HOMEWORK 1

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Problem 1

Maxwellian distribution is given as:

$$f_v(v) = \frac{1}{(2\pi)^{3/2} v_{th}^3} \exp(-\frac{v^2}{2v_{th}^2}).$$

Find

a. the average velocities of each component
Solution:
As known, the velocity vector has three components, for time being, let's stick to regular notation for the velocity $\vec{v} = (v_x, v_y, v_z)$. Maxwellian distribution given in the problem can be given explicitly in terms of these components as:

$$f_v(v_x, v_y, v_z) = \frac{1}{(2\pi)^{3/2} v_{th}^3} \exp(-\frac{v_x^2 + v_y^2 + v_z^2}{2v_{th}^2}). \tag{1}$$

To calculate the average velocity, first, let's find the number of particle in a unit volume as:

$$n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_v(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z. \tag{2}$$

Note that Maxwellian distribution is separable in terms of velocity components, therefore the integral 2 can be written as:

$$n = \frac{1}{(2\pi)^{3/2} v_{th}^3} \int_{-\infty}^{\infty} \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x \int_{-\infty}^{\infty} \exp(-\frac{v_y^2}{2v_{th}^2}) \mathrm{d}v_y \int_{-\infty}^{\infty} \exp(-\frac{v_z^2}{2v_{th}^2}) \mathrm{d}v_z. \tag{3}$$

In the integral 3, the result of each integral is same, therefore if we compute one, it will be enough to proceed. It is a well known integral but let's perform it explicitly anyway.

$$\int_{-\infty}^{\infty} \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x = v_{th} \int_{-\infty}^{\infty} \exp(-\frac{v^2}{2}) \mathrm{d}v = \sqrt{2\pi} v_{th}. \tag{4}$$

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After plugging the result into the integral 3, the number of particles in unit volume $n = 1$. It is unity, it facilitates the further operation. The average velocity in x direction can be found as:

(5)
$$< v_x >= \frac{1}{(2\pi)^{3/2} v_{th}^3} \int_{-\infty}^{\infty} v_x \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x \int_{-\infty}^{\infty} \exp(-\frac{v_y^2}{2v_{th}^2}) \mathrm{d}v_y \int_{-\infty}^{\infty} \exp(-\frac{v_z^2}{2v_{th}^2}) \mathrm{d}v_z.$$

Here is the result of the following integral:

$$\int_{-\infty}^{\infty} v_x \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x = -2v_{th}^2 \exp(-\frac{v_x^2}{2v_{th}^2}) \Big|_{-\infty}^{\infty} = 0. \tag{6}$$

Therefore, $< v_x >= 0$ Similarly the mean velocity in y direction is given as:

(7)
$$< v_y >= \frac{1}{(2\pi)^{3/2} v_{th}^3} \int_{-\infty}^{\infty} \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x \int_{-\infty}^{\infty} v_y \exp(-\frac{v_y^2}{2v_{th}^2}) \mathrm{d}v_y \int_{-\infty}^{\infty} \exp(-\frac{v_z^2}{2v_{th}^2}) \mathrm{d}v_z.$$

The result of the integral above is also same, as a result, $< v_y >= 0$. Finally, the mean velocity in z direction is defined as:

(8)
$$< v_z >= \frac{1}{(2\pi)^{3/2} v_{th}^3} \int_{-\infty}^{\infty} \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x \int_{-\infty}^{\infty} \exp(-\frac{v_y^2}{2v_{th}^2}) \mathrm{d}v_y \int_{-\infty}^{\infty} v_z \exp(-\frac{v_z^2}{2v_{th}^2}) \mathrm{d}v_z.$$

The integral 8 also vanishes so $< v_z >= 0$.

b. the average kinetic energy

Solution:

The average kinetic energy is defined as:

(9)
$$\frac{1}{2} n (< v_x^2 > + < v_y^2 > + < v_z^2 >) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v_x^2 + v_y^2 + v_z^2) f_v(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z.$$

Since the distribution function is isotropic meaning that it is independent of the direction, the average kinetic energy can also be given as:

$$\frac{3}{2} n < v_x^2 >= \frac{3}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x^2 f_v(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z. \tag{10}$$

