HOMEWORK 3

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PROBLEM 1

Show that for an electric field of the form:

$$\vec{E}(\vec{x},\tau,t) = \vec{E}_0(\vec{x},\tau)\cos(\omega t - \vec{k}\vec{x}).$$

the magnetic field is given by

$$\vec{B}(\vec{x},\tau,t) = -\frac{1}{\omega}\left\{\left[\vec{\nabla}\times\vec{E}_0(\vec{x},\tau)\right]\sin(\omega t - \vec{k}\vec{x}) - \left[\vec{k}\times\vec{E}_0(\vec{x},\tau)\right]\cos(\omega t - \vec{k}\vec{x})\right\}.$$

Solution:

by Maxwell's equation the relation between the magnetic field $\vec{B}$ and the electric field $\vec{E}$ can be given as:

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}. \tag{1}$$

Therefore, if we replace the electric field given in problem to equation 1, the rate of change of the magnetic field can be obtained as:

$$\frac{\partial\vec{B}}{\partial t} = -\left[\vec{\nabla}\times\vec{E}_0(\vec{x},\tau)\right]\cos(\omega t - \vec{k}\vec{x}) - \left[\vec{k}\times\vec{E}_0(\vec{x},\tau)\right]\sin(\omega t - \vec{k}\vec{x}). \tag{2}$$

Note that here τ is assumed to be slowly varying time, therefore during the time integration over one wave period $2\pi/\omega$, it is assumed to be constant, therefore the magnetic field $\vec{B}$ can be given as:

(3)

$$\vec{B} = \int\frac{\partial\vec{B}}{\partial t}\mathrm{d}t = -\frac{1}{\omega}\left\{\left[\vec{\nabla}\times\vec{E}_0(\vec{x},\tau)\right]\sin(\omega t - \vec{k}\vec{x}) - \left[\vec{k}\times\vec{E}_0(\vec{x},\tau)\right]\cos(\omega t - \vec{k}\vec{x})\right\}.$$

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Problem 2

An electron of charge $q_e = -e$ and mass m_e and an proton of charge $q_e = e$ and mass $m_i = m_p$ are initially at rest at $\mathbf{x} = (0, 0, 0)$ in a magnetic field $\vec{B} = B_0\hat{z}$. An electric field is then turned on at $t = 0$ and increased linearly until $t_1 = \frac{20\pi m_i}{eB_0}$ at which point electric field held constant,

$$\vec{E} = \begin{cases} 0 & t < 0 \\ E_0(t/t_1)\hat{y} & 0 \leq t \leq t_1 \\ E_0\hat{y} & t \geq t_1 \end{cases}$$

Find the total current as a function of time due to drifts of two particles (neglect the current due to the fast Larmor oscillation).

Solution:

To derive any arbitrary drift motion due to the presence of a magnetic field $\mathbf{B}$, let's write the equation of motion:

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4}$$

Under these condition, Since we have magnetic field only in $\hat{z}$ direction, it only causes an acceleration in $\hat{x}$ and $\hat{y}$ directions, therefore the equations of motion can be given as:

$$\dot{v}_x = qv_yB_0/m, \tag{5}$$

$$\dot{v}_y = q(E/m - v_xB_0/m). \tag{6}$$

Equation 5 and equation 6 can be combined by taking derivative of equation 6 and replacing equation 6 into it. The resultant equation can be obtained as:

$$\ddot{v}_x + (\frac{qB_0}{m})^2 v_x = \frac{q^2B_0E}{m^2}. \tag{7}$$

Therefore, the solution can be given as a combination of homogeneous and particular solution:

$$v_x = v_x^h + v_x^p. \tag{8}$$

The homogeneous part is a combination of cosine and sine function with a frequency $\omega = qB_0/m$ and the non-homogeneous part is a polynomial in form of $at + b$, therefore the solution can be explicitly given as:

$$v_x = c_1 \cos\omega t + c_2 \sin\omega t + at + b. \tag{9}$$

for $0 < t \leq t_1$, the particular solution can be given as:

$$v_x^p = \frac{E_0}{B_0t_1}t. \tag{10}$$

If we replace it into equation 5, the velocity in y direction can be found as:

$$v_y^p = \frac{E_0}{B_0t_1\omega}. \tag{11}$$

After imposing initial conditions, it comes out that the total velocity field can be given as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{E_0}{B_0 t_1} t - \frac{E_0}{B_0 t_1 \omega} \sin \omega t \\ \frac{E_0}{B_0 t_1 \omega}(1 - \cos \omega t) \end{bmatrix}. \tag{12}$$

