10}$$

where integration is replaced by summation for discrete components.

Definition 3.11. The exact possible coverage of $(1-\alpha)$ can be found by the highest posterior density of HPD credible set as the set:

$$C = \{\theta \in \Theta : p(\theta \mid y) \geq k(\alpha)\}. \tag{3.12}$$

where $k(\alpha)$ is the largest constant satisfying $P(C \mid y) \geq 1-\alpha$.

For 2-sided credible set, we can generally take the $\alpha/2$ and $(1-\alpha/2)$ quantiles of $p(\theta \mid y)$ as our $100(1-\alpha)\%$ credible set for θ. This equal tail credible set will be equal to the HPD credible set if the posterior is symmetric and unimodal, but will be a bit wider otherwise.

(c) Evaluating hypothesis: Suppose we wish to test a hypothesis H which asserts that θ lies in some region A. The Bayesian way to test to the hypothesis is simply to calculate the (posterior) probability that it is true: $P(\theta \in A \mid x)$.

3.4. Multivariate inference

All the above treated θ as a scalar parameter. If we have a vector θ, then in general we can consider inference of the above forms about any scalar function $\phi = g(\theta)$. The inferences are then derived simply from the marginal posterior distribution of ϕ.

LECTURE 4
Formal Inference

Suppose that we want to answer a question that falls neatly into the frequentist point estimation framework, "What is the best estimate of θ?". In the frequentist theory, we need to be explicit about what we would regard as good properties for an estimator in order to identify a best one.

The Bayesian approach also needs to know what makes a good estimate before an answer can be given. This falls into the framework of formal inference.

Formally, we would seek the estimate that minimises expected square error. So the expectation is derived from the posterior distribution. We want to minimise:

$$\begin{aligned} E((\hat{\theta}-\theta)^2 \mid x) &= \hat{\theta}^2 - 2\hat{\theta}E(\theta \mid x) + E(\theta^2 \mid x) \\ &= (\hat{\theta} - E(\theta \mid x))^2 + v(\theta \mid x). \end{aligned} \tag{4.1}$$

So the estimate $\hat{\theta}$ that minimises this expected squared error is $\hat{\theta} = E(\theta \mid x)$.

4.1. Utility and decisions

Formal inference aims to obtain optimal answer to inference questions. This is done with reference to a measure of how good or bad the various possible inferences would be deemed to be if we knew the true value of θ. This measure is a utility function. Formally, $u(d, \theta)$ defines the value of inference d if the true value of the parameter is θ. Formal inference casts an inference problem as a decision problem. A decision problem is characterized by:

- a set Ω of possible parameter values
- a probability distribution for $\theta \in \Omega$
- a set D of possible decisions

- utility function $u(d,\theta)$ for $d \in D$ and $\theta \in \Omega$.

The solution is

$$\boxed{d_{opt} = arg \quad max_d E_\theta(u(d,\theta))} \tag{4.2}$$

Here the distribution of θ is its prior distribution.

- In inference problems, we generally define a measure of badness of an inference, which we call a loss function $L(d,\theta)$. We can simply define utility to be negative loss, and then the optimal inference is the one which minimises posterior expected loss.

4.1.1. Formal Point Estimation

The set D is the set of all possible values of θ. We have seen that if we use squared error loss (which is implicitly the measure used by frequentist in considering mean-squared-error of the variance of an unbiased estimator), formally defining $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$, then the posterior mean is the optimal estimator. If we use absolute error loss $L(\hat{\theta},\theta) = \left|\hat{\theta}-\theta\right|$, then the optimal estimator is the posterior median.

4.1.2. Formal Interval Estimation

The possible inferences now are interval or, more generally, subsets of possible values of θ. A loss function will penalize an interval if it fails to contain the true value of θ.

R Codes

Listing 1. Triplot Code in R

```
#####################################################
#                                                #
#      A Sample Triplot by Anil Aksu                #
#   It is developed to show some basics of R        #
#                                    #
#####################################################

## the range of sampling
x=seq(−4,4,length=101)
## this function gets numbers from console
prior=dnorm(x, mean = 0.5, sd = 0.7, log = FALSE)
likelihood=dnorm(x, mean = 0.49, sd = 0.65, log = FALSE)
posterior=dnorm(x, mean = 0.52, sd = 0.5, log = FALSE)


## let's plot them
plot(range(x), range(c(likelihood,prior,posterior)), ...
   type='n', xlab=expression(paste(theta)), ...
   ylab=expression(paste("f(", theta, " )")))
lines(x, prior, type='l', col='blue')
lines(x, likelihood, type='l', col='red')
lines(x, posterior, type='l', col='green')

title("Prior, Likelihood and Posterior Distribution")
legend(
  "topright",
  lty=c(1,1,1),
  col=c("blue", "red", "green"),
  legend = c("prior", "likelihood","posterior")
)
```

Listing 2. Inference Plots Code in R

```
#####################################################
```

```
#                                                  #
#      Posterior, Perspective and Contour plots     #
#                  by Anil Aksu                     #
#                                    #
####################################################

## the range of sampling
x=seq(0,20,length=101)
## this function gets numbers from console
posterior=dnorm(x, mean = 7, sd = 1.5, log = FALSE)


## let's plot them
plot(range(x), range(c(posterior)), type='n', ...
    xlab=expression(paste(theta)), ...
    ylab=expression(paste("f(", theta, "|x )")))

lines(x, posterior, type='l', col='blue',lwd=5)

title("Posterior Distribution")
  legend = c("posterior")

## perspective plot
x <- seq(-10, 10, length= 30)
y <- x
f <- function(x, y) { r <- sqrt(x^2+y^2); 10 * ...
    sin(r)/r }
z <- outer(x, y, f)
z[is.na(z)] <- 1
op <- par(bg = "white")
persp(x, y, z, theta = 30, phi = 30, expand = 0.5, ...
    col = "lightblue")
persp(x, y, z, theta = 30, phi = 30, expand = 0.5, ...
    col = "lightblue",
      ltheta = 120, shade = 0.75, ticktype = "detailed",
      xlab = "X", ylab = "Y", zlab = "Sinc( r )"
) -> res
round(res, 3)

# contour plot
a <- expand.grid(1:20, 1:20)
b <- matrix(a[,1]^2 + a[,2]^2, 20)
filled.contour(x = 1:20, y = 1:20, z = b,
               plot.axes = { axis(1); axis(2); ...
                   points(10, 10) })


## bivariate posterior sampling

## the range of sampling
x=seq(-4,6,length=101)
## this function gets numbers from console
posterior=0.8*dnorm(x, mean = 0, sd = 1, log = ...
    FALSE)+0.2*dnorm(x, mean = 4, sd = 1, log = FALSE)

## let's plot them
plot(range(x), range(c(posterior)), type='n', ...
    xlab=expression(paste(theta)), ...
    ylab=expression(paste("f(", theta, "|x )")))

lines(x, posterior, type='l', col='blue',lwd=5)

# title("Bivariate Posterior Distribution")
  legend = c("posterior")
```

BIBLIOGRAPHY

1. Allen B. Dawney. *Think Bayes: Bayesian Statistics in Python.* O'REILLY, 2013.
2. Sheldon Ross. *Introduction to Probability Models*. Academic Press, Boston, 2014.