The integral 10 can be explicitly given as:

(11)
$$\frac{3}{2} n < v_x^2 >= \frac{3}{2(2\pi)^{3/2} v_{th}^3} \int_{-\infty}^{\infty} v_x^2 \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x \int_{-\infty}^{\infty} \exp(-\frac{v_y^2}{2v_{th}^2}) \mathrm{d}v_y \int_{-\infty}^{\infty} \exp(-\frac{v_z^2}{2v_{th}^2}) \mathrm{d}v_z.$$

The only result was not obtained previously in the integral 13 is :

(12)
$$\int_{-\infty}^{\infty} v_x^2 \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x = -v_{th}^2 v_x \exp(-\frac{v_x^2}{2v_{th}^2}) \Big|_{-\infty}^{\infty} + v_{th}^2 \int_{-\infty}^{\infty} \exp(-\frac{v_x^2}{2v_{th}^2}) \mathrm{d}v_x = \sqrt{2\pi} v_{th}^3.$$

After replacing this and the previous results, the average kinetic energy is obtained as:

$$\frac{3}{2}n < v_x^2 >= \frac{3}{2}v_{th}^2. \tag{13}$$

c. the average velocity $< |v| >$
Solution:
The average velocity is defined as:

$$< |v| >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{v_x^2+v_y^2+v_z^2}f_v(v_x,v_y,v_z)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z. \tag{14}$$

The easiest way to take this integral is to write it in spherical coordinates as follows:

$$< |v| >= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v f_v(v) v^2 \sin\phi \mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta. \tag{15}$$

The integral in v direction can be taken separately as:
(16)
$$\int_0^{\infty} v^3 \exp(-\frac{v^2}{2v_{th}^2})\mathrm{d}v = -v_{th}^2 v^2 \exp(-\frac{v^2}{2v_{th}^2})\Big|_0^{\infty} + 2v_{th}^2\int_0^{\infty} v\exp(-\frac{v^2}{2v_{th}^2})\mathrm{d}v_x = 2v_{th}^4.$$

After replacing this result into equation 15, it is found that

$$< |v| >= \frac{8\pi v_{th}^4}{(2\pi)^{3/2}v_{th}^3} = \sqrt{\frac{8}{\pi}}v_{th}. \tag{17}$$

d. the flux of charged particles $\Gamma_{p,x} = \frac{1}{2}n_0 < |v_x| >$ along x direction
Solution:

(18)
$$< |v_x| >= \frac{1}{(2\pi)^{3/2}v_{th}^3}\int_{-\infty}^{\infty}|v_x|\exp(-\frac{v_x^2}{2v_{th}^2})\mathrm{d}v_x\int_{-\infty}^{\infty}\exp(-\frac{v_y^2}{2v_{th}^2})\mathrm{d}v_y\int_{-\infty}^{\infty}\exp(-\frac{v_z^2}{2v_{th}^2})\mathrm{d}v_z.$$

Here is the result of the following integral:
(19)
$$\int_{-\infty}^{\infty}|v_x|\exp(-\frac{v_x^2}{2v_{th}^2})\mathrm{d}v_x = 2\int_0^{\infty} v_x\exp(-\frac{v_x^2}{2v_{th}^2})\mathrm{d}v_x = -4v_{th}^2\exp(-\frac{v_x^2}{2v_{th}^2})\Big|_0^{\infty} = 4v_{th}^2.$$

Therefore, the flux is obtained as:

$$\Gamma_{p,x} = \sqrt{\frac{2}{\pi}}v_{th}. \tag{20}$$

Problem 2

Find the plasma parameters λ_D, $\wedge$, ω_{pe}, ω_{pi}, ω_{ci}, ω_{ce} for the ionized gas given in table 1.