At $t = t_1$, the velocity field can be given as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{E_0}{B_0} - \frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \\ \frac{E_0}{B_0 t_1 \omega}(1 - \cos \omega t_1) \end{bmatrix}. \tag{13}$$

This can be directly used as an initial condition for $t > t_1$. Similar to the previous time interval, the total solution for $t > t_1$ can be given as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} c_1 \cos \omega t + c_2 \sin \omega t + \frac{E_0}{B_0} \\ -c_1 \sin \omega t + c_2 \cos \omega t \end{bmatrix}. \tag{14}$$

The velocities have to match at $t = t_1$, therefore:

$$\begin{bmatrix} \cos \omega t_1 & \sin \omega t_1 \\ -\sin \omega t_1 & \cos \omega t_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{E_0}{B_0} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \\ \frac{E_0}{B_0 t_1 \omega}(1 - \cos \omega t_1) \end{bmatrix} + \begin{bmatrix} \frac{E_0}{B_0} \\ 0 \end{bmatrix}. \tag{15}$$

After cancelling E_0/B_0 at both sides and inverting the matrix and multiplying it with the right hand side of the equation 15, constants c_1 and c_2 can be found as:

$$\begin{bmatrix} \cos \omega t_1 & -\sin \omega t_1 \\ \sin \omega t_1 & \cos \omega t_1 \end{bmatrix} \begin{bmatrix} -\frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \\ \frac{E_0}{B_0 t_1 \omega}(1 - \cos \omega t_1) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \\ \frac{E_0}{B_0 t_1 \omega}(\cos \omega t_1 - 1) \end{bmatrix}. \tag{16}$$

As a result, the velocity field after $t > t_1$ is obtained as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -\frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \cos \omega t + \frac{E_0}{B_0 t_1 \omega}(\cos \omega t_1 - 1) \sin \omega t + \frac{E_0}{B_0} \\ \frac{E_0}{B_0 t_1 \omega} \sin \omega t_1 \sin \omega t + \frac{E_0}{B_0 t_1 \omega}(\cos \omega t_1 - 1) \cos \omega t \end{bmatrix}. \tag{17}$$

The velocity field came out to be independent of the sign of the charge. The current is given as:

$$\mathbf{j} = n_e e(\vec{u}_e - \vec{u}_i). \tag{18}$$

Under this formulation, the only parameter different is the frequency due to the differences in mass which are m_e and m_i. Let's denote corresponding frequencies by ω_e and ω_i. Therefore, for $t < t_1$ the current can be given as:

$$\begin{bmatrix} j_x \\ j_y \end{bmatrix} = n_e e \begin{bmatrix} \frac{E_0}{B_0 t_1}\left(\frac{\sin \omega_i t}{\omega_i} - \frac{\sin \omega_e t}{\omega_e}\right) \\ \frac{E_0}{B_0 t_1}\left(\frac{(1-\cos \omega_e t)}{\omega_e} - \frac{(1-\cos \omega_i t)}{\omega_i}\right) \end{bmatrix}. \tag{19}$$

And also for $t > t_1$ the current can be given as:

$$\begin{bmatrix} j_x \\ j_y \end{bmatrix} = n_e e \begin{bmatrix} \frac{E_0}{B_0 t_1}\left(\frac{\sin \omega_i t_1 \cos \omega_i t}{\omega_i} - \frac{\sin \omega_e t_1 \cos \omega_e t}{\omega_e}\right) - \frac{E_0}{B_0 t_1}\left(\frac{\sin \omega_i t(\cos \omega_i t_1 - 1)}{\omega_i} - \frac{\sin \omega_e t(\cos \omega_e t_1 - 1)}{\omega_e}\right) \\ -\frac{E_0}{B_0 t_1}\left(\frac{\sin \omega_i t_1 \sin \omega_i t}{\omega_i} - \frac{\sin \omega_e t_1 \sin \omega_e t}{\omega_e}\right) - \frac{E_0}{B_0 t_1}\left(\frac{\cos \omega_i t(\cos \omega_i t_1 - 1)}{\omega_i} - \frac{\cos \omega_e t(\cos \omega_e t_1 - 1)}{\omega_e}\right) \end{bmatrix}. \tag{20}$$