Definitions:
Debye length $\lambda_D = \sqrt{\epsilon_0 T_e / n_e e^2}$
Number of particles in Debye sphere$\wedge = n\lambda_D^3$
electron plasma frequency $\omega_{pe} = \sqrt{4\pi n_e e^2 / m_e}$
ion plasma frequency $\omega_{pi} = \sqrt{4\pi n_i Z^2 e^2 / m_i}$
electron gyrofrequency $\omega_{ce} = eB/m_e c$
ion gyrofrequency $\omega_{ci} = ZeB/m_i c$

Note that $e = 1.602 \times 10^{-19} C$, $m_e = 9.109^{-31} kg$, $\epsilon_0 = 8.85^{-12} Cm^{-1}V^{-1}$ and $c = 3.00 \times 10^8 ms^{-1}$.

Plasma criteria are given as:

- the length scale $L >> \lambda_D$
- Number of particles in Debye sphere$\wedge >> 1$
- $\omega_{pi}\tau > 1$

Under these conditions, only Interstellar Gas and Solar Corona satisfy these criteria. Indeed, I could not check the last criteria as I don't know how to calculate the average time between collisions τ.

Table 1. Plasma Parameters of Some Ionized Gases

Ionized Gas	λ_D (m)	$\wedge$	ω_{pe}	ω_{pi}	ω_{ce}	ω_{ci}
Interstellar Gas	1.85×10^{10}	6.4×10^{36}	0.59		5.9×10^{-9}	
Solar Wind	1.85×10^{10}	6.4×10^{37}	1.88		5.9×10^{-7}	
Val ellen Belts	5.8×10^{9}	2.02×10^{38}	18.8		5.9×10^{-5}	
Earth's Ionosphere	1.85×10^{7}	6.4×10^{33}	188.1		1.7×10^{-3}	
Solar Corona	5.87×10^{7}	2.02×10^{36}	1881.6		5.8×10^{-8}	
Gas Discharge	2.6×10^{4}	2.02×10^{36}	5.9×10^{5}		0	
Process Plasmas	1.85×10^{5}	1.81×10^{32}	5.9×10^{5}		58.6	
Fusion Experiments	1.85×10^{5}	6.4×10^{35}	5.9×10^{6}		2.9×10^{3}	
Fusion Reactor	1.85×10^{5}	6.4×10^{35}	5.9×10^{5}		2.9×10^{3}	

```
% created by Anil Aksu to calculate plasma parameters of ...
    some ionized gases

clear all
format long

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                         %
%       Plasma related constants          %
%                                         %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% the speed of light
c=3.*10^8;
% the charge of electron
e=1.602*10^-19;
% the mass of electron
m=9.109*10^-31;
% permitivity of vacuum
eps=8.85*10^-12;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                         %
%       Ionized Medium Parameters         %
%                                         %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%  let's read them from excel document
Plasma = xlsread('PlasmaParameters.xlsx');

% the number of ionized gas
Nion=length(Plasma(:,1));
% the debye length array
Debye=zeros(Nion,1);
% the number of particles in debye sphere
webge=zeros(Nion,1);
% the electron plasma frequency
```

```
omega_pe=zeros(Nion,1);
% the electron gyro frequency
omega_ce=zeros(Nion,1);

% let's calculate them
for i=1:Nion
    % the debye length
    [ Debye(i,1) ] = getDebyeLength( ...
        eps,Plasma(i,3),Plasma(i,2),e );
    % the number of particles in debye sphere
    wedge(i,1)=Plasma(i,2)*Debye(i,1)^3.;
    % the plasma electron frequency
    [ omega_pe(i,1) ] = getPlasmaFrequency( ...
        eps,Plasma(i,2),e,m );
    % the plasma gyro frequency
    [ omega_ce(i,1) ] = getGyroFrequency(e,Plasma(i,4),m,c );
end
```

```
function [ l_d ] = getDebyeLength( eps,T,n,e )
%this function calculates the debye length under given ...
    parameters
l_d=sqrt(eps*T/(n*(e^2)));

end
```

```
function  [ omega_pe ] = getPlasmaFrequency( eps,n,e,m )
%this function calculates the plasma frequency under given ...
    parameters
omega_pe=sqrt(4.*pi*n*(e^2)/m);

end
```

```
function  [ omega_ce ] = getGyroFrequency(e,B,m,c )
%this function calculates the gyro frequency under given ...
    parameters
omega_ce=e*B/(m*c);

end
```